A UNIFIED VIEW OF INEQUALITIES FOR DISTANCE-REGULAR GRAPHS, PART I

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Abstract

In this paper we introduce language of a configuration and t-point counts, which on natural way represent the number of ordered t-uples of vertices. Every t-point count can be written as a sum of $(t-1)$-point counts, and with that we can obtain a system of linear equations and inequalities, i.e., a linear constraint satisfaction problem CSP, whose variables are the t-point counts. This language is perfect tool for better understanding combinatorial structure of distance-regular graphs. We obtain some old and new inequalities. Among else we prove the following. Let Γ be a distance-regular graph of diameter d and intersection array $\{b_0, \dots, b_{d-1}; c_1, \dots, c_d\}$. If $1 \leq i < j \leq d$ then

$$a_j > a_i \quad \Rightarrow \quad b_i > b_j \qquad \text{and} \qquad a_j < a_i \quad \Rightarrow \quad c_j > c_i.$$

If $c_q < c_{q+1}$ and $a_q \leq c_{q+1} - c_q$ (where $q \geq 2$) then

$$d \leq (k + 1 - c_{q+1})q + 1 \leq (k-1)q + 1.$$

This diameter bound is tight.

MSC: 05C12, 05C62, 05E30

Keywords: distance-regular graph, diameter bound, linear constraint satisfaction problem

Contents

1 Introduction to Part I (configuration and t-point count)

Let $\Gamma = (X, R)$ be a distance-regular graph of diameter $d = d_\Gamma$ with $v_\Gamma := |X|$ vertices and intersection array $i(\Gamma) = \{b_0, \dots, b_{d-1}; c_1, \dots, c_d\}$. We write

$$n := b_0, \quad a_i := n - b_i - c_i, \quad \lambda := a_1.$$

Thus the valency is n (conventionally denoted k) since we often use k (and s) as indices. Please refer to Section 2 for more background information.

In this paper we introduce language of a configuration and t-point counts, which on natural way represent the number of ordered t-uples of vertices (configurations of vertices of prescribed type, e.g., the number of 4-cycles, of induced quadrangles, ...). Every t-point count can be written as a sum of $(t-1)$-point counts, and with that we can obtain a system of linear equations and inequalities, i.e., a linear constraint satisfaction problem CSP, whose variables are the t-point counts.

Definition. 1.1 Let Γ denote graph with a vertex set $X = \{i_x, j_y, ..., h_w\}$. A **configuration** is a finite ordered list $\bar{z} = 1_x 2_y 3_z \dots (t-1)_p t_w$ of vertices of Γ. A t-point **type** is an undirected graph Δ with t nodes $x, y, z, \dots, p, w$ whose edges are labelled with integers $\in \{0, 1, \dots, d_\Gamma\}$; we fix one drawing Δ and a fixed labelling of the vertices of Δ as first, second, etc. A t-point type is called **complete** if any two nodes are joined by a labelled edge. A configuration $\bar{z}$ is of type Δ if

$$d(i_u, j_v) = \Delta_{uv} \quad \text{whenever } u \sim v \text{ in } \Delta.$$

Here Δ_{uv} denotes the label of the edge uv. In drawings, missing labels are taken as having the value 1. We define the rational number $[\Delta]$ as the number of configurations of type Δ divided by v_Γ,

$$[\Delta] = \frac{1}{v_\Gamma}\Big\{(1_x, 2_y, 3_z, ..., (t-1)_p, t_w)\,\Big|\, \partial(i_u, j_v) = \Delta_{uv},\ 1 \le i, j \le t,\ i \ne j\Big\}.$$

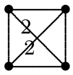
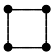
A **t-point count** is a number $[\Delta]$, where Δ is a complete t-point type (the number of ordered t-uples of vertices (that correspond to fixed labelling of the vertices of an undirected graph Δ) with the given ordered $\frac{t(t-1)}{2}$-list of distances divided by v_Γ),

$$[\Delta] = \frac{1}{v_\Gamma} \sum_{1_x, 2_y, 3_z, ..., (t-1)_p, t_w \in X} \chi_{\Gamma_{\Delta_{xy}}(1_x)}(2_y) \cdot \chi_{\Gamma_{\Delta_{xz}, \Delta_{yz}}(1_x, 2_y)}(3_z) \cdot \ ...$$

$$... \cdot \chi_{\Gamma_{\Delta_{xw}, \Delta_{yw}, \Delta_{zw}, ..., \Delta_{pw}}(1_x, 2_y, 3_z, ..., (t-1)_p)}(t_w) =$$

$$= \frac{1}{v_\Gamma} \sum_{1_x \in X} \sum_{2_y \in \Gamma_{\Delta_{xy}}(1_x)} \sum_{3_z \in \Gamma_{\Delta_{xz}, \Delta_{yz}}(1_x, 2_y)} \cdots \sum_{t_w \in \Gamma_{\Delta_{xw}, \Delta_{yw}, \Delta_{zw}, ..., \Delta_{pw}}(1_x, 2_y, 3_z, ..., (i-1)_p)} 1,$$

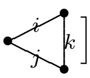
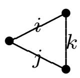
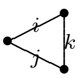
where $\chi_A(x)$ is characteristic function.

Thus an induced quadrangle is both of type  and of type , but the second type fits also 4-cycles that induce K_{211} or K_4. Clearly,

$$[\Delta] \ge 0 \quad \text{for all types } \Delta. \tag{1}$$ a1

Please refer to Section 3 and 4 for examples (in Section 3 we show how to compute 4-point count for a given graph and in Section 4 we consider examples when we fix three vertices in a given 4-point count. In fact, in second case we get triple intersection numbers of K. Coolsaet and A. Jurišić from [5]).

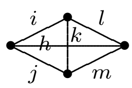
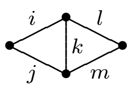
Let Γ be a distance-regular graph of diameter d. Note that t-point counts with $t \le 3$ are determined by the intersection array; in particular,

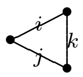
$\Big[$  $\Big] = \dfrac{v_\Gamma \cdot n_k \cdot p^k_{ij}}{v_\Gamma} = n_k p^k_{ij}$ or $\Big[$  $\Big] = n_i p^i_{jk}$ or $\Big[$  $\Big] = n_j p^j_{ik}$. (2) a2

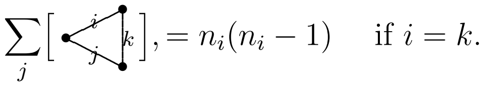
If an intersection array $i(\Gamma)$ determines Γ up to isomorphism then all t-point counts are determined by the intersection array. It is conceivable that in many concrete cases, the t-point counts with small $t > 3$ are already determined by equations and inequalities between counts valid for arbitrary distance-regular graphs.

Indeed, there are many such relations between different counts. The simplest ones are obtained by summing and elimination according to the following rules:

Sum over a distance: e.g.,

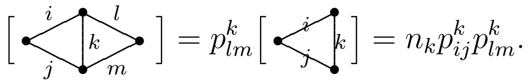
$\sum_h \Big[$  $\Big] = \Big[$  $\Big]$, (3) a3

$\sum_j \Big[$  $\Big], = n_i n_k \quad$ if $i \ne k$, (4) a4 and



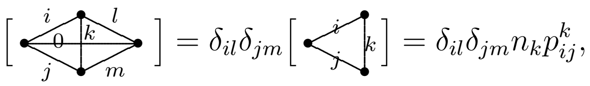
$\sum_j [\ \cdot\ ] = n_i(n_i - 1)$ if $i = k$. (5) `pu`

Elimination of nodes of valency 2: e.g.,



$[\ \cdot\ ] = p^k_{lm} [\ \cdot\ ] = n_k p^k_{ij} p^k_{lm}$. (6) `a5`

Elimination of distance 0: e.g.,



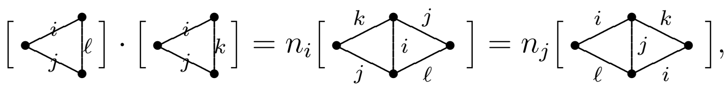
$[\ \cdot\ ] = \delta_{il}\delta_{jm} [\ \cdot\ ] = \delta_{il}\delta_{jm} n_k p^k_{ij}$, (7) `a6`

where δ_{ik} is the Kronecker symbol.

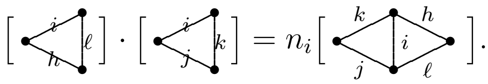
Moreover, isomorphic types clearly have the same count. As in the special case (3), every $[\Delta]$ can be written as a sum of t-point counts. Thus we obtain a system of linear equations and inequalities, i.e., a linear **constraint satisfaction problem** (**CSP**), whose variables are the t-point counts. We refer to this as the t**-point CSP**.

Using more general configuration algebra techniques, further nonlinear equations or inequalities can be added, but we shall not consider these here. The simplest ones are obtained by multiplication and by summing over a multiplication according to the following rules.

Multiplication: e.g.,

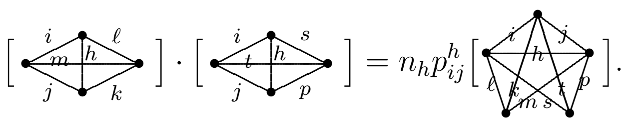


$[\ \cdot\ ] \cdot [\ \cdot\ ] = n_i [\ \cdot\ ] = n_j [\ \cdot\ ]$, (8) `a9`



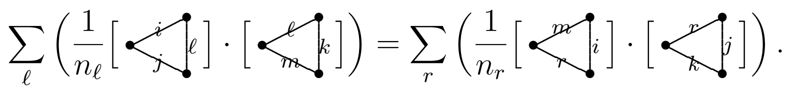
$[\ \cdot\ ] \cdot [\ \cdot\ ] = n_i [\ \cdot\ ]$. (9) `b1`

Moreover, if sizes of $|\Gamma_{\ell m k}(x, y, z)|$ and $|\Gamma_{spt}(x, y, z)|$ does not depend on $x \in X$, $y \in \Gamma_h(x)$ and $z \in \Gamma^h_{ij}(x, y, z)$ then



$[\ \cdot\ ] \cdot [\ \cdot\ ] = n_h p^h_{ij} [\ \cdot\ ]$. (10) `b2`

Sum over a multiplication: e.g.,



$\sum_\ell \left(\frac{1}{n_\ell} [\ \cdot\ ] \cdot [\ \cdot\ ]\right) = \sum_r \left(\frac{1}{n_r} [\ \cdot\ ] \cdot [\ \cdot\ ]\right)$. (11) `b3`

If the system of linear equations and inequality has no feasible solution then no graph with the assumed intersection array exists. If a feasible solution exists, it happens in tight cases that certain t-point counts are forced to vanish for *every* feasible solution. This implies the geometric statement that Γ contains no configurations of certain types. This information often enables further analysis which leads to uniqueness or nonexistence.

For specific intersection arrays $i(\Gamma)$ one can use numerical linear programming techniques to discover the set of constraints which are always active. To get more insight and more general results one has to look at the "most promising" counts and linear relations, and eliminate variables systematically.

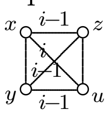
We now summarize our main results.

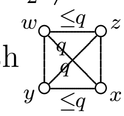
One of main purposes of this paper is to present language of a configurations and t-point counts - we begin that from Sections 3 and 4; here we reprove some well known inequalities using notation of this language. Recall that K. Coolsaet and A. Jurisić in [5] introduced notation of triple intersection numbers, and with help of them they proved nonexistence of certain distance-regular graphs. In our notation their triple intersection numbers are in fact our 4-point counts with three vertices fixed (see Subsection 4.1). In Subsection 4.2 we have 5-point counts with four vertices fixed i.e. we have quadruple intersection number. In Section 5 we re-prove Koolen's inequalities, if $2 \leq e \leq d$ then

$$c_e > c_{e-1} \qquad \Rightarrow \qquad c_e \geq c_i + c_{e-i} \quad \text{for all } i = 1, 2, ..., e-1$$

and

$$b_e > b_{e+1} \qquad \Rightarrow \qquad b_e \geq c_i + b_{e+i} \quad \text{for all } i = 1, 2, ..., d-1.$$

A quadruple $xyzu$ of vertices is called a parallelogram of length i if $xyzu$ is consistent with 

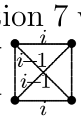
. The graph Γ is called m-parallelogram-free for some $m = 2, 3, ..., d$ if Γ does not contain any parallelogram of length at most m. In [12], H. Suzuki studied strongly closed subgraphs of diameter 2 in a parallelogram-free distance-regular graph Γ of diameter $d \geq 4$ such that $b_1 > b_2$ and $a_2 \neq 0$. A quadruple $wxyz$ of vertices is called a root of size q if $wxyz$ is consistent with 

. Related conditions called $(\mathrm{CR})_q$ and $(\mathrm{SS})_q$ are studied by Hiraki [7, 8]. Recall what are these two conditions

$$(\mathrm{CR})_q: \quad \{z\} \cup C(x,z) \cup A(x,z) = \{z\} \cup C(y,z) \cup A(y,z)$$

for any triple xyz of vertices with $d(x,z) = d(y,z) = q$ for which there exist three sequences of vertices $(x_0, x_1, ..., x_m = x)$, $(y_0, y_1, ..., y_m = y)$ and $(z_0, z_1, ..., z_m = z)$ such that $d(x_0, y_0) \leq 1$, $x_{i-1}z_{i-1}x_iz_i$ and $y_{i-1}z_{i-1}y_iz_i$ are roots of size q for all $1 \leq i \leq m$; and

$$(\mathrm{SS})_r: \quad \{z\} \cup C(x,z) \cup A(x,z) = \{z\} \cup C(y,z) \cup A(y,z)$$

for any triple of vertices xyz with $d(x,z) = d(y,z) = q$ and $d(x,y) \leq 1$. Under some conditions these conditions are a necessary and sufficient for the existence of a strongly closed subgraph which is $(c_{r+1} + a_{r+1})$-regular of diameter $r+1$ (where $r = r(\Gamma) = \max\{j \,|\, (c_j, a_j, b_j) = (c_1, a_1, b_1)\}$). In Section 6 we introduce 4-point counts containing two disjoint edges, which among else includes parallelogram of length m and root of size q. Using them we derive some inequalities and we explain what they mean in combinatorial seance for distance-regular graphs. In Section 7 we generalize 4-point counts from Section 6, and we derive some inequalities when $\Big[$ 

$\Big] > 0$ for some i $(1 \leq i \leq d)$. In Section 8 we prove that if $1 \leq i < j \leq d$ then

$$a_j > a_i \quad \Rightarrow \quad b_i > b_j$$

and

$$a_j < a_i \quad \Rightarrow \quad c_j > c_i.$$

In Section 9 we re-prove Hiraki's inequality

$$c_q < c_{q+1} \quad \text{and} \quad a_q \leq c_{q+1} - c_q \quad \Rightarrow \quad c_{q+i} \geq c_i + c_q \quad \text{for all } i \ (2 \leq i \leq d - q).$$

Finally in Section 10 using similar technique as A. Hiraki in [6] we prove that if $c_q < c_{q+1}$ and $a_q \leq c_{q+1} - c_q$ (where $q \geq 2$) then

$$d \leq (k + 1 - c_{q+1})q + 1 \leq (k - 1)q + 1.$$

This diameter bound is tight.

