On the structure of clouds

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Abstract. Clouds, recently introduced by the author, give – within the traditional probabilistic framework – a concept for imprecise probability, with which quantitative conclusions can be derived from uncertain probabilistic information. A cloud is to a random variable more or less what an interval is to a number.

In this paper, some structural results about clouds are derived. In particular, it is shown that every cloud contains some random variable.

Keywords: cloud, imprecise probability, jump, confidence region, uncertainty, interval

1 Introduction

A **cloud** (Neumaier [1]) over a set $\mathbb{M}$ is a mapping $\mathbf{x}$ that associates with each $\xi \in \mathbb{M}$ a (nonempty, closed and bounded) interval $\mathbf{x}(\xi)$ such that

$$]0,1[\ \subseteq \bigcup_{\xi \in \mathbb{M}} \mathbf{x}(\xi) \subseteq [0,1]. \tag{1}$$

$\mathbf{x}(\xi)$ is called the **level** of ξ in the cloud $\mathbf{x}$, and $\overline{\mathbf{x}}(\xi) - \underline{\mathbf{x}}(\xi)$ is called its **width**. A random variable x with values in $\mathbb{M}$ **belongs to** a cloud $\mathbf{x}$ over $\mathbb{M}$, written $x \in \mathbf{x}$, if

$$\Pr(\underline{x}(x) \geq \alpha) \leq 1 - \alpha \leq \Pr(\overline{x}(x) > \alpha) \quad \text{for all } \alpha \in [0,1]. \tag{2}$$

Here Pr denotes the probability of the statement given as argument, and it is required that the sets consisting of all $\xi \in \mathbb{M}$ with $\underline{x}(\xi) \geq \alpha$ (resp. $\overline{x}(\xi) > \alpha$) are measurable in the σ-algebra on $\mathbb{M}$ consisting of all sets $A \subseteq \mathbb{M}$ for which $\Pr(x \in A)$ is defined.

The definition implies that a cloud can be viewed as a nested family of inner and outer confidence regions to different confidence levels. As shown in [1], clouds capture a mix of probabilistic and fuzzy uncertainty that is able to handle multivariate uncertainty problems algorithmically by reducing calculations to global optimization. Clouds have close relations to histograms, likelihood ratios, and continuous cumulative distribution functions of real univariate random variables.

While the usefulness of clouds has been demonstrated in [1], an important theoretical aspect was left open. Indeed, it is not clear from the definition whether an arbitrary cloud must necessarily contain a random variable. In the present paper we show that this is indeed the case. Given the simplicity of the concept, the proof is surprisingly difficult.

Let $\mathbf{x}$ be a cloud over $\mathbb{M}$. The cloud $\mathbf{x}'$ over $\mathbb{M}'$ is called a **subcloud** of $\mathbf{x}$ if $\mathbb{M}' \subseteq \mathbb{M}$ and

$$\mathbf{x}'(\xi) \subseteq \mathbf{x}(\xi) \quad \text{for all } \xi \in \mathbb{M}'; \tag{3}$$

in this case we write $\mathbf{x}' \subseteq \mathbf{x}$. A cloud $\mathbf{x}$ is **minimal** if every subcloud of $\mathbf{x}$ is equal to $\mathbf{x}$.

In Section 2, we show that every cloud contains a minimal cloud. The proof is based on a characterization of minimal clouds in terms of jumps and regular points.

In Section 3, we show that every discrete or real-valued random variable is contained in some minimal cloud, and that every minimal cloud is associated with a unique monotone linear functional on a suitable function algebra. Together with the results from Section 2, it follows that every cloud contains some random variable.

