A UNIFIED VIEW OF INEQUALITIES FOR DISTANCE-REGULAR GRAPHS, PART II

Arnold Neumaier
Universität Wien
Fakultät für Mathematik
Oskar-Morgenstern-Platz 1
A-1090 Wien, Austria
Arnold.Neumaier@univie.ac.at

Safet Penjić
University of Primorska
Andrej Marušič Institute
Muzejski trg 2
6000 Koper, Slovenia
Safet.Penjic@iam.upr.si

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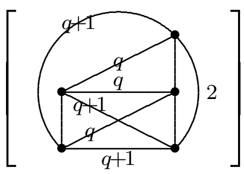
Abstract

Let Γ be a distance-regular graph of diameter d and intersection array $\{b_0, b_1, ..., b_{d-1}; c_1, c_2, ..., c_d\}$. In this paper we continue to study language of t-point counts and t-point sets. In particular, we begin to study open case from Part I, case when $c_q < c_{q+1}$ and $a_q > c_{q+1} - c_q$. Under additional assumption that $b_{q+1} = b_q$ and



$= 0$

we prove that $d \leq 2q+1$. We also study induced polygons of even and odd length, where independently of results from Part I, and under different assumptions we prove that Koolen's inequalities hold. Amonge else, in the end we obtain new inequality for graphs with even girth: If Γ has induced polygon of length $2h$ then

$$b_{i-1} + c_i - \lambda - 2 \geq \frac{c_{i-1}c_{i-2}\dots c_{i-h+1} + b_i b_{i+1}\dots b_{i+h-2}}{b_2 b_3 \dots b_{h-1}}$$

hold for any i $(h+1 \leq i \leq d-h+1)$.

MSC: 05C12, 05C62, 05E30

Keywords: distance-regular graph, diameter bound

Contents

11 Introduction to Part II

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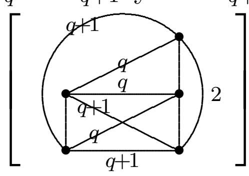
In Part I of the paper we proved that $c_q < c_{q+1}$ and $a_q \leq c_{q+1} - c_q$ (where $q \geq 2$) yield

$$d \leq (k + 1 - c_{q+1})q + 1 \leq (k - 1)q + 1.$$

It is our ultimate goal to get similar result under assumption that $c_q < c_{q+1}$ and $a_q > c_{q+1} - c_q$. To achieve this goal we systematically continue to study t-point counts. In Section 12 we consider 4-point counts containing path of length two. We use this counts first to prove that $c_q < c_{q+1}$, $a_q > c_{q+1} - c_q$ and $b_q = b_{q+1}$ yield $a_{q+1} \neq 0$. After that we make additional assumption that 5-point count



$= 0$ and we prove that

$$d \leq 2q + 1.$$

Using this bound, in Section 13 we obtain different kind of bounds using intersection numbers only.

In Section 15 we study induced polygons of even and odd length. Independently of results from Part I, we prove the following. If Γ has induced polygon of odd length (of length $2h + 3$ for some h $(1 \leq h \leq d)$) then

$$b_h \geq c_i + b_{i+h} \qquad \text{for any } i \ \ (1 \leq i \leq d - h),$$

which yield $b_h > b_{h+1}$. If Γ has induced polygon of of length $2h$ for some h $(2 \leq h \leq d)$ then

$$c_h \geq c_{h-i} + c_i \qquad \text{for any } i \ \ (1 \leq i \leq h - 1),$$

which yield $c_h \geq c_{h-1}$. If Γ has induced polygon of length $2h+2$ for some h $(2 \leq h \leq d)$ then

$$b_h \geq c_i + b_{i+h} \qquad \text{for any } i \ \ (2 \leq i \leq d-h+1)$$

which yield $b_h > b_{h+1}$. For example, one of immediate consequences of above results are: (i) If $b_1 = b_2$ then Γ does not have induced pentagon. (ii) If $b_2 = b_3$ or $c_2 = c_3$ then Γ does not have induced hexagon. (iii) If $b_2 = b_3$ then Γ does not have induced heptagon. (iv) If $b_3 = b_4$ or $c_3 = c_4$ then Γ does not have induced octagon.

In Section 17 we find a new inequality for distance-regular graphs involving the intersection numbers of Γ when the girth is even.

12 4-point counts R_i, S_i, T_i, U_i, V_i and W_i

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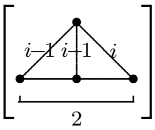
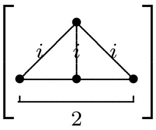
In this section we will consider the 4-point counts containing path of length two. We begin by defining rational numbers

$$v_i := \frac{1}{n_i c_i}$$

$$\geq 0 \qquad (1 \leq i \leq d).$$

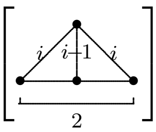
Note that by definition $v_1 = 0$.

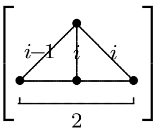
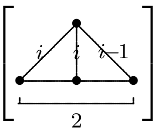
RS **Proposition. 12.1** *Let $\Gamma = (X, R)$ denote a distance regular graph of diameter d and valency $n \geq 3$. For any i $(2 \leq i \leq d)$ we have*

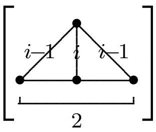
$$R_i :=$$

$$= n_i\Big(a_i(b_1 - c_i) + c_i(\lambda - a_{i-1}) + c_i v_i - b_i v_{i+1}\Big) =$$

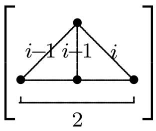
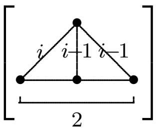
$$= n_i\Big(a_i b_1 - (a_{i-1} + a_i - a_1 - v_i)c_i - b_i v_{i+1}\Big) =$$

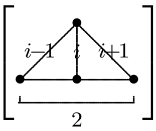
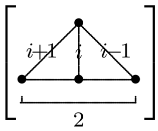
$$= n_i\Big(a_i(b_1 - c_i) + c_i(\lambda - a_{i-1})\Big) + V_i - V_{i+1},$$

$$S_i :=$$

$$= n_i c_i (b_1 - c_{i-1} - v_i),$$

$$T_i :=$$

$$=$$

$$= n_i c_i (a_{i-1} + a_i - a_1 - v_i),$$

$$U_i :=$$

$$= n_i c_i (c_i - 1 - a_{i-1} + v_i),$$

$$V_i :=$$

$$=$$

$$= n_i c_i v_i,$$

$$W_i :=$$

$$=$$

$$= n_i c_i b_i.$$

Proof. By summing over a distance we find

$$V_i + T_i = n_i c_i (a_{i-1} + a_i - \lambda),$$

$$W_{i-1} + V_i + S_i = n_i c_i b_1,$$
$$U_i + T_i + W_i = n_i c_i b_1,$$
$$T_i + R_i + V_{i+1} = n_i a_i b_1.$$

Solving the resulting linear system or equations gives the above formulas. Note that we also have $S_i + R_i + U_{i+1} = n_i(c_i(b_{i-1} - 1) + b_i(c_{i+1} - 1) + a_i(a_i - \lambda - 1))$. ∎

Corollary. 12.2 *With reference to Proposition 12.1, for any i $(2 \leq i \leq d)$ we have*

$$\max\{0, a_{i-1} + 1 - c_i\} \leq v_i \leq \min\{a_i + a_{i-1} - \lambda, b_1 - c_{i-1}\}, \tag{57}$$ B1

$$a_i b_1 \geq (a_{i-1} + a_i - a_1) c_i + b_i v_{i+1} - c_i v_i, \tag{58}$$ B2

$$a_i b_1 \leq (a_{i-1} + a_i - \lambda) c_i \qquad \Rightarrow \qquad V_i \geq V_{i+1}. \tag{59}$$ B3

$$v_i = 0 \qquad \Rightarrow \qquad c_i \geq a_{i-1} + 1, \tag{60}$$ B4

$$b_1 > b_2 \qquad \Rightarrow \qquad T_i > 0 \quad \text{or} \quad U_i > 0. \tag{61}$$ B5

Proof. (57) and (58) follow from nonnegative of expressions from Proposition 12.1. (59) follows from $R_i \geq 0$. (60) follows from $U_i \geq 0$. (61) follows from $0 \leq T_i + U_i = n_i c_i (b_1 - b_i)$. ∎

Corollary. 12.3 *With reference to Proposition 12.1, for any i $(2 \leq i \leq d)$ we have*

$$\lambda + 2 \leq b_{i-1} + c_i \tag{62}$$ B6

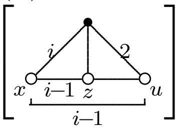
$$\lambda + 1 \leq a_i + c_i \tag{63}$$ B7

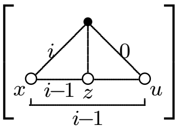
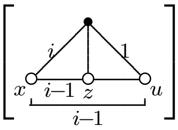
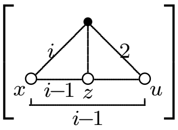
Proof. (62) follows from $0 \leq S_i + U_i = n_i c_i (c_i + b_{i-1} - \lambda - 2)$. (62) follows from (57). ∎

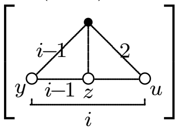
Inequality (62) is known from [3, Proposition 5.5.1(iii)].

a> **Corollary. 12.4** *With reference to Proposition 12.1, if $a_{i-1} \neq 0$ and $v_i = 0$ for some i $(2 \leq i \leq d-1)$ then*

$$b_{i-1} \leq \lambda, \qquad a_{i-1} \leq \lambda \qquad \text{and} \qquad c_i \geq a_{i-1} + 1 \geq 2.$$