2 Preliminaries

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A *graph* Γ is a pair (X, R), where $X = \{u, v, w, ...\}$ is a nonempty set and $R = \{uv, wz, ...\}$ is a collection of two element subsets of X. The elements of X are called the *vertices* of Γ, and the elements of R are called the *edges* of Γ. When $xy \in R$, we say that vertices x and y are *adjacent*, or that x and y are *neighbors*. Adjacency between vertices x and y will be denoted by $x \sim y$. A subset $C \subseteq X$ is called a *clique* if every distinct $x, y \in C$ are neighbors. A graph is *finite* if both its vertex set and edge set are finite. An edge with identical ends is called a *loop*, and a graph is *simple* if it has no loops and no two of its edges join the same pair of vertices.

Let $\Gamma = (X, R)$ be a graph. For any two vertices $x, y \in X$, a walk of length h from x to y is a sequence $[x_0, x_1, x_2..., x_h]$ ($x_i \in X$, $0 \leq i \leq h$) such that $x_0 = x$, $x_h = y$, and x_i is adjacent to x_{i+1} for all i ($0 \leq i \leq h - 1$). We say that Γ is connected if for any $x, y \in X$, there is a walk from x to y. From now on, assume that Γ is finite, simple and connected.

For any $x, y \in X$, the *distance* between x and y, denoted $d(x, y)$, is the length of the shortest walk from x to y. The *diameter* $d = d(\Gamma)$ is defined to be

$$d = \max\{d(u, v) \mid u, v \in X\}.$$

A walk in Γ is said to be *closed* if it starts and ends at the same vertex.

A *circuit* of length m is a sequence of distinct vertices $[x_0, x_1, x_2..., x_m]$ such that (x_{i-1}, x_i) is an edge of Γ for all $1 \leq i \leq m$, where $x_m = x_0$. A circuit of length m is called *reduced* if $m \geq 4$ and any proper subset of it does not form a circuit. A shortest reduced circuit is called a *minimal circuit*. The *numerical girth* of Γ, denoted by g, is the length of a minimal circuit.

A *polygon* of length m is a sequence of distinct vertices $p_1p_2p_3...p_{m+1}$ such that (p_{i-1}, p_i) is an edge of Γ for all $1 \leq i \leq m + 1$, where $p_{m+1} = p_1$. A polygon of length m is called *reduced* if $m \geq 4$ and any proper subset of it does not form a polygon. A shortest reduced polygon is called a *minimal polygon*. *Induced polygon* $p_1p_2p_3...p_mp_1$ is a reduced polygon such that if $i \leq j$ then $d(p_i, p_j) = \min\{|i - j|, |i + m - j|\}$ is implied.

Two graphs $\Gamma_1 = (X_1, R_1)$ and $\Gamma_2 = (X_2, R_2)$ are said to be isomorphic if there are bijections $\varphi : X_1 \to X_2$ and $\psi : R_1 \to R_2$ such that $e = uv$ if and only if $\psi(e) = \varphi(u)\varphi(v)$; such a pair (φ, ψ) of mappings is called an isomorphism between Γ_1 and Γ_2.

A simple graph in which each pair of distinct vertices is joined by an edge is called a *complete graph*. Up to isomorphism, there is just one complete graph on n vertices; it is denoted by K_n. A *bipartite* graph is one whose vertex set can be partitioned into two subsets X and Y, so that each edge has one end in X and one end in Y; such a partition (X, Y) is called a bipartition of the graph. A *complete bipartite graph* is a simple bipartite graph with bipartition (X, Y) in which each vertex of X is joined to each vertex of Y; if $|X| = m$ and $|Y| = n$, such a graph is denoted by $K_{m,n}$. A *k-partite graph* is a graph whose vertices can be decomposed into k disjoint sets such that no two vertices within the same set are adjacent; if there are p, q, ..., r vertices in the k sets, the complete k-partite graph is denoted $K_{p,q,...,r}$. For example $K_{2,1,1}$ is a graph with 4 vertices and with five edges.

Suppose that Y is a nonempty subset of X. The subgraph of $\Gamma = (X, R)$ whose vertex set is Y and whose edge set S is the set of those edges of Γ that have both ends in Y is called the subgraph of Γ *induced* by Y; we say that $\Delta = (Y, S)$ is an *induced subgraph* of $\Gamma = (X, R)$.

Let $\Gamma = (X, R)$ be a graph with diameter d. For a vertex $x \in X$ and any non-negative integer h not exceeding d, let $\Gamma_h(x)$ denote the subset of vertices in X that are at distance h from x. Put $\Gamma_{-1}(x) = \Gamma_{d+1}(x) := \emptyset$. For any two vertices x and y in X at distance h, let

$$
\begin{aligned}
C(x, y) &= C_h(x, y) := \Gamma_{h-1}(x) \cap \Gamma_1(y)\\
A(x, y) &= A_h(x, y) := \Gamma_h(x) \cap \Gamma_1(y)\\
B(x, y) &= B_h(x, y) := \Gamma_{h+1}(x) \cap \Gamma_1(y)
\end{aligned}
$$

We say Γ is regular with valency n if each vertex in Γ has exactly n neighbours. A graph Γ is called *distance-regular* if there are integers b_i, c_i $(0 \leq i \leq d)$ which satisfy $c_i = |C_i(x, y)|$ and $b_i = |B_i(x, y)|$ for any two vertices x and y in X at distance i. Clearly such a graph is regular of valency $n := b_0$. From this definition it is routine to show that Γ is distnce-regular if and only if for all triples h, i, j $(0 \leq h, j, i \leq D)$, and for all $x, y \in X$ with $\partial(x, y) = h$, the number

$$p^h_{ij} := |\Gamma_i(x) \cap \Gamma_j(y)|$$

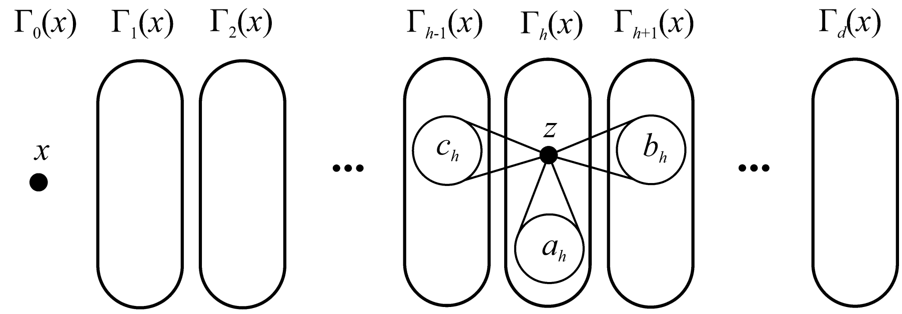
is independent of choice of x and y. The numbers p^h_{ij}, and with that c_i, b_i, and a_i, where

$$a_i := n - b_i - c_i \quad (0 \leq i \leq d)$$

is the number of neighbours of y in $\Gamma_i(x)$ for $x, y \in X$ at distance i, are called the *intersection numbers* of Γ. It is not hard to see that $a_i = p^i_{i1}$, $b_i = p^i_{i+1,1}$ and $c_i = p^i_{i-1,1}$. The array

$$i(\Gamma) = \{b_0, b_1, ..., b_{d-1}; c_1, c_2, ..., c_d\}$$

is called the intersection array of Γ.



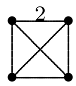
01 **Figure 1** Intersection diagram (of rank 0) with respect to x and illustration for coefficients c_h, a_h and b_h.

For vertices $x \in X$, $y \in \Gamma_h(x)$ we put

$$\Gamma_{ij}(x,y) = \Gamma_{ij}^h(x,y) = \Gamma_i(x) \cap \Gamma_j(y),$$

and for arbitrary $x, y, z \in X$

$$\Gamma_{ijk}(x,y,z) = \Gamma_i(x) \cap \Gamma_j(y) \cap \Gamma_k(z), \quad \text{etc.}$$

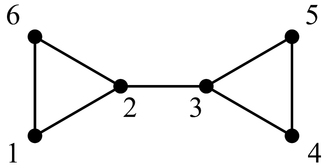
A distance-regular graph with no parallelogram of length two (that is with $\Big[$  $\Big] =$ 0) is called a *distance-regular graph of order* (s,t), where $s = \lambda + 1$ and $t = b_1/s$, because the incidence geometry defined on the set of vertices and the set of maximal cliques has a property that each line contains $s+1$ points and each point is on $t+1$ lines. For a distance-regular graph of order (s,t), *geometric girth* is defined to be the shortest length of a reduced circuit. If a distance-regular graph Γ satisfies the conditions $b_1 > b_2$ and $c_2 = 1$, then it is of order (s,t) and the geometric girth is five.

md **Definition. 2.1** Throughout the rest of this paper, $\Gamma = (X,R)$ shall refer to a fixed distance-regular graph with diameter d, valency n, intersection numbers p_{ij}^h $(0 \le h,i,j \le d)$ and intersection array $\{b_0,\ldots,b_{d-1}; c_1,\ldots,c_d\}$. Rational number $[\Delta]$ will denote a t-point count. For $x \in X$, $y \in \Gamma_h(x)$ we use following abbreviations $C(x,y) = C_h(x,y) = \Gamma_{h-1,1}^h(x,y) := \Gamma_{h-1}(x) \cap \Gamma_1(y)$, $A(x,y) = A_h(x,y) = \Gamma_{h,1}^h(x,y) := \Gamma_h(x) \cap \Gamma_1(y)$, $B(x,y) = B_h(x,y) = \Gamma_{h+1,1}^h(x,y) := \Gamma_{h+1}(x) \cap \Gamma_1(y)$, and for arbitrary $x,y,z \in X$, $\Gamma_{ij}(x,y) = \Gamma_i(x) \cap \Gamma_j(y)$, $\Gamma_{ijk}(x,y,z) = \Gamma_i(x) \cap \Gamma_j(y) \cap \Gamma_k(z)$, etc.

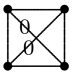
3 Simple examples, part I

hu In this section we give examples how to compute 4-point counts for a given graph, and we prove some equations that we use in the rest of the paper. Lets begin with simplest one.

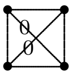
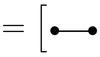
Example. 3.1 Let $\Gamma = (X,R)$ denote non-regular graph with vertex set $X = \{1,...,6\}$ and edge set $R = \{12, 16, 23, 26, 34, 35, 45\}$ (see Figure 2).

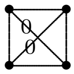


02 **Figure 2** Graph with vertex set $X = \{1, ..., 6\}$ and edge set $R = \{12, 16, 23, 26, 34, 35, 45\}$.

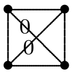
If we fix drawing of K_4 (as square with diagonals) and fix labelling of the vertices as, for example, first (left upper corner), second (left bottom corner), third (right bottom corner) and fourth (right upper corner) then the following two configurations 1212 and 2121 are of the type  . Note that

$$\left[\; \right] = \left[\; \right] = \frac{1}{6}(2+3+3+2+2+2) = \frac{7}{3}$$

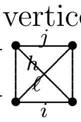
or since every edge have two different configurations of type  we can compute

$$\left[\; \right] = \frac{7 \cdot 2}{6} = \frac{7}{3}.$$



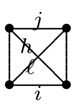
(we count the number of $z_1z_2z_3z_3$ with $z_1 = z_3$, $z_2 = z_4$ and adjacent $z_1 \sim z_2$, hence the number of ordered edges z_1z_2). ∎

Note that family of graphs that we have in next example is interesting for study on their own. This family is nonempty; for example every hypercube satisfy given condition.

Example. 3.2 Let $\Gamma = (X, R)$ denote distance-regular graph with diameter d, valency n and assume that for any i, j and h $(0 \leq i, j, h \leq d)$ size of $|\Gamma_{ijh}| = |\Gamma_{ijh}(x, y, z)| = |\Gamma_i(x) \cap \Gamma_j(y) \cap \Gamma_h(z)|$ do not depend on the considered vertices $x, y, z \in X$ but only on their distances $\partial(x, y)$, $\partial(x, z)$ and $\partial(y, z)$. (Note similarity between this condition and definition of distance-regular graph: $\Gamma = (X, R)$ is distance-regular graph if and only if for any i, j $(0 \leq i, j \leq d)$ size of $|\Gamma_{ij}| = |\Gamma_{ij}(x, y)| = |\Gamma_i(x) \cap \Gamma_j(y)|$ do not depend on the considered vertices $x, y \in X$ but only on the value of $\partial(x, y)$.) Then we can calculate 4-point count $[$  $]$ on $4! = 24$ different ways. Four of them are the following.

