Computer graphics, linear interpolation, and nonstandard intervals

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Abstract. This document is an assessment of the value of optimal linear interpolation enclosures and of nonstandard intervals, especially with respect to applications in computer graphics, and of the extent a future IEEE interval standard should support these.

It turns out that essentially all present applications of nonstandard intervals to practical problems can be matched by similarly efficient approaches based on standard intervals only. On the other hand, a number of applications were inspired by the use of nonstandard arithmetic.

This suggests the requirement of a minimal support for nonstandard intervals, allowing implementations of nonstandard interval arithmetic to be compatible with the standard, while a full support by making one of the conflicting variants required seems not appropriate.

Keywords: Bezier curves, de Boor algorithm, de Casteljau algorithm, interval arithmetic, Kaucher arithmetic, linear interpolation, modal arithmetic, monotonicity, nonstandard intervals, NURBS, optimal enclosure, overestimation, patents, range enclosure, ray tracing

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Contents

1 Introduction

This document is an assessment of the value of optimal linear interpolation enclosures and of nonstandard intervals, especially with respect to applications in computer graphics, and of the extent a future IEEE interval standard should support these.

Its purpose is to justify the decisions proposed in the Vienna proposal on interval standardization [54] regarding linear interpolation and nonstandard interval support. The proposal requires the availability of nonstandard intervals with which one can implement nonstandard interval arithmetic without violating the standard, but gives otherwise no support for nonstandard arithmetic. On the other hand, it provides optional support for optimal linear interpolation.

In the following, we use the notation as proposed in KEARFOTT et al. [37]. In particular, boldface letters such as $\mathbf{x}$ denote (standard or nonstandard) intervals or interval vectors; $\underline{x}$ and $\overline{x}$ denote their lower and upper bound, respectively. A **nonstandard interval** is a pair $\mathbf{x} = [\underline{x}, \overline{x}]$ of numbers where – unlike in standard intervals – the inequality $\underline{x} \leq \overline{x}$ is violated. Nonstandard intervals are currently used in three flavors, corresponding to Kahan arithmetic, Kaucher arithmetic, and modal arithmetic. The latter two variants are identical in some but not all respects.

The conclusion of the present analysis is that there are many signs of the lack of maturity of nonstandard intervals:

- For Kaucher intervals, there are conflicting definitions for binary operations that are not commutable and no definitions for division by intervals containing zero; for Kahan intervals, the definitions are completely different.
- the available results are somewhat poorly organized and difficult to understand;
- the current expositions of modal theory lack sufficient rigor to merit being trusted without detailed checking;
- their theory is little known in the interval community;
- as a result, there is not sufficiently widespread experience of its use;
- apart from some applications in linear algebra, the practical use of nonstandard intervals is still in an experimental stage;
- there is no evidence of usage on real-life problems beyond what can also be achieved without nonstandard intervals.

Many (and perhaps all) publicly available applications of interval analysis using nonstandard intervals (and in particular, all applications to computer graphics) can be handled with similar effficiency and more flexibility by interval analysis using ordinary intervals only. This is made explicit for the applications in computer graphics.

On the other hand, nonstandard intervals have some interesting applications, and were repeatedly the motivating context of algorithmic ideas. Therefore they should be compatible with any standard for interval arithmetic.

A stronger support is not appropriate in view of the above weaknesses together with the additional complexity of implementing nonstandard intervals and the resulting slowdown in software implementations, even for ordinary interval operations.

This does not exclude the possibility that, some time in the future, nonstandard intervals might occupy a more prominent place within interval analysis.

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I invited several influential people working with nonstandard intervals to give statements that could be quoted in this paper. I obtained such statements from Svetoslav Markov, Miguel Sainz, Sergey Shary, Alexandre Goldsztejn, and Luc Jaulin; they are appended at the end of the paper.

2 Interval techniques in computer graphics

In the recent patent [26] with the title
– *System and Method to Compute Narrow Bounds on a Modal Interval Polynomial Function*,
Nathan Hayes (cf. [23]) introduced a modal formula which produces an optimal range enclosure for linear interpolation. This allows one to compute fairly realistic interval enclosures (see Section 3 below) for Bezier curves and NURBS (nonuniform rational B-splines), two basic data structures in computational geometry. The quality of the enclosures looks impressive.

Nathan Hayes is the president of the company Sunfish Studio (cf. [91]) that specializes in surface rendering software for high resolution computer graphics. He holds several patents [30] relating to interval analysis: A patent on interval ray tracing technology [27] with the title

– *System and method of visible surface determination in computer graphics using interval analysis,*
four patents on modal arithmetic with the titles
– *System and Method to Compute Narrow Bounds on a Modal Interval Spherical Projection [29]*
– *Representation of Modal Intervals within a Computer [24]*
– *Reliable and Efficient Computation of Modal Interval Arithmetic Operations [25]*
– *System and Method to Compute Narrow Bounds on a Modal Interval Polynomial Function* (quoted above),
and one apparently still pending patent application with the title
– *Modal Interval Processor* [28].

There are several other patents or patent applications related to interval ray tracing. Apart from Hayes' patent mentioned above, I know of a patent by Piponi [60], with the title
– Reverse-rendering method for digital modeling,
and by Reshetov et al. [73, 74] a patent with the title
– Applications of interval arithmetic for reduction of number of computations in ray tracing problems.
and a patent application with the title
– Method and apparatus for multi-level ray tracing.

Another patent, by Gavriliu & Barr [15], with the title
– *Efficient method of identifying non-solution or non-optimal regions of the domain of a function*
concerns a general branch and bound method using Taylor models; the text mentions the potential use in computer graphics. Other patents related to branch and bound with intervals are by Ruetsch [75, 76],
– Method and apparatus for initializing interval computations through subdomain sampling,
– Using interval techniques to solve a parametric multi-objective optimization problem.

Surface rendering by interval methods, often called interval ray-tracing, is a branch-and-bound technique that recursively treats whole beams of infinitely many rays instead of sampling many individual rays as in standard ray-tracing methods for computer graphics. The search terms *interval arithmetic ray-tracing* give in scholar.google.com about 1380 hits to scientific papers mentioning this subject, showing the potential relevance of interval methods in this field.

An early reference to a complete interval ray-tracing algorithm for realistic computer graphics is Enger [4]; Wolfgang Enger was one of my Ph.D. students, and this paper is the summary of his thesis. Another, recent thesis by Flores [6] (not yet online) on interval ray-tracing of implicit surfaces uses modal arithmetic. Work related to this thesis (using ordinary intervals only) is in [7, 8, 9, 10]. For other recent work on interval ray-tracing, see, e.g., Hu et al. [33], Wang & Nnaji [98], and the references quoted there.

3 NURBS, Bezier curves, and linear interpolation

NURBS and Bezier curves are basic building blocks in computer-aided geometric design (CAGD). Useful background references include Piegl & Tiller [59], Prautzsch et al. [71], Seidel [82], and Sederberg [81]. In particular, Seidel discusses the de Boor algorithm and knot insertion in terms of multiaffine functions, and Sederberg treats interval Bezier curves, i.e., families of Bezier-curves defined by interval-valued control points in Chapter 12.

NURBS (the standard acronym for nonuniform rational B-splines) are projective versions of curves whose projective coordinates are defined by piecewise polynomial curves called splines. In CAGD, d-dimensional curves defined by splines are represented by a linearly ordered sequence of real numbers called knots and corresponding control points, which are vectors in $\mathbb{R}^d$. NURBS with weights $w_j > 0$ and control points $P_j \in \mathbb{R}^d$ have piecewise polynomial projective coordinates $C_j = \binom{w_j}{w_jP_j} \in \mathbb{R}^{d+1}$, which represent the control points of a spline with $d+1$ components. NURBS allow one to represent conic sections exactly, and give a projectively covariant representation of curves. NURBS surfaces are 2-dimensional versions of NURBS; their evaluation can be reduced to the recursive evaluation of NURBS. Rational Bezier curves and rational Bezier surfaces are the special cases of NURBS and NURBS surfaces in which the splines reduce to ordinary polynomials.

For d-component splines of degree n, with $2n-1+s$ knots $u_i \in \mathbb{R}$ $(i = 1-n : n-1+s)$ ordered in a increasing fashion and $n+s$ control points $C_j \in \mathbb{R}^d$ $(j = 1 : n+s)$, the **de Boor algorithm** defines the function value $f = f(x)$ at the point $x \in [u_{i-1}, u_i]$ for $i = 1 : s$ as follows (in a Matlab-like style):

$$
\begin{array}{l}
\text{for } j = i : i+n,\ c_j = C_j;\ \text{end;}\\
\text{for } k = 1 : n,\\
\quad l = n+1-k; \quad \text{for } j = i : i+n-k,\\
\qquad t = (x - u_{j-l})/(u_j - u_{j-l});\\
\qquad c_j = (1-t)c_j + tc_{j+1};\\
\quad \text{end;}\\
\quad B_k = c_{i+n-k};\\
\text{end;}\\
f = c_i;
\end{array}
$$

This defines $f(x)$ for all $x \in [u_0, u_s]$. Clearly, $f(x)$ is in each interval $[u_{i-1}, u_i]$ a polynomial of degree at most n. $f(x)$ agrees for $x \in [u_{i-1}, u_i]$ with the polynomial $f(x, \ldots, x)$ derived from the multiaffine function $f(x_1, \ldots, x_n)$ uniquely determined by the interpolation conditions

$$f(u_{j-1}, \ldots, u_{j-n}) = C_j \quad \text{for } j = i : i+n.$$

The de Boor algorithm may be viewed as an n-fold insertion of a knot at x between u_{i-1} and u_i. This view implies that it can be used to restrict the spline to the interval $[u_0, x]$ by using the knots u_l' and control points C_l' with

$$u_l' = \begin{cases} u_l & \text{for } l = 1-n : i-1, \\ x & \text{for } l = i : i+n-1, \end{cases} \quad C_l' = \begin{cases} C_l & \text{for } l = 1 : i, \\ B_{l-i} & \text{for } l = i+1 : i+n, \end{cases}$$

and to the interval $[x, u_s]$ by using the knots u_l'' and control points C_l'' with

$$u_l'' = \begin{cases} x & \text{for } = 1-n : 0, \\ u_{l+i} & \text{for } l = 1 : n-1+s-i, \end{cases} \quad C_l'' = \begin{cases} B_{n+1-l} & \text{for } l = 1 : n, \\ C_{l+i} & \text{for } l = n+1 : n+s-i. \end{cases}$$

This makes it ideal for use in branch and bound methods.

The special case where $s = 1$ and $u_j = 0$ for $j \leq 0$, $u_j = 1$ for $j > 0$ gives the **de Casteljau algorithm** for evaluating points on **Bezier curves**, i.e., the images of [0,1] under vector-valued polynomials written in a basis of Bernstein polynomials,

$$B(x) = \sum_{i=0}^{n} \binom{n}{i} (1-x)^{n-i} x^i C_{i+1}.$$

Interval version. In general, given an arithmetic expression, one can simply evaluate it by substituting intervals for each variable. This is called a **naive interval evaluation**, and is guaranteed to result in an interval containing the range. In many cases, naive interval evaluation gives quite poor results; i.e., the interval obtained may be much wider than the true range.

But there are many ways to massage an expression before or after applying interval arithmetic, and in the hands of experts (or of software programmed by experts) this enables one to get better enclosures, and sometimes even the exact range. An exposition of many of these techniques is given in the companion paper on improving interval enclosures (Neumaier [53]); see also Section 7 below.

We discuss here how monotonicity analysis improves the quality of bounds for the special case of the linear interpolation expression

$$\texttt{linearInt}(x, y, t) := (1-t)x + ty, \quad t \in [0, 1].$$

(Sederberg [81] considers `linearInt` for fixed t, calling it the affine map.) In the de Boor algorithm, each t is in $[0, 1]$, so the algorithm consists in recursive linear interpolation of adjacent c_j's.

Given enclosures $\mathbf{x}$ of x, $\mathbf{y}$ of y and $\mathbf{t}$ of t (with $\mathbf{t}$ contained in $[0, 1]$), the naive evaluation is $\mathbf{f} = (1 - \mathbf{t}) * \mathbf{x} + \mathbf{t} * \mathbf{y}$. For $\mathbf{x} = [1, 2]$, $\mathbf{y} = [2, 3]$ and $\mathbf{t} = [0, 1]$, the range is $[1, 3]$ but the naive evaluation gives $\mathbf{f} = [0, 1] * [1, 2] + [0, 1] * [2, 3] = [0, 5]$, which is 250% of the optimal width. A naive interval evaluation of the equivalent expression $x + t(y - x)$ gives $\mathbf{f} = [1, 2] + [0, 1] * [0, 2] = [1, 4]$ which is significantly better, only 150% of the optimal width.