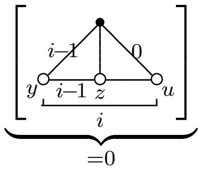
Proof. Pick $u \in X$ and $z \in \Gamma_1(u)$. Since $a_{i-1} \neq 0$ we can pick $x \in \Gamma_{i-1,i-1}$, and since $v_i = 0$ for every such x we have [] $= 0$. Thus we have

$b_{i-1} =$ [] (=0) + [] + [] (=0), that is all b_{i-1} vertices that are neighbours of z and on distance i from x must be in $\Gamma_{11}(u,z)$. This yield $b_{i-1} \leq \lambda$. Similarly, since $v_i = 0$, for any $y \in \Gamma_{i,i-1}(u,z)$ we have

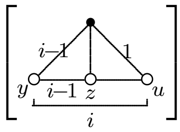


$= 0$ and this yield

$$a_{i-1} =$$



$+$



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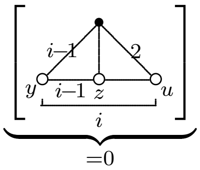

That is all a_{i-1} vertices that are neighbour of z and on distance $i-1$ from y must be in $\Gamma_{11}(u,z)$. This yield $a_{i-1} \leq \lambda$. Inequality $c_i \geq a_{i-1} + 1 \geq 2$ follows from (60) and assumptions. ∎

Corollary. 12.5 *With reference to Proposition 12.1, let t be index such that* $1 = c_t < c_{t+1}$. *If* $a_{t-1} \neq 0$ *then* $v_t \neq 0$ *and* $v_{t+1} \neq 0$.

Proof. Since $b_1 \geq b_2 \geq ... \geq b_t$ and $1 = c_1 = ... = c_t$ we have that $\lambda \leq a_2 \leq ... \leq a_t$. Also $a_{t-1} \neq 0$ yield $\lambda < a_t$. Now the results follow immediate from Corollary 12.4. ∎

Note that if $b_1 > b_2$ and $c_1 = c_t = 1$ for some $t \geq 2$ then $\lambda < a_2 \leq a_t$.

Proposition. 12.6 *With reference to Proposition 12.1, the following* (i)–(iii) *hold.*

(i) *If* $\lambda < b_{i-1} - 1$ *then* $S_i \neq 0$.

(ii) *If* $\lambda < a_{i-1}$ *then* $V_i \neq 0$.

(iii) *If* $i - 1 < d$ *then* $V_i \neq 0$ *or* $S_i \neq 0$.

Proof. Pick $u \in X$, $y \in \Gamma_i(u)$. Now we have c_i choices to pick $z \in C_i(y,u)$.

(i) For every z there exist $b_{i-1} - 1$ vertices $v \in B_{i-1}(y,z) - \{u\}$. If $S_i = 0$ then any such v must be neighbour of u which imply $b_{i-1} - 1 \leq \lambda$, a contradiction. The result follows.

(ii) For every z there exist a_{i-1} vertices $v \in A_{i-1}(y,z)$. If $V_i = 0$ then any such v must be neighbour of u which imply $a_{i-1} \leq \lambda$, a contradiction. The result follows.

(iii) Immediate from $0 \leq V_i + S_i = n_i c_i (b_1 - c_{i-1})$ and (12). ∎

12.1 Note when $c_{q+1} > c_q$ and $a_q > c_{q+1} - c_q$

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In this subsection we begin to study case when $c_{q+1} > c_q$ and $a_q > c_{q+1} - c_q$.

oa **Lemma. 12.7** *With reference to Proposition 12.1, let q be an integer with* $2 \leq q \leq d-1$. *If* $c_q < c_{q+1}$, $a_q > c_{q+1} - c_q$ *then the following* (i), (ii) *hold.*

(i) *If* $b_{q+1} = 1$ *then* $d < 2q + 2$.

(ii) *If* $b_{q+1} \neq 1$ *then* $S_{q+1} \neq 0$.

Proof. (i) Since $c_{q+1} > c_q \geq 1 = b_{q+1}$, the result follows from (12).

(ii) Assume that $S_{q+1} = 0$. Then, because of Proposition 12.1, $v_{q+1} = b_1 - c_q$. Since $T_{q+1} = n_{q+1}c_{q+1}(a_q + a_{q+1} - \lambda - v_{q+1}) \geq 0$ and $a_q > c_{q+1} - c_q$ we have

$$a_q + a_{q+1} - \lambda - v_{q+1} = a_q + a_{q+1} - \lambda - b_1 + c_q > c_{q+1} - c_q + a_{q+1} - \lambda - b_1 + c_q =$$

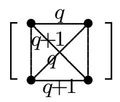
$$= c_{q+1} + a_{q+1} - k + c_1 = c_1 - b_{q+1} \geq 0$$

which yield $b_{q+1} = 1$, a contradiction. The result follows. ∎

ob **Lemma. 12.8** *With reference to Proposition 12.1, let* q *be an integer with* $2 \leq q \leq d-1$, *and assume that* $c_q < c_{q+1}$ *and* $a_q > c_{q+1} - c_q$. *If* $c_q = 1$ *then* $v_{q+1} \neq 0$.

Proof. To get a contradiction, assume that $v_{q+1} = 0$. Then $a_q > c_{q+1} - c_q$ and $U_{q+1} = n_{q+1}c_{q+1}(c_{q+1} - 1 - a_q - v_{q+1}) \geq 0$ yield $c_{q+1} \geq a_q + 1 > c_{q+1} - c_q + 1$, that is $c_q > 1$, a contradiction. The result follows. ∎

oc **Lemma. 12.9** *With reference to Proposition 12.1, let* q *be an integer with* $2 \leq q \leq d-1$, *and assume that* $c_q < c_{q+1}$ *and* $a_q > c_{q+1} - c_q$. *If* $b_q = b_{q+1}$ *then* $a_{q+1} \neq 0$.

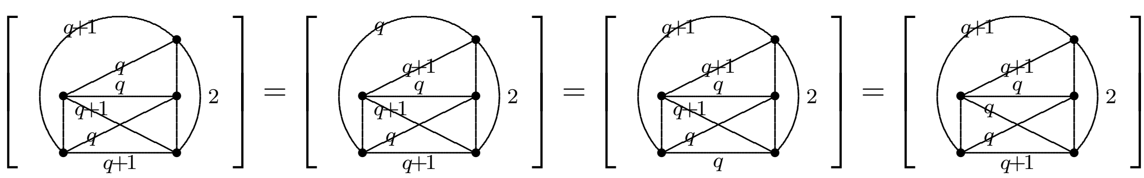
Proof. By Proposition 6.1 we have $B_{q+1} = \Big[$  $\Big] = n_{q+1}c_{q+1}\Big(a_q - (c_{q+1} - c_q - e_{q+1})\Big) = n_{q+1}c_{q+1}\Big(a_{q+1} - (b_q - b_{q+1} - e_{q+1})\Big)$. Since $b_q = b_{q+1}$ it follows by (30) that we have $e_{q+1} = 0$, which yield

$$B_{q+1} = n_{q+1}c_{q+1}\Big(a_q - (c_{q+1} - c_q)\Big) = n_{q+1}c_{q+1}a_{q+1}.$$

The result follows. ∎

12.2 Case $c_q < c_{q+1}$, $a_q > c_{q+1} - c_q$, $b_{q+1} = b_q$ and $\Phi_{q+1} = 0$

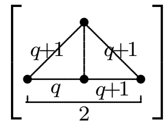
Consider 5-point count defined on the following way

$\Phi_{q+1} =$  .

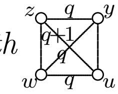
In this subsection we show that if $c_q < c_{q+1}$, $a_q > c_{q+1} - c_q$, $b_{q+1} = b_q$ and $\Phi_{q+1} = 0$ for some integer q $(2 \leq q \leq d-1)$ then

$$d \leq 2q + 1.$$

Lemma. 12.10 *With the notation of Definition 2.1, if* $\Phi_{q+1} \neq 0$ *then*

$B_{q+1} \neq 0, \qquad F_{q+1} \neq 0, \qquad V_{q+1} \neq 0, \qquad S_{q+1} \neq 0, \qquad$ *and* $\Big[$  $\Big] \neq 0.$

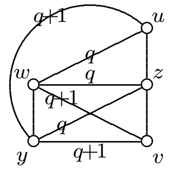
Proof. Immediate from definitions of Φ_{q+1} and B_{q+1}, F_{q+1}, V_{q+1}, S_{q+1}. ∎

o5 **Lemma. 12.11** *With the notation of Definition 2.1, assume that $b_q = b_{q+1}$ and $c_{q+1} > c_q$ so that we can choose $zuwy$ consistent with* 

. If $\Phi_{q+1} = 0$ then

$$\Gamma_{q+1}(w) \cap B_q(y,z) = \emptyset.$$

Proof. Assume to the contrary, that is assume $\Gamma_{q+1}(w) \cap B_q(y,z)$ is nonempty, so that we can choose $v \in \Gamma_{q+1}(w) \cap B_q(y,z)$. Then $wyvzu$ are consistent with



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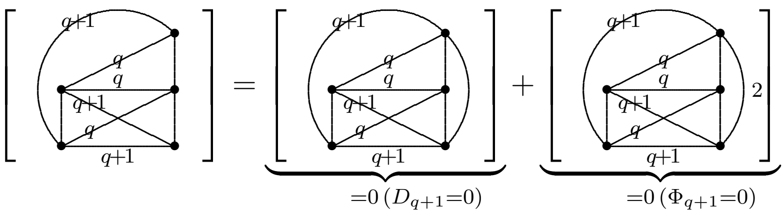
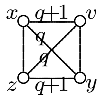
Now since 