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2 Minimal clouds

A **jump** of the cloud $\mathbf{x}$ over $\mathbb{M}$ is a point $\xi \in \mathbb{M}$ with thick level $\mathbf{x}(\xi)$ (i.e., $\overline{x}(\xi) > \underline{x}(\xi)$) such that for all $\xi' \in \mathbb{M} \setminus \{\xi\}$,

$$\text{either} \quad \underline{x}(\xi') \geq \overline{x}(\xi) \quad \text{or} \quad \overline{x}(\xi') \leq \underline{x}(\xi). \tag{4}$$

A **regular point** of $\mathbf{x}$ is a point $\xi \in \mathbb{M}$ with thin level $\mathbf{x}(\xi)$ (i.e., $\underline{x}(\xi) = \overline{x}(\xi) = \mathbf{x}(\xi)$) such that

$$\mathbf{x}(\xi) \in \ ]0,1[\tag{5}$$

and

$$\mathbf{x}(\xi) \in \mathbf{x}(\xi') \quad \Rightarrow \quad \xi' = \xi. \tag{6}$$

In general, clouds need not have any jumps or regular points.

2.1 Theorem. *A cloud having only jumps and regular points is minimal.*

Proof. Let $\mathbf{x}$ be a cloud over $\mathbb{M}$, and let $\mathbf{x}'$ be a subcloud of $\mathbf{x}$ over $\mathbb{M}'$.

If ξ is a jump of $\mathbf{x}$ and $\alpha \in \operatorname{int} \mathbf{x}(\xi)$ then (4) implies $\alpha \notin \mathbf{x}(\xi')$ for all $\xi' \neq \xi$. Since $\mathbf{x}'$ is a subcloud of $\mathbf{x}$ we have $\alpha \notin \mathbf{x}'(\xi')$ for all $\xi' \neq \xi$. Hence, since $\alpha \in \ ]0,1[$, the definition (1) for $\mathbf{x}', \mathbb{M}'$ in place of $\mathbf{x}, \mathbb{M}$ requires $\xi \in \mathbb{M}$ and $\alpha \in \mathbf{x}'(\xi)$. Thus $\mathbf{x}'(\xi)$ contains all interior points of $\mathbf{x}(\xi)$, and since both are closed intervals, (3) implies $\mathbf{x}'(\xi) = \mathbf{x}(\xi)$.

If ξ is a regular point of $\mathbf{x}$ then $\mathbf{x}(\xi)$ is thin and contains a unique number $\alpha \in \ ]0,1[$. By (6), $\alpha \notin \mathbf{x}(\xi')$ for all $\xi' \notin \xi$, and as before, we conclude that $\xi \in \mathbb{M}$ and $\alpha \in \mathbf{x}'(\xi)$. Thus (3) implies $\mathbf{x}'(\xi) = \mathbf{x}(\xi)$.

Thus $\mathbb{M}'$ contains all jumps and regular points of $\mathbb{M}$, and on these, $\mathbf{x}'$ agrees with $\mathbf{x}$. In particular, if all points of $\mathbb{M}$ are jumps or regular points of $\mathbf{x}$, $\mathbb{M}' = \mathbb{M}$ and $\mathbf{x}' = \mathbf{x}$. $\square$

2.2 Theorem.

(i) A minimal cloud contains only jumps and regular points.

(ii) Every cloud contains a minimal cloud.

Proof. Let $\mathbf{x}$ be a cloud over $\mathbb{M}$. We shall construct a subcloud $\mathbf{x}'$ of $\mathbf{x}$ over a suitable subset $\mathbb{M}'$ of $\mathbb{M}$ which has only jumps and regular points. By Theorem 1.1, $\mathbf{x}'$ is minimal, proving (ii). Moreover, if $\mathbf{x}$ is already a minimal cloud then $\mathbf{x}' = \mathbf{x}$ by definition of minimality, which implies (i).

To construct the subcloud, we first consider the special case where $\mathbf{x}$ has the following property:

(J) Every $\xi \in \mathbb{M}$ with thick $\mathbf{x}(\xi)$ is a jump.

