Handling uncertainty in higher dimensions with potential clouds towards robust design optimization

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Abstract

Robust design optimization methods applied to real life problems face some major difficulties: how to deal with the estimation of probability densities when data are sparse, how to cope with high dimensional problems and how to use valuable information in the form of unformalized expert knowledge. We introduce the *clouds* formalism as means to process available uncertainty information reliably, even if limited in amount and possibly lacking a formal description. We provide a worst-case analysis with confidence regions of relevant scenarios which can be involved in an optimization problem formulation for robust design.

Keywords. clouds, uncertainty modeling, confidence regions, robust design, design optimization

1 Background

Robust design optimization is the art of safeguarding reliably against uncertain perturbations while seeking an optimal design point. In every design process an engineer faces the task to qualify the object he has designed to be robust. That means the design should not only satisfy given requirements on functionalities, but should also work under uncertain, adverse conditions that may show up during employment of the designed object.

Hence the process of robust design optimization demands both the search of an optimal design with respect to a given design objective, and an appropriate method of handling uncertainties. In particular for early design phases, it is frequent engineering practice to assign and refine intervals or safety margins to the uncertain variables. These intervals or safety margins are propagated within the whole optimization process. Thus the design arising from this process is supposed to include robustness intrinsically. Note that the assessment of robustness is exclusively based on expert knowledge of the engineers who assign and refine the intervals. There is no quantification of reliability, no rigorous worst-case analysis involved.

Several methods exist to approach reliability quantification from a rigorous mathematical background, originating from classical probability theory, statistics, fuzzy theory or based on simulation techniques. However, real life applications of many methods disclose various problems. One of the most prominent is probably the fact that the dimension of many uncertain real life scenarios is very high. This can cause severe computational effort, also famous as the curse of dimensionality. Namely, even given the complete knowledge about the multivariate probability distributions of the uncertainties, the numerical computation of the error probabilities requires high dimensional integration and becomes very expensive. Moreover, if the amount of available uncertainty information is very limited, well-known current methods either do not apply at all, or are endangered to critically underestimate error probabilities. Also a simplification of the uncertainty model, e.g., a reduction of the problem to an interval analysis after assigning intervals to the uncertainties as described before (e.g., so called 3 σ boxes), entails a loss of valuable uncertainty information which would actually be available, maybe only unformalized, but not at all considered in the uncertainty model.

In the literature several discussions on the introduced problems and approaches to their solution can be found. Different approaches to design optimization can be studied, e.g., in [1]. A criticism on simulation techniques in case of limited uncertainty information can be found in [4], showing that the lack of information typically causes these techniques to underestimate the effects of the uncertain tails of the probability distribution. Uncertainty handling with fuzzy theory for engineering applications is investigated, e.g., in [16], simulation techniques, e.g., in [12]. Some works go into the arithmetic operations on random variables to bound the probability distributions of functions of random variables with p-boxes, e.g., [5], [17]. There are also attempts to generalize aspects of the different uncertainty approaches and put them into the framework of random sets, cf. [11]. As mentioned many methods are limited to low dimensional problems. Problems that come with the curse of dimensionality are described, e.g., in [9].

2 Introducing the new approach

Our work deals with a new approach based on the *clouds* formalism [14]. Clouds can process limited amounts of stochastic information in an understandable and computationally attractive way, even in higher dimensions, in order to perform a reliable worst-case analysis, reasonably safeguarded against perturbations that result from unmodeled or unavailable information. Since the strength of our new methodology lies especially in the application to real life problems with a very limited amount of uncertainty information available, we focus in particular on problem statements arising in early design phases where today's methods of handling the limited information are very immature. On the one hand, the information is usually available as bounds or marginal probability distributions on the uncertain variables, without any formal correlation information. On the other hand, unformalized expert knowledge will be captured to improve the uncertainty model adaptively by adding correlation constraints to exclude scenarios deemed irrelevant. The information can also be provided as real sample data, if available.