But it is easy to see that f is increasing in x and y, hence the lower bound is attained at $x = 1$, $y = 2$, and the upper bound at $x = 2$, $y = 3$. Thus to get the bounds, it suffices to compute an enclosure for $(1-t)x+ty$ at fixed x, y and $t \in \mathbf{t}$, and by rearranging the expression to $x + t(y - x)$, one gets (in exact arithmetic) the exact range of these expressions by inserting $\mathbf{t}$ in the now only occurrence of t. In the above case, one computes the lower bound from $\inf(1 + [0,1](2-1)) = \inf[1,2] = 1$, and the upper bound from $\sup(2 + [0,1](3-2)) = \sup[2,3] = 3$, confirming that the range is $[1,3]$.

Now in general, one needs only one bound each of these two enclosures; this allows one to save half of the work. The final result of our analysis is therefore an optimal computation without any explicit interval operation. With $\mathbf{x}$ = `[xl,xu]`, $\mathbf{y}$ = `[yl,yu]`, $\mathbf{t}$ = `[tl,tu]` $\subseteq [0,1]$, we get `linearInt`$(\mathbf{x},\mathbf{y},\mathbf{t})$ = `[l,u]`, where

```
set round down
dl = yl-xl;tl1=(tl if dl>=0 else tu);l=xl+tl1*dl;
set round up
du = yu-xu;tu1=(tu if du>=0 else tl);u=xu+tu1*du;
```

A switch of rounding modes can be avoided by representing lower bounds by their negatives. Thus, with $\mathbf{x}$ = `[-xn,xp]`, $\mathbf{y}$ = `[-yn,yp]`, $\mathbf{t}$ = `[-tn,tp]` $\subseteq [0,1]$, we get `linearInt`$(\mathbf{x},\mathbf{y},\mathbf{t})$ = `[-n,p]`, where

```
set round up
dn = yn-xn;tn1=(tp if dn>=0 else tn);n=xn-tn1*dn;
dp = yp-xp;tp1=(tp if dp>=0 else tn);p=xp+tp1*dp;
```

This ability to find efficiently the exact range in the linear interpolation problem lead Hayes in the patent mentioned in Section 2 to interval enclosures for Bezier curves and NURBS. There he derived an equivalent algorithm in terms of modal arithmetic, by appealing to a modal coercion theorem; cf. the discussion in Section 7.

The resulting algorithm is not the best possible, however.

For degree $n > 1$ the result of Hayes' algorithm may be worse than just using monotonicity in control points together with optimal enclosures of the Bernstein polynomials or the corresponding B-spline basis. The representation of a Bezier curve in explicit Bernstein form (or of NURBS in explicit B-spline basis form) gives sometimes more accurate results. Moreover, at least for fixed knot sequences (and always in the Bezier case), the explicit basis form is cheaper to compute than the recursive form used by Hayes for polynomials of degrees greater than 2.

Hayes' interval de Boor algorithm does not work directly when $\mathbf{t}$ contains a knot in its interior. However, in branch-and bound schemes it is natural anyway to split the t-domain initially at the distinct knots, so that this limitation is not serious.

In case of degree > 1, Hayes' algorithm does not have the quadratic approximation property, desirable for good behavior on narrow intervals. That an algorithm for enclosing ranges has the quadratic approximation property means that if the argument width goes to zero like $O(\varepsilon)$ then the overestimation is $O(\varepsilon^2)$ rather than the typical $O(\varepsilon)$. Centered forms have the quadratic approximation property; see NEUMAIER [52, Sections 8 and 9]. For Hayes' algorithm, an example is given by a Bezier curve with control points 1, 2, 1 in each component and $\mathbf{t} = [0.5 - \varepsilon, 0.5 + \varepsilon]$. The range has width $O(\varepsilon^2)$, the algorithm gives an enclosure with width $O(\varepsilon)$, hence the overestimation is $O(\varepsilon)$.

Applying the de Boor algorithm for the lower and upper bound of an interval gives a control point representation of the spline on this interval, so that the range can be enclosed by the hull of the new control polygon. This enclosure has the quadratic approximation property, and is often even optimal. Thus for narrow intervals (especially near an interior extremum), this algorithm is preferable to Hayes' algorithm, especially in branch and bound algorithms.

Fat arcs by SEDERBERG [80] even provide a cubic approximation property; for other good enclosures see PETERS [58]. For accurate linear underestimators of multivariate Bezier polynomials see GARLOFF & SMITH [14].

4 Linear interpolation in the Vienna proposal

The linear interpolation formula arises not only in the above application to computer graphics and, more generally, range enclosure for multivariate polynomials, but also in various other applications. This makes the operation important enough to consider its implementation as a special operation and that of its reverse mode.

In the Vienna proposal on interval standardization [54], the optional implementation of optimal linear interpolation is specified in Section 5.8, in both forward mode (5.8.1) and reverse mode (5.8.2 and 5.8.3).

```
5.8. Linear interpolation

  1. There may be an operation linearInt(xx,yy,tt) that provides,
     for any two compact intervals xx=[xl,xu], yy=[yl,yu], and any
     interval tt=[tl,tu] contained in [0,1] an enclosure of the set of
     all (1-t)*x+t*y such that x in xx, y in yy, t in tt, which is
     equivalent to or tighter than the result [l,u] of the following
     algorithm.
        set round down
        dl = yl-xl;tl1=(tl if dl>=0 else tu);l=xl+tl1*dl;
        set round up
        du = yu-xu;tu1=(tu if du>=0 else tl);u=xu+tu1*du;
```

The behavior for noncompact xx or yy and for tt not in [0,1] is not specified.

2. There may be an operation linearExt(xx,yy,tt) that provides, for any two compact intervals xx=[xl,xu], yy=[yl,yu], and any interval tt=[tl,tu] contained in [1,Inf] an enclosure of the set of all (1-t)*x+t*y such that x in xx, y in yy, t in tt, which is equivalent to or tighter than the result [l,u] of the following algorithm.
```
set round down
dl = yl-xu;tl1=(tl if dl>=0 else tu);l=xu+tl1*dl;
set round up
du = yu-xl;tu1=(tu if du>=0 else tl);u=xl+tu1*du;
```
   The behavior for noncompact xx or yy and for tt not in [1,Inf] is not specified.

3. There may be an operation linearInv(xx,yy,zz,tt) that provides, for any four compact intervals xx, yy, zz, and tt=[tl,tu] with tl>=0 an enclosure of the set of all t in tt such that z=(1-t)*x+t*y for some x in xx, y in yy, z in zz, which is equal to or tighter than
```
convexHull(timesInv(zz-xl,yy-xl,tt),timesInv(zz-xu,yy-xu,tt))
```
   with convexHull and timesInv defined in Sections 5.6 and 3.11, respectively.
   The behavior for noncompact xx, yy, or zz is not specified.

Remarks.

1. The extrapolation behavior for t in [-Inf,0] can be obtained by replacing t with 1-t. Cases where tt is an interval containing 0 or 1 in the interior are not needed in computational geometry; not requiring them saves case distinctions.

2. Note that the four equations
```
z=(1-t)*x+t*y,                y=(1-s)*x+s*z,
x=(1-p)*y+p*z=(1-q)*z+q*y,    t=(z-x)/(y-x)
```
   are equivalent if s=1/t, p=1/(1-t), q=1/(1-s) and these quotients are defined, as are the conditions
```
0<t<1, s>1, p>1, q<0
```
   and
```
t>1, 0<s<1, p<0, q>0
```
   This explains the sign restrictions used.

5 Nonstandard intervals

Nonstandard intervals are currently used in three flavors, Kahan arithmetic, Kaucher arithmetic, and modal arithmetic. Kahan arithmetic is completely different from the other two. Kaucher arithmetic and modal arithmetic are identical in many respects (the differences are much smaller that suggested by the confusing introductory remarks in SAINZ et al. [77]), but they differ in their treatment of binary operations (an observation due to GOLDSZTEJN [17]), although by a fortunate coincidence, this difference happens to vanish in case of the standard arithmetic operations.

Kahan arithmetic is a completion of interval arithmetic based on treating nonstandard intervals with reverse bound as intervals on the projectively closed real line, $[\underline{x}, \overline{x}] = [-\infty, \overline{x}] \cup [\underline{x}, \infty]$; they arise naturally when dividing in a set-theoretic sense an interval by an interval containing zero in the interior. This interpretation was introduced in KAHAN [35] with applications to the evaluation of continued fractions. An implementation is described in LAVEUVE [39]. There is little recent work on (or using) Kahan arithmetic, hence this variant is not discussed further.

Generalized intervals, today usually called Kaucher intervals (sometimes also directed intervals), were first mentioned in 1956 by WARMUS [99] without any resonance in the literature, then seriously studied in 1969 in an obscure paper by ORTOLF [57], which was taken up and made visible in 1973 in the Ph.D. thesis by KAUCHER [36]. This work was purely algebraic. Important for practical applications is the modal interpretation, introduced 1980–1985 by GARDEÑES et al. [12, 11]; these papers contain serious mathematical errors that were corrected in 1999 in [89]; see SHARY [88, p. 374]. A standard reference for modal arithmetic is the 2001 paper by GARDEÑES et al. [13]. A more readable account is given in the thesis by HERRERO [31].

Both Kaucher arithmetic and modal arithmetic are defined on the set of Kaucher intervals. Kaucher theory thinks in terms of proper and improper intervals, while modal theory thinks in terms of ordinary intervals with a 2-valued mode or direction attached to it; reversing the direction in the modal version is equivalent to reversing the bounds in the Kaucher version. Thus both theories can be mapped into each other, and in many respects, the theories are completely equivalent. However, there are some subtle differences. As historically developed, the theories make different proposals for interpreting general binary operations. It happens that these proposals are equivalent for the standard binary operations but not in general.

The following synopsis of the basic concepts of these variants of interval arithmetic concentrates on the features needed for the understanding of their merits and limitations in real applications, and does not attempt to give an exhaustive exposition. To be able to treat Kaucher arithmetic and modal arithmetic on the same footing, some notation not yet found in the literature is introduced to distinguish different possible definitions of binary operations.

A **Kaucher interval** is a pair $\mathbf{x} = [\underline{x}, \overline{x}]$ of real numbers $\underline{x}, \overline{x} \in \mathbb{R}$; it is called **proper** if $\underline{x} \leq \overline{x}$ and **improper** if $\overline{x} \leq \underline{x}$; thus each Kaucher interval is either proper or improper; the **thin** Kaucher intervals, i.e., those of the form $[x, x]$ with $x \in \mathbb{R}$, are both proper and improper, and are identified with real numbers x. (This terminology is customary for Kaucher intervals, though it conflicts with the standard notion that a proper interval is an ordinary interval containing more than one point; therefore, [54] uses the terminology *standard* and *nonstandard* for proper and improper, respectively.)