If (J) holds, we write X for the set of all jumps, and note that $\mathbf{x}(\xi)$ is thin for all $\xi \notin X$. We consider the sets

$$X(\alpha) = \{\xi \in \mathbb{M} \mid \alpha \in \mathbf{x}(\xi)\}, \quad \alpha \in \ ]0,1[, \tag{7}$$

which are nonempty by definition of a cloud. For α in

$$A := \ ]0,1[\ \setminus \bigcup_{\xi \in X} \mathbf{x}(\xi), \tag{8}$$

$X(\alpha)$ must be a subset of $\mathbb{M} \setminus X$, hence

$$\mathbf{x}(\xi) = \alpha \quad \text{for} \quad \xi \in X(\alpha), \tag{9}$$

and the $X(\alpha)$, $\alpha \in A$, are pairwise disjoint. By the axiom of choice, we can choose points $\xi(\alpha) \in X(\alpha)$ for every $\alpha \in A$. Let $\mathbf{x}'$ be the mapping obtained by restricting $\mathbf{x}$ to the set

$$\mathbb{M}' = X \cup \{\xi(\alpha) | \alpha \in A\}.$$

By construction, $\mathbf{x}'$ is a subcloud of $\mathbf{x}$, with jumps $\xi \in X$ and regular points $\xi(\alpha)$, $\alpha \in A$. Thus $\mathbf{x}'$ has only jumps and regular points, as required.

We conclude that if a cloud $\mathbf{x}$ contains a subcloud with property (J) then the theorem is true. To prove that this is indeed the case, suppose by way of contradiction that $\mathbf{x}$ contains no such subcloud. We suppose that we have already found a subcloud $\mathbf{x}_l$ of $\mathbf{x}$ over $\mathbb{M}_l \subseteq \mathbb{M}$ and a set $X_l \subseteq \mathbb{M}_l$ with the property that every $\xi \in X_l$ is a jump of $\mathbf{x}_l$ and, with the width

$$w_l(\xi) = \overline{x}_l(\xi) - \underline{x}_l(\xi),$$

$$w_l(\xi) \geq \frac{1}{2} w_l(\xi') \quad \text{ for all } \xi \in X_l, \ \xi' \in \mathbb{M}_l \setminus X_l. \tag{10}$$

Clearly, this holds for $l = 0$ with $\mathbf{x}_0 = \mathbf{x}$, $\mathbb{M}_0 = \mathbb{M}$, $X_0 = \emptyset$, Since, by assumption, $\mathbf{x}_l$ does not have property (J), $\mathbf{x}_l(\xi)$ is thick for some $\xi \in \mathbb{M}_l \setminus X_l$. Therefore,

$$d := \sup_{\xi \in \mathbb{M}_l \setminus X_l} (\overline{x}_l(\xi) - \underline{x}_l(\xi)) > 0. \tag{11}$$

Case 1. There is a jump ξ' with

$$\overline{x}_l(\xi') - \underline{x}_l(\xi') > d/2. \tag{12}$$

In this case, we put $\xi_l = \xi'$ and

$$X_{l+1} = X_l \cup \{\xi_l\}, \quad Y_{l+1} = \{\xi \in \mathbb{M}_l \setminus X_l \mid \mathbf{x}_l(\xi) \not\subseteq \mathbf{x}_l(\xi_l)\}, \tag{13}$$

$$\mathbb{M}_{l+1} = X_{l+1} \cup Y_{l+1}, \tag{14}$$

and define

$$\mathbf{x}_{l+1}(\xi) = \begin{cases} \mathbf{x}_l(\xi) & \text{if } \xi \in X_{l+1}, \\ \mathbf{x}_l(\xi) \setminus \text{int } \mathbf{x}_l(\xi_l) & \text{if } \xi \in Y_{l+1}. \end{cases} \tag{15}$$

Then $\mathbf{x}_{l+1}$ is a subcloud of $\mathbf{x}$, and every $\xi \in X_{l+1}$ is a jump of $\mathbf{x}_{l+1}$.

Case 2. There is no jump ξ' with (12). By (11), we can still choose a point $\xi_l \in \mathbb{M}_l \setminus X_l$ with (12). Let

$$X_l' = \{\xi \in \mathbb{M}_l \mid \mathbf{x}_l(\xi_l) \subseteq \text{int } \mathbf{x}_l(\xi)\}; \tag{16}$$

since X_l contains jumps only, we have $X_l' \cap X_l = \emptyset$. Since we are not in Case 1, ξ_l is not a jump; therefore $X_l' \neq \emptyset$. Now let put

$$\underline{\alpha}_l = \inf\{\underline{x}_l(\xi) \mid \xi \in X_l'\}, \quad \overline{\alpha}_l = \sup\{\overline{x}_l(\xi) \mid \xi \in X_l'\}. \tag{17}$$

