GLOBAL ATTRACTIVITY OF THE ZERO SOLUTION FOR WRIGHT'S EQUATION*

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Abstract. In 1955 E.M. Wright proved that all solutions of the delay differential equation $\dot{x}(t) = -\alpha \left(e^{x(t-1)} - 1\right)$ converge to zero as $t \to \infty$ for $\alpha \in (0, 3/2]$, and conjectured that this is even true for $\alpha \in (0, \pi/2)$. The present paper proves the conjecture for $\alpha \in [1.5, 1.5706]$ (compare with $\pi/2 = 1.570796...$). The first part of the proof verifies that it is sufficient to guarantee the nonexistence of slowly oscillating periodic solutions, and it shows that slowly oscillating periodic solutions with small amplitudes cannot exist. In the second part a computer-assisted proof is given to exclude slowly oscillating periodic solutions with large amplitudes.

Key words. Delayed logistic equation, Wright's equation, Wright's conjecture, slowly oscillating periodic solution, discrete Lyapunov functional, Poincaré–Bendixson theorem, verified computational techniques, computer-assisted proof, interval arithmetic

AMS subject classifications. 34K13, 34K20, 37L15, 65G20, 65L03

1. Introduction.

In 1955, Edward M. Wright [36], motivated by an unpublished note of Lord Cherwell about a heuristic approach to the density of prime numbers (see also [11], [37]), studied the delay differential equation

$$\dot{u}(t) = -\alpha u(t-1)[1 + u(t)], \quad \alpha > 0. \tag{1.1}$$

An equivalent form, the so-called delayed logistic equation or Hutchinson's equation

$$\dot{v}(t) = \alpha v(t)[1 - v(t-1)],$$

was introduced by Hutchinson [12] in 1948 for ecological models.

Considering only those solutions of equation (1.1) which have values in $(-1, \infty)$, the transformation $x = \log(1 + u)$ leads to the equation

$$\dot{x}(t) = f_\alpha(x(t-1)) \tag{1.2}$$

with $f_\alpha(\xi) = -\alpha(e^\xi - 1)$, $\xi \in \mathbb{R}$. Throughout this paper (1.2) is also called Wright's equation. It is one of the simplest nonlinear delay differential equations. Wright [36] was the first who obtained deep results for equation (1.2). He proved among others that all solutions of (1.2) approach zero as $t \to \infty$ provided $\alpha \le \frac{3}{2}$, and he made the following remark: *My methods, at the cost of considerable elaboration, can be used to extend this result to $\alpha \le \frac{37}{24}$ and, probably to $\alpha < 1.567...$ (compare with $\frac{\pi}{2} = 1.570796...$). But the work becomes so heavy for the last step that I have not completed it.*

For every $\alpha > \frac{\pi}{2}$, Wright [36] proved the existence of bounded solutions of equation (1.2) which do not tend to zero. If $\alpha < \frac{\pi}{2}$ then the roots of the characteristic equation

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$z+\alpha e^{-z} = 0$ of the linear variational equation $\dot{y}(t) = -\alpha y(t-1)$ of (1.2) have negative real parts. Thus the zero solution of (1.2) is locally attractive.

Based on the above facts the question of the global attractivity of the zero solution of (1.2) for parameter values $\alpha < \frac{\pi}{2}$ arises naturally, and it is known as Wright's conjecture.

CONJECTURE 1. *For every $\alpha < \frac{\pi}{2}$, the zero solution of equation (1.2) is globally attractive, i.e., all solutions approach zero as $t \to \infty$.*

The problem is still open, and, as far as we know, Wright's result, i.e., $\alpha \leq \frac{3}{2}$, is still the best one for the global attractivity of the zero solution. Walther [33] proved that the set of parameter values α, for which 0 is globally attracting, is an open subset of $(0, \frac{\pi}{2})$.

We mention that Wright's equation motivated the development of a wide variety of deep analytical and topological tools (see e.g. the monographs [8], [10]) to get more information about the dynamics of (1.2). For example, Jones [13] proved the existence of slowly oscillating periodic solutions of (1.2) for $\alpha > \frac{\pi}{2}$, where slow oscillation means that $|z_1 - z_2| > 1$ for each pair of zeros z_1, z_2 of the periodic solution. Chow and Mallet-Paret [5] showed that there is a supercritical Hopf bifurcation of slowly oscillating periodic solutions from the zero solution at $\alpha = \frac{\pi}{2}$.

Applying the Poincaré–Bendixson type result of Mallet-Paret and Sell [24] it can be shown that any solution of equation (1.2) approaches either a nontrivial periodic solution or zero as $t \to \infty$. Mallet-Paret and Walther [25] verified that slow oscillation is generic for equation (1.2) for all $\alpha > 0$, that is, for an open dense set of initial data from the phase space the solutions are eventually slowly oscillating. Mallet-Paret [22] obtained a Morse decomposition of the global attractor of (1.2); see also McCord and Mischaikow [27]. For several other results we refer to the monographs [8], [10]. Despite of the simplicity of equation (1.2) and the very intensive investigation since 1955, it seems that we are still far from the complete understanding of the dynamics of (1.2).

Conjecture 1 is not the only open question for equation (1.2). Recently, Lessard [21] made some progress toward the proof of Jones' conjecture [13]:

CONJECTURE 2. *For every $\alpha > \frac{\pi}{2}$, equation (1.2) has a unique slowly oscillating periodic orbit.*

In the work [20] the set $U(\alpha)$ was defined in the space of continuous functions from $[-1, 0]$ into $\mathbb{R}$, as the forward extension (by the semiflow) of a local unstable manifold at zero. Then the dynamic and geometric structure of its closure $\overline{U(\alpha)}$ were described. The results of [20] are valid for equations including (1.2). In [18] for equation (1.2) we formulated the so called generalized Wright's conjecture:

CONJECTURE 3. *For every $\alpha > 0$, the set $\overline{U(\alpha)}$ is the global attractor for equation (1.2).*

An affirmative answer for Conjecture 3 would mean a more or less complete understanding of the dynamics of equation (1.2). For example, for the equation $\dot{x}(t) = -ax(t) - b\tanh(cx(t-1))$ with $a \geq 0$, $b > 0$, $c > 0$, the analogue of Conjecture 3 is known to be valid [17].

In this paper we prove that Wright's conjecture is equivalent to the nonexistence of slowly oscillating periodic solutions, and we develop a reliable computational tool to exclude the existence of slowly oscillating periodic solutions with amplitude greater than a certain constant $\epsilon_0 > 0$. For $\frac{3}{2} \leq \alpha < \frac{\pi}{2}$, we show that for every slowly oscillating periodic solution p of (1.2), $\max_{t\in\mathbb{R}} p(t) > \log \frac{\pi}{2\alpha}$ holds. Since $\log \frac{\pi}{2\alpha} \to 0$ as $\alpha \to \frac{\pi}{2}$, we are able to prove Wright's conjecture only for those values of α for which

$\log \frac{\pi}{2\alpha} > \epsilon_0$. These results combined verify Wright's conjecture for $\alpha \in [1.5, 1.5706]$, i.e., give

THEOREM 1.1. *If* $\alpha \in [1.5, 1.5706]$*, then the zero solution of equation (1.2) is globally attractive.*

We have also completed a computational proof of the following assertion. Although it is not used in the present paper, it can be applied in further investigations on the Wright conjecture, once the existence of slowly oscillating periodic solutions of (1.2) with amplitude below 0.04 can be excluded.

THEOREM 1.2. *If* $\alpha \in [1.5, \pi/2]$ *and* $p^\alpha : \mathbb{R} \to \mathbb{R}$ *is a slowly oscillating periodic solution of (1.2), then* $\max_{t\in\mathbb{R}} |p^\alpha(t)| < 0.04$ *holds.*

Applying center manifold theory and local Hopf bifurcation techniques it is possible to find an $\epsilon_* > 0$ independently of α such that, for any $\alpha \in [1.5, \pi/2]$ and any slowly oscillating periodic solution $p^\alpha : \mathbb{R} \to \mathbb{R}$ of (1.2), the inequality $\max_{t\in\mathbb{R}} |p^\alpha(t)| > \epsilon_*$ holds [19]. However, ϵ_* is much smaller than the constant 0.04 obtained in Theorem 1.2, and thus the proof is still not complete for Conjecture 1.