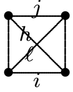
(i) Pick $x \in X$, $y \in \Gamma_1(x)$ and $z \in \Gamma^1_{hi}(x, y)$ and let $\Gamma_{j\ell 1} = \Gamma_{j\ell 1}(x, y, z)$. If we fix labelling of the vertices as first (left upper corner), second (left bottom corner), third (right bottom corner) and fourth (right upper corner) then

$$\left[\; \right] = \frac{v_\Gamma \cdot n \cdot p^1_{hi} \cdot |\Gamma_{j\ell 1}|}{v_\Gamma}.$$

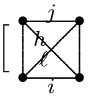


(ii) Pick $x \in X$, $y \in \Gamma_i(x)$ and $z \in \Gamma^i_{\ell 1}(x, y)$ and let $\Gamma_{1hj} = \Gamma_{1hj}(x, y, z)$. If we fix labelling of the vertices as first (left bottom corner), second (right bottom corner), third (right upper corner) and fourth (left upper corner) then

$$\left[\; \right] = \frac{v_\Gamma \cdot n_i \cdot p^i_{\ell 1} \cdot |\Gamma_{1hj}|}{v_\Gamma}.$$

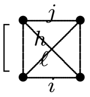


(iii) Pick $x \in X$, $y \in \Gamma_1(x)$ and $z \in \Gamma^1_{hj}(x,y)$ and let $\Gamma_{i\ell 1} = \Gamma_{i\ell 1}(x,y,z)$. If we fix labelling of the vertices as first (right bottom corner), second (right upper corner), third (left upper corner) and fourth (left bottom corner) then

$$\Big[$$ 

$$\Big] = \frac{v_\Gamma \cdot n \cdot p^1_{hj} \cdot |\Gamma_{i\ell 1}|}{v_\Gamma}.$$

(iv) Pick $x \in X$, $y \in \Gamma_j(x)$ and $z \in \Gamma^j_{\ell 1}(x,y)$ and let $\Gamma_{1hi} = \Gamma_{1hi}(x,y,z)$. If we fix labelling of the vertices as first (right upper corner), second (left upper corner), third (left bottom corner) and fourth (right bottom corner) then

$$\Big[$$ 

$$\Big] = \frac{v_\Gamma \cdot n_j \cdot p^j_{\ell 1} \cdot |\Gamma_{1hi}|}{v_\Gamma}.$$

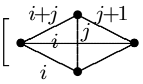
∎

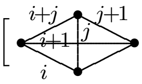
Lemma. 3.3 *The following* (i)–(iii) *hold.*

(i)

$$i + j \le d \qquad \Rightarrow \qquad c_i \le b_j, \tag{12}$$

and

$c_i = b_j$ *if and only if* $\Big[$ 

$\Big] = 0 = \Big[$ 

$\Big]$.

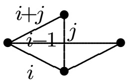
(ii)

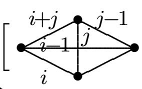
$$p^{h+1}_{ij} = \frac{1}{b_h}\Big(b_{j-1}p^h_{i,j-1} + (a_j - a_h)p^h_{ij} + c_{j+1}p^h_{i,j+1} - c_h p^{h-1}_{ij}\Big).$$

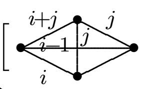
(iii)

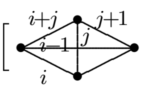
$$p^i_{j+1,h} = \frac{1}{c_{j+1}}\Big(p^i_{j,h-1}b_{h-1} + (a_h - a_j)p^i_{jh} + p^i_{j,h+1}c_h - b_{j-1}p^i_{j-1,h}\Big).$$

Proof. (i) Note that

$n_i p^i_{j,i+j} c_i = \Big[$ 

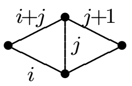
$\Big] = \underbrace{\Big[$ 

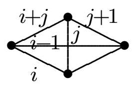
$\Big]}_{=0 \text{ (tr. ineq.)}} + \underbrace{\Big[$ 

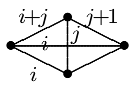
$\Big]}_{=0 \text{ (tr. ineq.)}} + \Big[$ 

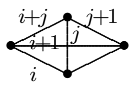
$\Big]$

and

$n_i p^i_{j,i+j} b_j = \Big[$ 

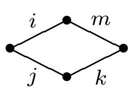
$\Big] = \Big[$ 

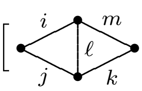
$\Big] + \Big[$ 

$\Big] + \Big[$ 

$\Big] \ge n_i p^i_{j,i+j} c_i$.

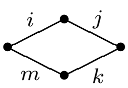
(ii) Since

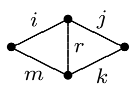
$\Big[$ 

$\Big] = \sum_{\ell=0}^{d} \Big[$ 

$\Big] = \sum_{\ell=0}^{d} \frac{v_\Gamma \cdot n_m \cdot p^m_{\ell k} \cdot p^\ell_{ij}}{v_\Gamma}$

and

$\Big[$ 

$\Big] = \sum_{r=0}^{d} \Big[$ 

$\Big] = \sum_{r=0}^{d} \frac{v_\Gamma \cdot n_m \cdot p^m_{ir} \cdot p^r_{jk}}{v_\Gamma}$

we have

$$\sum_{\ell} p_{ij}^{\ell} p_{\ell k}^{m} = \sum_{r} p_{ir}^{m} p_{jk}^{r}. \qquad (13)$$ du

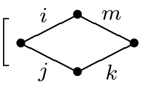
By (13) for $m=h$ and $k=1$ we have $\sum_{r=0}^{d} p_{ij}^{r} p_{r1}^{h} = \sum_{r=0}^{d} p_{ir}^{h} p_{j1}^{r}$ which yield

$$c_h p_{ij}^{h-1} + a_h p_{ij}^{h} + b_h p_{ij}^{h+1} = b_{j-1} p_{i,j-1}^{h} + a_j p_{ij}^{h} + c_{j+1} p_{i,j+1}^{h}$$

and with that

$$p_{ij}^{h+1} = \frac{1}{b_h}\Big(b_{j-1} p_{i,j-1}^{h} + (a_j - a_h) p_{ij}^{h} + c_{j+1} p_{i,j+1}^{h} - c_h p_{ij}^{h-1}\Big).$$

(ii) We have



$$\Big[\;\cdot\;\Big] = n_i \sum_{h=0}^{d} p_{mh}^{i} p_{jk}^{h} = n_m \sum_{h=0}^{d} p_{ih}^{m} p_{jk}^{h} =$$

$$= n_i \sum_{h=0}^{d} p_{jh}^{i} p_{mk}^{h} = n_j \sum_{h=0}^{d} p_{ih}^{j} p_{mk}^{h} = \sum_{h=0}^{d} n_h p_{im}^{h} p_{jk}^{h} = \sum_{h=0}^{d} n_h p_{ij}^{h} p_{mk}^{h}$$

which among else yield

$$\sum_{h=0}^{d} p_{mh}^{i} p_{jk}^{h} = \sum_{h=0}^{d} p_{jh}^{i} p_{mk}^{h}. \qquad (14)$$ eu

By (14) for $m=h$ and $k=1$ we have $\sum_{r=0}^{d} p_{rh}^{i} p_{1j}^{r} = \sum_{\ell=0}^{d} p_{j\ell}^{i} p_{1h}^{\ell}$ which yield

$$c_{j+1} p_{j+1,h}^{i} + a_j p_{jh}^{i} + b_{j-1} p_{j-1,h}^{i} = p_{j,h-1}^{i} b_{h-1} + p_{jh}^{i} a_h + p_{j,h+1}^{i} c_h$$

and with that

$$p_{j+1,h}^{i} = \frac{1}{c_{j+1}}\Big(p_{j,h-1}^{i} b_{h-1} + (a_h - a_j) p_{jh}^{i} + p_{j,h+1}^{i} c_h - b_{j-1} p_{j-1,h}^{i}\Big).$$

∎

1u **Lemma. 3.4** *We have*

$$p_{i,i-1}^{1} = \frac{c_i n_i}{n} = \frac{b_{i-1} n_{i-1}}{n}, \qquad p_{ii}^{1} = \frac{a_i n_i}{n} \qquad \textit{and} \qquad p_{i,i+1}^{1} = \frac{b_i n_i}{n} \qquad (1 \le i \le d);$$

$$p_{i+1,i-1}^{2} = p_{i-1,i+1}^{2} = \frac{b_2 b_3 \ldots b_i}{c_1 c_2 \ldots c_{i-1}} = \frac{n_i c_i b_i}{n b_1},$$

and

$$p_{i,i+1}^{2} = p_{i+1,i}^{2} = \frac{b_2 b_3 \ldots b_i}{c_1 c_2 \ldots c_i}(a_i + a_{i+1} - \lambda) = \frac{n_i c_i}{n b_1}(a_i + a_{i+1} - \lambda) \qquad (2 \le i \le d-1);$$

$$p_{22}^{2} = \frac{1}{c_2}(c_2 b_1 + a_2^2 + c_3 b_2 - n - \lambda a_2) = \frac{1}{c_2}(c_2(b_1 - 1) + b_2(c_3 - 1) + a_2(a_2 - \lambda - 1)) \qquad \textit{and}$$

$$p_{ii}^{2} = \frac{b_2 b_3 \ldots b_{i-1}}{c_1 c_2 \ldots c_i}(c_i b_{i-1} + a_i^2 + c_{i+1} b_i - n - \lambda a_i) \quad (3 \le i \le d-1);$$

$$p_{ij}^{i+j}=\frac{c_{i+1}...c_{i+j}}{c_1...c_j},\qquad p_{ij}^{i-j}=\frac{b_{i-1}...b_{i-j}}{c_1...c_j},\qquad p_{i,j+1}^{i+j}=p_{ij}^{i+j}\frac{a_i+..+a_{i+j}-a_1-...-a_j}{c_{j+1}},$$

$$\text{and}\qquad p_{i,j+1}^{i-j}=p_{ij}^{i-j}\frac{a_i+..+a_{i-j}-a_1-...-a_j}{c_{j+1}}.$$

Also

$$p_{2i}^{i}=\frac{1}{c_2}(c_ib_{i-1}+a_i(a_i-\lambda)+b_ic_{i+1}-n)\qquad(2\le i\le d-1).$$

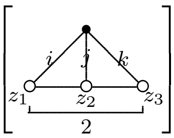
Proof. By definition $p_{0h}^i=\delta_{ih}$, $p_{j0}^i=\delta_{ij}$, $p_{j,d+1}^i=0$, $p_{1,i-1}^i=c_i$, $p_{1i}^i=a_i$, $p_{1,i+1}^i=b_i$ and $p_{1h}^i=0$ if $h\le i-2$ or $h\ge i+2$. Now use Lemma 3.3(ii)(iii) and induction on i. ∎

4 Simple examples, part II

ma In [5] K. Coolsaet and A. Jurisić introduce notation of triple intersection numbers, and with help of them they proved nonexistence of certain distance-regular graphs. In our notation their triple intersection number is in fact our 4-point count with three vertices fixed (see Subsection 4.1). In Subsection 4.2 we have 5-point counts with four vertices fixed i.e. we have quadruple intersection number.

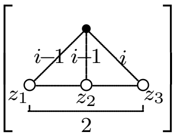
4.1 Induced path of length 2

mg Pick $z_1\in X$, $z_2\in\Gamma_1(z_1)$ and $z_3\in\Gamma_{21}(z_1,z_2)$ and abbreviate



$:= |\Gamma_{ijk}(z_1,z_2,z_3)|.$

8k **Lemma. 4.1** *For $z_1\in X$, $z_2\in\Gamma_1(z_1)$, $z_3\in\Gamma_{21}(z_1,z_2)$ we have*



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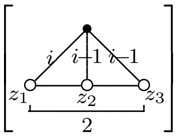
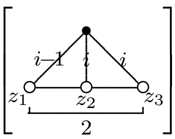


,



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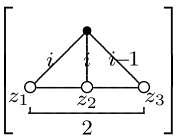
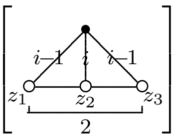
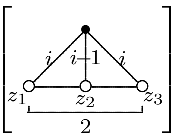


and

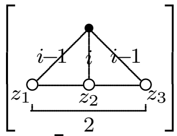


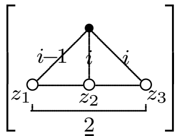
$+$

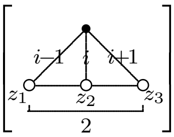


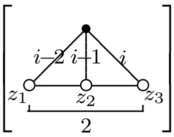
$= 2p_{i-1,i}^1-p_{i-1,i+1}^2-p_{i,i-2}^2-p_{i,i-1}^2$

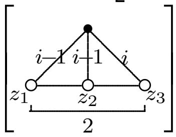
for any i $(2\le i\le d-1)$.