If $\underline{\alpha}_l = \underline{x}_l(\xi')$ or $\overline{\alpha}_l = \overline{x}_l(\xi')$ for some $\xi' \in X_l'$ then ξ' is a jump satisfying (12), contradicting Case 2. Thus neither the infimum nor the supremum in (17) are attained. We can therefore pick an infinite sequence of distinct points $\xi_1, \xi_2, \ldots \in X_l'$ so that

$$\inf_k \underline{x}_l(\xi_k) = \underline{\alpha}_l, \quad \sup_k \overline{x}_l(\xi_k) = \overline{\alpha}_l. \tag{18}$$

(We pick a sequence for the inf and one for the sup, and then interlace the two.) We construct a subsequence $\xi_{k_1}, \xi_{k_2}, \ldots$ as follows. Let $l_1 = 1$, and suppose we know already $k_1, \ldots k_j$. For odd j, we define

$$\mathbf{x}_l'(\xi_{k_j}) = \mathbf{x}_l(\xi_{k_j}) \cap [\overline{x}_l(\xi_l), \infty[$$

and pick k_{j+1} such that

$$\underline{x}_l(\xi_{k_{j+1}}) \leq \underline{x}_l(\xi_{k_j});$$

this is possible since the infimum in (18) is not attained. And for even j, we define

$$\mathbf{x}_l'(\xi_{k_j}) = \mathbf{x}_l(\xi_{k_j}) \cap \;] - \infty, \underline{x}_l(\xi_l)]$$

and pick k_{j+1} such that

$$\overline{x}_l(\xi_{k_{j+1}}) \geq \overline{x}_l(\xi_{k_j});$$

this is possible since the supremum in (18) is not attained. The construction guarantees that the cloud $\mathbf{x}_l'$ over $\mathbb{M}_l' = (\mathbb{M}_l \setminus X_l') \cup \{\xi_{k_1}, \xi_{k_2}, \ldots\}$ defined by this and

$$\mathbf{x}_l'(\xi) = \mathbf{x}_l(\xi) \quad \text{if } \xi \in \mathbb{M}_l \setminus X_l'$$

is a subcloud of $\mathbf{x}_l$, for which the set (16) with $\mathbf{x}_l, \mathbb{M}_l$ replaced by $\mathbf{x}_l', \mathbb{M}_l'$ is empty. After this replacement, we can again construct a subcloud $\mathbf{x}_{l+1}$ by (13) – (15).

If we repeat the procedure we find an infinite sequence of subclouds $\mathbf{x}_l$ over $\mathbb{M}_l$ with a set $X_l \subseteq \mathbb{M}_l$ of size l with the required properties. Now the cloud $\mathbf{x}_\infty$ over the set $\mathbb{M}_\infty$ defined by

$$\mathbf{x}_\infty(\xi) = \bigcap_{l \geq 0} \mathbf{x}_l(\xi) \quad \text{for } \xi \in \mathbb{M}_\infty = \bigcap_{l \geq 0} \mathbb{M}_l$$

is a subcloud of $\mathbf{x}$ for which every $\xi \in X_\infty = \bigcup_{l \geq 0} X_l$ is a jump and the property (10) holds for $l = \infty$. Since $X_\infty = \{\xi_1, \xi_2, \ldots\}$ is a countably infinite set of jumps whose widths add up to a number ≤ 1 (since the open intervals $\mathbf{x}_\infty(\xi_l)$ are pairwise disjoint), these widths cannot have a positve infimum. Therefore (10) implies that $w_\infty(\xi') = 0$ for all $\xi' \in \mathbb{M}_\infty \setminus X_\infty$. Thus $\mathbf{x}_\infty$ is a subcloud of $\mathbf{x}$ with property (J), contradiction. □