The paper is organized as follows. Sections 2, 3 and 4 contain the analytical part of the proof of Theorem 1.1. Preliminary results are collected in Section 2. It is shown in Section 3 that the global attractivity of the zero solution of (1.2) is equivalent to the nonexistence of slowly oscillating periodic solutions. An explicit lower bound is constructed in Section 4 for the amplitudes of the possible slowly oscillating periodic solutions in the case $\alpha < \pi/2$. The rigorous numerical computations are contained in Sections 5, 6 and 7. Section 5 describes the construction of some bounding functions for slowly oscillating periodic solutions. In Section 6 we define the pseudocode of the applied algorithms that were used for the computational proof. Finally, Section 7 provides the implementation details that are so critical for verified numerical procedures, and we discuss the computational results.

Some new ideas of the present paper made it possible to improve Wright's famous 3/2-result. First of all, the problem can be reduced to the nonexistence of slowly oscillating periodic solutions. The proof of this fact is based on a Poincaré–Bendixson type technique due to Mallet-Paret and Sell [24], and on a geometric idea of Walther [33] to estimate the amplitudes of slowly oscillating periodic solutions (see Sections 3, 4). The verified numerical calculations, to exclude slowly oscillating periodic solutions, apply interval arithmetic [1, 31]. Motivated by the original proof of Wright, a sequence of bounding functions, i.e., lower and upper bounds for the possible slowly oscillating periodic solutions, are constructed in an iterated way. Nonexistence of slowly oscillating periodic solutions with given maximum M and minimum $-m$, provided that M and m are away from zero, is verified if, for example, the maximum of the upper bound of an upper bounding function is less than M. In order to see why this technique can give an improvement compared to Wright's result we recall the idea of Wright's proof. Based on the result of Section 3, it is sufficient to show the nonexistence of slowly oscillating periodic solutions. Wright did not use this fact. We formulate Wright's technique to exclude the existence of slowly oscillating periodic solutions. Assume that y is a slowly oscillating periodic solution of equation (1.2) with maximal value M and with minimal value $-m$, where $m > 0$, $M > 0$. Let z be a time with $y(z) = 0$. Then, by the fact [24] that for periodic solutions of (1.2) there is a unique zero of y' between two consecutive zeros of y, and three consecutive zeros of y determine the minimal period, $z + 1$ must be a minimum or a maximum point of

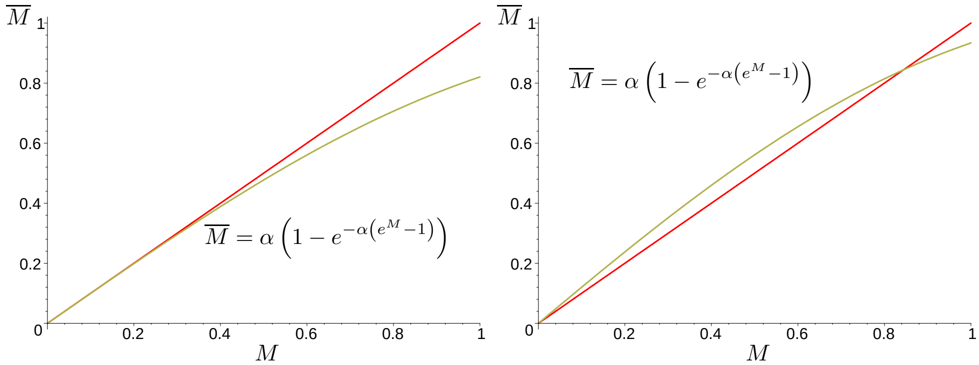

FIG. 1.1. *The simplest bounding function values compared to M for $\alpha = 1.0$ (left) and $\alpha = 1.1$ (right).*

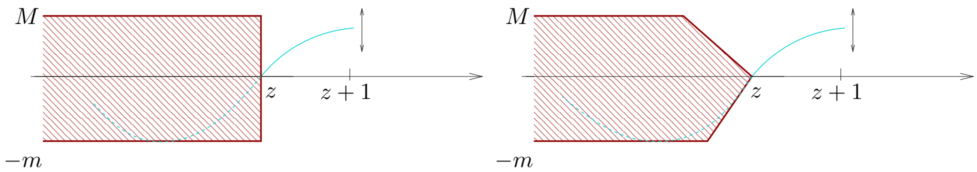

FIG. 1.2. *Regions where the solutions can be before z by using inequality (1.3).*

y. Consider first the case $y(z+1) = M$. Then

$$M = \int_z^{z+1} y'(t)\,dt = -\alpha \int_z^{z+1} \left(e^{y(t-1)} - 1\right)\,dt = -\alpha \int_{z-1}^{z} \left(e^{y(t)} - 1\right)\,dt$$

$$\leq -\alpha \int_{z-1}^{z} \left(e^{-m} - 1\right)\,dt = \alpha\left(1 - e^{-m}\right).$$

In other words, M can be bounded as $M \leq \alpha\left(1 - e^{-m}\right)$, and in a similar way $m \leq \alpha\left(e^M - 1\right)$. Now one can derive the inequality $M \leq \overline{M} := \alpha\left(1 - e^{-\alpha\left(e^M - 1\right)}\right)$ which is illustrated in Figure 1.1. For $\alpha = 1$, the respective bounding function is below M for positive values of M, i.e., $\overline{M} < M$ holds, which is a contradiction. Therefore, no slowly oscillating periodic solutions can exist. But for $\alpha > 1$ (illustrated in the figure for $\alpha = 1.1$) the inequality does not imply this statement, since not all positive values of M can be discarded.

The above reasoning was strengthened by Wright utilizing a better estimation of the possible solution up to z. Applying the bounds $-m \leq y(t) \leq M$, we obtain

$$-\alpha\left(e^M - 1\right) \leq y'(t) \leq -\alpha\left(e^{-m} - 1\right) \tag{1.3}$$

from (1.2) for all t. The bound (1.3) for the derivative of y allows us to exclude the existence of slowly oscillating periodic solutions for a wider set of values α. Figure 1.2 illustrates the regions where the slowly oscillating periodic solutions can be taking into account (1.3).

We skip here the technical details of Wright's paper [36], and just give the conditions obtained via the transformation $m = \log \frac{1}{1-v}$, $M = \log(1+u)$ from the inequalities (3.10), (3.9) and (3.11) of [36], respectively:

$$M \le \alpha\left(1-e^{-m}\right)+\frac{me^{-m}}{1-e^{-m}}-1 \quad \text{if} \quad m \le \alpha\left(1-e^{-m}\right), \tag{1.4}$$

$$M \le \alpha - \frac{1-e^{\alpha\left(e^{-m}-1\right)}}{1-e^{-m}}, \tag{1.5}$$

$$m \le \alpha\left(e^{M}-1\right) - M\frac{e^{M}}{e^{M}-1}+1. \tag{1.6}$$

These inequalities imply a new upper bound $\overline{M}$ for M, i.e., $M \le \overline{M}$ holds for all $M > 0$. Figure 1.3 shows the effects of the improved bounding inequalities. In case $\alpha = 1.1$ compare the two different bounding functions in Figures 1.1 and 1.3. For the new, sharper bounding function $\overline{M} < M$ holds for all $M > 0$. This is a contradiction, that is, for $\alpha = 1.1$ there is no slowly oscillating periodic solution. For $\alpha = 1.5$ one can see that the bounding function implies the nonexistence of slowly oscillating periodic solutions. In fact, Wright [36] proved that (1.4), (1.5), (1.6) imply $M = m = 0$ in case $1 \le \alpha \le 1.5$. For $\alpha > 1.5$ (illustrated in the figure for $\alpha = 1.55$ and $\alpha = \pi/2$), no conclusion can be drawn on the basis of these bounding functions. Our new iterative bounding scheme (see Section 5) extends the above ideas of Wright. However, the obtained bounding inequalities are too complicated to analytically derive sharp bounds. This is the place where interval arithmetic plays a part. The verified numerical calculations applying interval arithmetic guarantee that, for values M and m not too close to zero, a slowly oscillating periodic solution with maximum M and minimum $-m$ cannot exist (see Sections 6, 7).

Observe that, for any $\alpha \le \pi/2$, the inequalities $M \le \alpha\left(1-e^{-m}\right)$ and $m \le \alpha\left(e^{M}-1\right)$ imply

$$M < \frac{\pi}{2} \quad \text{and} \quad m < \frac{\pi}{2}\left(e^{\pi/2}-1\right) < 6. \tag{1.7}$$

The paper [3] investigated the problem with traditional verified differential equation solver algorithms, without the bounding schemes presented here. The technique worked only for $\alpha \in [1.5, 1.5 + 10^{-22}]$ to show that any solution is eventually in $[-0.075, 0.075]$. For larger values of α, in particular closer to $\pi/2$, the required CPU times exploded.