The set of all Kaucher intervals is denoted by $\mathbb{KR}$; it contains the set $\mathbb{IR}$ of all ordinary (proper) intervals and the set $\mathbb{R}$ of all real numbers, considered as thin intervals. The **dual** of a Kaucher interval $\mathbf{x}$ is the Kaucher interval

$$\mathbf{x}^* = \mathrm{dual}(\mathbf{x}) := [\overline{x}, \underline{x}].$$

This notation is extended componentwise to vectors of Kaucher intervals. The **proper interval** associated to $\mathbf{x} \in \mathbb{KR}$ is the interval $\mathbf{x}^{\perp} \in \mathbb{IR}$ defined by

$$\mathbf{x}^{\perp} = \mathrm{pro}(\mathbf{x}) := \begin{cases} \mathbf{x} & \text{if } \mathbf{x} \text{ is proper,} \\ \mathbf{x}^* & \text{otherwise,} \end{cases}$$

and **Kaucher membership** E is defined by

$$x \mathrel{\mathrm{E}} \mathbf{x} \quad :\Leftrightarrow \quad x \in \mathbf{x}^{\perp}.$$

Kaucher intervals are partially ordered by **Kaucher inclusion** $\sqsubseteq$, defined by

$$\mathbf{x} \sqsubseteq \mathbf{y} = \mathbf{y} \sqsupseteq \mathbf{x} \quad :\Leftrightarrow \quad \underline{x} \geq \underline{y}, \quad \overline{x} \leq \overline{y}.$$

Restricted to proper Kaucher intervals, this becomes the set theoretic inclusion of ordinary intervals. Note that in general,

$$\mathbf{x}^* \sqsubseteq \mathbf{y}^* \quad \Leftrightarrow \quad \mathbf{y} \sqsubseteq \mathbf{x},$$

$$\mathbf{x}^* \sqsubseteq \mathbf{y} \quad \Leftrightarrow \quad \mathbf{y}^* \sqsubseteq \mathbf{x},$$

and for $\mathbf{x}, \mathbf{y} \in \mathbb{IR}$,

$$\mathbf{x}^* \sqsubseteq \mathbf{y} \quad \Leftrightarrow \quad \mathbf{x} \cap \mathbf{y} \neq \emptyset \quad \Leftrightarrow \quad \mathbf{y}^* \sqsubseteq \mathbf{x}.$$

We define the **Kaucher union** of a function F defined on some interval $\mathbf{x} \in \mathbb{IR}$, with Kaucher interval values (in particular also for functions with real values or ordinary interval values) as

$$\bigsqcup_{x \in \mathbf{x}} F(x) := [\inf_{x \in \mathbf{x}} \underline{F}(x), \sup_{x \in \mathbf{x}} \overline{F}(x)],$$

and the **Kaucher intersection** as

$$\bigsqcap_{x\in\mathbf{x}} F(x) := [\sup_{x\in\mathbf{x}} \underline{F}(x), \inf_{x\in\mathbf{x}} \overline{F}(x)].$$

It is easily checked that Kaucher union and Kaucher intersection reduce to ordinary union and intersection when all $F(x)$ are proper intervals, except that in place of an empty intersection one gets an improper interval. Note that, for ordinary intervals $\mathbf{x}, \mathbf{x}'$ and Kaucher interval valued functions F and G,

$$F(x) \sqsubseteq G(x) \text{ for all } x \in \mathbf{x} \quad \Rightarrow \quad \bigsqcup_{x\in\mathbf{x}} F(x) \sqsubseteq \bigsqcup_{x\in\mathbf{x}} G(x),$$

$$F(x) \sqsubseteq G(x) \text{ for all } x \in \mathbf{x} \quad \Rightarrow \quad \bigsqcap_{x\in\mathbf{x}} F(x) \sqsubseteq \bigsqcap_{x\in\mathbf{x}} G(x),$$

$$\mathbf{x} \subseteq \mathbf{x}' \quad \Rightarrow \quad \bigsqcup_{x\in\mathbf{x}} F(x) \sqsubseteq \bigsqcup_{x\in\mathbf{x}'} F(x),$$

$$\mathbf{x} \subseteq \mathbf{x}' \quad \Rightarrow \quad \bigsqcap_{x\in\mathbf{x}} F(x) \sqsupseteq \bigsqcap_{x\in\mathbf{x}'} F(x),$$

$$\bigsqcup_{x\in\mathbf{x}} \bigsqcup_{y\in\mathbf{y}} F(x,y) = \bigsqcup_{y\in\mathbf{y}} \bigsqcup_{x\in\mathbf{x}} F(x,y), \tag{1}$$

$$\bigsqcap_{x\in\mathbf{x}} \bigsqcap_{y\in\mathbf{y}} F(x,y) = \bigsqcap_{y\in\mathbf{y}} \bigsqcap_{x\in\mathbf{x}} F(x,y), \tag{2}$$

$$\bigsqcup_{x\in\mathbf{x}} \bigsqcap_{y\in\mathbf{y}} F(x,y) \sqsubseteq \bigsqcap_{y\in\mathbf{y}} \bigsqcup_{x\in\mathbf{x}} F(x,y); \tag{3}$$

the last containment follows easily from

$$\sup_{x\in\mathbf{x}} \inf_{y\in\mathbf{y}} f(x,y) \le \inf_{y\in\mathbf{y}} \sup_{x\in\mathbf{x}} f(x,y), \tag{4}$$

which holds for real-valued functions defined on $\mathbf{x} \times \mathbf{y}$ since the functions $\underline{f}$ and $\overline{f}$ defined by $\underline{f}(x) := \inf_{y\in\mathbf{y}} f(x,y)$ and $\overline{f}(y) := \sup_{x\in\mathbf{x}} f(x,y)$ satisfy $\underline{f}(x) \le f(x,y) \le \overline{f}(y)$ for all x, y.

Both Kaucher union and Kaucher intersection may be combined in the operator defined for a Kaucher interval $\mathbf{x}$ by

$$\mathop{\square}_{x \,\mathrm{E}\, \mathbf{x}} F(x) := \begin{cases} \bigsqcup_{x\in\mathbf{x}} F(x) & \text{if } \mathbf{x} \text{ is proper,} \\ \bigsqcap_{x\in\mathbf{x}^*} F(x) & \text{if } \mathbf{x} \text{ is improper;} \end{cases}$$

clearly, there is no ambiguity in case that $\mathbf{x}$ is both proper and improper. (This operation might be called *unisection* or *interunion*; the English language has no natural symmetric name for it.) While this operation has no simple commutation properties, it is **Kaucher inclusion monotone**:

$$\mathbf{x} \sqsubseteq \mathbf{x}' \quad \Rightarrow \quad \mathop{\Box}_{x\,\mathrm{E}\,\mathbf{x}} F(x) \sqsubseteq \mathop{\Box}_{x\,\mathrm{E}\,\mathbf{x}'} F(x).$$

For the modal interpretation by means of quantifiers, the following implications (valid for ordinary intervals $\mathbf{x}$ and Kaucher intervals $\mathbf{F}$) are relevant:

$$\forall x \in \mathbf{x} : F(x) \sqsubseteq \mathbf{F} \quad \Leftrightarrow \quad \bigsqcup_{x\in\mathbf{x}} F(x) \sqsubseteq \mathbf{F}, \tag{5}$$

$$\exists x \in \mathbf{x} : F(x) \sqsubseteq \mathbf{F} \quad \Rightarrow \quad \bigsqcap_{x\in\mathbf{x}} F(x) \sqsubseteq \mathbf{F}. \tag{6}$$

Note that the reverse implication of (6) is in general false, even when $\mathbf{F}$ is assumed to be proper. Thus there is an inherent asymmetry in the Kaucher extension of unions and intersections, which causes later complications in the interpretation of a generalized arithmetic based on these.

We now consider the extension of (possibly only partially defined) unary and binary real-valued operations of real arguments to Kaucher intervals, by specializing the above to the case where F is a real-valued expression involving only a single arithmetic operation; regarded as Kaucher interval valued functions, the lower and upper bounds agree.

The **Kaucher extension** of a unary operation ϕ that is defined for all $x\,\mathrm{E}\,\mathbf{x}$ is defined by

$$\phi(\mathbf{x}) := \mathop{\Box}_{x\,\mathrm{E}\,\mathbf{x}} \phi(x) = \begin{cases} \displaystyle\bigsqcup_{x\in\mathbf{x}} \phi(x) & \text{if } \mathbf{x} \text{ is proper,} \\ \displaystyle\bigsqcap_{x\in\mathbf{x}^*} \phi(x) & \text{if } \mathbf{x} \text{ is improper.} \end{cases}$$

For binary operations, the lack of commutativity of $\bigsqcup$ and $\bigsqcap$ allows four different extensions to Kaucher intervals. The **Kaucher extension** of a binary operation $\circ$ that is defined for all $x\,\mathrm{E}\,\mathbf{x}$ and all $y\,\mathrm{E}\,\mathbf{y}$ is defined by

$$\mathbf{x} \circ_K \mathbf{y} := \mathop{\Box}_{x\,\mathrm{E}\,\mathbf{x}} \mathop{\Box}_{y\,\mathrm{E}\,\mathbf{y}} x \circ y = \begin{cases} \displaystyle\bigsqcup_{x\in\mathbf{x}} \bigsqcup_{y\in\mathbf{y}} x \circ y & \text{if } \mathbf{x} \text{ and } \mathbf{y} \text{ are proper,} \\ \displaystyle\bigsqcup_{x\in\mathbf{x}} \bigsqcap_{y\in\mathbf{y}^*} x \circ y & \text{if } \mathbf{x} \text{ is proper, } \mathbf{y} \text{ improper,} \\ \displaystyle\bigsqcap_{x\in\mathbf{x}^*} \bigsqcup_{y\in\mathbf{y}} x \circ y & \text{if } \mathbf{x} \text{ is improper, } \mathbf{y} \text{ proper,} \\ \displaystyle\bigsqcap_{x\in\mathbf{x}^*} \bigsqcap_{y\in\mathbf{y}^*} x \circ y & \text{if } \mathbf{x} \text{ and } \mathbf{y} \text{ are improper.} \end{cases}$$

The **anti-Kaucher extension** of a binary operation $\circ$ that is defined for all $x \mathrel{\mathrm{E}} \mathbf{x}$ and all $y \mathrel{\mathrm{E}} \mathbf{y}$ is defined by

$$\mathbf{x} \circ_L \mathbf{y} := \bigsqcap_{y \,\mathrm{E}\, \mathbf{y}} \bigsqcap_{x \,\mathrm{E}\, \mathbf{x}} x \circ y = \begin{cases} \bigsqcup_{x\in\mathbf{x}} \bigsqcup_{y\in\mathbf{y}} x \circ y & \text{if } \mathbf{x} \text{ and } \mathbf{y} \text{ are proper,} \\ \bigsqcap_{y\in\mathbf{y}^*} \bigsqcup_{x\in\mathbf{x}} x \circ y & \text{if } \mathbf{x} \text{ is proper, } \mathbf{y} \text{ improper,} \\ \bigsqcup_{y\in\mathbf{y}} \bigsqcap_{x\in\mathbf{x}^*} x \circ y & \text{if } \mathbf{x} \text{ is improper, } \mathbf{y} \text{ proper,} \\ \bigsqcap_{x\in\mathbf{x}^*} \bigsqcap_{y\in\mathbf{y}^*} x \circ y & \text{if } \mathbf{x} \text{ and } \mathbf{y} \text{ are improper;} \end{cases}$$

i.e., its definition differs from the Kaucher extension in case that one of $\mathbf{x}$ and $\mathbf{y}$ is proper and the other is improper. The **modal extension** ($*$-*semantic extension* in the terminology of GARDEÑES et al. [13]) of a binary operation $\circ$ that is defined for all $x \mathrel{\mathrm{E}} \mathbf{x}$ and all $y \mathrel{\mathrm{E}} \mathbf{y}$ is defined by

$$\mathbf{x} \circ_M \mathbf{y} := \begin{cases} \bigsqcup_{x\in\mathbf{x}} \bigsqcup_{y\in\mathbf{y}} x \circ y & \text{if } \mathbf{x} \text{ and } \mathbf{y} \text{ are proper,} \\ \bigsqcup_{x\in\mathbf{x}} \bigsqcap_{y\in\mathbf{y}^*} x \circ y & \text{if } \mathbf{x} \text{ is proper, } \mathbf{y} \text{ improper,} \\ \bigsqcup_{y\in\mathbf{y}} \bigsqcap_{x\in\mathbf{x}^*} x \circ y & \text{if } \mathbf{x} \text{ is improper, } \mathbf{y} \text{ proper,} \\ \bigsqcap_{x\in\mathbf{x}^*} \bigsqcap_{y\in\mathbf{y}^*} x \circ y & \text{if } \mathbf{x} \text{ and } \mathbf{y} \text{ are improper;} \end{cases}$$

i.e., its definition differs from the Kaucher extension in case that $\mathbf{y}$ is proper but $\mathbf{x}$ is not. The **dual modal extension** ($**$-*semantic extension* in the terminology of GARDEÑES et al. [13]) of a binary operation $\circ$ that is defined for all $x \mathrel{\mathrm{E}} \mathbf{x}$ and all $y \mathrel{\mathrm{E}} \mathbf{y}$ is defined by

$$\mathbf{x} \circ_N \mathbf{y} := \begin{cases} \bigsqcup_{x\in\mathbf{x}} \bigsqcup_{y\in\mathbf{y}} x \circ y & \text{if } \mathbf{x} \text{ and } \mathbf{y} \text{ are proper,} \\ \bigsqcap_{y\in\mathbf{y}^*} \bigsqcup_{x\in\mathbf{x}} x \circ y & \text{if } \mathbf{x} \text{ is proper, } \mathbf{y} \text{ improper,} \\ \bigsqcap_{x\in\mathbf{x}^*} \bigsqcup_{y\in\mathbf{y}} x \circ y & \text{if } \mathbf{x} \text{ is improper, } \mathbf{y} \text{ proper,} \\ \bigsqcap_{x\in\mathbf{x}^*} \bigsqcap_{y\in\mathbf{y}^*} x \circ y & \text{if } \mathbf{x} \text{ and } \mathbf{y} \text{ are improper;} \end{cases}$$