3 Clouds and random variables

3.1 Theorem. *Every real-valued random variable x for which* $\Pr(x = \xi)$ *and* $\Pr(x \leq \xi)$ *exist for all $\xi \in \mathbb{R}$ is contained in some minimal cloud over a subset $\mathbb{M}$ of $\mathbb{R}$ with*

$$\Pr(x \in \mathbb{M}) = 1. \tag{19}$$

Proof. We define (for $\xi \in \mathbb{R}$)

$$\overline{x}(\xi) := \Pr(x \leq \xi), \quad \underline{x}(\xi) := \overline{x}(\xi) - \Pr(x = \xi) = \Pr(x < \xi), \tag{20}$$

and note that $\overline{x}(\xi)$, the cumulative distribution function of x, is monotone increasing with limits

$$\lim_{\xi \to -\infty} \overline{x}(\xi) = 0, \qquad \lim_{\xi \to +\infty} \overline{x}(\xi) = 1,$$

and continuous from the right,

$$\lim_{\xi \downarrow \xi_0} \overline{x}(\xi) = \overline{x}(\xi_0).$$

Moreover,

$$\underline{x}(\xi) = \sup_{\xi' < \xi} \Pr(x \leq \xi') = \sup_{\xi' < \xi} \overline{x}(\xi'). \tag{21}$$

If $\alpha \in \ ]0, 1[$ then the above properties imply that

$$\xi_\alpha := \inf\{\xi \in \mathbb{R} \mid \overline{x}(\xi) \geq \alpha\} \tag{22}$$

exists and is finite. Since $\overline{x}(\xi_\alpha) \geq \alpha$ by (22) and right continuity, and $\underline{x}(\xi_\alpha) \leq \alpha$ by (21), we have

$$\alpha \in \mathbf{x}(\xi_\alpha). \tag{23}$$

Therefore $\mathbf{x}$ is a cloud on $\mathbb{R}$. Since (21) implies that

$$\xi > \xi_\alpha \quad \Rightarrow \quad \underline{x}(\xi) \geq \alpha, \tag{24}$$

and (22) implies that

$$\overline{x}(\xi) > \alpha \quad \Rightarrow \quad \xi \geq \xi_\alpha, \tag{25}$$

we have $\Pr(\underline{x}(x) \geq \alpha) \geq \Pr(x > \xi_\alpha) = 1 - \Pr(x \leq \xi_\alpha) = 1 - \overline{x}(\xi_\alpha) \leq 1 - \alpha$ and $\Pr(\overline{x}(x) > \alpha) \leq \Pr(x \geq \xi_\alpha) = 1 - \Pr(x < \xi_\alpha) = 1 - \underline{x}(\xi_\alpha) \geq 1 - \alpha$, hence (2) holds and the random variable x belongs to the cloud $\mathbf{x}$. From (20) we see that $\mathbf{x}(\xi)$ is thick iff $\Pr(x = \xi) > 0$; let X be the set of all such ξ. If $\xi \in X$ and $\xi' > \xi$ then

$$\underline{x}(\xi') = \Pr(x < \xi') \geq \Pr(x \leq \xi) = \overline{x}(\xi),$$

and if $\xi' \le \xi$ then

$$\overline{x}(\xi') = \Pr(x \le \xi') \le \Pr(x < \xi) = \underline{x}(\xi).$$

Hence (4) holds and all $\xi \in X$ are jumps of $\mathbf{x}$. We now let

$$X' = \Big\{\xi_\alpha \;\Big|\; \alpha \in \;]0,1[, \alpha \notin \bigcup_{\xi \in X} \mathbf{x}(\xi)\Big\}, \quad \mathbb{M} = X \cup X'. \tag{26}$$

Clearly, the restriction $\mathbf{x}'$ of $\mathbf{x}$ to $\mathbb{M}$ is still a cloud, and the $\xi \in X$ are also jumps of $\mathbf{x}'$. If $\xi_\alpha \in X'$ then $\xi_\alpha \notin X$, hence $\mathbf{x}(\xi_\alpha)$ is thin, and by (23),

$$\mathbf{x}(\xi_\alpha) = \alpha \quad \text{for } \xi_\alpha \in X'. \tag{27}$$

By definition of X', (5) holds and $\alpha \notin \mathbf{x}(\xi')$ for any $\xi' \in X$, hence (6) holds for $\xi = \xi_\alpha$. Thus all $\xi \in X'$ are regular points of $\mathbf{x}'$. Therefore $\mathbf{x}'$ only has jumps and regular points, and by Theorem 2.1, it is a minimal cloud.