Both the analytical and computational techniques of this paper can be easily modified to more general delay differential equations with monotone negative feedback, e.g., for equations $\dot{x}(t) = -\mu x(t) + f(x(t-1))$ where $\mu \ge 0$ and $f : \mathbb{R} \to \mathbb{R}$ is smooth with $f(0) = 0$ and $f' < 0$.

We remark that for the 2-dimensional Ricker map a technique analogous to that of this paper, i.e, a combination of analytical tools and interval arithmetic, was recently successfully applied in [2] to answer a global attractivity conjecture of S. Levin and R. May from 1976.

2. Preliminary results, notation. The results which we mention below without references are all well known, and can be found e.g. in [8] or [10].

The natural phase space for equation (1.2) is $C = C([-1,0],\mathbb{R})$ equipped with the supremum norm $||\cdot||$. By the method of steps, every $\phi \in C$ uniquely determines a

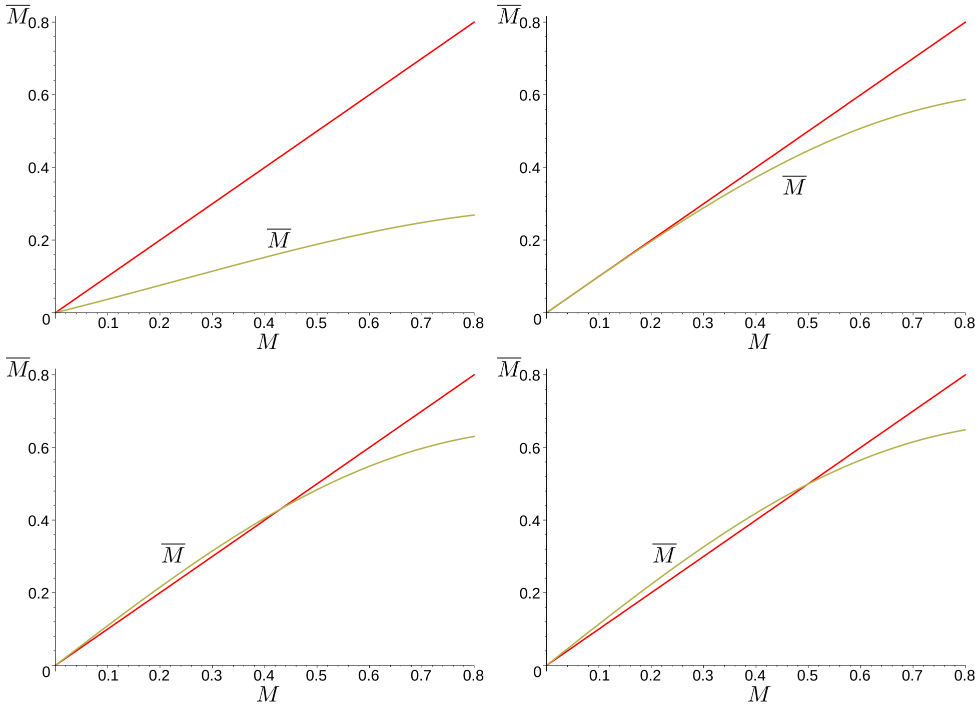

FIG. 1.3. *The second bounding function values compared to M for $\alpha = 1.1$ (top left), $\alpha = 1.5$ (top right), $\alpha = 1.55$ (bottom left), and $\alpha = \pi/2$ (bottom right).*

solution $x = x^\phi : [-1, \infty) \to \mathbb{R}$ of (1.2), i.e., a continuous function x so that $x|_{(0,\infty)}$ is differentiable, $x|_{[-1,0]} = \phi$, and x satisfies (1.2) for all $t > 0$. C^1 is the Banach space of all C^1-maps $\phi : [-1, 0] \to \mathbb{R}$, with norm $||\phi||_1 = ||\phi|| + ||\dot{\phi}||$. If $I \subset \mathbb{R}$ is an interval, $x : I \to \mathbb{R}$ is a continuous function, $t \in \mathbb{R}$ so that $[t - 1, t] \subset I$, then the segment $x_t \in C$ is defined by $x_t(s) = x(t + s)$, $-1 \leq s \leq 0$.

For every $\phi \in C$ the unique solution $x^\phi : [-1, \infty) \to \mathbb{R}$ is bounded. The map

$$F : \mathbb{R}^+ \times C \ni (t, \phi) \mapsto x_t^\phi \in C$$

defines a continuous semiflow. The element 0 of C is the only stationary point of F. All maps $F(t, \cdot) : C \to C$, $t \geq 0$ are injective. It follows that for every $\phi \in C$ there is at most one solution $x : \mathbb{R} \to \mathbb{R}$ of (1.2) with $x_0 = \phi$. We denote also by x^ϕ such a solution on $\mathbb{R}$ whenever it exists. For a given $\phi \in C$, the ω-limit set of ϕ is defined as

$$\begin{aligned}\omega(\phi) = \{\psi \in C : \text{there is a sequence } (t_n)_0^\infty \subset [0, \infty) \text{ so that} \\ t_n \to \infty \text{ and } F(t_n, \phi) \to \psi \text{ as } n \to \infty\}.\end{aligned}$$

Each map $F(t, \cdot)$, $t \geq 0$, is continuously differentiable. The operators $D_2F(t, 0)$, $t \geq 0$, form a strongly continuous semigroup. The spectrum of the generator of the semigroup $(D_2F(t, 0))_{t\geq 0}$ consists of the solutions $\lambda \in \mathbb{C}$ of the characteristic equation

$$\lambda + \alpha e^{-\lambda} = 0.$$

In case $\frac{1}{e} < \alpha \leq \frac{\pi}{2}$, all points in the spectrum form a sequence of complex conjugate pairs $(\lambda_j, \overline{\lambda_j})_0^\infty$ with

$$0 \geq \text{Re } \lambda_0 > \text{Re } \lambda_1 > \text{Re } \lambda_2 > \dots,$$

$$0 < \text{Im } \lambda_0 \leq \frac{\pi}{2}, \quad 2j\pi < \text{Im } \lambda_j < (2j+1)\pi \text{ for all } j \in \mathbb{N} \setminus \{0\}.$$

Let P denote the realified generalized eigenspace of the generator associated with the spectral set $\{\lambda_0, \overline{\lambda_0}\}$. Let Q denote the realified generalized eigenspace given by the spectral set of all $\lambda_k, \overline{\lambda_k}$ with $k \geq 1$. Then P and Q are positively invariant under $D_2F(t,0)$ for all $t \geq 0$, and $C = P \oplus Q$.

We recall the definition and some properties of a discrete Lyapunov functional

$$V : C \setminus \{0\} \to \mathbb{N} \cup \{\infty\}.$$

The version which we use was introduced in Mallet-Paret and Sell [23]. The definition is as follows. First, set $\text{sc}(\phi) = 0$ whenever $\phi \in C \setminus \{0\}$ is nonnegative or nonpositive, otherwise, for nonzero elements of C, let

$$\begin{aligned} \text{sc}(\phi) = \sup\big\{ k \in \mathbb{N} \setminus \{0\} : & \text{ there is a strictly increasing finite sequence} \\ & (s^i)_0^k \text{ in } [-1,0] \text{ with } \phi(s^{i-1})\phi(s^i) < 0 \text{ for all } i \in \{1,2,\dots,k\}\big\} \leq \infty. \end{aligned}$$

Then define

$$V(\phi) = \begin{cases} \text{sc}(\phi) & \text{if sc}(\phi) \text{ is odd or } \infty, \\ \text{sc}(\phi) + 1 & \text{if sc}(\phi) \text{ is even.} \end{cases}$$

Set

$$\begin{aligned} R = \{\phi \in C^1 : \quad & \phi(0) \neq 0 \text{ or } \dot{\phi}(0)\phi(-1) < 0, \\ & \phi(-1) \neq 0 \text{ or } \dot{\phi}(-1)\phi(0) > 0, \\ & \text{all zeros of } \phi \text{ in } (-1,0) \text{ are simple}\}. \end{aligned}$$

We list some basic properties of V [23], [24].

Proposition 2.1.