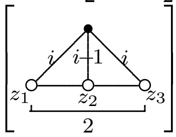
Proof. Just for a moment lets denote by $D =$ 

, $E =$ 

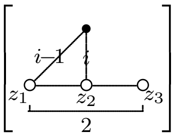
, $F =$ 

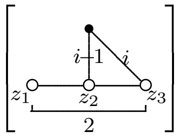
, $P =$ 

, $Q =$ 

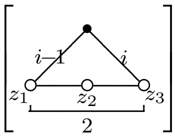
and $R =$ 

. Note that

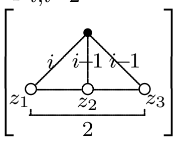
$p^1_{i-1,i} =$ 

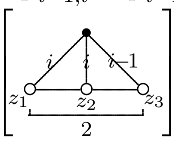
$= D + E + F,$ $p^1_{i-1,i} =$ 

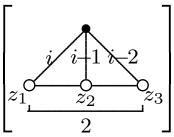
$= P + Q + R,$

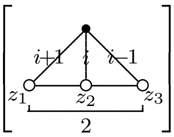
$F = p^2_{i-1,i+1},$ $p^2_{i,i-1} =$ 

$= E + Q,$ $P = p^2_{i,i-2},$

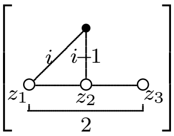
yield $D + E = p^1_{i,i-1} - p^2_{i-1,i+1}$, $Q + R = p^1_{i,i-1} - p^2_{i,i-2}$ and $D + R = 2p^1_{i-1,i} - p^2_{i-1,i+1} - p^2_{i,i-2} - p^2_{i,i-1}$. Similarly, if we abbreviate $M =$ 

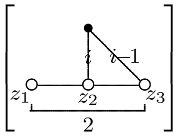
, $N =$ 

, $S =$ 

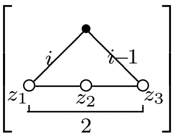
, $T =$ 

then

$p^1_{i-1,i} =$ 

$= S + M + R,$ $p^1_{i-1,i} =$ 

$= D + N + T,$

$T = p^2_{i-1,i+1},$ $p^2_{i,i-1} =$ 

$= M + N,$ $S = p^2_{i,i-2},$

yield $D + N = p^1_{i,i-1} - p^2_{i-1,i+1}$ and $M + R = p^1_{i,i-1} - p^2_{i,i-2}$. ∎

Corollary. 4.2 *We have*

$$c_i + b_{i-1} \geq \lambda + 2 \qquad (15)$$ 6u

for any i ($2 \leq i \leq d-1$).

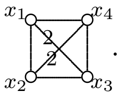
Proof. Lemma 4.1 yield $2p^1_{i-1,i} - p^2_{i-1,i+1} - p^2_{i,i-2} - p^2_{i,i-1} \geq 0$. Now the result follows from Lemma 3.4. ∎

Inequality (15) is due to D. E. Taylor & R. Levingston [13].

4.2 Induced quadrangle

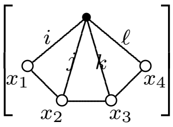
mc

Assume that there exists $x_1x_2x_3x_4$ consistent with the diagram



.

Now fix such quadrangle and abbreviate

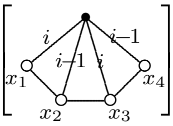
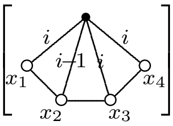
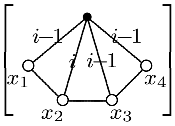
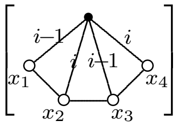
 $:= |\Gamma_{ijk\ell}(x_1, x_2, x_3, x_4)|.$

41 **Theorem. 4.3** (P. Terwilliger [14]) *If* Γ *contains an induced quadrangle then the following* (i), (ii) *hold.*

(i)

$$(c_i - c_{i-1}) + (b_{i-1} - b_i) \geq \lambda + 2. \tag{16}$$ 6k

and equality hold if and only if

 $=$  $=$  $=$  $= 0$

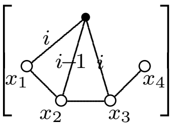
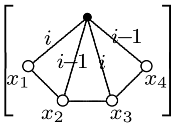
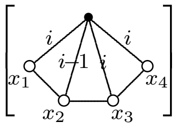
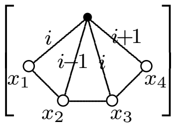
for all i $(1 \leq i \leq d)$.

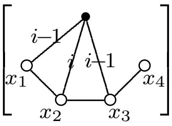
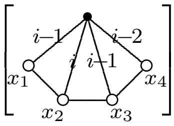
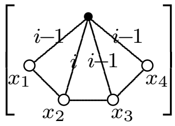
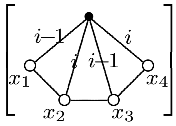
(ii)

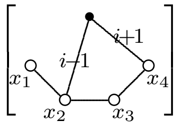
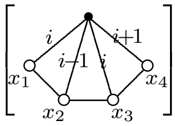
$$d \leq \frac{k + c_d}{\lambda + 2}$$

and equality hold if and only if equality hold in (16) *for every* $i = 1, 2, ..., d$.

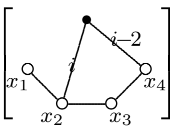
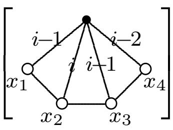
Proof. (i) We have

$|\Gamma_{i,i-1,i}(x_1, x_2, x_3)| =$  $=$  $+$  $+$ ,

$|\Gamma_{i-1,i,i-1}(x_1, x_2, x_3)| =$  $=$  $+$  $+$ ,

$p^2_{i-1,i+1} =$  $=$ ,

and

$p^2_{i,i-2} =$  $=$ .

This yield

$$p^2_{i-1,i+1} + p^2_{i,i-2} \leq |\Gamma_{i,i-1,i}(x_1,x_2,x_3)| + |\Gamma_{i-1,i,i-1}(x_1,x_2,x_3)|. \qquad (17)$$ 4k

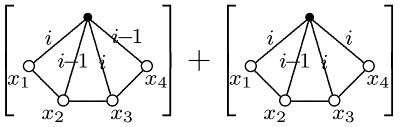
By Lemma 4.1 for any $x_1 \in X$, $x_2 \in \Gamma_1(x)$ and $x_3 \in \Gamma_{21}(x_1,x_2)$ we have

$$|\Gamma_{i,i-1,i}(x_1,x_2,x_3)| + |\Gamma_{i-1,i,i-1}(x_1,x_2,x_3)| = 2p^1_{i,i-1} - p^2_{i,i-1} - p^2_{i,i-2} - p^2_{i-1,i+1} \qquad (18)$$ 5k

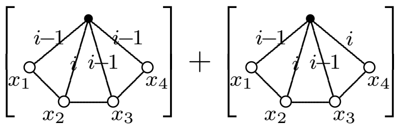
From (17) and (18) we have

$$2p^2_{i-1,i+1} + 2p^2_{i,i-2} \leq 2p^1_{i,i-1} - p^2_{i,i-1}$$

which yield (16). Equality hold in (17) (and with that in (16)) if and only if


$= 0$

and


$= 0.$

(ii) Sum the inequalities from (16) for $i = 1, 2, ..., d$. ∎

Remark. 4.4 The diameter bound of the above theorem is tight, and there are infinite families of distance-regular graphs achieving the bound. In fact characterization of equality case was done by P. Terwilliger in [15].

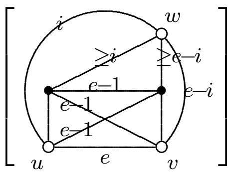
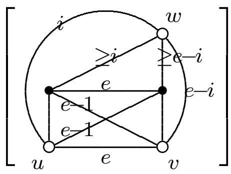
5 Koolen's inequalities

gu In [6, Lemmas 3.4, 3.5], Hiraki re-proved Koolen's inequalities from [9, 10]. In Lemmas 5.1, 5.3 we also re-proved these inequalities using t-point counts. In fact, in this section we present how to obtain Koolen's inequalities using 5-point counts with three vertices fixed. In Part II of sub-sequential paper we will get inequalities (19) and (22) under different assumptions i.e. we will get that induced polygon of length $2e+3$ yield (22) and that induced polygon of length $2e+2$ yield both (19) and (22).

lu **Lemma. 5.1** *If $c_e > c_{e-1}$ for some e $(2 \leq e \leq d)$ then*

$$c_e \geq c_i + c_{e-i} \qquad \textit{for all } i = 1, 2, ..., e-1. \qquad (19)$$ mu

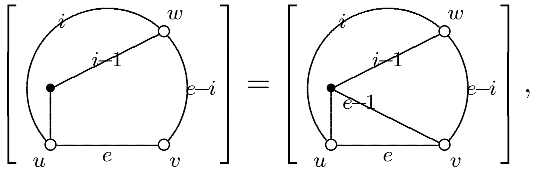
Moreover, in that case $c_i + c_{e-i} = c_e$ if and only if

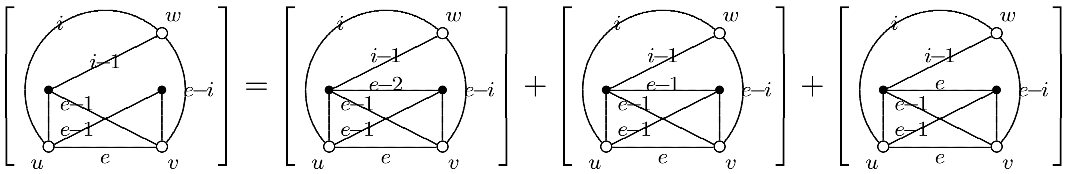

$= 0$ *and*

$= 0.$

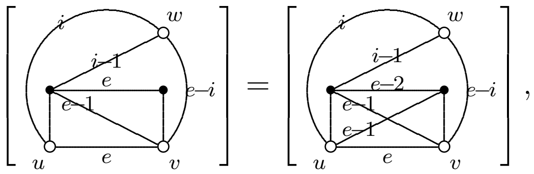
Proof. Pick $u \in X$, $v \in \Gamma_e(u)$ and $w \in \Gamma_{i,e-i}(u,v)$. We count the size of the set

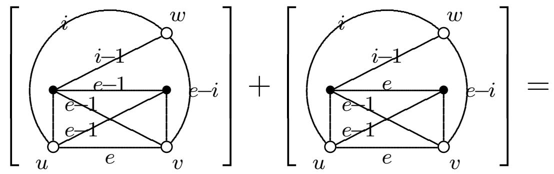
$$\Omega = \{(y,z) \in Y \times Z \,|\, Y := C_i(w,u) \text{ and } Z := C_e(u,v)/C_{e-i}(w,v)\}$$

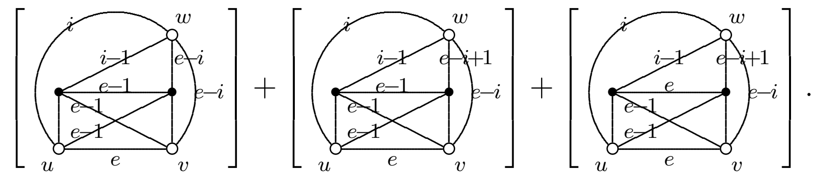
in two ways. Using triangle inequality, we can express $c_i(c_e - c_{e-1})$ via 5-point counts with three vertices fixed:

$$c_i = |Y| = |C_i(w,u)| =$$


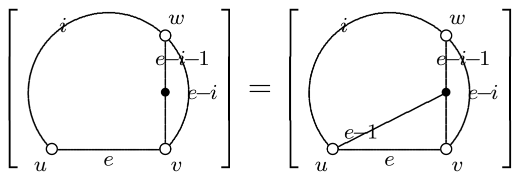
$$c_i c_e =$$


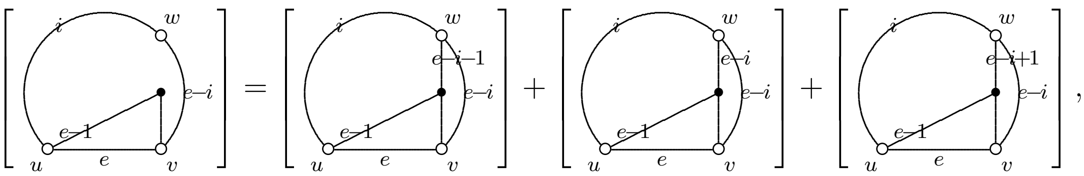
$$c_i c_{e-1} =$$


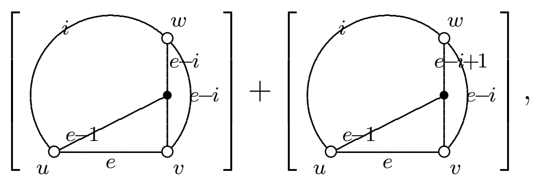
$$c_i(c_e - c_{e-1}) =$$


$$=$$

(20) nu

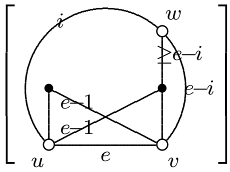
Similarly, we can express product $(c_e - c_{e-1})(c_e - c_{e-1})$:

$$c_{e-i} =$$


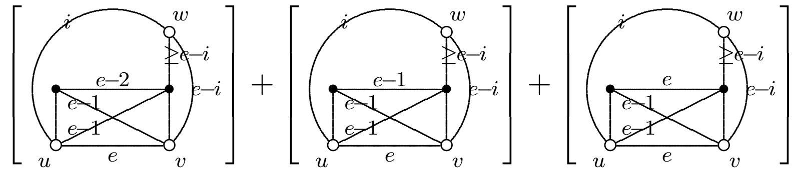
$$c_e =$$


$$c_e - c_{e-i} =$$


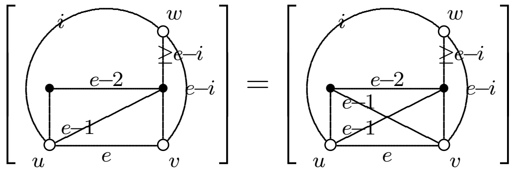
$$(c_e - c_{e-i})c_e =$$



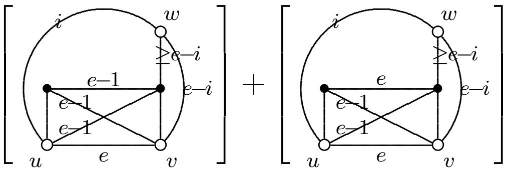
$$=$$



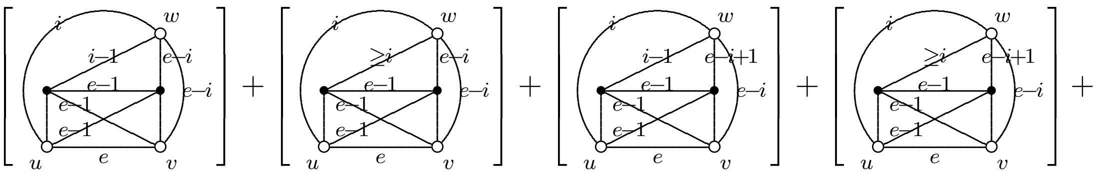
$$(c_e - c_{e-i})c_{e-1} =$$

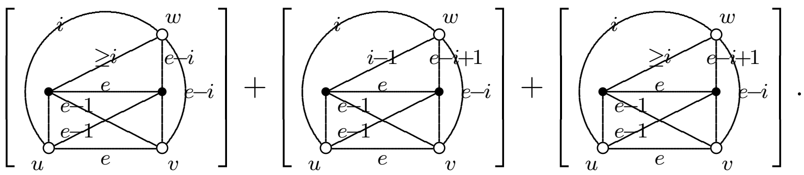


$$(c_e - c_{e-i})(c_e - c_{e-1}) =$$



$$=$$





(21) ou

Now it is not hard to see that (20) $\leq$ (21) i.e. that $c_i(c_e - c_{e-1}) \leq (c_e - c_{e-i})(c_e - c_{e-1})$. The result follows. ∎