It remains to be shown that (19) holds, so that the restriction $\mathbf{x}'$ of $\mathbf{x}$ to $\mathbb{M}$ still contains x. For $\alpha \in [0,1]$, the sets

$$X_\alpha = \{\xi \in \mathbb{R} \mid \alpha \in \mathbf{x}(\xi)\} = \{\xi \in \mathbb{R} \mid \Pr(x < \xi) \le \alpha \le \Pr(x \le \xi)\} \tag{28}$$

are (for $\alpha \ne 0,1$ nonempty) intervals, not necessarily open or closed. Since the $\xi \in X$ are jumps, $X_\alpha \cap X$ consists of at most one point of X_α, which must be an endpoint. Hence the sets

$$X'_\alpha = X_\alpha \setminus X \tag{29}$$

are also intervals, and since $\Pr(x = \xi) = 0$ for $\xi \notin X$, the definition (28) implies

$$\Pr(x < \xi) = \Pr(x \le \xi) = \alpha \quad \text{for } \xi \in X'_\alpha. \tag{30}$$

Choosing for ξ the endpoints of X'_α or, if these do not belong to X'_α, sequences of points converging to the endpoints, we find that $\Pr(x \in X'_\alpha) = 0$, so that all X'_α are null sets. If X'_α contains a single point only, (27) implies that $X_\alpha = \{\xi_\alpha\} \subseteq X'$. Therefore, the intervals X'_α of positive width cover the set $(\mathbb{R} \setminus X) \setminus X' = \mathbb{R} \setminus \mathbb{M}$. But the X'_α are pairwise disjoint intervals, hence only countably many of them can have positive width. Therefore $\mathbb{R} \setminus \mathbb{M}$ is covered by a countable union of null sets, and is therefore itself a null set. The proves (19), and shows that the restriction $\mathbf{x}'$ of $\mathbf{x}$ to $\mathbb{M}$ still contains x. □

3.2 Corollary. *Every random variable with discrete spectrum is contained in a minimal cloud over a countable set.*

Proof. We can map the spectrum bijectively to a set of real numbers and then apply Theorem 3.1. □

Note that, in both cases, the minimal cloud is not unique; e.g., if $\mathbf{x}$ is a minimal cloud containing x then the mirror cloud $\mathbf{x}'$ with $\mathbf{x}'(\xi) = 1 - \mathbf{x}(\xi) = [1 - \overline{x}(\xi), 1 - \underline{x}(\xi)]$ is another minimal cloud containing x.

3.3 Theorem. *For every minimal cloud* $\mathbf{x}$ *over* $\mathbb{M}$*, there is a unique probability measure* $d\mu$ *on* $\mathbb{M}$ *such that the sets consisting of all* $\xi \in \mathbb{M}$ *with* $\underline{x}(\xi) \geq \alpha$ *(resp.* $\overline{x}(\xi) > \alpha$ *are measurable, and* $x \in \mathbf{x}$ *iff* $\Pr(x \in A) = \mu(A)$ *for all measurable subsets* A *of* $\mathbb{M}$ *iff*

$$\langle f(x) \rangle = \int_{\mathbb{M}} d\mu(\xi) f(\xi) \tag{31}$$

for all bounded measurable real-valued functions f *on* $\mathbb{M}$*.*

Here $\langle f(x) \rangle$ denotes the expectation of $f(x)$. See Whittle [2] for an introduction to probability theory by means of expectations.