If we have a look at Figure 1, we see confidence levels on some two dimensional random variable ε. The curves displayed can be considered to be level sets of a function $V(\varepsilon) : \mathbb{R}^2 \to \mathbb{R}$, called the potential. The potential characterizes confidence regions $C_\alpha := \{\varepsilon \in \mathbb{R}^2 \mid V(\varepsilon) \leq V_\alpha\}$, where V_α is determined by the condition $\Pr(\varepsilon \in C_\alpha) = \alpha$. If the probability information

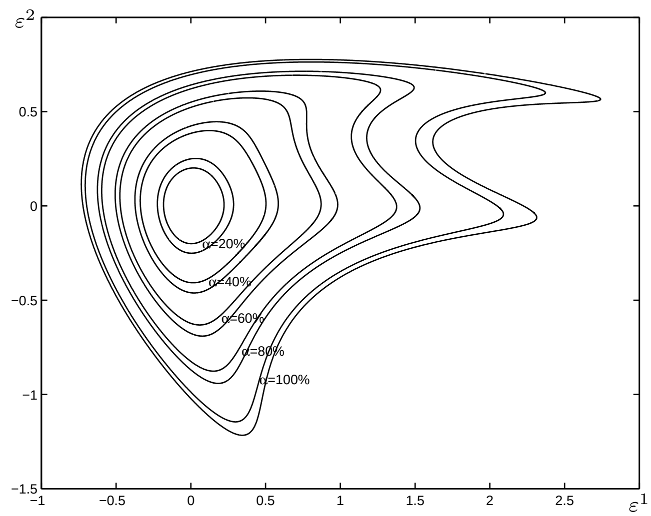

Figure 1: Nested confidence regions in two dimensions for confidence levels $\alpha = 0.2, 0.4, 0.6, 0.8, 1$.

is not precisely known nested regions are generated

$$\underline{C}_\alpha := \{\varepsilon \in \mathbb{R}^2 \mid V(\varepsilon) \leq \underline{V}_\alpha\},$$

where $\underline{V}_\alpha$ is largest such that $\Pr(\varepsilon \in \underline{C}_\alpha) \leq \alpha$, and

$$\overline{C}_\alpha := \{\varepsilon \in \mathbb{R}^2 \mid V(\varepsilon) \leq \overline{V}_\alpha\},$$

where $\overline{V}_\alpha$ is smallest such that $\Pr(\varepsilon \in \overline{C}_\alpha) \geq \alpha$. The information in $\underline{C}_\alpha$ and $\overline{C}_\alpha$ is called a *potential cloud*. The V_α, $\underline{V}_\alpha$, $\overline{V}_\alpha$ can be found from the cumulative distribution function (CDF) of $V(\varepsilon)$ and lower and upper bounds of it. These bounds in turn can be determined empirically using the Kolmogoroff-Smirnov (KS) distribution [10].

2.1 Potential cloud generation

We assume that the uncertainty information consists of given samples, boxes, non-formalized correlation bounds or continuous marginal CDFs F_i, $i \in I \subseteq \{1, 2, \ldots, n\}$, on the n-dimensional vector of uncertainties ε, without any formal knowledge about correlations or joint distributions. In case there is no sample provided or the given sample is very small, a sample S has to be generated. For these cases we first use a Latin Hypercube Sampling (LHS, cf. [13]) inspired method to generate S, i.e., the sample points $x_1, x_2, \dots, x_{N_S}$ are chosen from a grid satisfying $x_i^j \neq x_k^j \ \forall i, k \in \{1, 2, \dots, N_S\}, k \neq i, \forall j \in \{1, 2, \dots, n\}$, where x_i^j is the projection of x_i to the j^{th} coordinate. If only boxes for ε are given, then the grid is equidistant, if marginal distributions are given the grid is transformed with respect to them to ensure that each grid interval has the same marginal probability. Thus the generated sample represents the marginal distributions. However after a modification of S, e.g., by cutting off sample points as we will do later, an assignment of weights to the sample points is necessary to preserve the marginal CDFs. In order to do so the weights $\omega_1, \omega_2, \dots, \omega_{N_S} \in [0, 1]$, corresponding to the sample points $x_1, x_2, \dots, x_{N_S}$, are required to satisfy the following conditions (1)