oo **Corollary. 5.2** *If $c_{i+q} > c_{i+q-1}$ for some $i+q$ $(2 \leq i+q \leq d)$ then*

$$c_{i+q} \geq c_i + c_q \qquad \textit{for all } i = 1, 2, ..., i+q-1.$$

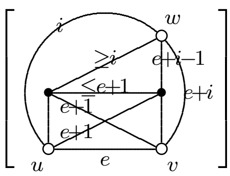
Proof. Immediate from Lemma 5.3. ∎

yu **Lemma. 5.3** *If $b_e > b_{e+1}$ for some e $(2 \leq e \leq d)$ then*

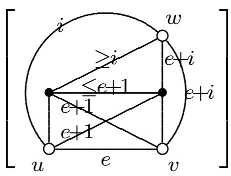
$$b_e \geq c_i + b_{e+i} \qquad \textit{for all } i = 1, 2, ..., d-e. \tag{22}$$

uu

Moreover, in that case $c_i + b_{e+i} = b_e$ if and only if



$= 0$ *and*

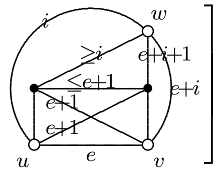


$= 0.$

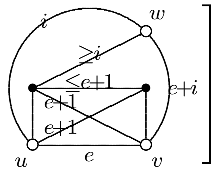
Proof. Pick $u \in X$, $v \in \Gamma_e(u)$ and $w \in \Gamma_{i,e+i}(u,v)$. Let $Y := B_{e+i}(w,v) = \Gamma_{e+i+1,1}(w,v)$ and $Z := B_e(v,u)\backslash C_i(x,u)$. By counting the size of the set

$$\{(y,z) \in Y \times Z \mid \partial(y,z) \le e+1\}$$

in two ways, similarly to the proof of Lemma 5.1 we have

$b_{e+i}(b_e - b_{e+1}) = \Big[$  $\Big]$ (23) wu

and

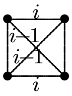
$(b_e - c_i)(b_e - b_{e+1}) = \Big[$  $\Big]$. (24) xu

The result follows. ∎

6 4-point counts A_i, B_i, C_i, D_i, E_i and F_i

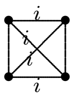
AB

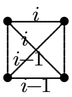
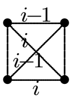
In this section we considering the 4-point counts containing two disjoint edges and define the rational numbers

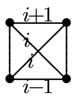
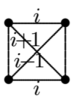
$e_i := \frac{1}{n_i c_i}\Big[$  $\Big] \ge 0.$ (25) b4

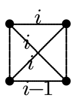
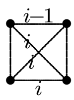
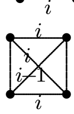
The names introduced for these 4-point counts will be used throughout.

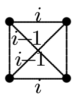
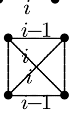
p1 **Proposition. 6.1** *With the notation of Definition 2.1, for any i $(2 \le i \le d)$ we have*

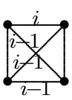
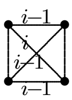
$A_i := \Big[$  $\Big] = n_i\Big(a_i^2 - c_i(b_{i-1} - b_i - e_i) - b_i(c_{i+1} - c_i - e_{i+1})\Big),$

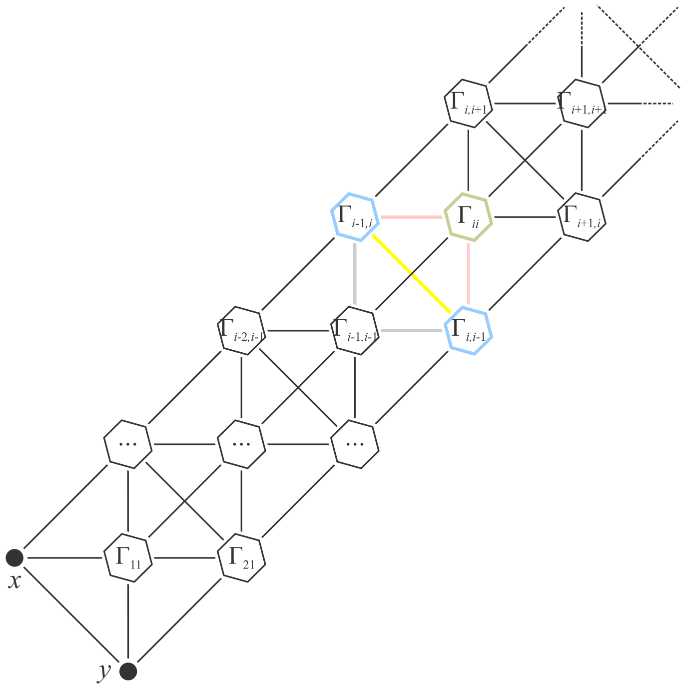
$B_i := \Big[$  $\Big] = \Big[$  $\Big] = n_i c_i\Big(a_{i-1} - (c_i - c_{i-1} - e_i)\Big) = n_i c_i\Big(a_i - (b_{i-1} - b_i - e_i)\Big),$

$C_i := \Big[$  $\Big] = \Big[$  $\Big] = n_i c_i b_i,$

$D_i := \Big[$  $\Big] = \Big[$  $\Big] = \Big[$  $\Big] = n_i c_i(b_{i-1} - b_i - e_i),$

$E_i := \Big[$  $\Big] = \Big[$  $\Big] = n_i c_i e_i,$

$F_i := \Big[$  $\Big] = \Big[$  $\Big] = n_i c_i(c_i - c_{i-1} - e_i).$



03 **Figure 3** Intersection diagram (of rank 1) with respect to (x, y) (that is $x \in X$, $y \in \Gamma_1(x)$, $\Gamma_{ij} = \Gamma_i(x) \cap \Gamma_j(y)$). Note that: (1) $e_i > 0$ iff there exist $x \in X$ and $y \in \Gamma_1(x)$ such that there is at least one edge between $\Gamma_{i-1,i}(x, y)$ and $\Gamma_{i,i-1}(x, y)$ (yellow edge exits). (2) $B_i > 0$ iff there exist $x \in X$ and $y \in \Gamma_1(x)$ such that there is at least one edge inside $\Gamma_{i-1,i}(x, y)$ or inside $\Gamma_{i,i-1}(x, y)$ (blue edge sets). (3) $A_i > 0$ iff there exist $x \in X$ and $y \in \Gamma_1(x)$ such that there is at least one edge inside $\Gamma_{ii}(x, y)$ (green edge set). (4) $D_i > 0$ iff there exist $x \in X$ and $y \in \Gamma_1(x)$ such that there is at least one edge between $\Gamma_{ii}(x, y)$ and $\Gamma_{i-1,i}(x, y) \cup \Gamma_{i,i-1}(x, y)$ (pink edges exist). (6) $F_i > 0$ iff there exist $x \in X$ and $y \in \Gamma_1(x)$ such that there is at least one edge between $\Gamma_{i-1,i-1}(x, y)$ and $\Gamma_{i-1,i}(x, y) \cup \Gamma_{i,i-1}(x, y)$ (grey edges exist).

Proof. By summing over a distance (see (3) and (6)) we find

$$n_{i+1}c_{i+1}c_i = n_i c_i b_i = C_i,$$

$$n_i c_i a_i = B_i + D_i,$$
$$n_i c_i a_{i-1} = F_i + B_i,$$
$$n_i c_i b_{i-1} = D_i + E_i + C_i,$$
$$n_i c_i^2 = C_{i-1} + E_i + F_i,$$
$$n_i a_i^2 = D_i + A_i + F_{i+1}$$

(for example $n_i a_i^2 = \sum_{h=0}^{d} \left[\boxtimes^{h}_{i}\right] = \left[\boxtimes^{i-1}_{i}\right] + \left[\boxtimes^{i}_{i}\right] + \left[\boxtimes^{i+1}_{i}\right]$). Now (25) implies that $E_i = n_i c_i e_i$, and solving the resulting triangular linear system of equations gives the above formulas. ∎

Remark. 6.2 If $e_i = 0$ then the counts B_i, D_i, F_i are all constant. This makes it easier to count bigger configurations.

A Terwilliger graph is a non-complete graph Γ such that, for any two vertices u, v at distance two, $\Gamma_1(u) \cap \Gamma_1(v)$ is a clique of size μ (for some fixed $\mu \geq 2$). The case $e_2 = 0$ corresponds to Terwilliger graphs which are very restricted in structure. In [15] Terwilliger showd that if Γ is distance-regular graph with diameter $d \geq \dfrac{k + c_d}{\lambda + 2}$ then one of the following (i)–(iii) holds: (i) Γ is a Terwilliger graph; (ii) Γ is strongly regular with smallest eigenvalue -2; (iii) Γ is a Hamming graph, a Doob graph, a locally Petersen graph, a Johnsnon graph, a half cube or the Gosset graph.

o1 **Corollary. 6.3** *With reference to Proposition 6.1, let* $x^+ = \max(x, 0)$. *Then*

$$b_{i-1} \geq b_i, \quad c_i \geq c_{i-1}, \tag{26}$$ b5

$$c_i(a_i - a_{i-1})^+ + b_i(a_i - a_{i+1})^+ \leq a_i^2, \tag{27}$$ b6

$$\max\Big(0, c_i - c_{i-1} - a_{i-1}, b_{i-1} - b_i - \frac{a_i^2}{c_i}, c_i - c_{i-1} - \frac{a_{i-1}^2}{b_i}\Big) \leq e_i, \tag{28}$$ b7

$$e_i \leq \min(b_{i-1} - b_i, c_i - c_{i-1}), \tag{29}$$ b8

$$b_{i-1} = b_i \iff D_i = E_i = 0, \tag{30}$$ b9

$$c_{i-1} = c_i \iff E_i = F_i = 0. \tag{31}$$ c1

Proof. (26), (30), (31), and (27) follow from

$$0 \leq D_i + E_i = n_i c_i (b_{i-1} - b_i),$$

$$0 \leq E_i + F_i = n_i c_i (c_i - c_{i-1}),$$

$$D_i - F_i = n_i c_i (a_i - a_{i-1}), \qquad D_{i+1} - F_{i+1} = n_i b_i (a_{i+1} - a_i),$$

$$(D_i - F_i)^+ + (F_{i+1} - D_{i+1})^+ \leq D_i + F_{i+1} + A_i = n_i a_i^2.$$

(29) follows from $D_i \geq 0$, $F_i \geq 0$, and (28) from $E_i, B_i, A_i, F_{i+1} \geq 0$ (for example, note that $e_i \geq \frac{b_i}{c_i}(c_{i+1} - c_i - e_{i+1}) + (b_{i-1} - b_i) - \frac{a_i^2}{c_i}$). ∎

Inequalities (26) and (27) are due to N. Biggs [2] and A.E. Brouwer & E.W. Lambeck [4], respectively.

Corollary. 6.4 (K. Nomura [11]) *With reference to Proposition 6.3, the following hold*

$$a_{i+1} \geq a_i(1 - \frac{a_i}{b_i}), \qquad a_i \geq a_{i+1}(1 - \frac{a_{i+1}}{c_{i+1}}),$$

$$0 < a_i < b_i \qquad \Rightarrow \qquad a_{i+1} > 0,$$

$$0 < a_{i+1} < c_{i+1} \qquad \Rightarrow \qquad a_i > 0.$$

Proof. Note that (27) yield $a_i^2 \geq b_i(a_i - a_{i+1})$ and $a_{i+1}^2 \geq c_{i+1}(a_{i+1} - a_i)$. The results follow. ∎

o2 **Corollary. 6.5** *With reference to Proposition 6.1, the following hold*

$$a_{i-1} \geq c_i - c_{i-1} - e_i, \tag{32}$$ A1

$$a_i \geq b_{i-1} - b_i - e_i, \tag{33}$$ A2

$$e_i = 0 \quad \Rightarrow \quad a_{i-1} \geq c_i - c_{i-1} \quad \text{and} \quad a_i \geq b_{i-1} - b_i, \tag{34}$$ A3

$$c_i e_i + b_i e_{i+1} \geq b_i(c_{i+1} - c_i) + c_i(b_{i-1} - b_i) - a_i^2, \tag{35}$$ A4

$$c_i = c_{i-1} \text{ or } b_i = b_{i-1} \quad \Rightarrow \quad e_i = 0, \tag{36}$$ A5

$$b_i = b_{i-1} \text{ and } a_i \neq 0 \quad \Rightarrow \quad a_{i-1} > c_i - c_{i-1}, \tag{37}$$ A6

$$c_i = c_{i-1} \text{ and } a_{i-1} \neq 0 \quad \Rightarrow \quad a_i > b_{i-1} - b_i. \tag{38}$$ A7

Proof. (32) and (33) follow from $B_i \geq 0$. (34) follows immediate from (32) and (33). (35) follows from $A_i \geq 0$. (36) follows from (29). If $b_i = b_{i-1}$ and $a_i \neq 0$ then $e_i = 0$ and $B_i > 0$. This yield (37). Similarly if $c_i = c_{i-1}$ and $a_{i-1} \neq 0$ then $e_i = 0$ and $B_i > 0$. This yield (38). ∎

6.1 Case when $B_i = 0$

oe

In this subsection we consiter the case when $B_i = 0$.

of **Lemma. 6.6** *With reference to Proposition 6.1, pick i $(2 \leq i \leq d-1)$ and assume that $B_i = 0$. Then the following* (i)–(iii) *hold.*

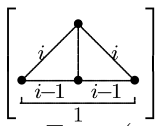
(i) *If $a_{i-1} \neq 0$ then $b_{i-1} \leq a_{i-1}$.*

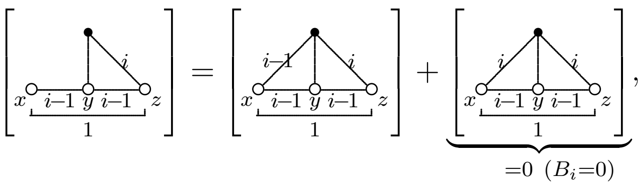
(ii) *Assume $a_{i-1} \neq 0$. Then*

$$b_{i-1} = a_{i-1} \qquad \text{if and only if} \qquad A_{i-1} = 0 \quad \text{and} \quad D_{i-1} = 0.$$

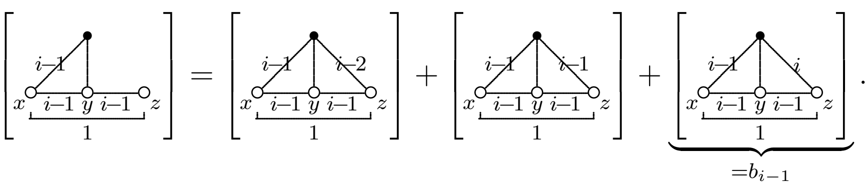
Moreover in this case $B_{i-1} \neq 0$.