(i) For every $\phi \in C \setminus \{0\}$ and for every sequence $(\phi_n)_0^\infty$ in $C \setminus \{0\}$ with $\phi_n \to \phi$ as $n \to \infty$,

$$V(\phi) \leq \liminf_{n\to\infty} V(\phi_n).$$

(ii) For every $\phi \in R$ and for every sequence $(\phi_n)_0^\infty$ in $C^1 \setminus \{0\}$ with $||\phi_n - \phi||_1 \to 0$ as $n \to \infty$,

$$V(\phi) = \lim_{n\to\infty} V(\phi_n) < \infty.$$

(iii) Let an interval $I \subset \mathbb{R}$, and continuous functions $b : I \to (-\infty, 0)$ and $z : I + [-1,0] \to \mathbb{R}$ be given so that $z|_I$ is differentiable with

$$\dot{z}(t) = b(t)z(t-1) \tag{2.1}$$

for $\inf I < t \in I$, and $z(t) \neq 0$ for some $t \in I + [-1,0]$. Then the map $I \ni t \mapsto V(z_t) \in \mathbb{N} \cup \{\infty\}$ is monotone nonincreasing. If $t \in I$, $t-3 \in I$ and $z(t) = 0 = z(t-1)$, then $V(z_t) = \infty$ or $V(z_{t-3}) > V(z_t)$. If $t \in I$ with $t-4 \in I$ and $V(z_{t-4}) = V(z_t) < \infty$, then $z_t \in R$.

(iv) If $b : \mathbb{R} \to (-\infty, 0)$ is continuous and bounded, $z : \mathbb{R} \to \mathbb{R}$ is differentiable and bounded, z satisfies (2.1) for all $t \in \mathbb{R}$, and $z(t) \neq 0$ for some $t \in \mathbb{R}$, then $V(z_t) < \infty$ for all $t \in \mathbb{R}$.

We remark that solutions of (1.2), differences of solutions of (1.2), and solutions of the linear variational equation satisfy an equation of the form (2.1) with a suitable coefficient $b(t)$. For example, for solutions x, $\hat{x}$ of (1.2), the difference $y = x - \hat{x}$ satisfies (2.1) with

$$b(t) = \int_0^1 f'_\alpha(sx(t-1) + (1-s)\hat{x}(t-1))\, ds.$$

The following result is a consequence of a more general Poincaré–Bendixson type theorem of Mallet-Paret and Sell [24] applied for equation (1.2). A nontrivial solution $x : \mathbb{R} \to \mathbb{R}$ and the corresponding orbit $\{x_t : t \in \mathbb{R}\}$ are called homoclinic to zero if $\lim_{|t|\to\infty} x(t) = 0$ and $\lim_{|t|\to\infty} x_t = 0$, respectively.

PROPOSITION 2.2. *For every $\phi \in C$, the ω-limit set $\omega(\phi)$ is either $0 \in C$ or a periodic orbit, or a set in C containing 0 and orbits homoclinic to 0.*

3. Attractivity and periodic solutions. Recall that a solution x of (1.2) is called slowly oscillating if $|z_1 - z_2| > 1$ for each pair of zeros of x.

The aim of this section is to reduce Conjecture 1 to the nonexistence of slowly oscillating periodic solutions.

THEOREM 3.1. *The zero solution of (1.2) is globally attracting if and only if (1.2) has no slowly oscillating periodic solution.*

For the proof of Theorem 3.1 we need the following result.

PROPOSITION 3.2. *Suppose $0 < \alpha \leq \pi/2$. Then (1.2) has no homoclinic orbit to zero.*

Proof. If $\alpha < \pi/2$ then 0 is locally asymptotically stable, and there is no homoclinic orbit to zero.

Suppose $\alpha = \pi/2$, and assume that $x : \mathbb{R} \to \mathbb{R}$ is a nontrivial solution of (1.2) with $x(t) \to 0$ as $|t| \to \infty$.

Let $\hat{x} : \mathbb{R} \to \mathbb{R}$ be another solution of (1.2) with $\hat{x}(t) \to 0$ as $|t| \to \infty$, and $\hat{x} \not\equiv x$. For example, $\hat{x} \equiv 0$, or $\hat{x}(\cdot) = x(\cdot + \tau)$ for some $\tau \neq 0$.

The function $y = x - \hat{x}$ satisfies

$$\dot{y}(t) = b(t)y(t-1) \quad (t \in \mathbb{R})$$

with

$$b(t) = \int_0^1 f'_{\pi/2}(sx(t-1) + (1-s)\hat{x}(t-1))\, ds.$$

From $y(t) \to 0$ as $t \to -\infty$ and $y \not\equiv 0$ it follows that there is a sequence $(t_n)_0^\infty$ such that $t_n \to -\infty$ as $n \to \infty$, and

$$|y(t_n)| = \sup_{t \leq 0} |y(t_n + t)| \quad (n \in \mathbb{N}).$$

For the functions

$$y^n(t) = \frac{y(t_n + t)}{|y(t_n)|} \quad (t \in \mathbb{R}),$$

we have

$$\dot{y}^n(t) = b(t_n + t)y^n(t-1) \quad (t \in \mathbb{R}) \tag{3.1}$$

and

$$1 = |y^n(0)| \geq |y^n(t)| \quad (t \leq 0). \tag{3.2}$$

Observe that

$$b(t_n + t) \to -\frac{\pi}{2} \quad \text{as } n \to \infty \text{ uniformly in } t \in (-\infty, 0]. \tag{3.3}$$

By (3.1), (3.2), and (3.3) there is a uniform bound for $|\dot{y}^n(t)|$, $n \in \mathbb{N}$, $t \leq 0$. Then the Arzela–Ascoli theorem and the diagonalization process gives a subsequence (n_k) and a continuous function $z : (-\infty, 0] \to \mathbb{R}$ such that

$$y^{n_k}(t) \to z(t) \text{ as } k \to \infty$$

uniformly on compact subsets of $(-\infty, 0]$, and

$$|z(t)| \leq |z(0)| = 1 \quad (t \leq 0).$$

Considering (3.1) and (3.3), we find that

$$\dot{y}^{n_k}(t) \to -\frac{\pi}{2}z(t-1) \text{ as } k \to \infty$$

uniformly on compact subsets of $(-\infty, 0]$. It follows that z is differentiable,

$$\dot{y}^{n_k}(t) \to \dot{z}(t) \text{ as } k \to \infty$$

uniformly on compact subsets of $(-\infty, 0]$, and

$$\dot{z}(t) = -\frac{\pi}{2}z(t-1) \quad (t \leq 0).$$

Recall the decomposition $C = Q \oplus P$. Let Pr_Q denote the projection of C onto Q along P. It is well known [10] that there are $K \geq 1$ and $\kappa > 0$ so that

$$||D_2F(t,0)\mathrm{Pr}_Q\phi|| \leq Ke^{-\kappa t}||\mathrm{Pr}_Q\phi||$$

for all $t \geq 0$ and $\phi \in C$. Then, for $-\infty < s < t \leq 0$, we have

$$\mathrm{Pr}_Q z_t = \mathrm{Pr}_Q D_2F(t-s,0)z_s = D_2F(t-s,0)\mathrm{Pr}_Q z_s$$

and

$$||\mathrm{Pr}_Q z_t|| \leq Ke^{-\kappa(t-s)}||\mathrm{Pr}_Q z_s|| \leq Ke^{-\kappa(t-s)}||\mathrm{Pr}_Q||.$$

Letting $s \to -\infty$, $\mathrm{Pr}_Q z_t = 0$ follows. Therefore, $z_t \in P \setminus \{0\}$ for all $t \leq 0$. The subspace P contains segments of $a\cos\frac{\pi}{2}t + b\sin\frac{\pi}{2}t$, $a, b \in \mathbb{R}$. Consequently, $V(\phi) = 1$ for all $\phi \in P \setminus \{0\}$, and $V(z_t) = 1$ for all $t \leq 0$. Proposition 2.1 (iii) gives $z_0 \in R$. As $||y_0^{n_k} - z_0||_1 \to 0$, Proposition 2.1 (ii) implies $V(y_0^{n_k}) = 1$ for all sufficiently large k. The definition of y and the monotone property of V in Proposition 2.1 (iii) combined yield

$$V(y_t) = 1 \quad \text{for all } t \in \mathbb{R}.$$

Hence, by Proposition 2.1 (iii) again,

$$(y(t), y(t-1)) \neq (0,0) \quad \text{for all } t \in \mathbb{R}. \tag{3.4}$$

Applying (3.4) with $\hat{x}(t) = x(t+\tau)$, $t \in \mathbb{R}$, for all $\tau \neq 0$, it follows that the curve

$$\gamma : \mathbb{R} \ni t \mapsto (x(t), x(t-1)) \in \mathbb{R}^2$$

is injective.