$$\sum_{j=1}^{k} \omega_{\pi_i(j)} \in [F_i(x^i_{\pi_i(k)}) - d, F_i(x^i_{\pi_i(k)}) + d], \qquad \sum_{k=1}^{N_S} \omega_k = 1. \tag{1}$$

for all $i \in I, k = 1, \dots, N_S$, where π_j is a sorting permutation of $\{1, \dots, N_S\}$, such that $x^j_{\pi_k(1)} \leq x^j_{\pi_k(2)} \leq \dots \leq x^j_{\pi_k(N_S)}$, and I the index set of those entries of the uncertainty vector ε where a marginal CDF $F_i, i \in I$ is given. The constraints (1) require the weights to represent the marginal CDFs with some reasonable margin d. In practice, one chooses d with KS statistics.

We determine bounds on the CDF of $V(\varepsilon)$ by $\overline{F} := \min(\widetilde{F} + D, 1)$ and $\underline{F} := \max(\widetilde{F} - D, 0)$, where $\widetilde{F}(\xi) := \sum_{\{j | V(x_j) \leq \xi\}} \omega_j$ the weighted empirical distribution for $V(\varepsilon)$, and D is again chosen with KS statistics. Finally we fit the two step functions $\underline{F}$, $\overline{F}$ to smooth, monotone lower bounds $\underline{\alpha}$ and upper bounds $\overline{\alpha}$.

2.2 Choice of the potential

We see that given a potential the corresponding potential cloud is easy to estimate, even for high dimensional data, and its interpretation is unambiguous in spite of the uncertainty about the full multidimensional probability distribution. The choice of the potential is dictated by the shape of the points set defined by the sample of available ε. We are looking for a way to find a good choice of V that gives the possibility to improve the potential iteratively and allows for a simple computational realization of the confidence regions, e.g., by linear constraints. This leads us to the investigation of polyhedron-shaped potentials. A polyhedron potential centered at $m \in \mathbb{R}^n$ can be defined as:

$$V_p(\varepsilon) := \max_k \frac{(A(\varepsilon - m))^k}{b^k}, \tag{2}$$

where $\varepsilon, b \in \mathbb{R}^n$, $A \in \mathbb{R}^{n \times n}$, $(A(\varepsilon - m))^k, b^k$ the k^{th} component of the vectors $A(\varepsilon - m)$ and b, respectively.

But how to achieve a polyhedron that reflects the given uncertainty information in the best way? As mentioned we assume the uncertainty information to consist of given samples, boxes or marginal distributions, and unformalized correlation constraints. After providing a sample S as described in Section 2.1 we define a box b_0 containing 100% of the sample points, and we define our potential $V_0(\varepsilon)$ box-shaped taking the value 1 on the margin of b_0. Based on expert knowledge, a user-defined variation of V_0 can be performed afterwards by cutting off sample points deemed irrelevant for the worst-case, cf. Figure 2. Thus an expert can specify the uncertainty information in the form of correlation bounds adaptively, even if the expert knowledge is only little formalized, resulting in a polyhedron shaped potential.

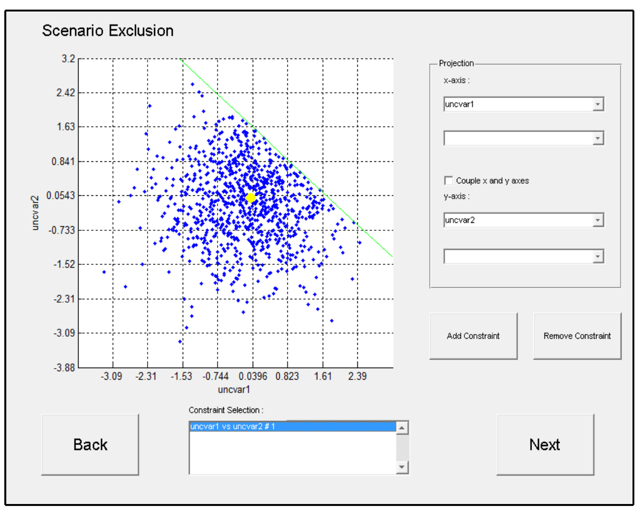

Figure 2: Graphical user interface for an interactive scenario exclusion.