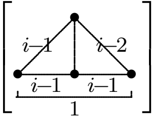
(iii) *$a_{i-1} \neq 0$ if and only if $F_i \neq 0$.*

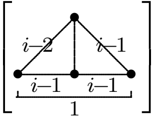
Proof. By assumption $B_i =$  $= 0$. Since $a_{i-1} \neq 0$ then we can pick $x \in X$, $y \in \Gamma_{i-1}(x)$ and $z \in A_{i-1}(y,x) = \Gamma_{i-1,1}(y,x)$. We have

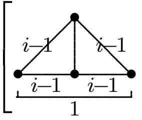
$b_{i-1} =$ 

and

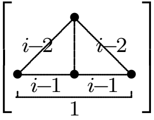
$a_{i-1} =$ 

This yield (i). Note that $D_{i-1} =$ [

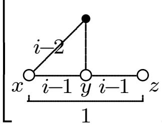
] $=$ [

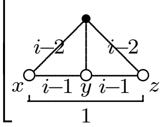
], $A_{i-1} =$ [

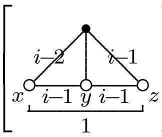
],

$B_{i-1} =$ [

] and

$c_{i-1} =$ [

] $=$ [

] $+$ [

].

This yield (ii). Claim (iii) follows from $B_i + F_i = n_i c_i a_{i-1}$. ∎

og **Lemma. 6.7** *With reference to Proposition 6.1, pick i $(2 \leq i \leq d-1)$ and assume that $B_i = 0$. Then the following* (i)–(iii) *hold.*

(i) *If $a_i \neq 0$ then $c_i \leq a_i$.*

(ii) *Assume $a_i \neq 0$. Then*

$$c_i = a_i \qquad \textit{if and only if} \qquad A_i = 0 \quad \textit{and} \quad F_{i+1} = 0.$$

Moreover in this case $B_{i+1} \neq 0$.

(iii) *$a_i \neq 0$ if and only if $D_i \neq 0$.*

Proof. Similar to the proof of Lemma 6.6. ∎

oh **Corollary. 6.8** *With reference to Proposition 6.1, pick i $(2 \leq i \leq d-1)$ and assume that $B_i = 0$. Then the following* (i), (ii) *hold.*

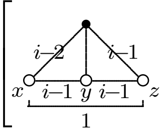
(i) *If $a_{i-1} \neq 0$ and $a_i \neq 0$ then*

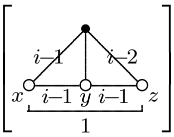
$$c_i > c_{i-1} \qquad \textit{and} \qquad b_{i-1} > b_i.$$

(ii) *$e_i = 0$ if and only if $c_i = a_{i-1} + c_{i-1}$ and $b_{i-1} = a_i + b_i$.*

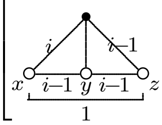
Proof. Immediate from (30), (31) and Lemmas 6.6(iii) and 6.7(iii). ∎

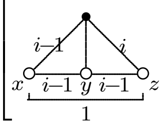
oi **Proposition. 6.9** *With reference to Proposition 6.1, pick i $(2 \leq i \leq d-1)$ and assume that $a_{i-1} \neq 0$ so that we can pick $x \in X$, $y \in \Gamma_{i-1}(x)$ and $z \in \Gamma_{i-1,1}(y, x)$. The following* (i)–(iv) *hold.*

(i) *If $B_{i-1} = 0$ then $c_{i-1} =$* [

] $=$ [

].

(ii) *If* $B_i = 0$ *then* $b_{i-1} =$ [

] $=$ [

].

(iii) *If* $A_{i-1} = B_{i-1} = B_i = 0$ *then*

$$a_{i-1} = c_{i-1} + b_{i-1}.$$

(iii) *If* $e_i = A_{i-1} = B_{i-1} = B_i = 0$ *then*

$$d < 2i - 1.$$

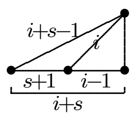
Proof. (i)–(iii) Similar to the proof of Lemma 6.6.

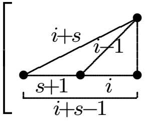
(iv) By Corollary 6.8(ii) we have $a_{i-1} = c_i - c_{i-1}$, and because of (iii) this yield $2c_{i-1} + b_{i-1} = c_i$. Thus $b_{i-1} < c_i$. The result now follows from (12). ∎

7 4-point counts B_i^s, C_i^s, D_i^s, E_i^s and F_i^s

BC

For every i ($0 \leq i \leq d$) and $s \geq 0$ define the rational numbers

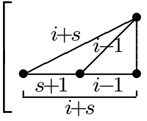
$e_i^s := \frac{1}{n_{i+s}p^{i+s}_{i-1,s+1}}$ [

] $= \frac{1}{n_{i+s}p^{i+s}_{i-1,s+1}}$ [

] $\geq 0.$ (39) c2

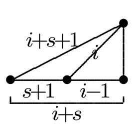
Note that $e_i^0 = e_i$.

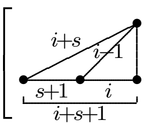
Bs **Proposition. 7.1** *With the notation of Definition 2.1, for any* i ($2 \leq i \leq d$) *and* $s \geq 0$ *we have*

$B_i^s :=$ [

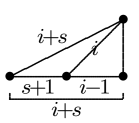
] $= n_{i+s}p^{i+s}_{i-1,s+1}\Big(a_{i+s} - (b_{i-1} - b_{i+s} - e_i^s)\Big) =$

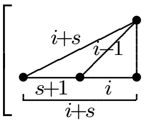
$$= n_{i+s}p^{i+s}_{i-1,s+1}\Big(a_{i-1} - (c_{i+s} - c_{i-1} - e_i^s)\Big),$$

$C_i^s :=$ [

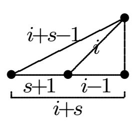
] $=$ [

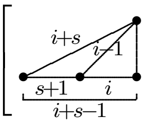
] $= n_{i+s}p^{i+s}_{i-1,s+1}b_{i+s}, \qquad C_{i-1}^s = n_{i+s}p^{i+s}_{i-1,s+1}c_{i-1},$

$D_i^s :=$ [

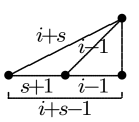
] $=$ [

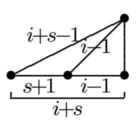
] $= n_{i+s}p^{i+s}_{i-1,s+1}(b_{i-1} - b_{i+s} - e_i^s),$

$E_i^s :=$ [

] $=$ [

] $= n_{i+s}p^{i+s}_{i-1,s+1}e_i^s,$

$F_i^s :=$ [

] $=$ [

] $= n_{i+s}p^{i+s}_{i-1,s+1}(c_{i+s} - c_{i-1} - e_i^s).$

Note that $B_i^0 = B_i$, $C_i^0 = C_i$, $D_i^0 = D_i$, $E_i^0 = E_i$ and $F_i^0 = F_i$.

Proof. By summing over a distance (see (3) and (6)) we find

$$\begin{aligned}
&B_i^s + D_i^s = n_{i+s} p^{i+s}_{i-1,s+1} a_{i+s},\\
&F_i^s + B_i^s = n_{i+s} p^{i+s}_{i-1,s+1} a_{i-1},\\
&C_i^s = n_{i+s} p^{i+s}_{i-1,s+1} b_{i+s},\\
&D_i^s + E_i^s = n_{i+s} p^{i+s}_{i-1,s+1} (b_{i-1} - b_{i+s}),\\
&E_i^s + F_i^s = n_{i+s} p^{i+s}_{i-1,s+1} (c_{i+s} - c_{i-1}),
\end{aligned}$$

Now (39) implies that $E_i^s = n_{i+s} p^{i+s}_{i-1,s+1} e_i^s$, and solving the resulting triangular linear system of equations gives the above formulas. ∎

o3 **Corollary. 7.2** *With reference to Proposition 7.1, if $i + s \le d$ then*

$$\max\left\{0, c_{i+s} - c_{i-1} - a_{i-1}, b_{i-1} - b_{i+s} - a_{i+s}\right\} \le e_i^s \le \min\{b_{i-1} - b_{i+s}, c_{i+s} - c_{i-1}\}$$

Proof. Immediate from nonnegativity of 4-point counts from Proposition 7.1. ∎

o4 **Corollary. 7.3** *With reference to Proposition 7.1, let i and s denote positive integers with $2 \le i \le d-1$ and $i + s \le d-1$. Then the following (i), (ii) hold.*

(i) *If $b_{i-1} = b_{i+s}$ and $a_{i+s} \neq 0$ then $a_{i-1} > c_{i+s} - c_{i-1}$.*

(ii) *If $c_{i-1} = c_{i+s}$ and $a_{i-1} \neq 0$ then $a_{i+s} > b_{i-1} - b_{i+s}$.*

Proof. If $b_{i-1} = b_{i+s}$ or $c_{i-1} = c_{i+s}$ then $e_i^s = 0$. Now if ($b_{i-1} = b_{i+s}$ and $a_{i+s} \neq 0$) or ($c_{i-1} = c_{i+s}$ and $a_{i-1} \neq 0$) then $B_i^s > 0$ and the results follow. ∎

+s **Corollary. 7.4** *With reference to Proposition 7.1, if $i + s \le d$ then the following* (i)–(iv) *hold.*

(i) $c_{i+s} > c_{i-1}$ *if and only if* $F_i^s \neq 0$ *or* $E_i^s \neq 0$.

(ii) $b_{i-1} > b_{i+s}$ *if and only if* $E_i^s \neq 0$ *or* $D_i^s \neq 0$.

(iii) $a_{i+s} > 0$ *if and only if* $B_i^s \neq 0$ *or* $D_i^s \neq 0$.

(iv) $a_{i-1} > 0$ *if and only if* $F_i^s \neq 0$ *or* $B_i^s \neq 0$.

Proof. Immediate from the proof of Proposition 7.1. ∎

D0 **Corollary. 7.5** *With reference to Proposition 7.1, the following* (i)–(iii) *hold.*

(i) $c_i = c_{2i-1}$ *if and only if* $F_{i+1}^{i-2} = 0$ *and* $E_{i+1}^{i-2} = 0$.

(ii) $b_i = b_{2i-1}$ *if and only if* $D_{i+1}^{i-2} = 0$ *and* $E_{i+1}^{i-2} = 0$.

(iii) *If* $c_{i-1} = c_{2i-1}$ *and* $b_{i-1} > b_i = b_{2i-1}$ *then* $F_i^{i-1} = 0,\ E_i^{i-1} = 0,\ D_i^{i-1} \neq 0$.

Proof. Immediate from Corollary 7.4 ∎

So, for example, if we want to show that given graph Γ with the property $c_{i-1} = c_{2i-1}$ and $b_{i-1} > b_i = b_{2i-1}$ is not possible, we need to prove that $D_i^{i-1} = 0$, which is a contradiction with Corollary 7.5(iii).