i.e., its definition differs from the Kaucher extension in case that $\mathbf{x}$ is proper but $\mathbf{y}$ is not. Using the properties of Kaucher union and Kaucher intersection, it is not difficult to see that

$$(\mathbf{x}^*)^* = \mathbf{x}, \quad \phi(\mathbf{x}^*) = \phi(\mathbf{x})^*,$$
$$(\mathbf{x} \circ_K \mathbf{y})^* = \mathbf{x}^* \circ_K \mathbf{y}^*, \quad (\mathbf{x} \circ_L \mathbf{y})^* = \mathbf{x}^* \circ_L \mathbf{y}^*,$$

$$(\mathbf{x} \circ_M \mathbf{y})^* = \mathbf{x}^* \circ_N \mathbf{y}^*, \quad (\mathbf{x} \circ_N \mathbf{y})^* = \mathbf{x}^* \circ_M \mathbf{y}^*,$$

and that all extensions are **Kaucher inclusion monotone**, i.e.,

$$\mathbf{x} \sqsubseteq \mathbf{x}' \quad \Rightarrow \quad \phi(\mathbf{x}) \sqsubseteq \phi(\mathbf{x}'),$$

$$\mathbf{x} \sqsubseteq \mathbf{x}' \quad \Rightarrow \quad \mathbf{x} \circ_X \mathbf{y} \sqsubseteq \mathbf{x}' \circ_X \mathbf{y}, \quad \mathbf{y} \circ_X \mathbf{x} \sqsubseteq \mathbf{y} \circ_X \mathbf{x}' \quad \text{for } X \in \{K, L, M, N\}.$$

This implies that it makes sense to implement the operations in finite-precision arithmetic with outward rounding, i.e., the lower bound of each operation is rounded down, the upper bound is rounded up. The result of an exact expression evaluation at Kaucher intervals is then always Kaucher contained in the expression evaluation with outward rounding. Thus all results whose interpretation does not depend on the exact values of an evaluation but for which Kaucher containment suffices remain valid in finite-precision arithmetic with outward rounding.

On the other hand, since finite-precision numbers may overflow, this requires the consideration of **extended Kaucher intervals** whose lower bound is $-\infty$ and/or whose upper bound is ∞. All statements in this section remain valid in this extension; it is not clear, however, whether this also holds for the coercion theorems of Section 7, which give the formal computation with Kaucher intervals a meaning useful for applications.

We also note the rules

$$\phi(\mathbf{x})^\perp = \phi(\mathbf{x})^\perp,$$
$$(\mathbf{x} \circ_X \mathbf{y})^\perp \sqsubseteq \mathbf{x}^\perp \circ \mathbf{y}^\perp,$$

valid in exact arithmetic only.

6 Kaucher arithmetic and modal arithmetic

Kaucher arithmetic consists of the arithmetic on Kaucher intervals by means of operations defined through Kaucher extensions of unary and binary operations. **Modal arithmetic** and **dual modal arithmetic** use the Kaucher extensions for unary operations but for binary operations the modal extensions and dual modal extensions, respectively. Restricted to proper Kaucher intervals, one gets in all cases ordinary interval arithmetic. In general, it follows directly from the above and (3) that

$$\mathbf{x} \circ_M \mathbf{y} \sqsubseteq \mathbf{x} \circ_K \mathbf{y} \sqsubseteq \mathbf{x} \circ_N \mathbf{y}, \quad \mathbf{x} \circ_M \mathbf{y} \sqsubseteq \mathbf{x} \circ_L \mathbf{y} \sqsubseteq \mathbf{x} \circ_N \mathbf{y}.$$

That proper containment is possible can be seen by considering the operation defined by $x \circ y := |x - y|$ with $\mathbf{x} = [-1, 1]$ and $\mathbf{y} = [1, -1]$.

However, the four extensions of binary operations agree in the important case of **commutable** operations, characterized by the minimax property

$$\sup_{x \in \mathbf{x}} \inf_{y \in \mathbf{y}} x \circ y = \inf_{y \in \mathbf{y}} \sup_{x \in \mathbf{x}} x \circ y \tag{7}$$

for all $\mathbf{x}, \mathbf{y} \in \mathbb{IR}$ such that $x \circ y$ is defined for all $x \in \mathbf{x}$ and all $y \in \mathbf{y}$. In particular, this holds if there is a **saddle point**, i.e., a point $(x_0, y_0) \in \mathbf{x} \times \mathbf{y}$ such that (7) equals $f(x_0, y_0)$. General criteria for the existence of saddle points, based on quasiconcavity and quasiconvexity, have been given by Sion [90]. In [89], it is shown (although the argument given there is far too short to be convincing) that a saddle point always exists for continuous and monotone binary operations; here $\circ$ is called **monotone** if $x \circ y$ is monotone in x for any fixed y and monotone in y for any fixed x. It turns out that a saddle point, and hence the common value of the two sides in (7), can be found by a uniform rule:

6.1 Theorem. (Commutability conditions)
If the operation $\circ$ is continuous on $\mathbf{x} \times \mathbf{y}$, monotone in x for any fixed $y \in \mathbf{y}$ and monotone in y for any fixed $x \in \mathbf{x}$, then a saddle point (x_0, y_0) exists, and we have one of the following cases:

(i) If $\circ$ is decreasing in x for all y and increasing in y for $x = \underline{x}$, or increasing in y for all x and decreasing in x for $y = \underline{y}$ then $(x_0, y_0) = (\underline{x}, \underline{y})$ is a saddle point.

(ii) If $\circ$ is decreasing in x for all y and decreasing in y for $x = \underline{x}$, or decreasing in y for all x and decreasing in x for $y = \overline{y}$ then $(x_0, y_0) = (\underline{x}, \overline{y})$ is a saddle point.

(iii) If $\circ$ is increasing in x for all y and increasing in y for $x = \overline{x}$, or increasing in y for all x and increasing in x for $y = \underline{y}$ then $(x_0, y_0) = (\overline{x}, \underline{y})$ is a saddle point.

(iv) If $\circ$ is increasing in x for all y and decreasing in y for $x = \overline{x}$, or decreasing in y for all x and increasing in x for $y = \overline{y}$ then $(x_0, y_0) = (\overline{x}, \overline{y})$ is a saddle point.

(v) Otherwise, the set of saddle points (x_0, y_0) forms an axiparallel rectangle contained in the interior of $\mathbf{x} \times \mathbf{y}$, and the operation has a constant value on the union of all such crosses. Moreover, in each of the four boxes obtained by removing from $\mathbf{x} \times \mathbf{y}$ the points on the cross defined by the two lines $x = x_0$ and $y = y_0$, the operation $\circ$ is uniformly monotone, i.e., has a uniform direction of monotonicity in x and a uniform direction of monotonicity in y.

The example $\mathbf{x} = \mathbf{y} = [-1, 1]$ and $x \circ y := 1$ if $x, y \geq 0$ or $x, y < 0$ and $x \circ y := 0$ otherwise shows that that the continuity assumption is necessary for part (v) of this classification. If $x \circ y$ is uniformly monotone in x or y then case (v) does not occur, and the continuity assumption is not needed.

Proof. If $\circ$ has a uniform direction of monotonicity in x for all y (i.e., if this direction does not depend on the value of y) or in y for all x then one of the assumptions of (i)–(iv) holds, and in each case the saddle point property is easily verified.

Thus it remains to discuss the case where the direction of monotonicity changes in both directions. The somewhat lengthy argument is sketched only; I would like to hear about a concise argument for settling this case. In this case, there are numbers $x_1 < x_2$ in $\mathbf{x}$ and

$y_1 < y_2$ in $\mathbf{y}$ such that the direction of monotonicity in x is different for $y = y_1$ and $y = y_2$, and the direction of monotonicity in y is different for $x = x_1$ and $x = x_2$. Consideration of the values at the four corners of the rectangle $R := [x_1, x_2] \times [y_1, y_2]$ shows that in two of the four possibilities for the direction of monotonicity, the function is forced to be constant on R. Using this property and the fact that a constant function is both monotone increasing and monotone decreasing, it is easy to establish that, when varying x (or y), there can be at most one change in the direction of monotonicity, apart from regions where $\circ$ is constant and the direction of monotonicity is arbitrary. Using the continuity of $\circ$, it is now straightforward to show that we must have the situation described in case (v). $\square$

Since the standard binary arithmetic operations $\circ \in \{+, -, *, /, \min, \max, \text{power}\}$ (with division $\mathbf{x}/\mathbf{y}$ restricted to $\mathbf{y}$ with $0 \not\sqsubseteq \mathbf{y}$, and the power $\mathbf{x}^{\mathbf{y}}$ restricted to $0 \leq \mathbf{x}^{\perp}$, and to $0 < \mathbf{x}^{\perp}$ if $\mathbf{y}^{\perp}$ is not positive) are monotone, they are commutable. Thus, when restricted to the case where all binary operations are commutable, and in particular for the standard arithmetic operations, Kaucher arithmetic, modal arithmetic, and dual modal arithmetic are identical, and the suffix K, L, M, or N to the binary operation can be dropped.

The theorem gives easily derivable explicit formulas for the endpoints of the resulting Kaucher intervals when exactly one argument is improper, and the case of two improper arguments is reduced by duality to the case of two proper arguments, where the formulas from ordinary interval arithmetic apply. The reader is invited to do this for the operations $\circ \in \{+, -, *, /, \min, \max, \text{power}\}$.

The binary operation $\operatorname{atan2}(x, y)$, the principal value of the argument of the complex number $y + ix$ if x or y are nonzero, and zero if $x = y = 0$, is monotone on boxes where it is continuous, but not everywhere. I do not know whether it is commutable in general.

The discussion can be extended to ternary operations. There are now eight possible cases; a ternary operation is called **commutable** if all eight cases give identical results. Simple monotone continuity seems to be no longer sufficient, but operations that are uniformly monotonic in each variable are commutable.

7 Coercion theorems and monotonicity

From the modal theory, we review here only two basic coercion theorems. The first coercion theorem is a combination of Gardeñes et al. [13, Theorem 4.1 and Theorem 4.8].

An arithmetic expression is called **uni-incident** in a list of variables if each of these variables occurs exactly once in the expression.

7.1 Theorem. (First coercion theorem)
Let $f(x, y) := e(x, y, y)$, where $e(x, y, z)$ is an arithmetic expression that is uni-incident in

z. Suppose that all operations occurring in this expression are continuous and commutable, and suppose that $\mathbf{f} \in \mathbb{KR}$ satisfies $e(\mathbf{x}, \mathbf{y}, \mathbf{y}^*) \sqsubseteq \mathbf{f}$. Then:

(i) If $\mathbf{f}$ is proper then

$$\forall x \in \mathbf{x}\ \exists y \in \mathbf{y}:\ f(x, y) \in \mathbf{f}.$$

(ii) If $\mathbf{f}$ is improper then

$$\forall v \in \mathbf{f}^*\ \forall x \in \mathbf{x}\ \exists y \in \mathbf{y}:\ f(x, y) = v.$$

Without commutability, the first coercion theorem does not hold. A counterexample is the arithmetic expression $e(x) = x \circ x$ where $\circ$ is the operation defined by $x \circ y := |x - y|$, applied to $\mathbf{x} = [-1, 1]$. The range $[0, 0]$ is not Kaucher contained in the coerced expression $\mathbf{x} \circ \mathbf{x}^* = [1, 0]$. (This is a simplified version of a counterexample by GOLDSZTEJN [17, Appendix B].)

If $\mathbf{x}$ is a vector of proper intervals then the first coercion theorem may be applied to $\mathbf{x}^*$ to get a Kaucher interval $\mathbf{f}$. Then $\mathbf{f}^*$ is an **inner approximation** of the range of $f(x)$ for $x \in \mathbf{x}$, i.e., $\mathbf{f}^*$ is Kaucher contained in the range. The inner approximation may be quite poor even for very harmless expressions; e.g., for $f(x) = x^2 - x$, one obtains with $\mathbf{x} = [0, 1]$ the inner approximations $(\mathbf{x}^*)^2 - \mathbf{x} = [0, 0]$ or $x^2 - x^* = [0, 0]$.