If (3.4) is used with $\hat{x} \equiv 0$ then $(x(t), x(t-1)) \neq (0,0)$, $t \in \mathbb{R}$, is obtained. Then by (1.2) all zeros of x are simple. In addition, γ transversally intersects the half line $v_+ = \{(0, v) : v > 0\}$. Indeed, if $\gamma(t) \in v_+$ for some $t \in \mathbb{R}$, then $x(t) = 0$, $x(t-1) > 0$, and $\dot{x}(t) < 0$.

The homoclinic solution x has arbitrarily large negative zeros. Otherwise, there is $T \in \mathbb{R}$ so that either $x(t) > 0$ and $\dot{x}(t) < 0$ for all $t < T$, or $x(t) < 0$ and $\dot{x}(t) > 0$ for all $t < T$. Both cases contradict $\lim_{t\to-\infty} x(t) = 0$.

Let t_1, t_2, t_3 be consecutive zeros of x with $\dot{x}(t_1) < 0$, $\dot{x}(t_2) > 0$, $\dot{x}(t_3) < 0$. Set

$$L = \{(1-s)\gamma(t_1) + s\gamma(t_3) : 0 < s < 1\}$$

and

$$\Gamma = \{\gamma(t) : t_1 \leq t \leq t_3\} \cup L.$$

Then Γ is a simple closed curve. By the Jordan curve theorem, $\mathbb{R}^2 \setminus \Gamma$ has two disjoint, open and connected components. The bounded component is the interior $\mathrm{int}(\Gamma)$ of Γ, and the unbounded component is the exterior $\mathrm{ext}(\Gamma)$ of Γ. Clearly, $(0,0) \in \mathrm{int}(\Gamma)$. $\gamma(t_1) \neq \gamma(t_3)$ because of the injectivity of γ. Suppose $\gamma(t_3) < \gamma(t_1)$ in the natural ordering of v_+.

The transversal intersection of γ and v_+ implies that γ can cross L only from outside of Γ to inside of Γ, that is, if $\gamma(t) \in L$ for some $t \in \mathbb{R}$, then

$$\gamma(t-s) \in \mathrm{ext}(\Gamma), \quad \gamma(t+s) \in \mathrm{int}(\Gamma)$$

for all sufficiently small $s > 0$.

Observe $\gamma(t_1 - s) \in \mathrm{ext}(\Gamma)$ for all small $s > 0$. Combining this fact, the injectivity of γ, the Jordan curve theorem and the fact that through L the curve γ can only enter into $\mathrm{int}(\Gamma)$, we conclude

$$\gamma(t) \in \mathrm{ext}(\Gamma) \quad \text{for all } t < t_1.$$

This contradicts $\lim_{t\to-\infty} \gamma(t) = (0,0)$.

The case $\gamma(t_3) > \gamma(t_1)$ analogously leads to a contradiction. This completes the proof of Proposition 3.2. ◻

Proof. [of Theorem 3.1] 1. It is obvious that if (1.2) has a slowly oscillating periodic solution then not all solutions approach zero as $t \to \infty$.

2. Suppose that (1.2) has no slowly oscillating periodic solution. Our aim is to show $\lim_{t\to\infty} x(t) = 0$ for all solutions of (1.2).

It is known [13] that for $\alpha > \pi/2$ equation (1.2) has a slowly oscillating periodic solution. Therefore, in the remaining part of the proof we may assume $\alpha \leq \pi/2$.

By Proposition 3.2, (1.2) cannot have an orbit which is homoclinic to zero. Using this fact, Proposition 2.2 implies that for any $\phi \in C$ the ω-limit set $\omega(\phi)$ is either $0 \in C$ or a periodic orbit.

In order to complete the proof it suffices to show that (1.2) cannot have nontrivial periodic solutions.

2.1. Let $\psi \in C$ be given by $\psi(\theta) = \alpha$, $\theta \in [-1, 0]$. Consider the solution $x = x^{\psi}$ of (1.2). We claim that $x^{\psi}(t) \to 0$ as $t \to \infty$. Assume the contrary. Then $\omega(\psi) = \{q_t : t \in \mathbb{R}\}$ for some nontrivial periodic solution q which cannot be slowly oscillating by our assumption. Clearly, $V(\psi) = 1$. The monotone property of V implies $V(x_t^{\psi}) = 1$, $t \geq 0$. There are $\eta \in \omega(\psi)$ and a sequence (t_n) with $t_n \to \infty$ such that $x_{t_n}^{\psi} \to \eta$ as $n \to \infty$. Proposition 2.1 (i) implies

$$\liminf_{n\to\infty} V(x_{t_n}^{\psi}) \geq V(\eta).$$

Therefore, $V(\eta) = 1$. The periodicity of q and results of Proposition 2.1 combined yield $V(q_t) = 1$, $q_t \in R$ for all $t \in \mathbb{R}$. It follows that q is a slowly oscillating periodic solution, a contradiction. Consequently, $x^{\psi}(t) \to 0$ as $t \to \infty$.

2.2. Let $p : \mathbb{R} \to \mathbb{R}$ be a nontrivial periodic solution of (1.2). Set $M = \max_{t\in\mathbb{R}} p(t)$. Choose t_0 with $p(t_0) = M$. Then $\dot{p}(t_0) = 0$, and by (1.2), $p(t_0 - 1) = 0$. Therefore

$$M = p(t_0) - p(t_0 - 1) = \int_{t_0-1}^{t_0} \dot{p}(t)\, dt = \int_{t_0-2}^{t_0-1} f_\alpha(p(t))\, dt < \alpha.$$

By the definition of ψ in 2.1 and $M < \alpha$, $V(x_0^{\psi} - p_0) = 1$ follows. The monotone property of V gives $V(x_t^{\psi} - p_t) = 1$ for all $t \geq 0$. If $T > 0$ is the minimal period of p, then by part 2.1

$$x_{nT}^{\psi} - p_{nT} \to -p_0 \quad \text{as } n \to \infty,$$

and Proposition 2.1 (i) implies

$$1 = \liminf_{n\to\infty} V(x_{nT}^{\psi} - p_{nT}) \geq V(-p_0) = V(p_0).$$

From $V(p_0) = 1$, from the periodicity of p, and from the monotone property of V, $V(p_t) = 1$ follows for all $t \in \mathbb{R}$. Consequently, p is a slowly oscillating periodic solution, and it is a contradiction. This completes the proof. ∎

4. Nonexistence of small slowly oscillating periodic solutions. We prove a slightly more general result which was motivated by a paper of Walther [34].

THEOREM 4.1. *Suppose $a > 0$, $b > 0$, $g \in C^1((-a, b), \mathbb{R})$ with $g(0) = 0$ and*

$$0 < g'(\xi) < \frac{\pi}{2} \quad \text{for all } \xi \in (-a, b) \setminus \{0\}. \tag{4.1}$$

Then equation

$$\dot{x}(t) = -g(x(t-1)) \tag{4.2}$$

has no slowly oscillating periodic solution x with $x(\mathbb{R}) \subset (-a, b)$.

Proof. Assume that x is a slowly oscillating periodic solution of (4.2) with $x(\mathbb{R}) \subset (-a, b)$. It is well known (see e.g. the paper of Mallet-Paret and Sell [24]) that the minimal period $T > 0$ of x is given by 3 consecutive zeros of x, and thus $T > 2$. Moreover, if $t_0 \in \mathbb{R}$ with $x(t_0) = \min_{t\in\mathbb{R}} x(t)$, then there exists a unique $t_1 \in (t_0, t_0+T)$ with $x(t_1) = \max_{t\in\mathbb{R}} x(t)$, and $\dot{x}(t) > 0$ for all $t \in (t_0, t_1)$, $\dot{x}(t) < 0$ for all $t \in (t_1, t_0 + T)$. It follows that $(x(t), \dot{x}(t)) \neq (0, 0)$ for all $t \in \mathbb{R}$.

The function $\mathbb{R} \ni t \mapsto k \sin \frac{\pi}{2}(t + t_*) \in \mathbb{R}$ is a solution of

$$\dot{z}(t) = -\frac{\pi}{2} z(t-1) \tag{4.3}$$

for any $k \in \mathbb{R}$ and $t_* \in \mathbb{R}$.

Define the simple closed curves

$$X : [0, T] \ni t \mapsto (x(t), \dot{x}(t)) \in \mathbb{R}^2,$$

and

$$Y_l^\tau : [0, 4] \ni t \mapsto l \left(\sin \frac{\pi}{2}(t + \tau), \frac{\pi}{2} \cos \frac{\pi}{2}(t + \tau) \right) \in \mathbb{R}^2$$

for $l > 0$, $\tau \in \mathbb{R}$. Let $|X|$, $|Y_l^\tau|$ denote the images of X, Y_l^τ, respectively, and $\operatorname{int}(Y_l^\tau)$, $\operatorname{ext}(Y_l^\tau)$ the interior, exterior of Y_l^τ, respectively. Clearly, $|Y_l^\tau| = \{(u, v) \in \mathbb{R}^2 : u^2 + \left(\frac{2}{\pi}\right)^2 v^2 = l^2\}$ is an ellipse, and it is independent of τ.