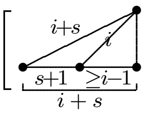
In (40) we have a generalization of half of the Brouwer–Lambeck inequality (27):

Proposition. 7.6 *With reference to Proposition 7.1*

$$c_i(a_{i+s} - a_{i-1}) \leq a_{i+s}(a_i + \ldots + a_{i+s} - a_1 - \ldots - a_s). \quad (40)$$ c3

Proof. Note that

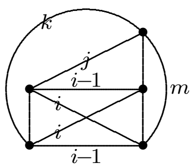
$$n_{i+s}p^{i+s}_{i-1,s+1}(a_{i+s} - a_{i-1}) = D_i^s - F_i^s \leq D_i^s \leq \left[\ \cdot\ \right] = n_{i+s}p^{i+s}_{i,s+1}a_{i+s}.$$



The result now follow from Lemma 3.4. ∎

7.1 Case when $e_i > 0$ for some i $(1 \leq i \leq d)$

Note that (28) involve e_i, hence is not a feasibility condition for the intersection array. However, when the lower bound in (28) is positive, we may apply the following results. We use the new counts



$$=: \begin{cases} H_{jk} & \text{if } m = 2, \\ K_{jk} & \text{if } m = 1. \end{cases} \quad (41)$$ c4

te **Theorem. 7.7** *With reference to Proposition 6.1, we have*

$$e_i > 0,\ c_{s+1} > \min(c_i - c_{i-1}, b_{i-1} - b_i) \quad \Rightarrow \quad c_{i+s} > c_i, \quad (42)$$ c5

$$e_i > 0, \quad i > 1 \quad \Rightarrow \quad c_{2i-1} > c_i. \quad (43)$$ c6

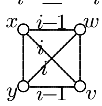
$$e_i > 0 \quad \Rightarrow \quad b_1 \geq b_i + c_{i-1}, \quad (44)$$ c7

$$e_i > 0 \quad \Rightarrow \quad b_{i-1} - b_i + c_i - c_{i-1} \geq \lambda + 2. \quad (45)$$ c8

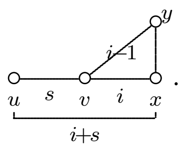
Proof. If $e_i > 0$ then by (29),

$$e := \min(c_i - c_{i-1}, b_{i-1} - b_i) \geq e_i > 0.$$

and with that $c_i \geq c_{i-1}$ and $b_{i-1} > b_i$. Also $e_i > 0$ imply that we may choose $xyvw$ consistent with

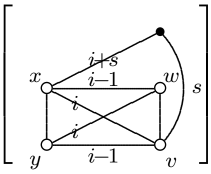


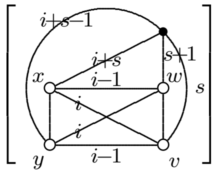
(since $E_i > 0$). Now consider vertex u consistent with



.

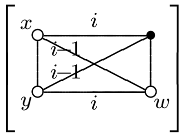
Note that

$p^i_{i+s,s} =$ 

$=$ 

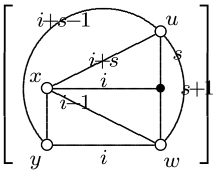
.

All $p^i_{i+s,s}$ choices for u yield $d(u,y) = i+s-1$ and $d(w,u) = s+1$. Now the number of $v' \in \Gamma_{1s}(w,u)$ with $d(v',y) = i-1$ is $\leq e$, i.e.



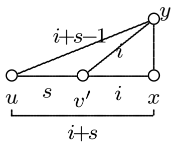
$\leq e.$

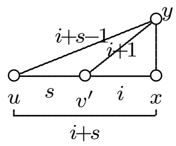
Also, note that

$c_{s+1} =$ 

.

If $c_{s+1} > e$, there is some v' with $d(v',y) \geq i$. With another words, there is v' such that $uv'xy$ is consistent with



or with


That is, we have

$$F^{s-1}_{i+1} > 0 \qquad \text{or} \qquad E^{s-1}_{i+1} > 0.$$

This yield $c_{i+s} > c_i$ (see Proposition 7.4(i)). Thus (42) holds, and (43) follows by taking $s = i-1 > 0$.

To derive the remaining statement we note that (41) gives

$$H_{ii+1} = E_i b_i, \quad H_{i-2i-1} = E_i c_{i-1},$$

$$H_{i-1i-1} + H_{i-1i} + H_{ii-1} + H_{ii} = E_i(b_1 - b_i - c_{i-1}), \tag{46}$$ c9

$$K_{i-1i-1} + H_{i-1i-1} + K_{ii-1} + H_{ii-1} = E_i(c_i - c_{i-1} - 1), \tag{47}$$ d1

$$K_{ii-1} + H_{ii-1} + K_{ii} + H_{ii} = E_i(b_{i-1} - b_i - 1), \tag{48}$$ d2

$$K_{i-1i-1} + K_{i-1i} + K_{ii-1} + K_{ii} = E_i\lambda. \tag{49}$$ d3

Swapping the two edges gives

$$K_{i-1i} = K_{ii-1}. \tag{50}$$ d4

Now (44) follows from the nonnegativity of (46), and (45) follows from (47)+(48)≥(49) which holds in view of (50). ∎

It would be interesting to know which graphs have property $e_i = 0$ for all i ($1 \leq i \leq d$).

Research problem. 7.8 *Classify all distance-regular graphs for which $e_i = 0$ for all i ($1 \leq i \leq d$).*

8 Note about case when $a_i > a_j$

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Pick i $(2 \le i \le d)$. In this section we show that

$$a_i > a_{i-1} \quad \Rightarrow \quad b_{i-1} > b_i$$

and

$$a_i < a_{i-1} \quad \Rightarrow \quad c_i > c_{i-1}.$$

Moreover, we have

$$a_j > a_i \quad \Rightarrow \quad b_i > b_j$$

and

$$a_j < a_i \quad \Rightarrow \quad c_j > c_i$$

hold for any i, j $(1 \le i < j \le d)$. By our knowledge, these two claims are new!

Lemma. 8.1 *Let* Γ *denote a distance-regular graph. Then the following* (i), (ii) *hold.*

(i) $a_i > a_{i-1} \quad \Rightarrow \quad b_{i-1} > b_i$.

(ii) $a_i < a_{i-1} \quad \Rightarrow \quad c_i > c_{i-1}$.

Proof. We prove claim (i). The proof of (ii) is similar.

From Proposition 6.1 we have $D_i - F_i = n_i c_i (a_i - a_{i-1})$, and since $a_i > a_{i-1}$ we have $D_i > F_i \ge 0$. This yield that $n_i c_i (b_{i-1} - b_i - e_i) = D_i > 0$ and the result follows. ∎

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Theorem. 8.2 *Let* Γ *denote a distance-regular graph and pick* i, j $(1 \le i < j \le d)$. *Then the following* (i), (ii) *hold.*

(i) $a_j > a_i \quad \Rightarrow \quad b_i > b_j$.

(ii) $a_j < a_i \quad \Rightarrow \quad c_j > c_i$.

Proof. Pick s such that $i + s = j$. From Proposition 7.1 we have $D^{s-1}_{i+1} - F^{s-1}_{i+1} = n_{i+s} p^{i+s}_{i,s} (a_{i+s} - a_i) = n_j p^j_{i,j-i} (a_j - a_i)$. If $a_j > a_i$ then $D^{s-1}_{i+1} > F^{s-1}_{i+1}$ which yield $b_i - b_j > e^{s-1}_{i+1} \ge 0$ and the result follows. The proof for the case $a_j < a_i$ is similar. ∎

Research problem. 8.3 *Using t-point counts explain under which restrictions on intersection numbers (and indices i, j) the following hold*

$$a_j > a_i \quad \Rightarrow \quad b_{i+1} > b_j,$$

$$a_j < a_i \quad \Rightarrow \quad c_j > c_{i+1}.$$

9 Hiraki's first inequality

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In [6, Lemma 3.1] A. Hiraki proved that if $2 \le q \le d-1$, $c_{q+1} > c_q$ and $a_q \le c_{q+1} - c_q$ then $c_q + c_i \le c_{q+1}$ for all $2 \le i \le d-q$. In this section, using t-point counts we re-prove Hiraki's inequality and we investigate different sub-cases. Hiraki's second inequality we prove in Section 16.

na **Lemma. 9.1** *Let $\Gamma = (X, R)$ be a distance regular graph of diameter d, valency $k \ge 3$, and let q be an integer with $2 \le q \le d-1$. Suppose $c_q < c_{q+1}$. The following* (i), (iii) *hold.*

(i) *If $a_q \le c_{q+1} - c_q$ then*

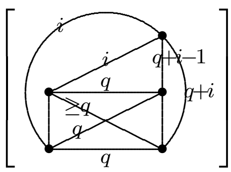
$$c_{q+i} \ge c_i + c_q \qquad \textit{for all } i \ (2 \le i \le d-q).$$

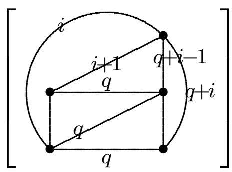
(ii) *If $a_q > c_{q+1} - c_q$ then*

$$c_{q+i} > \frac{1}{\eta} c_i + c_q \qquad \textit{for all } i \ (2 \le i \le d-q)$$

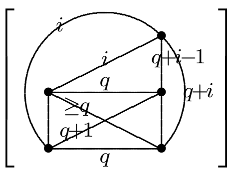
where $\eta = \frac{a_q}{c_{q+1} - c_q}$.

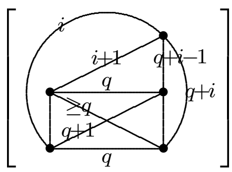
Moreover, if $a_q = c_{q+1} - c_q$ then $c_{q+i} = c_i + c_q$ if and only if



$= 0,$


$= 0,$



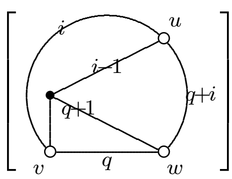
$= 0$ *and*


$= 0.$

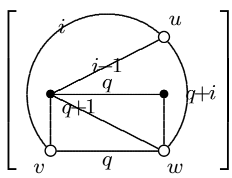
Proof. Proof of (i) can be found in [6]. Because of completeness, use of t-point counts technique and to see from where inequality comes from, we give it also here. Pick $u \in X$, $v \in \Gamma_i(u)$ and $w \in \Gamma_{q+i,q}(u, v)$. We consider sets $Y := C_i(u, v)$ and $Z := C_{q+i}(u, w) \backslash C_q(v, w)$, and we count size of the set

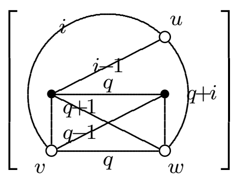
$$\Omega = \{(y, z) \in Y \times Z \mid d(y, z) = q\}$$

in two ways. From

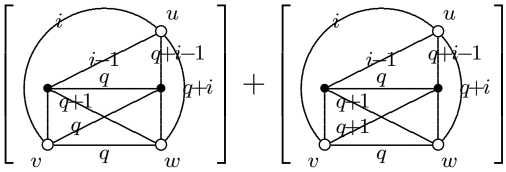
$c_i = |Y| = |C_i(u, v)| =$


,

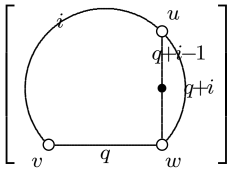
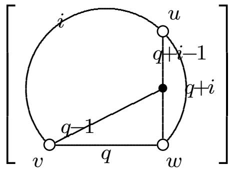
$c_i c_{q+1} =$


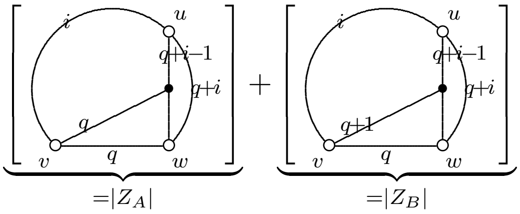
, $c_i c_q =$


,

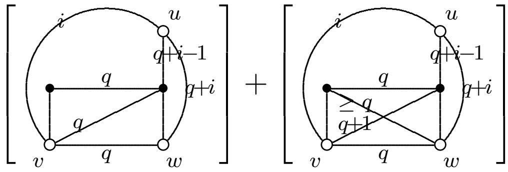
$$c_i(c_{q+1} - c_q) =$$ 

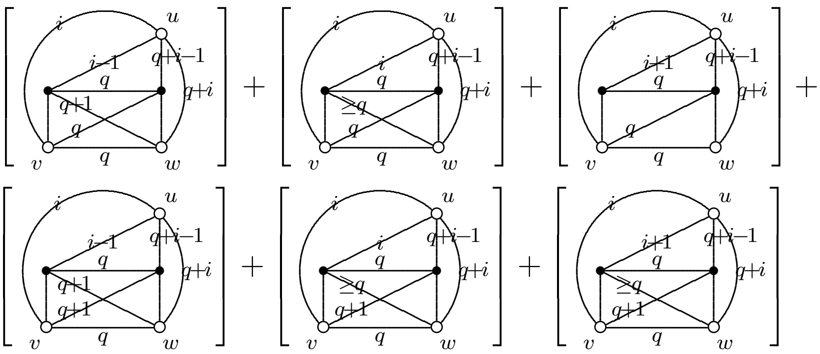
and

$$c_{q+i} = |C_{q+i}(u,w)| =$$  , $$c_q = |C_q(v,w)| =$$  ,

$$|Z| = c_{q+i} - c_q = |C_{q+i}(u,w) \backslash C_q(v,w)| =$$ 

(note that $|Z| = |Z_A| + |Z_B|$),

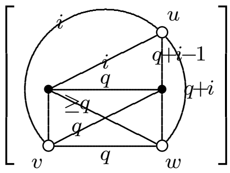
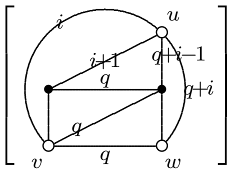
$$|Z_A|a_q + |Z_B|(c_{q+1} - c_q) =$$  =

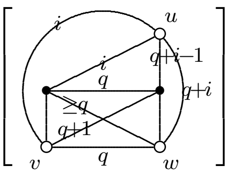
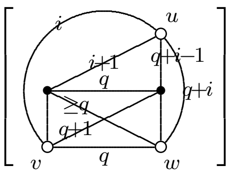
= 

it is not hard to see that

$$c_i(c_{q+1} - c_q) = |\Omega| \leq |Z_A|a_q + |Z_B|(c_{q+1} - c_q). \qquad (51)$$ no

Note that $c_i(c_{q+1} - c_q) = |Z_A|a_q + |Z_B|(c_{q+1} - c_q)$ if and only if

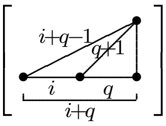
 $= 0,$  $= 0,$

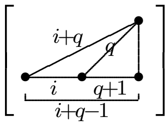
 $= 0$ and  $= 0.$

(i) If $a_q \le c_{q+1} - c_q$ then, by (51), we have $c_i(c_{q+1} - c_q) \le (|Z_A| + |Z_B|)(c_{q+1} - c_q) = (c_{q+i} - c_q)(c_{q+1} - c_q)$, and the result follows.