The above coercion theorem gives, as all theorems of Section 4 in [13], range enclosures only if all components of the argument are proper, and in that case, the result is identical with naive interval evaluation.

More interesting range enclosures can be obtained with modal arithmetic in certain cases by modifying some instances of some variables from proper to improper by dualizing the interval. (Again, further massaging may be possible.) Under the right circumstances, this provides valid and tighter intervals enclosing the range of the function defined by an expression, and sometimes even the exact range.

Our second coercion theorem is a combination of [13, Theorem 4.1 and Theorem 5.2], restricted for the sake of simplicity to the case where only a range enclosure is wanted. It is equivalent to Theorem 2.2 in POPOVA [66].

Let us call an arithmetic expression $e(x)$ in the components of x, defined in a box $\mathbf{x}$, **totally monotone** in x_i if
(i) it is monotone in x_i for all values of the other variables, and
(ii) for each individual occurrence of x_i in the expression, the arithmetic expression $e(x, t)$ obtained by substituting this occurrence by t is either monotone increasing for all values of $x \in \mathbf{x}$, or monotone decreasing for all these values. In this case, the occurrence is called **contrary** if the direction of monotonicity of $e(x, t)$ in t is opposite to that of $e(x)$ in x_i. (Clearly, this is possible only if x_i occurs more than once in $e(x)$.)

7.2 Theorem. (Second coercion theorem)
Let $e(x)$ be an arithmetic expression in the components of x all of whose operations are continuous and commutable, and let $e(x, x')$ be the arithmetic expression in which all contrary occurrences of some component x_i are replaced by the variable x'_i. Then the range of the function defined by the expression for $x \in \mathbf{x}$ is contained in the interval $e(\mathbf{x}, \mathbf{x}^*)$.

Note that Kaucher inclusion monotonicity implies that outward rounding gives correct results in finite precision.

Under certain additional conditions, the results of the second coercion theorem can even be proved to give (in exact arithmetic) the exact range [13, Theorem 5.4].

Relations to monotonicity. Modal coercion theorems may give much better enclosures than naive interval evaluation; but so do centered forms and monotonicity arguments.

In Neumaier [53], a number of standard interval recipes for range enclosures are given that improve upon the naive interval evaluation. We quote here two of them.

7.3 Theorem. (Ranges for partial monotone functions)
Let $e(x, y, z)$ be an expression in the components of x, y, z. Suppose that $e(x, y, z)$ can be evaluated for all $x, y, z \in \mathbf{x}$ and that the resulting function is monotone increasing in y and monotone decreasing in z. Then the range of the function f defined by $f(x) := e(x, x, x)$ for $x \in \mathbf{x}$ is contained in the interval

$$\mathbf{f} := [\inf_{x\in\mathbf{x}} e(x, \underline{x}, \overline{x}), \sup_{x\in\mathbf{x}} e(x, \overline{x}, \underline{x})].$$

If $e(x, y, z)$ is independent of x then $\mathbf{f}$ is the exact range.

7.4 Theorem. (Fractional multilinear ranges)
Let $e(x)$ be an expression in the components of x that, for some subset K of indices, is fractional linear (i.e., equivalent to the form $(at + b)/(ct + d)$ with a, b, c, d independent of t) in the components $t = x_k$ $(k \in K)$ of x. If $e(x)$ is defined for all $x \in \mathbf{x}$ then the range of $e(x)$ for $x \in \mathbf{x}$ is contained in the convex hull of the ranges on the subboxes of $\mathbf{x}$ obtained by specializing in all possible ways the bounds x_k $(k \in K)$ to one of their endpoints. If K contains all indices then $\mathbf{f}$ is the exact range.

Together with the rearrangement of expressions, the reuse of common subexpressions, and savings that come from the need to evaluate only a lower or an upper bound of the ranges of $e(x, \underline{x}, \overline{x})$ or $e(x, \overline{x}, \underline{x})$, these theorems cover all uses of modal arithmetic to improved range enclosures that I know of. The efficiency is comparable to that obtained with modal intervals, while the range of applicability is much wider.

Some instances of these theorems can be duplicated exactly by modal arithmetic. For example, the modal arithmetic expressions $\mathbf{x} + \mathbf{t} * (\mathbf{y} - \mathbf{x}^*)$ and $\mathbf{x} - \mathbf{t} * (\mathbf{x}^* - \mathbf{y})$ (the first one used in HAYES [26]) both provide alternative ways of computing the exact range in the linear interpolation problem of Section 3. The enclosure property is a consequence of the second coercion theorem stated above, and optimality can be seen either by reference to [13, Theorem 5.4], or by expressing the formula in terms of lower and upper bounds – one finds exactly the formulas derived in Section 3. Thus, here we have several different ways of getting the optimal enclosure, one without and two with using modal arithmetic, with two different derivations.

Another case of interest in computer graphics is the function $f(x,y,z) := x/\sqrt{x^2+y^2+z^2}$, which can be optimally enclosed at $(\mathbf{x},\mathbf{y},\mathbf{z})$ with similar efficiency either by monotonicity arguments using $\mathbf{w} = \mathbf{y}^2 + \mathbf{z}^2$ and the enclosure

$$\mathbf{f} = \left[\underline{x}/\sqrt{\underline{x}^2+\overline{w}}\,,\ \overline{x}/\sqrt{\overline{x}^2+\underline{w}}\right],$$

or by the modal procedure described in the recent patent by HAYES [29].

Modal coercion theorems for obtaining more narrow enclosures of ranges require total monotonicity (i.e., fixed direction monotonicity with respect to each occurrence of the relevant variables), whereas simple monotonicity suffices in the standard interval approach. For example, the range of the function defined by the expression $e(x,y) = (xy-1)/(x+y+1)$ for $x \in \mathbf{x} \geq 0$ and $y \in \mathbf{y} \geq 0$ can be found exactly using Theorem 7.4, by taking the hull of the function values at the four corners of the box. But for $\mathbf{x} = \mathbf{y} = [0,2]$, modal theory is not applicable to the expression since it is not totally monotonic. However, as Nathan Hayes (personal communication) observed, the equivalent expression $e'(x,y) = xy/(x+y+1) - 1/(x+y+1)$ leads to the optimal modal enclosure $\mathbf{f} = \mathbf{x} * \mathbf{y}/(\mathbf{x}^* + \mathbf{y}^* + 1) - 1/(\mathbf{x}+\mathbf{y}+1)$. Similar cumbersome reformulations are possible for more complex fractional multilinear expressions such as $(x-1)(y-1)(z-1)/(x+y+z+1)$ for $x,y,z \in [0,3]$.

Moreover, there are variants of this monotonicity argument using bicentered forms (NEUMAIER [50, p. 59]), which work for expressions which are not monotone and cannot be treated by modal theory.

POPOVA [64] and POPOVA & KRAEMER [68] (see also TONON [92]) give improved enclosures for the solution of linear systems, in which they use Kaucher arithmetic. As described in more detail in NEUMAIER [53], this technique works as well with other methods for improving range enclosures, and is then more generally applicable since no total monotonicity is required. Thus replacing coercion theorems by monotonicity theorems makes their results stronger, while bypassing modal theory completely.

For the cases where modal theory applies, the question is therefore one of trade-off between the cost and the quality of different techniques. In particular, for applications involving a huge number of evaluations of ranges of the same expression (as in computer graphics), the evaluation must be fast.

The savings of modal arithmetic can usually be realized by looking at common subexpressions and evaluating these only once, and by checking whether from an interval operation only one of the bounds must be calculated. This can be accomplished automatically by a compiler or preprocessor.

In special situations, Kaucher intervals might be an efficient way of implementing this reasoning, and their use may simplify the programming if an implementation of modal arithmetic is available.

8 Discussion of nonstandard intervals

The preceding two sections gave a concise, general definition of nonstandard interval arithmetic for Kaucher intervals and presented precise versions of some key theorems of the associated modal theory. In this section, we look at nonstandard interval arithmetic and its applications from a more global persoective, giving references only in place of details, and concentrating on the assessment of the value of nonstandard interval arithmetic from an applied perspective.

Algebraic properties. Some algebraic properties of Kaucher arithmetic are better than those for interval arithmetic: Kaucher intervals form a group under addition, and Kaucher intervals not containing zero form a group under multiplication.

These group properties allow one to manipulate formal equations involving Kaucher intervals and to find formal solutions of Kaucher interval equations; see, e.g., Markov [42], Popova [63], Sainz et al. [77, 78]. In simple cases, these formal solutions can be given a useful interpretation. For example, the Kaucher solution $\mathbf{x} = \mathbf{b}/\mathbf{a}^*$ of the formal equation $\mathbf{a} * \mathbf{x} = \mathbf{b}$ with ordinary intervals $\mathbf{a}, \mathbf{b}$ provides (if the result is an ordinary interval) a set such that $ax \in \mathbf{b}$ for all $a \in \mathbf{a}, x \in \mathbf{x}$.

However, this recipe fails when $0 \in \mathbf{a}$; for example, as mentioned by Shary (cf. Appendix C), if $\mathbf{a} = [-1, 1]$ and $\mathbf{b} = [0, 1]$ then the formal equation has no solution but $\mathbf{x} = [0, 0]$ satisfies $ax \in \mathbf{b}$ for all $a \in \mathbf{a}, x \in \mathbf{x}$. This calls for a generalization of Kaucher division by intervals containing zero, but it seems that this has not been explored in the literature.

The example generalizes to linear problems with Kaucher interval coefficients, see the survey by Shary [88]; but except in such simple cases, the solutions of such equations are not interpretable or have only very weak interpretations not flexible enough for applications. Similarly, algebraic manipulations such as generalized LU-decompositions (Goldsztejn & Chabert [19]) become possible; but again the results are hardly interpretable.

Therefore, the improved algebraic properties are difficult to exploit in real applications. The practically useful properties of Kaucher arithmetic all arise from their modal semantics, not from the fact that one has these two groups around. Thus the group properties are nice but seem to be nearly irrelevant in applications.

Applications of formal algebraic equations to range enclosures can usually be formulated and solved without the explicit use of nonstandard interval arithmetic. For example, the method derived by SHARY [87] using Kaucher arithmetic was reanalyzed in NEUMAIER [51] in terms of standard interval arithmetic (Kaucher intervals were used there only to provide the equivalence with Shary's work), leading to stronger theoretical properties of the method.

Other algebraic properties of interval arithmetic become worse for Kaucher intervals: Unlike in ordinary interval arithmetic, all intervals containing zero in their interior become zero-divisors. The subdistributivity of ordinary interval arithmetic only holds in an algebraically unattractive, modified version. (See POPOVA [65] for a complete discussion of distributivity properties.) In particular, unlike for integral domains whose completion results in a field, the "completion" of interval arithmetic by Kaucher intervals results in an algebraic structure far from a field.

Further discussions of algebraic properties can be found, e.g., in DIMITROVA et al. [3] and MARKOV [40, 41, 43].

Note also that it is possible to modify interval arithmetic such that the distributive law remains valid (NEUMAIER [47]), but again, this has only theoretical significance, as it introduces extra pessimism in the enclosures.

Inner approximations and quantified constraint satisfaction. The second coercion theorem implies that (in exact arithmetic) if e is an expression in x in which all variables are uni-incident then $Y \subseteq e(\mathbf{x})$ implies that, for all $y \in Y$ there is an $x \in \mathbf{x}$ such that $y = f(x)$. However, in finite-precision arithmetic, this does not hold with the ordinary outward-rounded interval arithmetic but needs inner rounding to get an inner approximation. If outward-rounding Kaucher or modal arithmetic is available then such an inner approximation can be found by computing $e(\mathbf{x}^*)^*$; thus the correct test is one for $Y \subseteq e(\mathbf{x}^*)^*$; cf. POPOVA [64, pp.129-130], POPOVA & KRAEMER [68].

Applications of Kaucher intervals to the computation of inner approximations of solution sets (tolerance analysis) or the hull of solution sets of systems of equations are discussed in GOLDSZTEJN [16], GOUBAULT & PUTOT [21], POPOVA & KRAEMER [68].

The solution of more general linear systems with quantified coefficients, generalizing bouth outer and inner solution sets, is discussed in terms of AE-solution sets and Kaucher arithmetic by SHARAYA [83] and SHARY [84, 85, 86, 88]. For solving quantified nonlinear constraints using Kaucher intervals and modal arithmetic, see GRANDÓN et al. [22], HERRERO [31], SAINZ et al. [77, 78], and SHARY [88, p. 378]. The requirements on nonlinear systems seem to be quite restrictive.