Proof. For any interval $I \subseteq [0,1]$ we define

$$X|_I := \{\xi \in \mathbb{M} \mid \mathbf{x}(\xi) \subseteq I\}. \tag{32}$$

Let X' be the set of regular points of $\mathbf{x}$. Regularity implies

$$\Pr(\mathbf{x}(x) < \alpha) = \Pr(\mathbf{x}(x) \leq \alpha) = \alpha \quad \text{if } \xi_\alpha \in X', \tag{33}$$

and from this we can deduce for any interval $I \subseteq [0,1]$ that

$$\Pr(x \in X|_I) = \sup I - \inf I \quad \text{if } X|_I \subseteq X'. \tag{34}$$

Indeed, if $X|_{[\beta,\gamma]} \subseteq X'$ then

$$\Pr(x \in X|_{]\beta,\gamma[}) = \Pr(\beta < \mathbf{x}(\xi) < \gamma) = \Pr(\mathbf{x}(\xi) < \gamma) - \Pr(\mathbf{x}(\xi) \leq \beta) = \gamma - \beta,$$

and by analogous reasoning, this also holds for $]\beta,\gamma]$, $[\beta,\gamma[$, or $[\beta,\gamma]$ in place of $]\beta,\gamma[$. But since we can write any interval I with $X|_I \subseteq X'$ as a disjoint union of countably many subintervals J with $X|_{\overline{J}} \subseteq X'$ (where $\overline{J}$ denotes the closure of J), (34) holds generally by σ-additivity.

Let X be the set of jumps of $\mathbf{x}$. By definition, the intervals $\mathbf{x}(\xi)$ for $\xi \in X$ are thick and can overlap in at most one point, hence their interiors form a disjoint union of nonempty open interval contained in $[0,1]$. This implies that X is countable, and we can enumerate the elements of X as $\xi = \xi_1, \xi_2, \ldots$. Let $n \in \mathbb{N}_0 \cup \{\infty\}$ denote the cardinality of X, and put

$$\alpha_l = \underline{x}(\xi_l), \; \beta_l = \overline{x}(\xi_l) \quad \text{for } \xi_l \in X,$$

$$\beta_0 = 0, \; \alpha_{n+1} = 1,$$
$$\gamma_l = \inf\{\alpha_k \mid \alpha_k \geq \beta_l\}, \quad l = 0, \ldots, n,$$
$$X_l = \{\xi \in \mathbb{M} \mid \beta_l < \mathbf{x}(\xi) < \gamma_l\} = X|_{]\beta_l,\gamma_l[}.$$

By Theorem 1.2(i), all $\xi \in \mathbb{M}$ are either jumps or regular, hence

$$\mathbb{M} = X \cup X', \quad X' = \bigcup_{l=0}^{n+1} X_l. \tag{35}$$

The fact that $\mathbf{x}$ is a cloud and X_l consists of regular points only implies that the mapping $\xi \to \alpha = \mathbf{x}(\xi)$ is a bijection from X_l to $]\beta_l, \gamma_l[$; let $\alpha \to \xi = \xi_\alpha$ denote its inverse, so that

$$\mathbf{x}(\xi_\alpha) = \alpha \quad \text{for } \alpha \in \;]\beta_l, \gamma_l[. \tag{36}$$

Property (34) now says that $\mathbf{x}(x)$, restricted to $x \in X_l$, is uniformly distributed, with total probability $\gamma_l - \beta_l$. Using the inverse mapping, this fixes for each l the distribution of x conditional to $x \in X_l$. The distribution of x conditional to $x \in X$ is the discrete measure with

$$\Pr(x = \xi_l) = \beta_l - \alpha_l \quad \text{for} \quad \xi_l \in X. \tag{37}$$

Indeed, by (4) and (6), $x = \xi_l$ iff $\underline{x}(\xi) \geq \alpha_l$ and $\overline{x}(\xi) \leq \beta_l$, hence

$$\Pr(x = \xi_l) = \Pr(\underline{x}(\xi) \geq \alpha_l) - \Pr(\overline{x}(\xi) > \beta_l) \leq (1 - \alpha_l) - (1 - \beta_l) = \beta_l - \alpha_l.$$

If (37) is false, we must have strict inequality for some l, hence.