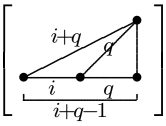
(ii) If $a_q > c_{q+1} - c_q$ then, by (51), we have $c_i(c_{q+1} - c_q) < (|Z_A| + |Z_B|)a_q = (c_{q+i} - c_q)a_q$, and the result follows. ∎

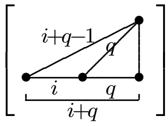
ru **Corollary. 9.2** *With reference to Lemma 9.1, suppose $c_q > c_{q+1}$. For $2 \le i \le d - q$ let*

$E^{i-1}_{q+1} :=$ 

$=$ 

and

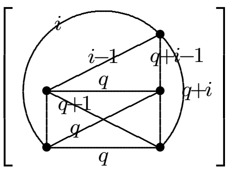
$F^{i-1}_{q+1} =$ 

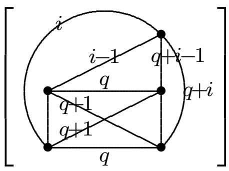
$=$ 

be as in Proposition 7.1. *Fix i $(2 \le i \le d - q)$. Then*

$$c_i(c_{q+1} - c_q) \le \frac{1}{n_{i+q}p^{i+q}_{i,q}} \left(F^{i-1}_{q+1}a_q + E^{i-1}_{q+1}(c_{q+1} - c_q)\right). \qquad (52)$$ su

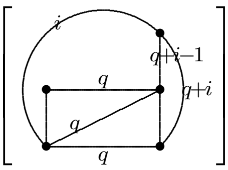
Proof. On the similar way as in the proof of Lemma 9.1 it is not hard to show that

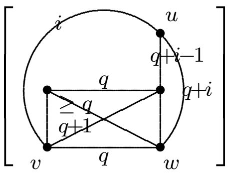
$n_{i+q}p^{i+q}_{i,q}c_i(c_{q+1} - c_q) =$ 

$+$ 

,

and

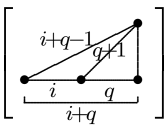
$F^{i-1}_{q+1}a_q + E^{i-1}_{q+1}(c_{q+1} - c_q) =$ 

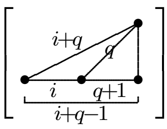
$+$ 

.

The result follows. ∎

nr **Corollary. 9.3** *With reference to Lemma 9.2, suppose $c_q > c_{q+1}$. For $2 \le i \le d - q$ let*

$e^{i-1}_{q+1} := \frac{1}{n_{i+q}p^{i+q}_{i,q}}$ 

$= \frac{1}{n_{i+q}p^{i+q}_{i,q}}$ 

$\ge 0 \qquad (53)$ ns

be as in (39). *Fix i $(2 \le i \le d - q)$. The following* (i)-(iv) *hold.*

(i) *If $c_{i+q} - c_q > e^{i-1}_{q+1}$ then $F^{i-1}_{q+1} > 0$.*

(ii) *If $c_{i+q} - c_q = e^{i-1}_{q+1}$ then $F^{i-1}_{q+1} = 0$.*

(iii) *If $c_{i+q} - c_q > e^{i-1}_{q+1}$ and $a_q < c_{q+1} - c_q$ then $c_{i+q} > c_i + c_q$.*

(iv) *If* $c_{i+q} - c_q = e^{i-1}_{q+1}$ *then* $c_{i+q} \geq c_i + c_q$.

(v) *If* $e^{i-1}_{q+1} = 0$ *then* $a_q \neq 0$ *and* $c_{q+i} \geq \frac{c_{q+1} - c_q}{a_q} c_i + c_q$.

Proof. Claim (i) and (ii) follow immediate from Proposition 7.1. If $c_{i+q} - c_q > e^{i-1}_{q+1}$ then F^{i-1}_{q+1} is nonzero. Note that

$$\frac{1}{n_{i+q} p^{i+q}_{i,q}} \left(F^{i-1}_{q+1} + E^{i-1}_{q+1} \right) = c_{i+q} - c_q.$$

This together with (52) yield (iii). If $c_{i+q} - c_q > e^{i-1}_{q+1}$ then $F^{i-1}_{q+1} = 0$ so (52) become

$$c_i (c_{q+1} - c_q) \leq \frac{1}{n_{i+q} p^{i+q}_{i,q}} E^{i-1}_{q+1} (c_{q+1} - c_q) =$$

$$= \frac{1}{n_{i+q} p^{i+q}_{i,q}} \left(F^{i-1}_{q+1} + E^{i-1}_{q+1} \right) (c_{q+1} - c_q) = (c_{q+i} - c_q)(c_{q+1} - c_q)$$

which imply (iv). Finally, if $e^{i-1}_{q+1} = 0$ then E^{i-1}_{q+1} is nonzero. This together with (52) yield (v). ∎

10 New diameter bound in case when $c_{q+1} > c_q$ and $a_q \leq c_{q+1} - c_q$

mf

In this section we show that if Γ is a distance-regular graph such that $c_q < c_{q+1}$ and $a_q \leq c_{q+1} - c_q$ (where $q \geq 2$) then

$$d \leq (k + 1 - c_{q+1})q + 1 \leq (k - 1)q + 1.$$

This bound is tight. Let's recall some well known diameter bounds. The following result for bipartite case is due to A. Hiraki [6].

Theorem. 10.1 ([6, Theorem 1.2]) *Let* Γ *be a bipartite distance-regular graph of diameter* d*, valency* $k \geq 3$ *and* $c_{r+1} > c_r$ *where* $r \geq 2$*. Suppose* Γ *is not the doubled Odd graph. Then*

$$d \leq \left[\frac{k+2}{2} \right] r + 1$$

where $[n]$ *denotes the maximal integer* m *such that* $m \leq n$*.*

Next result, also for bipartite case, is duo to J. H. Koolen [9].

Theorem. 10.2 ([9, Theorem 6]) *Let* Γ *be a bipartite distance-regular graph of diameter* d*, valency* $k \geq 3$ *and girth* $2s > 6$*. If* Γ *is not the doubled Odd graph then*

$$d \leq (s - 1)(k - 1) - \left[\frac{k-3}{2} \right].$$

If girth of Γ is $2s \geq 6$ then $c_s > c_{s-1}$ ($s \geq 3$), but converse is not true. The following result of P. Terwilliger [16, 17] for bipartite distance-regular graphs is well known.

np **Theorem. 10.3** ([16, 17]) *Let Γ be a bipartite distance regular graph of diameter $d \geq 3$, valency $k \geq 3$ and $c_{r+1} > c_r$. Then*

$$d \leq (k-1)r + 1. \tag{54}$$ ni

Bound (54) is tight. The hypercubes and the doubled Odd graphs satisfy $d = (k-1)r + 1$ with $r = 1, 2$, respectively.

Our diameter bound is same as (54), but our graph does not need to be bipartite.

Best known general bound that can be related with condition $c_q < c_{q+1}$ (without any restriction on a_q) is bound given by S. Bang, A Hiraki and J.H. Koolen in [1].

Theorem. 10.4 ([1, Corollary 1.4]) *Let Γ be a distance regular graph of diameter $d \geq 2$, valency $k \geq 3$, and assume that $c_{r+1} > c_r = 1$. Then*

$$d \leq \frac{1}{2}k^{\alpha}r + 1$$

where $\alpha := \min\{x > 0 \,|\, 4^{\frac{1}{x}} - 2^{\frac{1}{x}} \leq 1\}$ (note that $1.44 < \alpha < 1.441$).

To prove our claim we need Lemma 10.5.

ne **Lemma. 10.5** *With reference to Lemma 9.1, let t be positive integer and let q be an integer with $2 \leq q \leq d-1$. If $c_q < c_{q+1}$, $a_q \leq c_{q+1} - c_q$ and $tq + 1 \leq d$ then*

$$c_{tq+1} \geq t + 1.$$

Proof. Note that $c_{q+1} \geq 2$. By assumption $a_q \leq c_{q+1} - c_q$, and from Lemma 9.1(i) we have $c_{i+q} \geq c_i + c_q$ for all i ($2 \leq i \leq d-q$).

We prove the inequality by mathematical induction on t. For basis of induction, note that for $t = 1$ we have $c_{tq+1} = c_{q+1} \geq 2 = t + 1$. For $t = 2$, $c_{tq+1} = c_{2q+1} \geq c_{q+1} + c_q \geq c_{q+1} + 1 \geq 3 = t + 2$. For induction step, assume that $c_{(s-1)q+1} \geq s$ for every $2 \leq s \leq t-1$ and lets prove that inequality also hold for t. Indeed $c_{tq+1} = c_{(t-1)q+1+q} \geq c_{(t-1)q+1} + c_q \geq c_{(t-1)q+1} + 1 \geq t + 1$ and the result follows. ∎

Note some similarity between Corollary 10.5 and [6, Lemma 4.2(2)]. Our proof of Theorem 10.6 go with the same lines as proof of [6, Theorem 1.2] (of bipartite case).

nf **Theorem. 10.6** *With reference to Lemma 9.1, let q be an integer with $2 \leq q \leq d-1$. If $c_q < c_{q+1}$ and $a_q \leq c_{q+1} - c_q$ then*

$$d \leq (k + 1 - c_{q+1})q + 1. \tag{55}$$ ng

Proof. Let $t+1 := k+1-c_{q+1}$ (i.e. $k = t + c_{q+1}$). Suppose $(t+1)q+2 \leq d$ to derive a contradiction. Then $tq+1 \leq d-q-1$. Thus we have $b_{tq+1} \geq b_{d-q-1} \geq c_{q+1}$ and $c_{tq+1} \geq t+1$ by Lemma 10.5. It follows that

$$t+1+c_{q+1} \leq c_{tq+1} + b_{tq+1} \leq k \leq t + c_{q+1}.$$

This is a contradiction. The theorem is proved. ∎

Remark. 10.7 Some of graphs of diameter $d \geq 5$ that are not bipartite, and that satisfy conditions of Theorem 10.6 are Odd graph on 13 points, Odd graph on 11 points, generalized dodecagon of order $(2,1)$ and dodecahedron. Bound (55) is tight for the hypercubes and the doubled Odd graphs. For these two families we have $d = (k-1)r+1$ with $r = 1, 2$, respectively (as we mention earlier).

10.1 Case when $c_{i+q} - c_q > e_{q+1}^{i-1}$ for $i \in \{q+1, 2q+1, ..., (m-1)q+1\}$

In this section we show that if Γ is a distance-regular graph such that $c_q < c_{q+1}$, $a_q < c_{q+1} - c_q$ $(q \geq 2)$ and $c_{i+q} - c_q > e_{q+1}^{i-1}$ for $i \in \{q+1, 2q+1, ..., (m-1)q+1\}$ (m is largest positive integer such that $mq+1 \leq d$) then

$$d \leq \left[\frac{k+4-c_{q+1}}{2}\right] q + 1 \leq \left[\frac{k+2}{2}\right] q + 1$$

where $[n]$ denotes the maximal integer m such that $m \leq n$.

nb **Lemma. 10.8** *Let $\Gamma = (X, R)$ be a distance regular graph of diameter d, valency $k \geq 3$ and let q be an integer with $2 \leq q \leq d-1$. If $c_q < c_{q+1}$, $a_q < c_{q+1} - c_q$, $c_{i+q} - c_q > e_{q+1}^{i-1}$ for $i \in \{q+1, 2q+1, ..., (m-1)q+1\}$ where $mq+1 \leq d$ then*

$$c_{tq+1} \geq 2t \qquad (1 \leq t \leq m).$$

Proof. Note that $c_{q+1} \geq 2$. By assumption $a_q < c_{q+1} - c_q$, $c_{i+q} - c_q > e_{q+1}^{i-1}$ for $i \in \{q+1, 2q+1, ..., (m-1)q+1\}$, so Corollry 9.3(iii) yield $c_{i+q} > c_i + c_q$ for all i ($i \in \{q+1, 2q+1, ..., (m-1)q+1\}$). This yield $c_{i+q} \geq c_i + c_q + 1$ for all i ($i \in \{q+1, 2q+1, ..., (m-1)q+1\}$).

We prove the inequality by mathematical induction on t. For basis of induction, note that for $t = 1$ we have $c_{tq+1} = c_{q+1} \geq 2 = 2t$. For $t = 2$, $c_{tq+1} = c_{2q+1} \geq c_{q+1} + c_q + 1 \geq c_{q+1} + 2 \geq 4 = 2t$. For induction step, assume that $c_{(s-1)q+1} \geq 2(s-1)$ for every $2 \leq s \leq t-1$ and lets prove that inequality also hold for t. Indeed $c_{tq+1} = c_{(t-1)q+1+q} \geq c_{(t-1)q+1} + c_q + 1 \geq c_{(t-1)q+1} + 2 \geq 2(t-1) + 2$ and the result follows. ∎

nc **Theorem. 10.9** *With reference to Lemma 10.8, assume that $c_q < c_{q+1}$ and $c_{i+q} - c_q > e_{q+1}^{i-1}$ for $i \in \{q+1, 2q+1, ..., (m-1)q+1\}$ where $mq+1 \leq d$. If $a_q < c_{q+1} - c_q$ then*

$$d \leq \left[\frac{k+4-c_{q+1}}{2}\right] q + 1. \qquad (56)$$ nd

Proof. Let $t := \left[\frac{k+4-c_{q+1}}{2}\right] - 1$, where $[n]$ denotes the maximal integer m such that $m \leq n$. This imply $k \leq 2t - 1 + c_{q+1}$. Suppose $(t+1)q + 2 \leq d$ to derive a contradiction. Then $tq + 1 \leq d - q - 1$. Thus we have $b_{tq+1} \geq b_{d-q-1} \geq c_{q+1}$ and $c_{tq+1} \geq 2t$ from Lemma 10.8. It follows that

$$2t + c_{q+1} \leq c_{tq+1} + b_{tq+1} \leq k \leq 2t - 1 + c_{q+1}.$$

This is a contradiction. The theorem is proved. ∎

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