Fix $\tau \in \mathbb{R}$. There exists a $k > 0$ so that

$$|X| \subset \operatorname{int}(Y_l^\tau) \quad \text{for all } l > k$$

and

$$|X| \cap |Y_k^\tau| \neq \emptyset.$$

Set $z(t) = k \sin \frac{\pi}{2}(t + \tau)$, $t \in \mathbb{R}$, and $Z = Y_k^\tau$. Clearly, $|X| \cap \operatorname{ext}(Z) = \emptyset$.

By the definition of k there are t_0, t_1 in $\mathbb{R}$ with

$$X(t_0) = Z(t_1) \in |X| \cap |Z|. \tag{4.4}$$

Replacing $x(\cdot)$ and $z(\cdot)$ with $x(\cdot + t_0)$ and $z(\cdot + t_1)$, respectively, we may assume $t_0 = t_1 = 0$, that is

$$(x(0), \dot{x}(0)) = (z(0), \dot{z}(0)).$$

Obviously, $(z(0), \dot{z}(0)) \neq (0, 0)$.

Suppose $\dot{x}(0) = \dot{z}(0) = 0$. Then $x(0) = z(0) = c \neq 0$. We consider only the case $c > 0$ as the case $c < 0$ is analogous. Clearly, $c = k$. From equation (4.2), condition (4.1), and $\dot{x}(0) = 0$, one finds $x(-1) = 0$. The monotone property of x and $x(0) = c > 0$, $\dot{x}(0) = 0$ imply

$$c = x(0) = \max_{t \in \mathbb{R}} x(t), \;\; \dot{x}(t) > 0 \quad \text{for all } t \in [-1, 0).$$

By $z(0) = c > 0$ and $\dot{z}(0) = 0$, we have $z(t) = c \cos \frac{\pi}{2} t$, $t \in \mathbb{R}$.

Let $\tau_x : [0, c] \to [-1, 0]$ and $\tau_z : [0, c] \to [-1, 0]$ denote the inverses of $x|_{[-1,0]}$ and $z|_{[-1,0]}$, respectively. The functions

$$\phi_x : [0, c] \ni u \mapsto \dot{x}(\tau_x(u)) \in \mathbb{R}, \;\; \phi_z : [0, c] \ni u \mapsto \dot{z}(\tau_z(u)) \in \mathbb{R}$$

satisfy $\phi_x(c) = \phi_z(c) = 0$, and $\phi_x(u) > 0$ for all $u \in [0, c)$, $\phi_z(u) > 0$ for all $u \in [0, c)$. The arcs

$$\Omega_x = \{X(t) : t \in [-1, 0]\} \;\text{ and } \Omega_z = \{Z(t) : t \in [-1, 0]\}$$

coincide with the graphs

$$\{(u,\phi_x(u)) : u \in [0,c]\} \text{ and } \{(u,\phi_z(u)) : u \in [0,c]\},$$

respectively. From the inclusions $|X| \subset \operatorname{int}(Z) \cup |Z|$ and $\Omega_x \subset |X| \cap \{(u,v) \in \mathbb{R}^2 : v \geq 0\}$, $\Omega_z \subset |Z| \cap \{(u,v) \in \mathbb{R}^2 : v \geq 0\}$ it follows that

$$0 \leq \phi_x(u) \leq \phi_z(u) \quad \text{for all } u \in [0,c].$$

From the definition of ϕ_x and ϕ_z, we obtain

$$\dot{x}(s) = \phi_x(x(s)),\ \ \dot{z}(s) = \phi_z(z(s)) \quad s \in [-1,0].$$

Hence

$$1 = \lim_{\epsilon \to 0+} (1-\epsilon) = \lim_{\epsilon \to 0+} \int_{-1}^{-\epsilon} \frac{\dot{x}(s)}{\phi_x(x(s))}\, ds = \int_0^c \frac{du}{\phi_x(u)}$$

where the last integral is improper. Similarly,

$$1 = \int_0^c \frac{du}{\phi_z(u)}.$$

Then

$$\int_0^c \frac{du}{\phi_x(u)} = \int_0^c \frac{du}{\phi_z(u)}.$$

The last equality and $0 < \phi_x(u) \leq \phi_z(u)$, $0 \leq u < c$, combined imply

$$\phi_x(u) = \phi_z(u) \quad \text{for all } u \in [0,c].$$

As

$$\frac{d}{du}\tau_x(u) = \frac{1}{\dot{x}(\tau_x(u))} = \frac{1}{\phi_x(u)} = \frac{1}{\phi_z(u)} = \frac{d}{du}\tau_z(u)$$

for all $u \in [0,c)$, and $\tau_x(0) = -1 = \tau_z(0)$, we conclude $\tau_x(u) = \tau_z(u)$ for all $u \in [0,c]$, and $x(t) = z(t)$ for all $t \in [-1,0]$.

As a consequence, $\dot{x}(-1) = \dot{z}(-1) = \frac{\pi}{2}c$. From equation (4.2) at $t = -1$, the equality $g(x(-2)) = -\frac{\pi}{2}c$ follows. Hence $x(-2) \in (-a,0)$. By the mean value theorem there is $\xi \in (x(-2),0) \subset (-a,0)$ with $g'(\xi)x(-2) = g(x(-2)) = -\frac{\pi}{2}c$. By (4.1), $x(-2) < -c$ follows, that is, $X(-2) \in \operatorname{ext}(Z)$, a contradiction. Therefore $\dot{x}(0) = \dot{z}(0) \neq 0$.

Then, for sufficiently small $\delta > 0$, x and z have inverses t_x and t_z in $(-\delta,\delta)$, respectively. Define

$$\eta_x : x((-\delta,\delta)) \ni u \mapsto \dot{x}(t_x(u)) \in \mathbb{R},\ \ \eta_z : z((-\delta,\delta)) \ni u \mapsto \dot{z}(t_z(u)) \in \mathbb{R}.$$

Then

$$\eta_x{}'(u) = \frac{\ddot{x}(t_x(u))}{\dot{x}(t_x(u))}, \quad \eta_z{}'(u) = \frac{\ddot{z}(t_z(u))}{\dot{z}(t_z(u))}.$$

In particular at $u = d = x(0) = z(0)$,

$$\eta_x{}'(d) = \frac{\ddot{x}(0)}{\dot{x}(0)}, \quad \eta_z{}'(d) = \frac{\ddot{z}(0)}{\dot{z}(0)}.$$

The smooth arcs

$$\{(u, \eta_x(u)) : u \in x((-\delta, \delta))\} \subset |X|, \quad \{(u, \eta_z(u)) : u \in z((-\delta, \delta))\} \subset |Z|$$

intersect at $u = d$. As $|X| \subset \operatorname{int}(Z) \cup Z$, it follows that $\eta_x{}'(d) = \eta_z{}'(d)$. So

$$\frac{\ddot{x}(0)}{\dot{x}(0)} = \frac{\ddot{z}(0)}{\dot{z}(0)}.$$

From $\dot{x}(0) = \dot{z}(0) \neq 0$, $\ddot{x}(0) = \ddot{z}(0)$ follows. Applying these equalities, from equations (4.2) and (4.3) one gets

$$g(x(-1)) = \frac{\pi}{2} z(-1) \tag{4.5}$$

and

$$g'(x(-1))\dot{x}(-1) = \frac{\pi}{2}\dot{z}(-1). \tag{4.6}$$

We have $x(-1) \neq 0$, $z(-1) \neq 0$, since otherwise $\dot{x}(0) = 0$, $\dot{z}(0) = 0$ from equations (4.2), (4.3), respectively.

Consequently, by (4.1) and (4.5),

$$0 < |z(-1)| < |x(-1)|, \quad x(-1)z(-1) > 0. \tag{4.7}$$

If $\dot{x}(-1) = 0$ or $\dot{z}(-1) = 0$, then by (4.1) and (4.6) we find $\dot{x}(-1) = \dot{z}(-1) = 0$, and (4.7) implies $X(-1) \in \operatorname{ext}(Z)$, a contradiction. Thus, $\dot{x}(-1) \neq 0$, $\dot{z}(-1) \neq 0$.