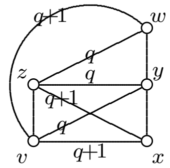
= 0, we have a contradiction. The result follows. ∎

o6 **Lemma. 12.12** *With the notation of Definition 2.1, assume that $b_{q+1} = b_q$ and $a_{q+1} \neq 0$ so that we can choose $zvxy$ consistent with* 

. If $\Phi_{q+1} = 0$ then

$$\Gamma_{q+1}(v) \cap A_q(z,y) = \emptyset.$$

Proof. Assume to the contrary, that is assume $\Gamma_{q+1}(v) \cap A_q(z,y)$ is nonempty, so that we can choose $w \in \Gamma_{q+1}(v) \cap A_q(z,y)$. Then $zvxyw$ are consistent with



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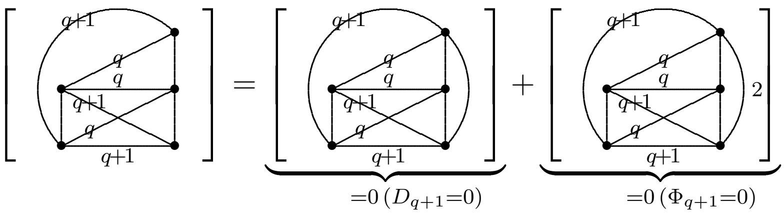
Now since 

= 0, we have a contradiction. The result follows. ∎

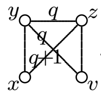
o7 **Lemma. 12.13** *With the notation of Definition 2.1, assume that $b_{q+1} = b_q$ and pick $z \in X$, $y \in \Gamma_q(z)$, $u \in B_q(y,z)$ and $x \in B_q(z,y)$. Then the following* (i)–(iii) *hold.*

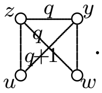
(i) $|\Gamma_{q,q+1,q}(y,z,x)| = c_{q+1} - c_q$ *and* $|\Gamma_{q,1,q}(z,y,u)| = c_{q+1} - c_q$.

(ii) $A_q(y,z) = \Gamma_{q,1,q+1}(y,z,x) \cup \Gamma_{q,1,q}(y,z,x)$ *and* $A_q(z,y) = \Gamma_{q,1,q+1}(z,y,u) \cup \Gamma_{q,1,q}(z,y,u)$.

(iii) $B_q(z,y) = \Gamma_{q+1,1,q+1}(z,y,u)$ *and* $B_q(y,z) = \Gamma_{q+1,1,q+2}(y,z,x)$.

Proof. (i) Since $b_{q+1} = b_q$ we have $e_{q+1} = 0$ and $F_{q+1} = n_{q+1}c_{q+1}(c_{q+1} - c_q)$. The result follows.

(ii) Pick $v \in A_q(y,z)$. We have that $xywz$ are consistent with 

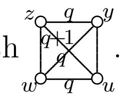
. By triangle inequality it is not hard to see that $d(x,v) \in \{q, q+1\}$. Similar for $w \in A_q(z,y)$ and 

.

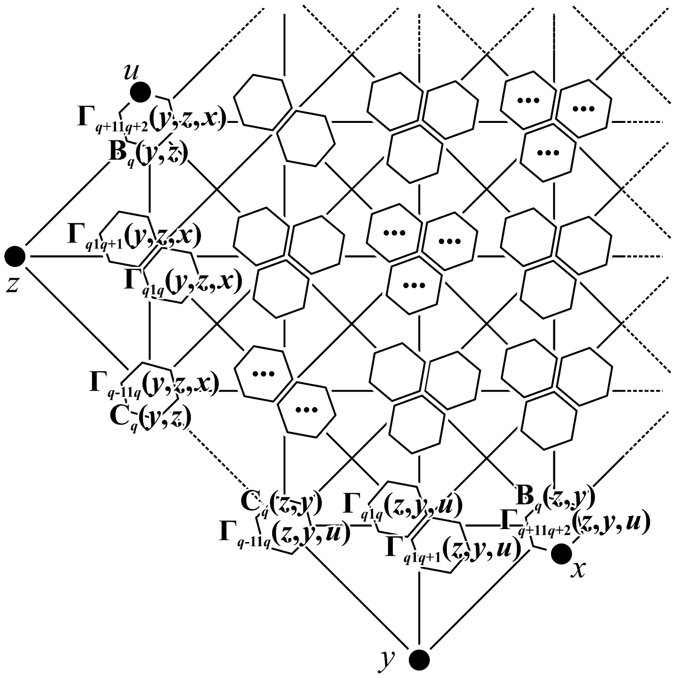
(iii) Consider $z \in X$, $y \in \Gamma_q(z)$, $u \in B_q(y,z)$ and pick $w \in B_q(z,y) = \Gamma_{q+1}(z) \cap \Gamma_1(y)$. Note that because of triangle inequality $d(u,w) \in \{q, q+1, q+2\}$. Now $d(u,w) = q$ is not possible since $E_{q+1} = 0$, and $d(u,w) = q+1$ is not possible since $D_{q+1} = 0$. Thus we must have $d(u,w) = q+2$, and the first equation follow. Similarly for $B_q(y,z) = \Gamma_{q+1,1,q+2}(y,z,x)$. ∎

o8 **Lemma. 12.14** *With the notation of Definition 2.1, assume that* $b_{q+1} = b_q$ *and pick* $z \in X$, $y \in \Gamma_q(z)$, $u \in B_q(y,z)$ *and* $x \in B_q(z,y)$. *If* $c_{q+1} > c_q$, $a_{q+1} \neq 0$ *and* $\Phi_{q+1} = 0$ *then there exists* $w \in \Gamma_{q,1,q}(z,y,u)$ *such that*

$$B_q(w,z) \subseteq \Gamma_{q,1,q}(y,z,x).$$

Proof. By Lemma 12.13(i), $|\Gamma_{q,1,q}(z,y,u)| = c_{q+1} - c_q$, so we can pick $w \in \Gamma_{q,1,q}(z,y,u)$. Vertices $zuwy$ are consistent with 

. By Lemma 12.11 this yield $\Gamma_{q+1}(w) \cap B_q(y,z) = \emptyset$. Since $\Gamma_{q+1}(w) \cap C_q(y,z) = \emptyset$, we must have $\Gamma_{q+1}(w) \subseteq A_q(y,z)$. By Lemma 12.13(ii), $A_q(y,z) = \Gamma_{q,1,q+1}(y,z,x) \cup \Gamma_{q,1,q}(y,z,x)$. For any $v \in \Gamma_{q,1,q+1}(y,z,x)$, by Lemma 12.12 $\Gamma_{q+1}(v) \cap A_q(z,y) = \Gamma_{q+1}(v) \cap \Big[\Gamma_{q,1,q+1}(z,y,u) \cup \Gamma_{q,1,q}(z,y,u)\Big] = \emptyset$ which yield $\Gamma_{q+1}(w) \subseteq \Gamma_{q,1,q}(y,z,x)$ and the result follows. ∎



09 **Figure 4** Intersection diagram (of rank q). The set $\Gamma_1(z)$ is partitioned with respect to (y, z, x) and the set $\Gamma_1(y)$ is partitioned with respect to (z, y, u), in both cases under assumption $b_{q+1} = b_q$. Note that $z \in X$, $y \in \Gamma_q(z)$, $x \in \Gamma_{q+1,1}(z, y)$ and $u \in \Gamma_{1,q+1}(u, y)$.

v6 **Theorem. 12.15** ([3, proof of Lemma 5.9.2]) *With the notation of Definition 2.1, assume that $b_{q+1} = b_q$, $c_{q+1} > c_q$, $a_{q+1} \neq 0$ and $\Phi_{q+1} = 0$. Then the following* (i), (ii) *hold.*

(i) $b_q + c_q \leq c_{q+1}$

(ii) $d < 2q + 1$.

Proof. (i) Immediate from Lemmas 12.13(i) and 12.14.

(ii) Immediate from (i) and (12). ∎

oj **Corollary. 12.16** *With the notation of Definition 2.1, assume that $b_q = b_{q+1}$, $c_{q+1} > c_q$, $a_q > c_{q+1} - c_q$ and $\Phi_{q+1} = 0$. Then*

$$d < 2q + 1.$$

Proof. By Lemma 12.9, $a_{q+1} \neq 0$. The result now follows immediate from Theorem 12.15. ∎

13 Diameter bounds for case when $1 = c_q < c_{q+1}$, $a_q > c_{q+1} - c_q$, $b_{q+1} = b_q$ and $\Phi_{q+1} = 0$

pc In this section we use Corollary 12.16 together with Theorem 13.1 to get diameter bounds for different sub-cases.

pa **Theorem. 13.1** ([2, Lemma 5.1(iii)]) *Let* Γ *be a sistance-regular graph of diameter* $d \geq 2$ *and valency* $n \geq 3$*. Let* $r = r(\Gamma) = |\{i \,|\, 1 \leq i \leq d-1$ *and* $(c_i, b_i) = (c_1, b_1)\}|$*. If* $b_{i-1} = 2$ *and* $c_i = 1$ *then* $i \leq 2r + 2$*.*

pb **Theorem. 13.2** *Let* Γ *be a distance-regular graph of diameter* $d \geq 2$*, valency* $n \geq 3$ *and* $1 = c_q < c_{q+1}$*. Assume that* $b_q = b_{q+1}$*,* $a_q > c_{q+1} - c_q$ *and* $\Phi_{q+1} = 0$*. Then the following* (i)–(iv) *hold.*

(i) *If* $b_{q-1} \geq 3$ *then*

$$d \leq 1 + 2\log_3(b_0 b_1 ... b_{q-1}).$$

(ii) *If* $b_{q-1} = 2$ *and* $b_1 \leq 2$ *then*

$$d \leq 9.$$

(iii) *If* $b_{q-1} = 2$ *and* $b_1 \geq 3$ *then*

$$d \leq 1 + 4\log_6(b_0 b_1 ... b_{q-1}).$$

(iv) *If* $b_{q-1} = 1$ *then* $b_1 \geq 2$*. Moreover, if* $b_1 = 2$ *then* $d \leq 9$*. If* $b_1 = 3$ *then* $d \leq 7$*. If* $b_1 \geq 4$ *then*

$$d \leq 1 + 4\log_8(b_0 b_1 ... b_{q-1}).$$

Proof. Our proof is with lines of [2, proof of Theorem 1.5].