Assume the linear constraints $A(\varepsilon - \mu) \leq b$ represent the exclusion of sample points and the box constraint from b_0, we define our polyhedron shaped potential as in (2) with $m = \mu$.

This potential, originating from a box potential, is suitable for symmetric samples, but: If some uncertain variables are described by asymmetric marginal probability densities, a better choice V_t of the potential could be achieved by an appropriate coordinate transformation ψ, i.e., $V_t(\varepsilon) := V_p(\psi(\varepsilon))$. An appropriate transformation would be, e.g., a logarithmic transformation of ε^i if $F_i : \mathbb{R}^+ \to [0, 1]$.

2.3 Robust design optimization problem

In terms of robust design the regions $\underline{C}_\alpha$ and $\overline{C}_\alpha$ yield safety constraints, as a design is called safe if all $\varepsilon \in \overline{C}_\alpha$ satisfy the design requirements, and it is called unsafe if one $\varepsilon \in \underline{C}_\alpha$ fails to satisfy these requirements. These safety constraints can easily be incorporated in an optimization problem formulation:

Provided an underlying model of a given structure to be designed, with several inputs and outputs, we denote as x the vector containing all output variables, and as z the vector containing all input variables. Let θ be a design point, i.e., it fully defines the design. Let T be the set of all possible designs. The input variables z consist of design variables which depend on the design θ, e.g., the thrust of a thruster, and external inputs with a nominal value that cannot be controlled for the underlying model, e.g., a specific temperature.

The input variables are affected by uncertainties. Let ε denote the related vector of uncertain errors. One can formulate the optimization problem as a mixed-integer, bi-level problem of the following form:

$$
\begin{array}{llr}
\min\limits_{\theta} \ \max\limits_{x,z,\varepsilon} & g(x) & \text{(objective functions)} \\
\text{s.t.} & G(x,z) = 0 & \text{(functional constraints)} \\
& z = Z(\theta) + \varepsilon & \text{(input constraints)} \\
& \theta \in T & \text{(selection constraints)} \\
& V_t(\varepsilon) \leq \underline{V}_\alpha & \text{(cloud constraint)}
\end{array}
\tag{3}
$$

where the design objective $g(x)$ is a function of the output variables of the underlying model. The functional constraints express the functional relationships defined in the underlying model. The input constraints assign to each design θ a vector z of input variables whose value is the nominal entry from $Z(\theta)$ plus its error ε with uncertainty specified by the cloud. The selection constraints specify which design points are allowed for θ. The cloud constraint involves the potential function $V = V_t(\varepsilon)$ as described in the Section 2.2 and models the worst-case relevant region $\underline{C}_\alpha$.

3 Summary

We present a new methodology based on clouds to provide confidence regions for safety constraints in robust design optimization. We can process the uncertainty information from expert knowledge towards a reliable worst-case analysis, even if the information is limited in amount and high dimensional. May the information be formalized or not, we do not lose valuable unformalized information.

A first study on the application of the potential clouds theory in design optimization occurs in [15]. This study presented an initial step on what clouds are capable of, applied to the case of uncertainty handling in spacecraft design. A significant further step is given in [6] showing that clouds can be successfully applied to real life problems. For a detailed view on the theoretical basis of uncertainty handling with potential clouds see [7].

The adaptive nature of our uncertainty model, i.e., the possibility of manually adding correlation bounds, is one of the key features. The iteration steps significantly improve the uncertainty information and we are able to process the new information to an improved uncertainty model.

All in all, the presented approach offers an attractive novel point of view on uncertainty handling and its involvement to robust design optimization.

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