Inner approximations to the hull of solution sets can also be found with ordinary interval analysis by means of centered forms, where they serve to bound the overestimation of an outer enclosure. See, e.g, Chapter 5.5 of my book [50] and the papers NEUMAIER [52] and NEUMAIER & POWNUK [56, Sections 2 and 5]. For the linear case, see also NEUMAIER [49], where Theorem 2 can easily be transformed into inner enclosures with the quadratic approximation property, which therefore are excellent when, as often is the case, the input intervals are narrow.

For inner approximations of solution sets (tolerance analysis) with ordinary intervals, see, e.g., NEUMAIER [48], and NEUMAIER & MERLET [55], and the book by FIEDLER et al. [5]. See also KREINOVICH et al. [38].

For solving quantified nonlinear constraints using ordinary interval arithmetic, see., e.g., RATSCHAN [72].

Developed theory. Interval analysis is a beautiful and rich theory. Its beauty comes in case of intervals from providing rigor (something not available in real number computations without intervals) and from general theorems which apply universally. My 1990 book [50] contains over 200 pages of theory, summarizing what had been developed in the first 30 years since 1959 when MOORE & YANG [46] seriously started interval analysis. There are a lot of nice results, and interesting relations to other fields of mathematics.

On the other hand, what is highly developed in Kaucher theory is far too algebraically oriented to be useful in real applications, and modal theory is quite shallow. The most useful modal interpretation and coercion theorems need very stringent requirements (uni-incidence or strong monotonicity properties) in their assumptions. In the same time span of 30 years since Kaucher and Gardeñes laid the foundations, there is hardly any further theory with nice results, and practically no analysis of algorithms. All applications are crafted by hand.

Together with the fact that Kaucher intervals do not provide any new level of quality (as intervals did with the step from approximate to rigorous reasoning), since what they achieve can be duplicated within the standard interval paradigm, it is clear that Kaucher intervals do not play the fundamental role ascribed to them by their proponents.

The centered form for Kaucher intervals introduced in GOLDSZTEJN [18] is a noticable improvement for narrow intervals when optimality theorems do not apply. It leads to the quadratic approximation property well-known from centered forms for ordinary intervals. However, the Kaucher centered form has almost completetly gotten rid of Kaucher arithmetic since only one inner product is performed in it, and one can easily remove this last trace of it.

The existence proving power is very weak in more than one dimension when applied naively; see [17], Example 13.2. More powerful results are obtained using centered forms [18], but they just reproduce an old existence test of Hansen and Sengupta.

In general, it seems that once one tries to do something more powerful with Kaucher intervals one ends up with something covered by classical interval analysis...

Use in standard interval analysis. Sometimes, alternative proofs or algorithms concerning results in standard interval analysis are available through modal theory. An example is GOLDSZTEJN & JAULIN [20, Lemma A.5]. See also DIMITROVA et al. [3].

Applications. The patent mentioned in the beginning of Section 2 is about a recursive application of an optimal enclosure of the linear interpolation operator in terms of modal arithmetic – rather than in terms of real arithmetic with directed rounding, as presented in Section 3 – to the de Casteljau algorithm and the de Boor's algorithm.

Modal intervals recently have generated a number of interesting applications, not only in computer vision but also to robotics and control. The thesis by HERRERO [31] treats some applications in robotics and control.

Other recent applications can be found on the web site [45] of the Modal Intervals and Control Engineering (MICE) group in Girona (Spain). Unfortunately, much of this stuff is in Spanish and I haven't thoroughly explored this site. The publication page only allows to locate publications known by key words, which makes it difficult to get an overview of their work. From email exchanges with Miguel Sainz and Josep Vehi from this group, I obtained a number of papers on applications, including the papers [1, 2, 10, 32, 79].

I also obtained from them a thesis by FLOREZ [6] on interval ray-tracing of implicit surfaces. The application of modal arithmetic in this thesis seems to be restricted to the test on p.79 of [6]. But there, the modal arithmetic is superfluous, as in his expression fR only proper intervals appear. An ordinary interval version equivalent to his equation (5.6) (with Inn corrected to Out) is, in the terminology of [6]: *If an ordinary outward rounded interval extension $f(X, Y, t_k)$ has definite and opposite signs (≤ 0 resp ≥ 0) for two values of t_k then we have a complete hit.* This follows directly from the intermediate value theorem, without invoking any modal theory.

Thus, the applications to implicit ray tracing in [6] do not really depend on nonstandard intervals.

POPOVA et al. [67] gives applications of modal intervals to uncertainty propagation in mechanical systems. There are also applications by GOUBAULT & PUTOT [21] on linear discrete dynamical systems and by WANG [93, 94, 95, 96, 97] on applications of modal intervals to tolerance analysis and imprecise probability.

Doubts on rigor. The theory on Kaucher arithmetic is presented in the literature in a mathematically precise form. However, the same cannot be said of modal theory. The basic reference to modal theory in recent publications is GARDEÑES et al. [13]. But this paper and previous papers that it summarizes suffer from weaknesses in formal rigor (imprecise formulations and often only proof sketches) which make it difficult to understand what is valid under which assumptions.

There are many subtle issues in applying the interpretability and coercion theorems, which makes their application error prone without very clear foundations. The theoretical conditions under which the coercion theorems hold for intervals with infinite endpoints are not known; note that some of the proofs require the application of fixed-point theorems where boundedness may be essential. (See MARKOV [40] for some theory on unbounded Kaucher intervals.)

The result of a Kaucher computation has a meaning only in special cases delineated in so-called interpretability or coercion theorems. Different versions of the operations for non-commutable binary operations seem to require commutability assumptions in various places. But the terminology of rational functions in [13] is ambiguous, sometimes referring to expressions in arbitrary operations, sometimes to expressions in commutable operations. This makes the contents of the theorems difficult to interpret correctly. The term "totally commutable" used in Theorem 4.7 of [13], and some other notation earlier, such as $\min(x, A)$ or X_f and X_i is completely undefined.

GOLDSZTEJN [17, 18] puts part of modal theory on a more rigorous footing, giving clear and detailed proofs of many results for the case when only the four arithmetic operations $+, -, *, /$ are involved.

There has been almost no discussion about the needs in case of only partially defined unary or binary operations. In particular, the behavior of the arithmetic for operations like division that are not everywhere defined have been explored so far only in POPOVA [61] (at the end of Section 2), where the result is allowed to consist of two intervals, thus leaving the domain of Kaucher intervals.

Finite-precision versions and implementations. In finite-precision arithmetic, rounded versions of the arithmetic are needed. In modal theory, modal arithmetic seems to be coupled to outward rounding, dual modal arithmetic to inward rounding.

A point that received little attention in the literature on solving formal interval equations using Kaucher arithmetic is the fact that formal equations can generally be solved only approximately on a computer; how to interpret approximate solutions in terms of rigorous statements about the results is not clear.

There are several implementations of Kaucher arithmetic for the standard arithmetic operation $+, -, *, /$ available: One by POPOVA [61] in Pascal-XSC, one by POPOVA & ULLRICH [69] in MATHEMATICA (which can be used in a symbolic mathematics environment for formula manipulations), and one by the SIGLA/X group [89] in C++. The MATHEMATICA implementation also supports lists of Kaucher intervals, which enables an unrestricted optimal division operation (different from the one suggested in the Vienna Proposal [54] for interval standardization).

The papers [61] and Popova & Ullrich [70] treat the rounding properties of Kaucher arithmetic in finite-precision arithmetic. The latter paper contains a formal specification of Kaucher arithmetic, compatible with the IEEE-754 floating-point standard; in particular, it addresses the empty interval and infinity and NaN as bounds, but not division by intervals containing zero. The only binary arithmetic operations covered in these papers are $+, -, *, /$; in addition, there are a variety of monadic operations and functions, relational operations, lattice operations, utility functions, etc.. Popova [62] discusses the efficiency of implementations of Kaucher multiplication versus standard interval multiplication.

The patents by Hayes [24, 25, 28] detail an implementation of modal arithmetic for the standard arithmetic operation $+, -, *, /$, which therefore coincides with Kaucher arithmetics.

Possible hardware support. Hayes [28] advocates a modal interval processor for hardware modal arithmetic. However, having (s suggested by Hayes) (i) a Kaucher/modal arithmetic pipeline and the usual floating-point pipeline will probably have a similar hardware complexity and throughput speed as having (as suggested in the companion paper Neumaier [53]) (ii) three floating-point pipelines, each one dedicated to one particular rounding mode (up, down, nearest), with the directed modes following Part 7 of my proposal, on problems where both can be used.

Both options (i) and (ii) will have similar costs in hardware. With (i) being available but not (ii), it will pay to adapt the algorithms to optimally use the modal processor. Conversely, with (ii) being available but not (i), it will pay to adapt the algorithms to optimally use the directed rounding pipelines.

If, in both cases, adaptation is done optimally. there will probably be not much difference in efficiency on problems where both a modal and a non-modal approach are possible. But (ii) has more flexibility and hence will work in more cases. If hardware support for Kaucher arithmetic is available, the modal techniques will of course become part of the collection of tricks discussed in Neumaier [53] for improving range enclosures.

Apart from this isssue, special hardware may be fine-tuned to get optimal performance for very specific applications. However, it doesn't make much sense to ask for special hardware for each function someone may be using. Rather, the techniques for improved range enclosures based on general techniques – whether they are based on modal arithmetic, or monotonicity, or other transformations such as those discussed in Neumaier [53] – should be integrated into (possibly hardware-dependent) preprocessors that create automatically or semiautomatically customized pieces of software that execute fast on whatever hardware is available.

9 Nonstandard intervals in the Vienna proposal

In the Vienna proposal on interval standardization [54], nonstandard arithmetic is accounted for in Sections 1,2, 2.4, and 2.5.

```
1.2. The set of intervals

Textbook intervals are closed and connected sets of real
numbers, the closed intervals familiar from mathematical textbooks.
Nonempty textbook intervals are represented by two bounds, their
infimum and their supremum. (See also Section 1.6.)

Standard intervals denote textbook intervals whose infimum
and supremum are representable by two B-numerals.
Nonstandard intervals do not denote textbook intervals.

[... This] also excludes the consideration of arithmetic based on
nonstandard intervals (Kahan arithmetic, Kaucher arithmetic,
modal arithmetic).

However, certain applications merit a minimal support of these not
mutually compatible nonstandard variants of interval arithmetic.
The availability of nonstandard intervals allows (but does not force)
implementors to extend the functionality of interval arithmetic to
handle either Kahan arithmetic, or Kaucher arithmetic, or modal
arithmetic consistent with their traditional interpretation.


2.4. Nonstandard intervals

An interval [l,u] is nonstandard if it is not a standard interval.
No specification is made for the meaning of nonstandard intervals.

In particular, [-Inf,-Inf] and [+Inf,+Inf] are nonstandard intervals;
intervals with exactly one bound NaN, and intervals [l,u] with l>u
are also nonstandard.


2.5. Constructors and operations checking the semantics

Here for brevity 1 = true, 0 = false.
```

```
 3. There is an operation isStandard(xx) that outputs 1 or 0
    depending on whether or not the input interval xx is standard.

 5. There is an operation standardInterval(l,u) that returns the
    tightest interval containing the values of the two numerals
    l and u if l<=u, and Empty otherwise.

    l and u may be numerals of different type, in which case
    type overloading is needed.

 6. There is an operation anyInterval(l,u) that creates the
    interval [l,u] from two B-numerals l and u without
    checking whether or not it is standard.

12. There is an operation dual(xx) that returns for any (standard or
    nonstandard) interval xx=[l,u] the (standard or nonstandard)
    interval [u,l].
```

Whether operations on nonstandard intervals are done according to Kahan arithmetic, Kaucher arithmetic, modal arithmetic, or remain unused is left to the implementor.

10 Summary

Nonstandard intervals have some interesting applications, and were repeatedly the motivating context of algorithmic advances. Therefore they should be compatible with any standard for interval arithmetic.

A stronger support is not appropriate in view of the above weaknesses together with the additional complexity of implementing nonstandard intervals and the resulting slowdown in software implementations, even for ordinary interval operations.

Thecurrent successes of nonstandard arithmetic are not primarily due to the nonstandard arithmetic, since they can be duplicated by arguments from the standard toolbox of interval analysis.

The most interesting of the current applications can be implemented without explicit nonstandard arithmetic and without loss of efficiency. I looked at a number of real applications, especially at those in computer graphics (where several patents indicate commercial interest), and found that (unlike in more theoretical work) nonstandard arithmetic was never needed to derive the relevant results.