(i) We have $\frac{n_i}{n_{i-1}} = \frac{b_{i-1}}{c_i} \geq 3$ for all i $(1 \leq i \leq q)$, which yield $n_q \geq 3^q$. By Corollary 12.16,

$$d \leq 1 + 2q \leq 1 + 2\log_3 n_q$$

and the result follows.

(ii) Since $c_q = 1$ and $b_{q-1} = 2$, by (15) we have $\lambda \leq 1$, wich yield that $k \leq 4$. The result follows from [1] and [4].

(iii) Let $r = r(\Gamma) = |\{i \,|\, 1 \leq i \leq d-1$ and $(c_i, b_i) = (c_1, b_1)\}|$. Since $c_q = 1$ and $b_{q-1} = 2$ by Theorem 13.1, $q \leq 2r + 2$ and with that $q - r - 1 \leq r + 1$ and $\frac{q}{2} \leq r + 1$. We have

$$n_q = \prod_{j=1}^{q} b_{j-1} \geq n b_1^r 2^{q-1-r} \geq 3^{r+1} 2^{q-1-r} \geq 6^{\frac{q}{2}}.$$

Therefore we have $log_6 n_q \geq \frac{q}{2}$ and thus

$$d \leq 2q + 1 = 4\frac{q}{2} + 1 = 1 + 4\log_6 n_q$$

holds.

(iv) By (15), $c_q = 1$ and $b_{q-1} = 1$ yield $\lambda = 0$. If $b_1 = 2$ then $n = 3$ and the result follows from [1]. If $b_1 = 3$ then $n = 4$ and the result follows from [4].

Now we assume that $b_1 \geq 4$. Then

$$n_q = \prod_{j=1}^{q} b_{j-1} \geq n b_1^r 2^{q-1-r} \geq 4^{r+1} 2^{q-1-r} \geq 8^{\frac{q}{2}}.$$

The result follows. ∎

Corollary. 13.3 *Let Γ be a distance-regular graph of diameter $d \geq 2$, valency $n \geq 3$ and $1 = c_q < c_{q+1}$. Assume that $b_q = b_{q+1}$, $a_q > c_{q+1} - c_q$ and $\Phi_{q+1} = 0$. Then*

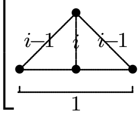
$$d \leq 1 + \frac{4}{3}\log_2 |X|.$$

Proof. Immediate from Theorem 13.2 and the fact that $|X| > n_q$. ∎

14 4-point counts X_i, Y_i and Z_i

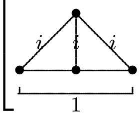
YZ

In this section we consider the 4-point counts containing triangle and their connection with 4-point counts which contain path of length two (see Section 12). We begin by defining rational numbers

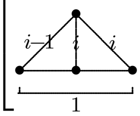
$$z_i := \frac{1}{n_i c_i}\Bigg[$$ 

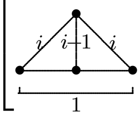
$$\Bigg] \geq 0.$$

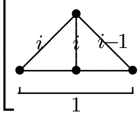
XY **Proposition. 14.1** *With the notation of Definition 2.1, for any i $(1 \leq i \leq d)$ we have*

$$X_i := \Bigg[$$ 

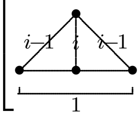
$$\Bigg] = n_i\Big((a_i - c_i)\lambda + c_i z_i - b_i z_{i+1}\Big),$$

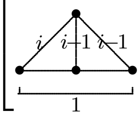
$$Y_i := \Bigg[$$ 

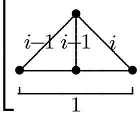
$$\Bigg] = \Bigg[$$ 

$$\Bigg] = \Bigg[$$ 

$$\Bigg] = n_i c_i(\lambda - z_i),$$

$$Z_i := \Bigg[$$ 

$$\Bigg] = \Bigg[$$ 

$$\Bigg] = \Bigg[$$ 

$$\Bigg] = n_i c_i z_i.$$

Proof. Note that

$$Z_i + Y_i = n_i c_i \lambda,$$
$$Y_i + X_i + Z_{i+1} = n_i a_i \lambda.$$

The results follow. ∎

R2 **Proposition. 14.2** *With reference to Propositions 12.1 and 14.1, for any i $(2 \leq i \leq d)$ we have*

$$R_i := n_i\Big(a_i(a_i - 1 - \lambda) + b_i z_{i+1} + c_i(\lambda - z_i)\Big),$$
$$S_i := n_i c_i(b_{i-1} - 1 - \lambda + z_i),$$
$$T_i := n_i c_i(a_i - \lambda + z_i),$$
$$U_i := n_i c_i(c_i - 1 - z_i),$$
$$V_i := n_i c_i(a_{i-1} - z_i),$$
$$W_i := n_i c_i b_i.$$

Proof. Note that

$$
\begin{aligned}
&W_i = n_i c_i b_i,\\
&Z_i + V_i = n_i c_i a_{i-1},\\
&n_i c_i + Z_i + U_i = n_i c_i c_i,\\
&Y_i + T_i = n_i a_i c_i,\\
&n_i c_i + Y_i + S_i = n_i c_i b_{i-1},\\
&n_i a_i + X_i + R_i = n_i a_i a_i,
\end{aligned}
$$

The results follow. ∎

Corollary. 14.3 *With reference to Propositions 14.2*

$$\max(0, \lambda - a_i, \lambda + 1 - b_{i-1}) \le z_i \le \min(\lambda, a_{i-1}, c_i - 1), \qquad (64) \quad \boxed{\texttt{j7}}$$

$$\lambda(a_i - c_i) - a_i(a_i - 1) \le b_i z_{i+1} - c_i z_i \le \lambda(a_i - c_i). \qquad (65) \quad \boxed{\texttt{j8}}$$

Proof. The expressions in Propositions 14.1 and 14.2 are nonnegative. ∎

Note similarity between (65) and [3, (6) from page 178].

With reference to Propositions 6.1 and 14.1, clearly $Z_2 = F_2$, so we have

$$z_2 = \mu - 1 - e_2. \qquad (66) \quad \boxed{\texttt{g1}}$$

Further relations between the z_i and e_i are obtained by counting the number of certain 5-point and 6-point configurations.

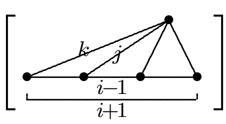
Proposition. 14.4 *With reference to Propositions 14.2, for any i $(1 \le i \le d)$ we have*

$$0 \le z_{i+1} - z_i \le \min(c_{i+1} - c_i - e_{i+1},\ b_{i-1} - b_i - e_i), \qquad (67) \quad \boxed{\texttt{g2}}$$

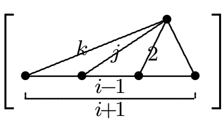
$$c_i - c_{i-1} - e_i \le \min(a_{i-1} - \lambda + z_i,\ a_{i-1} - z_i), \qquad (68) \quad \boxed{\texttt{g3}}$$

$$b_{i-1} - b_i - e_i \le \min(a_i - \lambda + z_i,\ a_i - z_i). \qquad (69) \quad \boxed{\texttt{g4}}$$

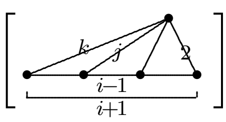
Proof. Define, for fixed i,

$L_{jk} :=$ 

,

$M_{jk} :=$ 

,

$N_{jk} :=$ 

.