Then (4.1) and (4.6) combined yield

$$0 < |\dot{z}(-1)| < |\dot{x}(-1)|, \quad \dot{x}(-1)\dot{z}(-1) > 0. \tag{4.8}$$

It is easy to see that (4.7) and (4.8) lead to the contradiction

$$X(-1) \in \operatorname{ext}(Z).$$

This completes the proof. □

We can apply Theorem 4.1 in the case $g(\xi) = \alpha(e^{\xi} - 1)$. If $0 < \alpha < \frac{\pi}{2}$, then for $-\infty < \xi < \log \frac{\pi}{2\alpha}$ we have $g'(\xi) < \frac{\pi}{2}$. So, we obtained

Corollary 4.2. *If $0 < \alpha < \frac{\pi}{2}$ and $p^{\alpha} : \mathbb{R} \to \mathbb{R}$ is a slowly oscillating periodic solution of equation (1.2) then*

$$\max_{t \in \mathbb{R}} p^{\alpha}(t) \geq \log \frac{\pi}{2\alpha} > 1 - \frac{2\alpha}{\pi}.$$

The last inequality in the corollary is a consequence of the elementary inequality $\log \xi > 1 - \frac{1}{\xi}$, $\xi > 1$.

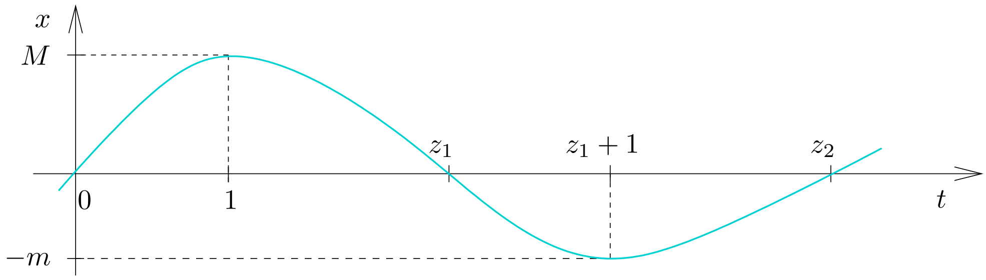

FIG. 5.1. *The typical shape of a slowly oscillating periodic solution.*

5. Construction of bounding functions. In this section we describe a sequence of bounding functions which can be applied to prove that certain slowly oscillating periodic solutions of equation (1.2) cannot exist. In Sections 6 and 7 technical details of the rigorous numerical part of the proof are given.

Define

$$B = \{\phi : [-1, 0] \to \mathbb{R} \mid \phi \text{ is bounded and integrable}\}.$$

If I is an interval, $t \in I$, $t-1 \in I$, and $u : I \to \mathbb{R}$ is a bounded and locally integrable function, then $u_t \in B$ is defined by $u_t(s) = u(t+s)$, $-1 \le s \le 0$. For elements ϕ, ψ in B we write $\phi \le \psi$ provided $\phi(s) \le \psi(s)$ for all $s \in [-1, 0]$.

Suppose that M, m are given positive numbers and $p : \mathbb{R} \to \mathbb{R}$ is a slowly oscillating periodic solution of (1.2) such that

$$\max_{t \in \mathbb{R}} p(t) = M, \quad \min_{t \in \mathbb{R}} p(t) = -m.$$

If we want to emphasize the dependence on α, M, m then we write $p(\alpha, M, m)$. By [24] without loss of generality we may assume that p has the shape as shown in Figure 5.1. That is, there are reals $z_1 > 1$ and $z_2 > z_1 + 1$ such that z_2 is the minimal period of p, and

$$p(0) = 0,\ p(z_1) = 0,\ p(z_2) = 0,\ p(1) = M,\ p(z_1 + 1) = -m,$$

$$p'(t) > 0 \text{ for all } t \in [0, 1) \cup (z_1 + 1, z_2],$$

$$p'(t) < 0 \text{ for all } t \in (1, z_1 + 1).$$

The elements

$$y_+^0,\ y_-^0,\ y_+^1,\ y_+^2,\ y_-^2,\ y_-^3$$

of B are called bounding functions of the periodic solution p if

$$\begin{aligned} y_-^0 \le\ & p_1 \le y_+^0 \\ & p_{z_1} \le y_+^1 \\ y_-^2 \le\ & p_{z_1+1} \le y_+^2 \\ y_-^3 \le\ & p_{z_2}, \end{aligned} \tag{5.1}$$

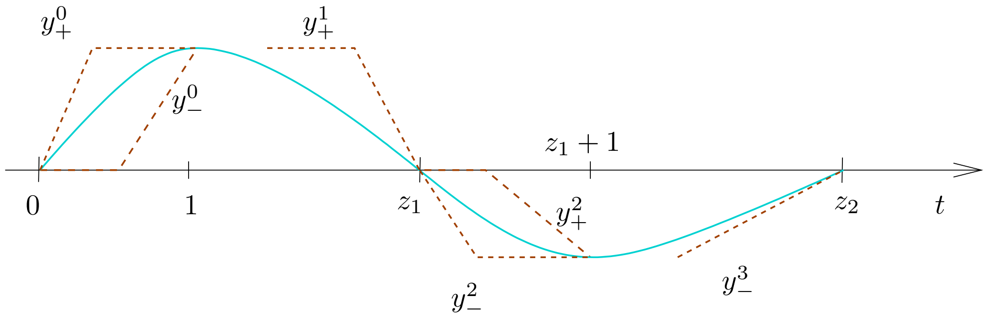

FIG. 5.2. *The bounding functions, denoted by dashed lines, are shifted to the part of the period where they bound p.*

where p_1, p_{z_1}, p_{z_1+1}, and p_{z_2} are the respective shifted parts of the periodic solution p. The roles of these functions are illustrated in Figure 5.2. In Figure 5.2 the bounding functions $y^0_\pm,\ y^1_+,\ y^2_\pm,\ y^3_-$ are shifted to $1,\ z_1,\ z_1+1,\ z_2$, respectively, to show where they bound the corresponding part of the periodic solution p.

The idea is that we construct in an iterative way a finite sequence of bounding functions for p. In each step the bounding functions are improved, i.e., the inequalities (5.1) are sharpened. After each step we check whether

$$y^0_+(0) < M \quad \text{and} \quad y^2_-(0) > -m \tag{5.2}$$

hold or not. If at least one of inequalities (5.2) is satisfied, then we stop the iteration process. In this case the conclusion is that there is no slowly oscillating periodic solution $p(\alpha, M, m)$. If none of the inequalities in (5.2) holds, then we construct the next element of the sequence of bounding functions.

The iteration process goes as follows. Initially we set

$$\begin{aligned} y^0_- &= y^2_+ = 0 \\ y^0_+ &= y^1_+ = M \\ y^2_- &= y^3_- = -m. \end{aligned} \tag{5.3}$$

Suppose that after k steps we obtained the bounding functions $y^0_\pm,\ y^1_+,\ y^2_\pm,\ y^3_-$ in B satisfying (5.1). We describe how to construct the new bounding functions

$$\hat{y}^0_\pm,\ \hat{y}^1_+,\ \hat{y}^2_\pm,\ \hat{y}^3_-.$$

For a $\phi \in C$ with $\phi(0) = 0$ the unique solution $x = x^\phi$ of equation (1.2) satisfies

$$x(t) = \int_0^t f_\alpha(x(u-1))\,du = \int_{-1}^{t-1} f_\alpha(\phi(u))\,du, \quad 0 \le t \le 1,$$

or equivalently

$$x_1(s) = \int_{-1}^{s} f_\alpha(\phi(u))\,du, \quad -1 \le s \le 0.$$

If $\psi \in B$ and $\eta \in B$ with $\psi \le \phi \le \eta$, then the monotone decreasing property of f_α can be used to obtain

$$\int_{-1}^{s} f_\alpha(\eta(u))\,du \le x_1(s) \le \int_{-1}^{s} f_\alpha(\psi(u))\,du, \quad -1 \le s \le 0. \tag{5.4}$$

Choosing $\phi = p_0 = p_{z_2}$ and $\psi = y_-^3$, we have $x(t) = p(t)$, and in this case (5.4) gives

$$p_1(s) \le \int_{-1}^{s} f_\alpha(y_-^3(u))\,du, \quad -1 \le s \le 0.$$

Similarly, if $\phi = p_{z_1}$ and $\eta = y_+^1$, then $x(t) = p(z_1 + t)$ and

$$\int_{-1}^{s} f_\alpha(y_+^1(u))\,du \le p_{z_1+1}(s), \quad -1 \le s \le 0.$$

Then the new bounds $\hat{y}_+^0$ and $\hat{y}_-^2$ are defined by

$$\hat{y}_+^0(s) = \min\left\{y_+^0(s), \int_{-1}^{s} f_\alpha(y_-^3(u))\,du\right\} \tag{5.5}$$

and

$$\hat{y}_-^2(s) = \max\left\{y_-^2(s), \int_{-1}^{s} f_\alpha(y_+^1(u))\,du\right\} \tag{5.6}$$

for each $s \in [-1, 0]$.