In particular, modal theory just seems to be a particular wrapper around certain monotonicity arguments from interval analysis. One can apparently get all benefits of modal theory just from monotonicity considerations and ordinary interval arithmetic without any extra theoretical baggage (which proved error prone and difficult to learn in the past), and without the need for any special implementation. The extra theoretical and algorithmic complexity needs to be justified by impressive performance in important cases that go beyond what can be done with standard tools from interval analysis.

Moreover, the semantics of the result of expression evaluation in Kaucher or modal arithmetic is incomplete; one needs restrictive interpretability conditions, and there is the need for coercion. In addition, the correct assumptions for some of the interpretation and coercion theorems seem to be in doubt.

In summary, there are many signs of the lack of maturity of nonstandard interval arithmetic:

- For Kaucher intervals, there are conflicting definitions for binary operations that are not commutable and no definitions for division by intervals containing zero; for Kahan intervals, the definitions are completely different.
- the available results are somewhat poorly organized and difficult to understand;
- the current expositions of modal theory lack sufficient rigor to merit being trusted without detailed checking;
- their theory is little known in the interval community;
- as a result, there is not sufficiently widespread experience of its use;
- apart from some applications in linear algebra, the practical use of nonstandard intervals is still in an experimental stage;
- there is no evidence of usage on real-life problems beyond what can also be achieved without nonstandard intervals.

This does not exclude the possibility that, some time in the future, nonstandard intervals might occupy a more prominent place within interval analysis.

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A A quote from Svetoslav Markov

I asked Svetoslav Markov (Bulgaria), who worked on Kaucher arithmetic for now over 30 years, and whose many papers on the subject include [3, 40, 41, 42, 43, 44], for a statement to include in this report. He sent me the following:

```
The IEEE standard for interval arithmetic should support Kaucher
interval arithmetic and midpoint-radius presentation.

Kaucher interval arithmetic (KIA) [Kaucher80] is the algebraic
extension to standard interval arithmetic (SIA). All operations in
KIA are obtained at no (or little) additional cost in compare to SIA.
The formulae for addition and inclusion remain the same, care should
be given only to multiplication.  KIA is the algebraic foundation to
modal and Kahan interval theories, theories which provide  semantic
interpretations for various problems in interval analysis.
KIA can be used for the formulation of algorithms involving modal or
Kahan intervals. In addition, via KIA, one can easily implement the
inner interval operations for standard intervals [Markov95].
Using the directed form of Kaucher  intervals [Markov95] one can
find numerous new algebraic properties in standard interval
arithmetic [DimitrovaMarkovPopova92]. Using KIA one can manipulate
with interval expressions/equations in a  manner similar to the one
with numbers. A demonstration of the use of these  properties for
algebraic manipulations (transformations) can be found e. g. in
[Markov99]. In Kauchers theory there is a distributive-like law,
which is easily applicable in the practice, especially when used in
the compact form using equality relation (not inclusion).
```

As originally developed by E. Kaucher, KIA involves the ''ring-like'' interval algebraic structure, that is the structure based on addition, multiplication and inclusion. Recently the ''vector-like'' part of the theory (using multiplication by scalars) has been accomplished [Markov05]. This essential part of KIA theory shows that interval spaces are direct sums of two spaces of different nature, implying that midpoint-radius presentation is the algebraically natural presentation of intervals. The theory nicely corresponds to engineering and computational practice where an usual interpretation of intervals is the one of approximate numbers. In accordance to this interpretation intervals are presented by two numbers of different nature (midpoint and radius). The presentation of the midpoint (which in the above mentioned theory belongs to a linear space) is the same as the one of usual real, resp. fp-numbers. However, the radius does not belong to a linear space and its presentation need not be the same as the one for the midpoint. An important practical reason for such distinction is that the radius is interpreted as an upper bound for the error and does not need to be a fp-number with a long mantissa. In practice a mantissa of length one (or maximum two) suffices, cf. [van Emden04]. Allowing such short mantissa for the radius may greatly accelerate the performance of IA operations. Midpoint-radius presentation allows the use of very narrow intervals, cf. [Rump99]. FP-realization of midpoint-radius arithmetic presents no problems [Rump99]. Formulae for MR-multiplication that are valid for Kaucher intervals are given in [Markov99, Markov06]. It is recommendable that MR-multiplication is realized also in the ''centred'' version [Rump99, KulpaMarkov03]. Centred multiplication can be extended for Kaucher intervals.

KIA can be easily implemented in inf-sup form as well. Since this form is commonly used by specialists in interval analysis, it is recommendable to be supported by the IEEE standard. Fp-presentations of KIA operations/relations are well known [Popova98].

A common argument against KIA is that any method using KIA can be formulated in terms of SIA. This is of course true, as KIA is defined by SIA as algebraic extension. The situation is similar to the one with numbers. For example, one can avoid the use of negative numbers, as they are defined via nonnegative ones. Similar arguments can be used against the use of vectors and matrices (methods using vectors/matrices can be formulated in terms of their components). However, there are already sufficiently many applications showing that the use of KIA is very convenient and fruitful, and the implementation of KIA is at (almost) no additional cost. A standard which does not support KIA will certainly be soon replaced by one that supports it. Same is true for the support of MR-presentation.

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B A quote from Miguel Sainz

Miguel Sainz, the head of the Modal Intervals and Control Engineering (MICE) group in Girona (Spain) – which is the main promotor of modal theory –, and whose papers on the subject include [1, 2, 7, 8, 9, 10, 13, 32, 77, 78, 79], sent me the following summary of the essence of modal intervals:

```
Modal Interval Analysis is a completion of the Classical Interval
Analysis which includes it as a particular case.  Let I*(R) the set of
modal intervals, of which elements we denoted by A = [a,b],
to distinguish of the set of classical intervals I(R) of which
elements are denoted by A' = [a,b]'.

Modal intervals I*(R) were created by Ernest Gardenyes in the 90'
to overcome a set of shortcomings that classical intervals I(R) have:

1) I*(R) completes the semiplane of Moore and define a one to one
correspondence between the set of points of the plane and I*(R).

2) Equations A' + X' = [0,0]' and A' * X'  = [1,1]'  have no solution
on I(R) because the difference is not the opposite operation to the
addition on I(R).
This anomaly can be solved by adding certain elements to the system
I(R), the dual intervals (by ex. the new element [2,1]:=Dual([1,2])),
which extend to the new system the preliminary structure
(I(R),+,-,*,/) by means of compatibility criteria. In I*(R) the
equations A + X = [0, 0] and A * X = [1,1] have unique solution.
So the completed structure (I*(R),+,-,*,/) is a group for the operation
of addition, as well as for the multiplication of intervals not
containing zero.

3) The equation A' + X' = B' has no solution in I(R) when
w(A') < w(B'). But even when w(A') < w(B') its solution cannot be
```

obtained by a rational computation in I(R). In I*(R) the solution of A + X = B is X = B - Dual(A). Similarly for the equation A * X = B, when A' does not contain 0.

4) Concerning to the lattice operation called meet, in I(R) is not always defined (e.g. [-5,-3]' meet [6,8]' is not defined).
But in I*(R)

[-5,-3] meet [6,8] = [6,-3]

which is a modal interval.

5) The intervals of I(R) are able to be rounded outside, but this rounding is not always enough to compute other results rounded outside. For example, an outer solution of the equation A'+ X ' = B' could be obtained from the transformed equation Inn(A')+ Y' = Out(B') but not from Ext(A') + Z' = Out(B') The problem is that the inner rounding is not always defined in I(R). So

Out([1/3,1/3]) = [0.3332,0.3334] (similar result than in I(R))
Inn([1/3,1/3]) = [0.3334,0.3332] which is an element of I*(R) and not of I(R).

6) Concerning the modes of selection of the values delimited by an interval, the information corresponding to the elements of system I(R) is already saturated once that they are given as subsets of R by its lower and higher bounds. To specify the logical mode of this interpretation, and doing it accessible to the analysis, one introduces an essential tool for the analysis by modal intervals: quantifiers universal, U, and existential, E. The scope of the failure, of not having the additional information provided by the association of quantifiers to interval bounds, can be illustrated by the following example of four interval statements referring to the relation a + x = b on the 'real line' and keeping the requirements a in [1,2]' and b in [3,7]:

1. U(a,[1,2]') U(x,[2,5]') E(b,[3,7]') a + x = b;
2. U(a,[1,2]') U(b,[3,7]') E(x,[1,6]') a + x = b;
3. U(x,[1,6]') E(a,[1,2]') E(b,[3,7]') a + x = b;
4. U(b,[3,7]') E(a,[1,2]') E(x,[2,5]') a + x = b.

Statements 1. and 4. hold for the well known solution X = [2,5]' of the classic set-theoretical interval equation [1,2]' + X' = [3,7]'. Statements 2. and 3., in spite of making full sense, are out of reach for this classic interval equation of the same form as the 'real'

one a + x = b. In case the addition was a digital operation, the practical interval equation would be, indeed,
[1,2]' + X' \subseteq [2.9,7.1]' with the only possible rounding for set-theoretical digital intervals: the classic 'outer rounding'.
In this case the statement 1. would become

1'. U(a,[1,2]') U(x,[2,5]') E(b,[2.9,7.1]') a + x = b

but statement 4. would cease to be valid, since the outer rounding of [3,7]' would be incompatible with the U-quantifier. Moreover, we should not forget that, even in the very simple case of the statement 1., the solution X' of the corresponding interval equation [1,2]' +X' \subseteq [3,7]' could not be obtained by any operation within the system of the classic set-theoretical intervals.
With modal intervals these logical shortages are overcome:

1. The solution of the equation [1,2] + X = [3,7] is

X = [3,7] - Dual([1,2]) = [2,5]

so that [1,2] + [2,5] = [3,7]. In accordance with the semantic theorems of the Modal Interval Analysis, it is equivalent to

U(a,[1,2]') U(x,[2,5]') E(b,[3,7]') a + x = b.

2. The solution of the equation [1,2] + X = [7,3] is

X = [7,3] - Dual([1,2]) = [6,1]

so that [1,2] + [6,1] = [7,3]. In accordance with the semantic theorems of the Modal Interval Analysis, it is equivalent to

U(a,[1,2]') U(b,[3,7]') E(x,[1,6]') a + x = b.

3. The solution of the equation [2,1] + X = [3,7] is

X = [3,7] - Dual([2,1]) = [1,6]

so that [2,1] + [1,6] = [3,7]. In accordance with the semantic theorems of the Modal Interval Analysis, it is equivalent to

U(x,[1,6]') E(a,[1,2]') E(b,[3,7]') a + x = b.

4. The solution of the equation [2,1] + X = [7,3] is

X = [7,3] - Dual([2,1]) = [5,2]

so that [2,1] + [5,2] = [7,3]. In accordance with the semantic theorems of the Modal Interval Analysis, it is equivalent to

U(b,[3,7]') E(a,[1,2]') E(x,[2,5]') a + x = b.

So I*(R) is an geometrical, algebraical and logical completion of I(R) in a similar way complex numbers are a completion of real numbers which provide solution to any quadratic equations. A ''proper'' modal interval ([a,b] with a<=b) is identifiable to a classical interval [a,b]'. So, all the results of the Classical Interval Analysis are results of the Modal Interval Analysis. Moreover an interval must be, by default, a modal interval.

In my humble opinion, the modal arithmetic (which includes the classical one) deserves to be implemented in the IEEE standard, not only for practical reasons but to have available the arithmetic of I*(R), which is the complete set of the intervals.

C A quote from Sergey Shary

Sergey Shary (Russia), who developed the use of Kaucher arithmetic in problems of linear algebra involving quantified uncertainty, and whose many papers on nonstandard intervals include [84, 85, 86, 87, 88], sent me the following comments:

1) You ask

> It seems to me that what can be done with modal intervals can be
> done with similar efficiency with ordinary intervals and monotonicity
> considerations.

I do not think so. A trivial example: one cannot move terms from one side of an equation (inequality, inclusion, etc.) to the other side in classical interval arithmetic, while this is possible in Kaucher arithmetic. So, the technique of algebraic transformations in Kaucher interval arithmetic is richer. The same with order relation, when we can take minima and maxima, with respect to inclusion, without unnecessary hampering restrictions.