Then $L_{jk} = 0$ for $(j,k) \ne (i-1, i), (i,i), (i, i+1)$, and

$$L_{ii+1} = c_i Y_{i+1}, \quad L_{i-1i} = b_i Z_i,$$

$$0 \le L_{ii} = c_i Z_{i+1} - b_i Z_i = n_i b_i c_i (z_{i+1} - z_i),$$

$$0 \le M_{ii} = c_i F_{i+1} - L_{ii} = n_i b_i c_i \Big(c_{i+1} - c_i - e_{i+1} - (z_{i+1} - z_i) \Big),$$

$$0 \le N_{ii} = b_i D_i - L_{ii} = n_i b_i c_i \Big(b_{i-1} - b_i - e_i - (z_{i+1} - z_i) \Big),$$

$$0 \le M_{ii+1} = c_i B_{i+1} - L_{ii+1} = n_i b_i c_i \Big(a_i - (c_{i+1} - c_i - e_{i+1}) - (\lambda - z_{i+1}) \Big) =$$

$$= n_i b_i c_i \Big(a_{i+1} - (b_i - b_{i+1} - e_{i+1}) - (\lambda - z_{i+1}) \Big),$$

$$0 \le N_{i-1i} = b_i B_i - L_{i-1i} = n_i b_i c_i \Big(a_{i-1} - (c_i - c_{i-1} - e_i) - z_i \Big) =$$

$$= n_i c_i b_i \Big(a_i - (b_{i-1} - b_i - e_i) - z_i \Big).$$

This implies (67), (68) and (69). ∎

ok **Theorem. 14.5** ([3, Proposition 5.5.6]) *With the notation of Definition 2.1, if $\lambda > 0$ then for any i $(1 \le i \le d-1)$ we have*

$$\min(b_i, c_i) \le a_i,$$

$$a_i(\lambda + 1 - a_i) \le \lambda \max(b_i, c_i).$$

Proof. Using the upper bound of (65), and the lower bound of (67) we have

$$(b_i - c_i) z_i \le \lambda (a_i - c_i). \tag{70}$$ ol

Since $0 \le z_i \le \lambda$ (by (64)), if $\lambda > 0$ we find

$$b_i \ge c_i \quad \stackrel{(70)}{\Longrightarrow} \quad a_i \ge c_i,$$

$$b_i < c_i \quad \Rightarrow \quad b_i - c_i < 0 \quad \Rightarrow \quad (b_i - c_i)\lambda \le (b_i - c_i) z_i \quad \stackrel{(70)}{\Longrightarrow} \quad b_i \le a_i$$

that is

$$\min(b_i, c_i) \le a_i.$$

Similarly, the inequality $\lambda(a_i - c_i) \le (b_i - c_i) z_{i+1} + a_i(a_i - 1)$ yields

$$a_i(\lambda + 1 - a_i) \le \lambda \max(b_i, c_i).$$

∎

Using (64) and lower bound of (67) it is easy to deduce the inequalities

$$\lambda \le \min(2a_i, a_i + a_{i-1}, b_{i-1} + c_i - 2),$$

of Brouwer et al. [5, Chapter 5.5].

Corollary. 14.6 *With the notation of Definition 2.1, if* $\lambda > 0$ *then*

$$d-2 \leq \frac{2(a_2+a_3+...+a_{d-1})}{\lambda}.$$

Thus, if $a_i \leq c_{i+1}-c_i$ *for all* $i=2,...,d-1$ *then* $d-2 \leq \frac{2(c_d-c_2)}{\lambda}$*, and if* $a_i \leq b_{i-1}-b_i$ *for all* $i=1,2,...,d$ *then* $d-2 \leq \frac{2(b_1-b_{d-1})}{\lambda}$.

Proof. Using (64) and lower bound of (67) we have $\lambda - a_i \leq z_i \leq z_{i+1} \leq a_i$ $(1 \leq i \leq d-1)$ and with that $\lambda \leq 2a_i$ $(2 \leq i \leq d-1)$. Summing these inequalities for $i = 2,...,d-1$ we have $(d-2)\lambda \leq 2(a_2+...+a_{d-1})$, and the result follows. ∎

15 On induced polygons of even and odd length

om In Section 5 we re-proved Koolen's inequalities $c_e \geq c_i + c_{e-i}$ $(1 \leq i \leq e-1)$ and $b_e \geq c_i + b_{e+i}$ $(1 \leq i \leq d-e)$ under assumption that $c_e > c_{e-1}$ and $b_e > b_{e+1}$ for some e $(2 \leq e \leq d)$, respectively. Note that using Proposition 6.1, it is possible to prove same inequalities under different assumptions, focusing only on cases when $D_i > 0$, $E_i > 0$ or $F_i > 0$. Also, note that if we know that $c_e > c_{e-1}$ or $b_e > b_{e+1}$ we can not say anything about existence of induced polygons of even or odd length.

Generally, we are always interested in induced subgraphs of distance-regular graphs. In this section we study t-point counts under assumption that Γ has induced polygon of odd or even length. Independently from Sections 5 and 6 we prove the following. ....

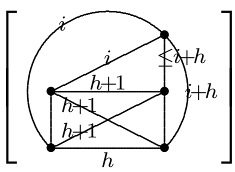
15.1 Induced polygon of odd length - case when $D_{h+1} > 0$

Assume that there exists reduced polygon $p_1p_2...p_{2h+2}p_{2h+3}$ (of length $2h+3$). In this section, among else, we prove that then Koolen's inequality (22) hold.

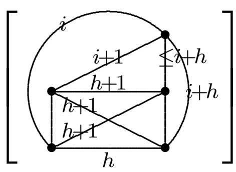
on **Theorem. 15.1** *Assume that* Γ *has induced polygon* $p_1p_2...p_{2h+2}p_{2h+3}$ *of odd length* $2h+3$ $(1 \leq h \leq d)$*. Then*

$$b_h \geq c_i + b_{h+i} \qquad \Big(\Longleftrightarrow \qquad (c_{h+i}-c_i) + a_{h+i} \geq a_h + c_h \Big)$$

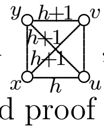
for all i $(1 \leq i \leq d-h)$*, and equality hold if and only if*



$=$

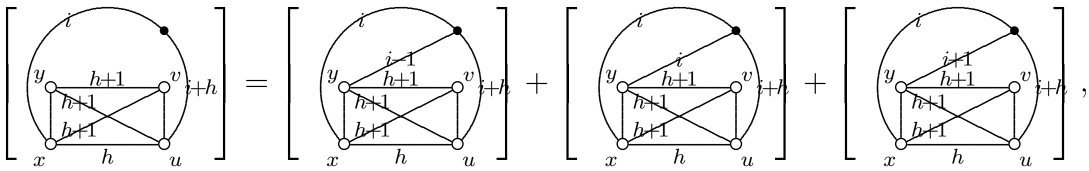


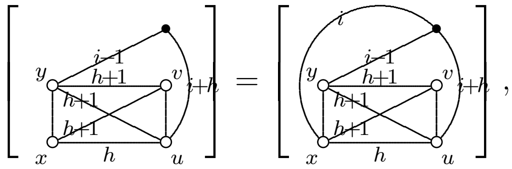
$= 0.$

Proof. Since Γ has induced polygon $p_1p_2...p_{2h+2}p_{2h+3}$ (of length $2h+3$) we can pick $x = p_1$, $u = p_{h+1} \in \Gamma_h(x)$, $v = p_{h+2} \in \Gamma_{h+1,1}(x,u)$ and $y = p_{2h+3} \in \Gamma_{1,h+1,h+1}(x,u,v)$. So $xuvy$ is consistent with 

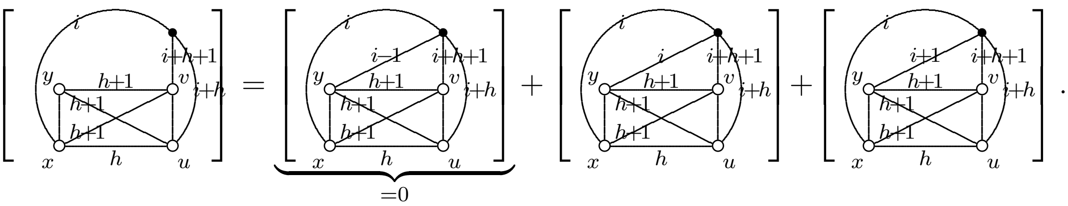
, and the result follows from Lemma 5.3 and Proposition 6.1. Here we give second proof i.e. we use different method from the proof of Lemma 5.3.

We have

$p^h_{i,i+h} =$ 

$p^{h+1}_{i-1,i+h} =$ 

and

$p^{h+1}_{i,i+h+1} =$ 

It is not hard to see that $p^h_{i,i+h} \geq p^{h+1}_{i-1,i+h} + p^{h+1}_{i-1,i+h}$. The result now follows from Lemma 3.4. ∎

3u **Corollary. 15.2** *The following* (i)–(iii) *hold.*

(i) *If* Γ *has induced pentagon then* $c_{i-1} + b_i \leq b_1$ *for any* i $(2 \leq i \leq d-1)$.

(ii) *If* Γ *has induced heptagon then* $c_{i-1} + b_{i+1} \leq b_2$ *for any* i $(2 \leq i \leq d-2)$.

(iii) *If* Γ *has induced nonagon then* $c_{i-1} + b_{i+2} \leq b_3$ *for any* i $(2 \leq i \leq d-3)$.

Proof. Immediate from Theorem 15.1. ∎

81 **Remark. 15.3 (on induced pentagon)** Dodecahedron has diameter $d = 5$, intersection array $\{3, 2, 1, 1, 1; 1, 1, 1, 2, 3\}$, so we have $0 = a_1 < a_2 = 1$ and

$$b_2 + c_1 = b_1, \qquad b_3 + c_2 = b_1, \qquad b_4 + c_3 = b_1, \qquad b_5 + c_4 = b_1$$

which yield that inequity from Theorem 15.1 is tight. Also note that $p^2_{i-1,i+1} + p^2_{i,i-2} \leq p^1_{i,i-1}$ yield $b_i + c_{i-1} \leq b_1$. This imply that in general case, without some additional restriction, it will not be possible to prove something like

$$p^2_{i-1,i+1} + p^2_{i,i-2} \leq p^1_{i,i-1} - \frac{1}{2} p^2_{i,i-1}$$

as in Theorem 4.3. ∎

op **Lemma. 15.4** *If* Γ *has induced pentagon, and if* $b_{i-1} - b_i \geq a_i$ *for every* $i = 1, 2, ..., d$ *then*

$$d \leq \frac{k + c_d}{\lambda + 1}.$$

Proof. Our assumption and Theorem 15.1 yield that for any i $(1 \leq i \leq d)$ we have

$$(c_i - c_{i-1}) + (b_{i-1} - b_i) \geq (c_i - c_{i-1}) + a_i \geq \lambda + 1.$$

Summing the above inequalities for $i = 1, 2, ..., d$, the result follows ∎

Remark. 15.5 If Γ has induced pentagon, and if diameter is large enough then Γ can have induced heptagon, induced nonagon and so on. As we mention in Remark 15.3 inequality $c_{i-1} + b_i \leq b_1$ is tight for dodecahedron (for every i $(2 \leq i \leq d)$). It is not hard to see that if $c_3 = c_2 = c_1 = 1$ and $b_1 = b_2 > b_3$ then Γ has induced heptagon. This yield that Coxeter graph (which intersection array is $\{3, 2, 2, 1; 1, 1, 1, 2\}$) has induced heptagon. For this graph inequality $c_{i-1} + b_{i+1} \leq b_2$ is tight for $i = 2$.