$$\Pr(x \in X) = \sum_l \Pr(x = \xi_l) < \sum_{l=1}^{n} (\beta_l - \alpha_l).$$

Now

$$\Pr(x \in X') = \sum_l \Pr(x \in X_l) = \sum_{l=0}^{n+1} (\gamma_l - \beta_l),$$

$$\begin{aligned} 1 &= \Pr(x \in \mathbb{M}) = \Pr(x \in X) + \Pr(x \in X') \\ &< \sum_{l=1}^{n} (\beta_l - \alpha_l) + \sum_{l=0}^{n} (\gamma_l - \beta_l) = \sum_{l=0}^{n} \gamma_l - \sum_{l=1}^{n} \alpha_l \\ &= \sum_{l=1}^{n+1} \alpha_l - \sum_{l=1}^{n} \alpha_l = \alpha_{n+1} = 1, \end{aligned}$$

contradiction. Thus (26) holds indeed. But now the distribution of x is completely fixed, and we get from (34), (35) and (37) the explicit representation

$$\langle f(x) \rangle = \sum_{l=1}^{n} f(\xi_l)(\beta_l - \alpha_l) + \sum_{l=0}^{n+1} \int_{\beta_l}^{\gamma_l} f(\xi_\alpha) d\alpha \tag{38}$$

if the integrals exist and the sums are absolutely convergent. In particular, the σ-algebra Σ on $\mathbb{M}$ consisting of all sets $A \subseteq \mathbb{M}$ for which $\Pr(x \in A)$ is defined is the smallest σ-algebra containing all singletons and all $X|_{]\beta,\gamma[}$ with $]\beta, \gamma[\ \subseteq \ ]\beta_l, \gamma_l[$ for some l, and the measure μ defined by $\mu(A) = \Pr(x \in A)$ for all $A \in \Sigma$ is canonically generated by the conditions

$$\mu(\{\xi\}) = \Pr(x = \xi) = \overline{x}(\xi) - \underline{x}(\xi), \tag{39}$$

$$\mu(X|_{]\beta,\gamma[}) = \gamma - \beta \quad \text{if} \quad]\beta, \gamma[\ \subseteq \ ]\beta_l, \gamma_l[. \tag{40}$$

□

3.4 Corollary. *If x, y are random variables with values in $\mathbb{M}$ contained in a minimal cloud $\mathbf{x}$ over $\mathbb{M}$ then $\Pr(\mathrm{St}(x)) = \Pr(\mathrm{St}(y))$ for all statements for which these probabilities are defined.*

3.5 Example. On the Cantor set

$$\mathbb{M} = \left\{ \xi = \sum_{i=1}^{\infty} 3^{-i} \xi_i \;\middle|\; \xi_i \in \{0, 2\}, \text{ infinitely many } \xi_i = 0 \right\},$$

we define the mapping $\mathbf{x} : \mathbb{M} \to [0,1[$ with

$$\mathbf{x}(\xi) := \sum_{i=1}^{\infty} 2^{-i}(\xi_i/2).$$

It is a strictly monotone bijection, with inverse

$$\xi_\alpha = \sum_{i=1}^{\infty} 3^{-i}(2\alpha_i) \quad \text{if } \alpha = \sum_{i=1}^{\infty} 2^{-i}\alpha_i \quad (\alpha_i \in \{0,1\})$$

(where the α_i contain infinitely many zeros), and hence defines a minimal cloud with only regular points. By Theorem 3.3, the expectation of functions of a random variable $x \in \mathbf{x}$ is therefore given by

$$\langle f(x) \rangle = \int_0^1 f(\xi_\alpha) d\alpha.$$

Since the Cantor set has Lebesgue measure zero, a random variable $x \in \mathbf{x}$ has singular spectrum. Indeed, the example is essentially equivalent to that given in Exercise 2.3.25.7 of THIRRING [3], though there no explicit formula for the expectation is given.

3.6 Theorem. *For every cloud* $\mathbf{x}$ *over* $\mathbb{M}$ *there exists a random variable* x *with values in* $\mathbb{M}$ *contained in* $\mathbf{x}$.

Proof. By Theorem 2.2(ii), every cloud contains a minimal cloud, and by Theorem 3.3, every minimal cloud contains a random variable. □

References

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[2] P. Whittle, Probability via Expectation, Springer Texts in Statistics, Springer, Berlin 1992.

[3] W. Thirring, A Course in Mathematical Physics 3. Quantum Mechanics of Atoms and Molecules, Springer, New York 1979.