Most of us admit negative numbers as something obvious, so what about their analogs among intervals? They are helpful for (at least) the same reasons as negative numbers - in easy algebraic manipulations that facilitate 'algebraic' ways of solution of our problems. In particular, when finding 'formal' (aka 'algebraic') solutions to interval equations. At this point I agree with what Miguel Sainz told in items 1)-4) of his letter.

One more justification of Kaucher complete interval arithmetic is that it has minimax nature, and the problems where it may be especially useful are those involving minimaxes and maximins, conflicts of interests, games, etc. How many such problems do we solve at present? The potential of Kaucher interval arithmetic has not been exposed yet ...

An example that is convincing for me (but may prove not sufficiently telling for someone not ''in the context'') is that practical techniques for estimation (inner or outer) of AE-solution sets to interval systems of equations are based on using Kaucher interval arithmetic, not on classical interval arithmetic. The efficiency here is gained mainly by going out of classical intervals. As a short explanation I can add that the ''tolerance problem'' for interval systems of equations and its further generalizations are subsumed under this point.

2) About the term ''modal intervals''

The algebraic structure that forms the basis of the ''modal interval theory'' coincides for the standard arithmetic operations with that invented by Edgar Kaucher in 1973 (in his PhD thesis) as a joint algebraic and order completion of the classical interval arithmetic. At least 6 years before the first results of Spanish group lead by E.Gardenyes. So, even from a historical standpoint we have to call this arithmetic by an appropriate name - ''Kaucher interval arithmetic'', - not connected with the obscure concept of ''modal intervals''.

My friend Miguel Sainz claims in his answer to Arnold Neumaier's questions that ''modal intervals I*(R) were created by Ernest Gardenyes in the 90' to overcome a set of shortcomings that classical intervals I(R) have''.

I should have corrected some dates and accents in this statement, but further historical remarks would not hurt too. The first papers presenting the results by E.Gardenyes and his co-workes appeared in 1979-1982. For example,

Gardenyes, E.; Trepat, A.:
The Interval Computing System SIGLA-PL/1 (0),
Freiburger Intervall-Berichte, No. 79/8 (1979),

Gardenyes, E and Trepat, A.:
Fundamentals of SIGLA, an interval computing system over the completed set of intervals,
Computing, 24 (1980), pp. 161-179,

Gardenyes, E., Trepat, A. and Janer, J.M.:
SIGLA-PL/1 development and applications,
in: Nickel K.L.E. (ed.), Interval Mathematics 1980,
Academic Press, 1980, pp. 301-315,

Gardenyes, E., Trepat, A. and Janer, J.M.:
Approaches to simulation and to the linear problem in the SIGLA system,
Freiburger Intervall-Berichte, No. 81/8 (1981), pp. 1-28,

Gardenyes, E., Trepat, A. and Mielgo, H.:
Present perspective of the SIGLA interval system,
Freiburger Intervall-Berichte, No. 82/9 (1982), pp. 1-65.

These texts presented a substantial piece of theory and applications of Kaucher interval arithmetic (called by another name, of course), and, fortunately, did not use the terms 'modal', 'modality' in connection with 'directed' intervals. However, they came after Kaucher's seminal PhD dissertation of 1973 and his two papers:

Kaucher, E.:
Ueber metrische und algebraische Eigenschaften einiger beim numerischen Rechnen auftretender Raeume,
Ph.D. dissertation, Universitaet Karlsruhe, Karlsruhe, 1973.

Kaucher, E.:
Algebraische Erweiterungen der Intervallrechnung unter Erhaltung der Ordnungs- und Verbandsstrukturen,
Computing Supplement, 1 (1977), pp. 65-79.

Kaucher, E.:
Interval analysis in the extended interval space IR,
Computing Supplement, 2 (1980), pp. 33-49.

Let me point out that items 1)-5) from the list organized by Miguel Sainz in his answer to substantiate the usefulness of ''modal intervals'' affect only algebraic and order properties of interval arithmetic and can be handled using Kaucher interval arithmetic without invoking any logical reasons.

The most spacious item 6) from the list is logical, but, to illustrate his theses, Miguel Sainz repeatedly tells ''In accordance with the semantic theorems of modal interval analysis ... '', thus employing advanced results of the ''modal interval theory''. And my natural question is: well, what would happen if we retain only these ''semantic theorems'' and ''interpretability theorems'' from the whole ''modal interval theory'' giving up the dubious concept of ''modal interval'' and the related things? I suppose that the resulting theory will become much more transparent, understandable and correct.

To sum up, Kaucher interval arithmetic is a core of the whole ''modal interval theory'', and, to my mind, should be put as a cornerstone for their correct substantiation.

On terminology, let me quote the paper
E.Gardenyes, M.A.Sainz, L.Jorba, R.Calm, R.Estela, H.Mielgo, A.Trepat:
''Modal Intervals'',
Reliable Computing, 7 (2001), pp. 77-111:

''A modal interval X is defined by a pair formed by a classical interval and a quantifier: X = (X',QX)''.

The ''modal interval theory'' in the form presented by its creators starts from joining the 'direction' of an interval with a quantifier it ''bears''. But after several paragraphs of the theory, special ''interpretability theorems'' are derived to establish logical meaning of this or that result of an expression involving these 'directed' intervals. WHY? One has already determined the sense of the 'direction' of an interval at the beginning of the ''modal interval theory'', once and forever! So, the logical meaning of the results of computations involving 'directed' intervals should be automatically derived from these formulas, and the so-called interpretability theorems are unnecessary!

Put differently, if one still needs the ''interpretability theorems'', with their conditions and reservations, then the stiff coupling of logical quantifiers with intervals is of no use, and we had better not to fix it at all, in order not to confuse potential ''customers'' of our theory.

The term ''directed intervals'' used by Svetoslav Markov for some time seems to me more correct and adequate to the essence of our intervals that may be either proper or improper.

3) What about ''modal interval analysis''?

Its characteristic feature, as one can guess from its expositions, is that we can formulate problems broadly using logical formulas, applying quantifiers, etc.

But the same things can be done in usual Interval Analysis too. For example, the so-called interval tolerance problem (very familiar to me), involving in its formulation various quantifiers in the left-hand side and right-hand side of an equation, was first formulated and solved without ''modal'' (Kaucher, directed) intervals. So, the first observation is that ''modal interval analysis'' does not present something radically new in the problem statements as compared with classical Interval Analysis. Moreover, I would say that ''modal interval analysis'' borrows its subject matter from usual interval analysis. And not completely, but only partially, since every problem treated by the ''modal interval theory'' can be formulated and posed in our usual verbal form without applying the ''modal language'' involving logical formulas, predicates, quantifiers, etc. And not vice versa!

The method of ''modal interval analysis'' is, no doubt, original, and stems from the wish to express logical consructions and logical meanings by algebraic tools. I myself have made much to develop this technique for various algebraic problems, having experienced both fun and pleasure as the result of my work with it. But the technique is not omnipotent, which is demonstrated by an example of interval linear tolerance problem for the simplest equation [-1,1] x = [0,1]. The problem is: find all such x's that the product Ax falls into the interval [0,1] of the right-hand side for every A from [-1,1]. Let someone try to solve it by ''modal interval analysis'' (I myself would call it ''formal-algebraic approach''). One will not be able to solve this simplest equation without additional tricks (not covered by ''modal interval analysis'') since the left-hand side multiplier [-1,1] contains zero in the interior. Interval Analisys solves this problem ... So, the claim that ''all the results of the Classical Interval Analysis are results of the Modal Interval Analysis'' should be classified as somewhat rash.

Furthermore, in the technique employed by ''modal interval analysis'', much is taken from that of classical Interval Analysis! We can see the same ''treating uncertainty sets as entire objects through establishing arithmetical and analytical operations, relations, etc., between them''. Exploiting the same concepts, such as inclusion monotonicity, set-theoretical operations and their extentions, etc. I daresay that the resemblence in the methods of Interval Analysis and ''modal interval analysis'' is huge, while the differences are too little for ''modal interval analysis'' to be called a separate branch of science!

To summarize. ''Modal interval analysis'' is merely a method within Interval Analysis, quite powerful and general, but it does not constitute a separate science (not to say more), insofar as its subject matter does not differ from that of Interval Analysis itself. I.e., ''modal interval analysis'' does not solve problems that are different from those Interval Analysis solves. And some of such problems can be solved by traditional methods better than by ''modal interval analysis''. So, the term ''modal interval analysis'' should be turned down, as pretentious and misleading about real place occupied by this technique and its real value. I would suggest something like ''modal theory'', but not ''analysis''.

4) Concluding remarks

To conclude, I must say that my criticism does not prevent me from acknowledging great significance of some particular accomplishments obtained within ''modal interval technique''. For example, the theorems on interpretation of results of the expressions with proper and improper intervals as arguments (corrected and polished if necessary) will remain forever, as a firm basis of computations with Kaucher interval arithmetic when computing minimaxes and so on. The modern interval analysis does not solve such problems widely, but, as the time passes, the importance and practicality of these results will only grow.

Overall, I'm very optimistic about the future of the techniques based on Kaucher complete interval arithmetic and refined and updated results of ''modal interval theory''. I can forsee that they will become cornerstones of the forthcoming interval analysis.

D A quote from Alexandre Goldsztejn

I asked Alexandre Goldsztejn, a young French researcher who wrote his 2005 thesis on generalized intervals, and whose papers on the subject include [16, 17, 18, 19, 20], for a statement to include in this report. He sent me the following:

Moore's ''original interval arithmetic'' (which was defined only for intervals included inside the definition domain of the real functions) has been extended in several ways. Among other extensions of the ''original interval arithmetic'' we find ''the cset arithmetic'', the ''Kaucher arithmetic'', and the ''modal arithmetic''.

I have come to the conclusion that the modal intervals (defined as pairs made of a classical interval and a quantifier) are not the right way of obtaining improved interpretations in terms of quantified propositions. I have shown that we can obtain the same enhanced interpretations defining a modal arithmetic on Kaucher intervals directly, advantageously getting rid of pairs made of a classical interval and a quantifier. Thus, I think it makes sense to consider a modal arithmetic defined on Kaucher intervals and that this arithmetic should be discussed for inclusion in the standard.

In my opinion, the IEEE working group is working on defining a new extension of the ''original interval arithmetic'' that is reasonable to call the (IEEE) ''standard interval arithmetic''. With this point of view, it is clear to me that neither the ''Kaucher arithmetic'' nor the ''modal arithmetic'' (nor the ''modal arithmetic on Kaucher intervals'') is an extension of the ''standard interval arithmetic''. For example,

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sqrt([1,-1]) and 1/[1,-1] are not defined in the Kaucher or the modal
arithmetics. On the other hand, the ''standard interval arithmetic''
raises difficult issues: For example, we need a flag to be raised when
an expression is evaluated for arguments that not included inside its
definition domain (e.g. for the application of existence theorems).
These issues would all have to be analyzed in the context of Kaucher
and modal arithmetics.

Is it possible to define a modal arithmetic on Kaucher intervals that
would extend the ''standard interval arithmetic''? This question is not
answered today, and I think this is not the right moment to try
answering it: We need today a ''standard interval arithmetic'' that is
accepted by everybody in our community, and this is a hard enough task
by itself.
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E A quote from Luc Jaulin

I asked Luc Jaulin (France), coauthor of the book "Applied interval analysis" [34], and author of work in the applications of intervals to important robotics problems, including several papers [20, 32, 34] involving modal arithmetic:

What is your assessment of modal intervals as regards their future? It seems to me that what can be done with modal intervals can be done with similar efficiency with ordinary intervals and monotonicity considerations. At present I am trying to assess the significance for and impact on real applications to be able to make a competent proposal (either way is possible) that has a chance to find a majority.

With his permission, I quote his reply.

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> What is your assessment of modal intervals as regards their future?

I am pessimistic about their future. The theory is nice but it is too
difficult to handle properly.


> It seems to me that what can be done with modal intervals can be
> done with similar efficiency with ordinary intervals and monotonicity
> considerations.

I agree. You should know that I am not a specialist
of modal intervals.
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> At present I am trying to assess the significance for and impact
> on real applications to be able to make a competent proposal
> (either way is possible) that has a chance to find a majority.


My feeling is that it should not be included for the following reasons.
(i) The modal theory is not easy to deal with and may lead
    to misunderstandings if the semantic is not clearly understood.
(ii) It is easy to add a layer over a classical interval arithmetic
    library to make an efficient modal interval library.
(iii) To me, an emptyset should be considered as an interval,
   the intersection of two intervals should always be an interval,
   an interval should be a subset of R.
   To my knowledge, it is not consistent with the modal theory.
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