Corollary. 15.6 *The Ivanov-Ivanov-Faradjev graph (intersection array* $\{7, 6, 4, 4, 4, 1, 1, 1; 1, 1, 1, 2, 4, 4, 6, 7\}$*) does not have induced heptagon and does not have induced nonagon.*

Proof. Immediate from Corollary 15.2. ∎

Corollary. 15.7 *If* Γ *has induced polygon of length* $2h + 3$ *(for some* h $(1 \leq h \leq d)$ *then*

$$b_h > b_{h+1}.$$

Proof. Assume to the contrary, that is assume that $b_h = b_{h+1}$. Since Γ has induced polygon of length $2h + 3$, by Theorem 15.1 we have

$$c_i + b_{i+h} \leq b_h$$

for any i $(1 \leq i \leq d - h)$. If we set $i = 1$ and since $b_h = b_{h+1}$ we have

$$c_1 + b_{h+1} \geq b_h$$

a contradiction. ∎

Remark. 15.8 If $b_1 = b_2$ then $\Gamma = (X, R)$ does not have induced pentagon. So for example if in additional we have $\lambda \neq 0$ and $c_3 > c_2$, then for any $x, y \in X$ we don't have edges between $\Gamma_{32}(x, y)$ and $\Gamma_{22}(x, y)$.

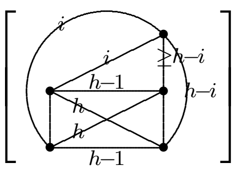
15.2 Induced polygon of even length - case when $E_h > 0$

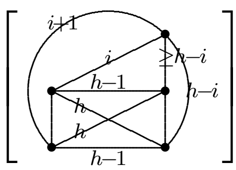
Assume that there exists reduced polygon of even length. In this section, among else, we prove that then Koolen's inequalities (19) and (22) hold.

os **Theorem. 15.9** *Assume that* Γ *has induced polygon* $p_1 p_2 ... p_{2h}$ *of even length* $2h$ $(2 \leq h \leq d)$. *Then*

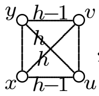
$$c_h \geq c_{h-i} + c_i$$

for all i $(1 \leq i \leq h - 1)$, *and equality hold if and only if*

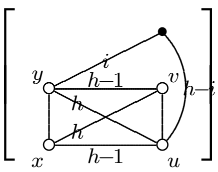
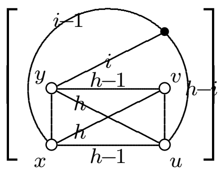
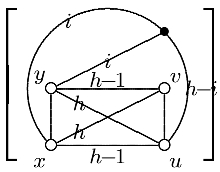
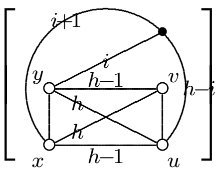


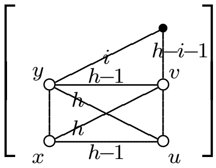
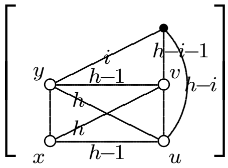
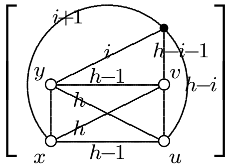
=


= 0.

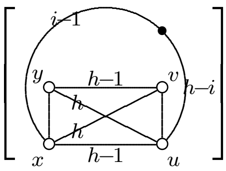
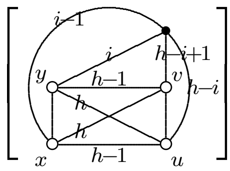
Proof. Since Γ has induced polygon $p_1p_2...p_{2h-1}p_{2h}$ (of length $2h$) we can pick $x = p_1$, $u = p_h \in \Gamma_{h-1}(x)$, $v = p_{h+1} \in \Gamma_{h,1}(x,u)$ and $y = p_{2h} \in \Gamma_{1,h,h-1}(x,u,v)$. So $xuvy$ is consistent with , and the result follows from Lemma 5.3 and Proposition 6.1. Here we give second proof i.e. we use different method from the proof of Lemma 5.3.

We have

$p^h_{i,h-i} =$  $=$  $+$  $+$ ,

$p^{h-1}_{i,h-i-1} =$  $=$  $=$ 

and

$p^{h-1}_{i-1,h-i} =$  $=$ .

The above three equalities yield

$$p^h_{i,h-i} \geq p^{h-1}_{i,h-i-1} + p^{h-1}_{i-1,h-i}$$

and by Lemma 3.4

$$\frac{c_{i+1}c_{i+2}...c_{h-1}c_h}{c_1c_2...c_{h-i-1}c_{h-i}} \geq \frac{c_{i+1}c_{i+2}...c_{h-2}c_{h-1}}{c_1c_2...c_{h-i-2}c_{h-i-1}} + \frac{c_ic_{i+1}...c_{h-2}c_{h-1}}{c_1c_2...c_{h-i-1}c_{h-i}}.$$

The result follows. ∎

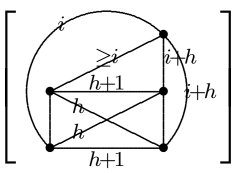
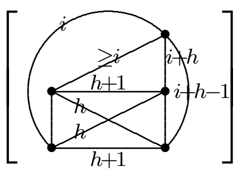
Corollary. 15.10 *If $c_3 = c_2$ then Γ does not have induced hexagon.*

Proof. Immediate from the proof of Theorem 15.9. ∎

$\boxed{\texttt{or}}$ **Theorem. 15.11** *Assume that Γ has induced polygon $p_1p_2...p_{2h+2}$ of odd length $2h+2$ $(1 \leq h \leq d)$. Then*

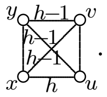
$$b_h \geq c_i + b_{h+i} \qquad \Big(\iff \qquad (c_{h+i} - c_i) + a_{h+i} \geq a_h + c_h\Big)$$

for all i $(1 \leq i \leq d-h)$, and equality hold if and only if

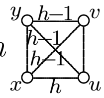
 $=$  $= 0.$

Proof. Similarly to the proof of Theorem 15.1. ∎

15.3 Induced vertices consistent with the given diagram - case when $F_h > 0$

From previous two subsections it is clear that we can use t-point counts starting from some set of vertices on given distances. In this section we consider vertices $xuvy$ consistent with diagram 

.

ot **Theorem. 15.12** *Assume that* Γ *has vertices* $xuvy$ *consistent with diagram* 

. Then

$$c_h \geq c_{h-i} + c_i$$

for all i $(1 \leq i \leq h-1)$.

Proof. Similarly to the proof of Theorem 15.1. ∎

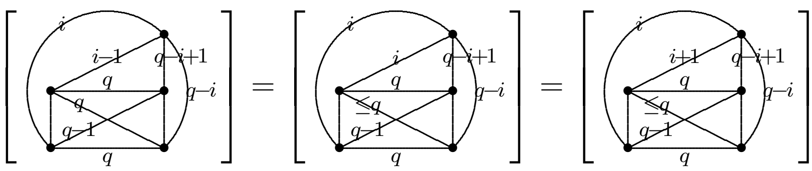
16 Hiraki's second inequality

oz In order to get diameter bound, we need to start from something - for example we can choose smallest q such that $c_q > c_{q-1}$ or $b_q < b_{q-1}$, and use one of these these two inequalities to obtain new one, that will in long run give some bounds on d. In this section we re-prove A. Hiraki result from [6], using t-point counts.

Lemma. 16.1 *Let* $\Gamma = (X, R)$ *be a distance regular graph of diameter* d*, valency* $k \geq 3$*, and let* q *be an integer with* $2 \leq q \leq d-1$*. Suppose* $b_q < b_{q-1}$*. If* $a_q \leq b_{q-1} - b_q$ *then*

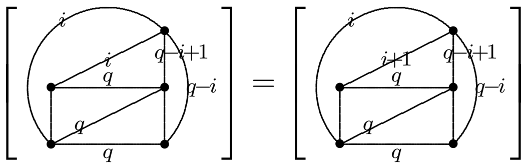
$$b_q + c_i \leq b_{q-i} \qquad \textit{for all } i \ (2 \leq i \leq q-1).$$

Moreover, if $a_q = c_{q+1} - c_q$ *then* $b_q + c_i = b_{q-i}$ *if and only if*



$= 0$

and

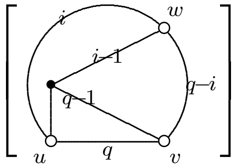


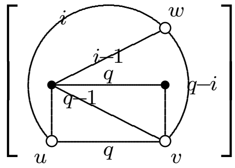
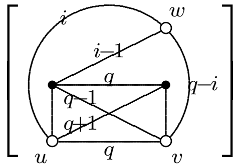
$= 0.$

Proof. Proof of (i) can be found in [6, Lemma 3.3]. We give proof also here because of several reasons: completeness, use of t-point counts technique and to see from where inequality comes from. Pick $u \in X$, $v \in \Gamma_q(u)$ and $w \in \Gamma_{i,q-i}(u, v)$. We consider sets $Y := C_i(w, u)$ and $Z := B_{q-i}(w, v) \backslash B_q(u, v)$, and we count size of the set

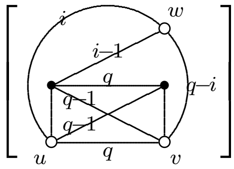
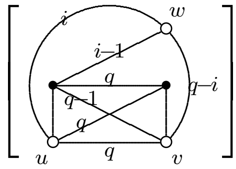
$$\Omega = \{(y, z) \in Y \times Z \mid d(y, z) = q\}$$

in two ways. From

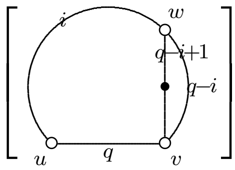
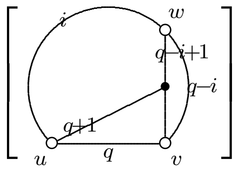
$c_i = |Y| = |C_i(u,v)| =$  ,

$c_i b_{q-1} =$  and $c_i b_q =$  ,

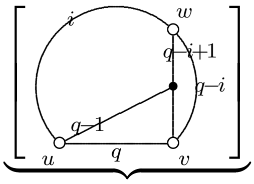
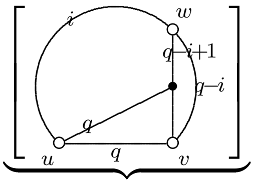
we have

$c_i(b_{q-1} - b_q) =$  $+$  .

Similarly

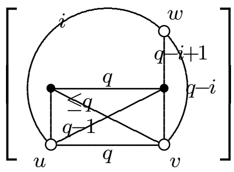
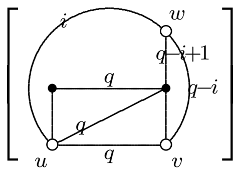
$b_{q-i} = |B_{q-i}(w,v)| =$  , $b_q = |B_q(u,v)| =$  ,

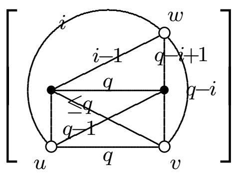
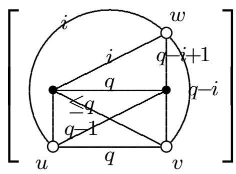
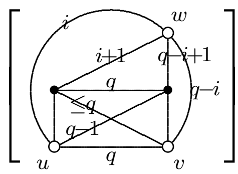
and

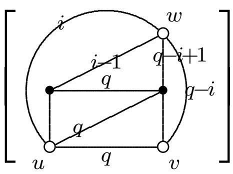
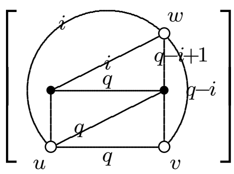
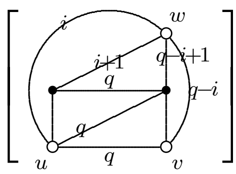
$b_{q-i} - b_q = |B_{q-i}(w,v) \backslash B_q(u,v)| =$  $+$ 

$=|Z_A|$ $=|Z_B|$

yield

$|Z_A|(b_{q-1} - b_q) + |Z_B| a_q =$  $+$  $=$

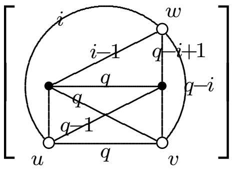
 $+$  $+$  $+$

 $+$  $+$  .

Now it is not hard to see that

$$c_i(b_{q-1} - b_q) = |\Omega| \le |Z_A|(b_{q-1} - b_q) + |Z_B| a_q$$

and equality hold if and only if



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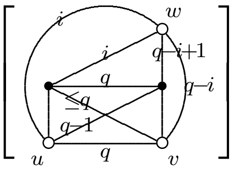

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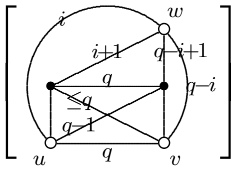
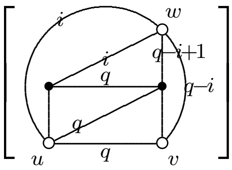
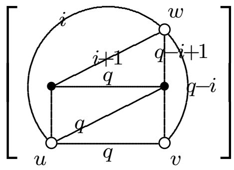


$= 0$

and



$=$


$= 0.$

The result follows. ∎

17 Generalized Terwilliger inequalities

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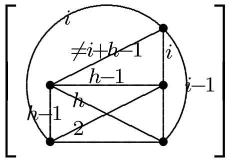
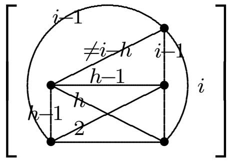
P. Terwilliger [7, 8, 9] explored inequalities obtained by counting configurations having two adjacent edges. Here we rederive and generalize some of his results. In particular, we shall find a new inequality for distance-regular graphs involving the intersection numbers of Γ when the girth is even.

Theorem. 17.1 *If Γ has induced polygon $p_1p_2...p_{2h}$ (of length $2h$) then*

$$b_{i-1} + c_i - \lambda - 2 \geq \frac{c_{i-1}c_{i-2}\ldots c_{i-h+1} + b_ib_{i+1}\ldots b_{i+h-2}}{b_2b_3\ldots b_{h-1}} \qquad (71)$$ ox

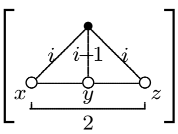
for any i $(h+1 \leq i \leq d-h+1)$, and equality hold if and only if

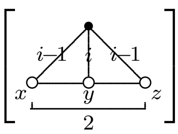


$= 0$ *and*


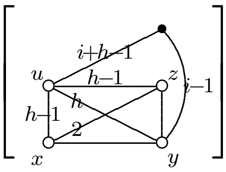
$= 0.$

Proof. Pick $x = p_{2h} \in X$, $y = p_1 \in \Gamma_1(x)$, $z = p_2 \in \Gamma_{2,1}(x,y)$ and $u = p_{h+1} \in \Gamma_{h-1,h,h-1}(x,y,z)$. We have

$|\Gamma_{i,i-1,i}(x,y,z)| =$


, $\quad |\Gamma_{i-1,i,i-1}(x,y,z)| =$


,

$p^h_{i-1,i+h-1} = |\Gamma_{i-1,i+h-1}(y,u)| =$


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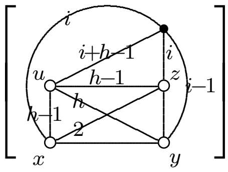
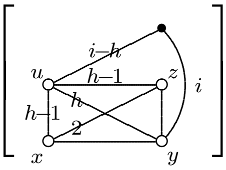
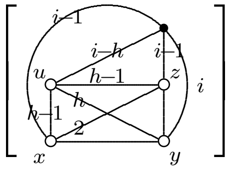


,

$p^h_{i,i-h} = |\Gamma_{i,i-h}(y,u)| =$


$=$


.

The above four equalities yield

$$p^h_{i-1,i+h-1} + p^h_{i,i-h} \le |\Gamma_{i,i-1,i}(p_{2h}, p_1, p_2)| + |\Gamma_{i-1,i,i-1}(p_{2h}, p_1, p_2)|$$

and by Lemma 4.1

$$p^h_{i-1,i+h-1} + p^h_{i,i-h} \le 2p^1_{i-1,i} - p^2_{i-1,i+1} - p^2_{i,i-2} - p^2_{i,i-1}.$$

By Lemma 3.4

$$(p^h_{i+h-1,i-1} + p^h_{i,i-h})\frac{nb_1}{n_i c_i} = \frac{b_i b_{i+1} ... b_{i+h-2} + c_{i-1}c_{i-2}...c_{i-h+1}}{b_2 b_3 ... b_{h-1}}$$

and

$$2p^1_{i,i-1} - p^2_{i,i-1} - p^2_{i,i-2} - p^2_{i-1,i+1} = \frac{c_i n_i}{nb_1}(2b_1 - (a_i + a_{i-1} - \lambda) - c_{i-1} - b_i).$$

The result now follows. ∎

In Subsection 4.2 we proved that

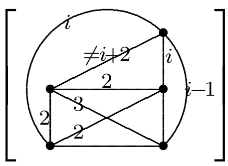
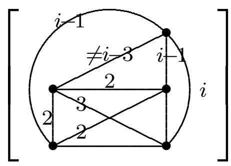
$$b_i + c_{i-1} \le c_i + b_{i-1} - \lambda - 2$$

(see inequality (16)). Note similarity between (16), (71) and (72).

Corollary. 17.2 *Assume that Γ has induced hexagon $p_1p_2p_3p_4p_5p_6$. Then*

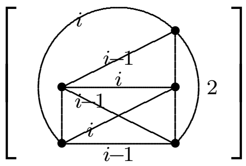
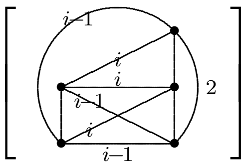
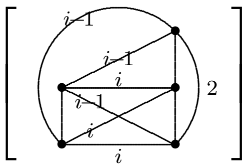
$$\frac{1}{b_2}(b_i b_{i+1} + c_{i-1}c_{i-2}) \le c_i + b_{i-1} - \lambda - 2 \tag{72}$$ oy

for any i $(4 \le i \le d-2)$, and equality hold if and only if

 $= 0$ *and*  $= 0.$

18 Note when $b_{i-1} > b_i$, $c_i = c_{i-1}$, $a_{i-1} \neq 0$ and $\Xi_i = 0$

In this section we study 5-point count

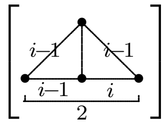
$\Xi_i =$  $=$  $=$  .

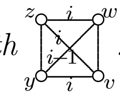
We show that if $b_{i-1} > b_i$, $c_i = c_{i-1}$, $a_{i-1} \neq 0$ and $\Xi_i = 0$ for some integer $i \ge 2$ then

$$c_i + b_i \le b_{i-1}.$$

Note that 5-point count Ξ_i contains 5 different 4-point counts.

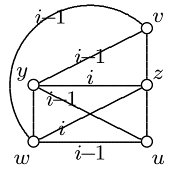
r1 **Lemma. 18.1** *With the notation of Proposition 12.1, if* $\Phi_i \neq 0$ *then*

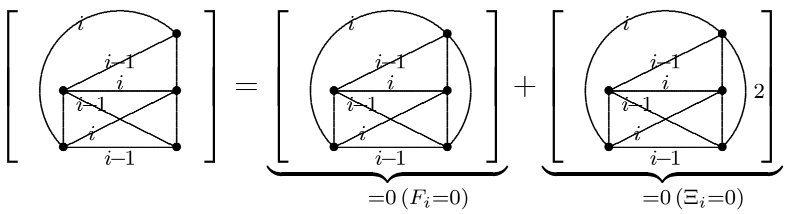
$B_i \neq 0, \qquad D_i \neq 0, \qquad U_i \neq 0, \qquad T_i \neq 0 \qquad$ *and*  $\neq 0.$

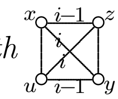
r2 **Lemma. 18.2** *With the notation of Definition 2.1, assume that* $c_i = c_{i-1}$ *and* $b_{i-1} > b_i$ *so that we can choose* $ywzv$ *consistent with* . *If* $\Xi_i = 0$ *then*

$$\Gamma_{i-1}(w) \cap C_i(y, z) = \emptyset.$$

Proof. Assume to the contrary, that is assume $\Gamma_{i-1}(w) \cap C_i(y, z)$ is nonempty, so that we can choose $u \in \Gamma_{i-1}(w) \cap C_i(y, z)$. Then $ywuzv$ are consistent with

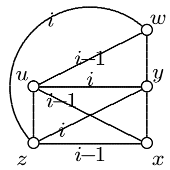
.

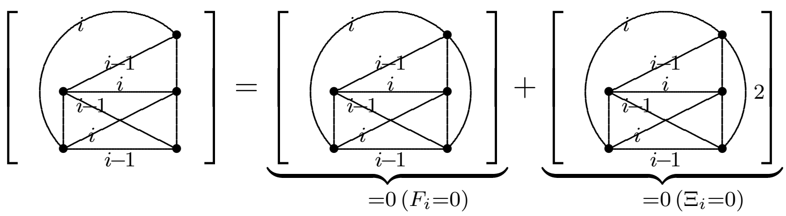
Now since  $= 0$, we have a contradiction. The result follows. ∎

r3 **Lemma. 18.3** *With the notation of Definition 2.1, assume that* $c_i = c_{i-1}$ *and* $a_{i-1} \neq 0$ *so that we can choose* $uzxy$ *consistent with* . *If* $\Xi_i = 0$ *then*

$$\Gamma_{i-1}(u) \cap A_i(z, y) = \emptyset.$$

Proof. Assume to the contrary, that is assume $\Gamma_{i-1}(u) \cap A_i(z, y)$ is nonempty, so that we can choose $w \in \Gamma_{i-1}(u) \cap A_i(z, y)$. Then $uzxyw$ are consistent with

.

Now since  $= 0$, we have a contradiction. The result follows. ∎

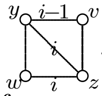
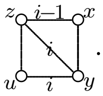
r4 **Lemma. 18.4** *With the notation of Definition 2.1, assume that $c_i = c_{i-1}$ and pick $z \in X$, $y \in \Gamma_i(z)$, $x \in C_i(z, y)$ and $v \in C_i(y, z)$. Then the following* (i)–(iii) *hold.*

(i) $|\Gamma_{1ii}(z, y, x)| = b_{i-1} - b_i$ *and* $|\Gamma_{1ii}(y, z, v)| = b_{i-1} - b_i$.

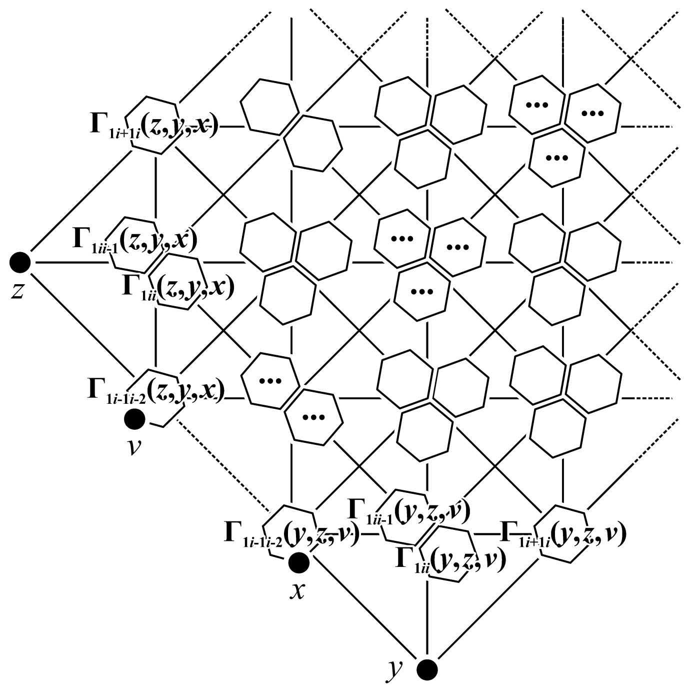
(ii) $A_i(y, z) = \Gamma_{1,i,i-1}(z, y, x) \cup \Gamma_{1ii}(z, y, x)$ *and* $A_i(z, y) = \Gamma_{1,i,i-1}(y, z, v) \cup \Gamma_{1ii}(y, z, v)$.

(iii) $C_i(z, y) = \Gamma_{i-2,1,i-1}(z, y, u)$ *and* $C_i(y, z) = \Gamma_{i-2,1,i-1}(y, z, x)$.

Proof. (i) Since $c_i = c_{i-1}$ we have $e_i = 0$ and $D_i = n_i c_i(b_{i-1} - b_i)$. The result follows.

(ii) Pick $w \in A_i(z, y)$. We have that $ywzv$ are consistent with . By triangle inequality it is not hard to see that $d(v, w) \in \{i - 1, i\}$. Similar for $u \in A_i(y, z)$ and .

(iii) Similar to the proof of Lemma 12.13(iii). ∎



10 **Figure 5** *Intersection diagram (of rank i) where $\Gamma_1(z)$ is with respect to (z, y, x) and $\Gamma_1(y)$ is with respect to (y, z, v), in case when $c_i = c_{i-1}$. Note that $z \in X$, $y \in \Gamma_i(z)$, $x \in \Gamma_{i-1,1}(z, y)$ and $v \in \Gamma_{1,i-1}(z, y)$.*

r5 **Lemma. 18.5** *With the notation of Definition 2.1, assume that $c_i = c_{i-1}$ and pick $z \in X$, $y \in \Gamma_i(z)$, $x \in C_i(z, y)$ and $v \in C_i(y, z)$. If $b_{i-1} > b_i$, $a_{i-1} \neq 0$ and $\Xi_i = 0$ then there exists $w \in \Gamma_{1ii}(y, z, v)$ such that*

$$C_i(w, z) \subseteq \Gamma_{1ii}(z, y, x).$$

Proof. By Lemma 18.4(i), $|\Gamma_{1ii}(y, z, v)| = b_{i-1} - b_i$, and since $b_{i-1} > b_i$ there exists $w \in \Gamma_{1ii}(y, z, v)$. By Lemma 18.3 for any $u \in \Gamma_{1,i,i-1}(z, y, x)$ we have $\Gamma_{i-1} \cap A_i(z, y) = \emptyset$.

This yield that $\Gamma_{i-1}(w) \cap \Gamma_{1,i,i-1}(z,y,x) = \emptyset$. Now since $\Gamma_{i-1}(w) \cap \Gamma_{1,i-1,i-2}(z,y,x) = \emptyset$ (by Lemma 18.2) and $\Gamma_{i-1}(w) \cap \Gamma_{1,i+1,i}(z,y,x) = \emptyset$ we have

$$\Gamma_{i-1}(w) \subseteq \Gamma_{1ii}(z,y,x).$$

The result follows. ∎

Corollary. 18.6 ([3, proof of Lemma 5.9.2]) *With the notation of Definition 2.1, assume that $c_i = c_{i-1}$, $b_{i-1} > b_i$, $a_{i-1} \neq 0$ and $\Xi_i = 0$. Then the following hold*

$$c_i + b_i \leq b_{i-1}.$$

Proof. Immediate from Lemmas 18.4(i) and 18.5. ∎

19 Acknowledgments

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