For $\phi \in C$ the unique solution $x = x^\phi$ of equation (1.2) satisfies

$$x(1) - x(t) = \int_t^1 \dot{x}(u)\,du = \int_{t-1}^{0} f_\alpha(\phi(u))\,du, \quad 0 \le t \le 1,$$

or equivalently

$$x_1(s) = x(1) - \int_s^0 f_\alpha(\phi(u))\,du, \quad -1 \le s \le 0.$$

If ψ and η are in B with $\psi \le \phi \le \eta$, then by using the monotone increasing property of $-f_\alpha$ we obtain

$$x(1) - \int_s^0 f_\alpha(\psi(u))\,du \le x_1(s) \le x(1) - \int_s^0 f_\alpha(\eta(u))\,du, \quad -1 \le s \le 0. \tag{5.7}$$

Applying inequality (5.7) in the cases $\phi = p_0$, $\psi = y_-^3$ and $\phi = p_{z_1}$, $\eta = y_+^1$, respectively, the new bounds $\hat{y}_-^0$ and $\hat{y}_+^2$ are defined by

$$\hat{y}_-^0(s) = \max\left\{y_-^0(s), M - \int_s^0 f_\alpha(y_-^3(u))\,du\right\} \tag{5.8}$$

and

$$\hat{y}_+^2(s) = \min\left\{y_+^2(s), -m - \int_s^0 f_\alpha(y_+^1(u))\,du\right\} \tag{5.9}$$

for $s \in [-1, 0]$.

The above definitions of $\hat{y}_\pm^0$ and $\hat{y}_\pm^2$ follow the original idea of Wright [36] although Wright constructed only the first two elements of the bounding sequence. The construction of the bounds $\hat{y}_+^1$ and $\hat{y}_-^3$ is slightly more complicated. It seems to be

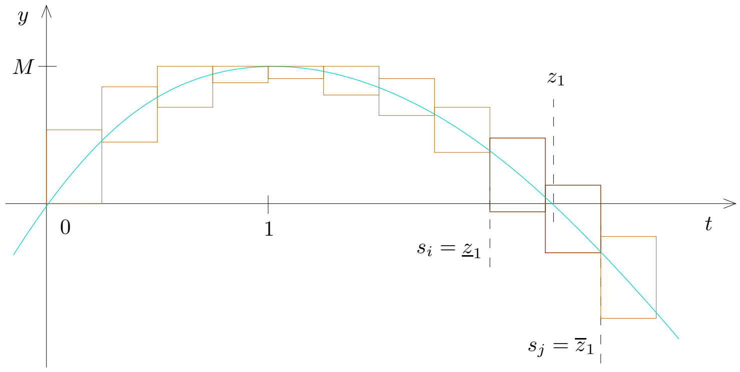

FIG. 5.3. *Illustration of the bounding procedure for the zero z_1 of p.*

new, it does not appear in Wright's paper [36]. The difficulty of the construction of $\hat{y}^1_+$ and $\hat{y}^3_-$ is that the zeros z_1 and z_2 of the periodic solution p are not known. Below we describe the definition of $\hat{y}^1_+$. The function $\hat{y}^3_-$ can be obtained analogously.

Starting from the bounds $y^0_\pm$, and applying a reliable numerical integration method, we get lower and upper bounds $\underline{P}$ and $\overline{P}$ for p. The functions $\underline{P}$ and $\overline{P}$ are step functions as illustrated in Figure 5.3. Let $h > 0$ denote the step size of the numerical integration, and denote $s_k = kh$ the mesh points. Both $\overline{P}$ and $\underline{P}$ are nondecreasing functions on $[0, 1]$. $\overline{P}$ and $\underline{P}$ are nonincreasing at least on the intervals $[1, s_i]$ and $[1, s_j]$, respectively, where i is the smallest positive integer so that $\underline{P}(t) < 0$ for all $t \in (s_i, s_i + h)$, and the positive integer j is the smallest one such that $\overline{P}(t) < 0$ for all $t \in (s_j, s_j + h)$. Setting

$$\underline{z}_1 = s_i, \quad \overline{z}_1 = s_j,$$

obviously $[\underline{z}_1, \overline{z}_1]$ is a verified enclosing interval for the zero z_1 of p. Set $\Delta = \overline{z}_1 - \underline{z}_1$. For $\Delta \le 1$ define the function $q : [-1, \underline{z}_1] \to \mathbb{R}$ (see Figure 5.4) by

$$q(t) = \begin{cases} 0 & \text{if } -1 \le t < -\Delta \\ \overline{P}(t + \Delta) & \text{if } -\Delta \le t < 1 - \Delta \\ M & \text{if } 1 - \Delta \le t < 1 \\ \overline{P}(t) & \text{if } 1 \le t \le \underline{z}_1. \end{cases}$$

For the case $1 < \Delta \le 2$

$$q(t) = \begin{cases} \overline{P}(t + \Delta) & \text{if } -1 \le t < 1 - \Delta \\ M & \text{if } 1 - \Delta \le t < 1 \\ \overline{P}(t) & \text{if } 1 \le t \le \underline{z}_1. \end{cases}$$

For $2 < \Delta$

$$q(t) = \begin{cases} M & \text{if } -1 \le t < 1 \\ \overline{P}(t) & \text{if } 1 \le t \le \underline{z}_1. \end{cases}$$

Clearly,

$$q(t) \ge p(t) \quad \text{for all } t \in [-1, \underline{z}_1].$$

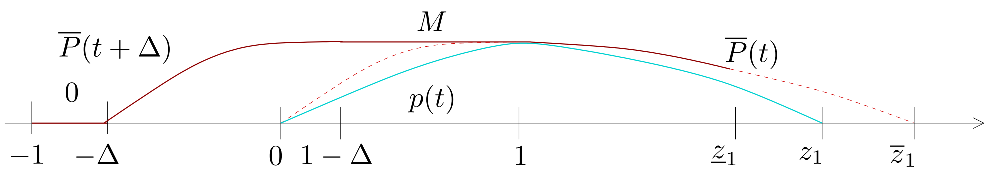

Fig. 5.4. *The definition of q.*

The step function q can be numerically computed, and without knowing the location of the zero z_1 of p function q gives an upper bound for p on the interval $z_1 - 2, z_1$. This is the content of the next statement.

Claim 1.

$$q(\underline{z}_1 + s) \geq p(z_1 + s) \quad \text{for all } s \in [-2, 0].$$

Proof. Assume that $\Delta \leq 1$. Setting $\delta = z_1 - \underline{z}_1$, the claim is equivalent to

$$q(t) \geq p(t+\delta) \quad \text{for all } t \in [\underline{z}_1 - 2, \underline{z}_1]. \tag{5.10}$$

From $z_1 \in [\underline{z}_1, \overline{z}_1]$, $\delta \in [0, \Delta]$ follows.

If $t \in [1, \underline{z}_1]$ then $q(t) = \overline{P}(t) \geq p(t)$. Function p is decreasing on $[1, \underline{z}_1 + \delta] = [1, z_1]$. Therefore $p(t+\delta) \leq p(t)$, $t \in [1, \underline{z}_1]$, and thus

$$q(t) \geq p(t+\delta) \quad \text{for all } t \in [1, \underline{z}_1]$$

follows. Inequality (5.10) clearly holds on $[1-\Delta, 1]$ since $q(t) = M$ for $t \in [1-\Delta, 1]$.

For $t \in [-\Delta, 1-\Delta)$ inequality (5.10) is equivalent to

$$\overline{P}(t+\Delta) \geq p(t+\delta) \quad \text{for all } t \in [-\Delta, 1-\Delta),$$

which is equivalent to

$$\overline{P}(t) \geq p(t - (\Delta - \delta)) \quad \text{for all } t \in [0, 1).$$

The last inequality is obvious since $p(s) < 0$ for $s \in (-\Delta, 0)$, p is increasing on $[0, 1]$, $\Delta \geq \delta$, and thus

$$\overline{P}(t) \geq p(t) \geq p(t - (\Delta - \delta)) \quad \text{for all } t \in [0, 1].$$

If $\underline{z}_1 - 2 < -\Delta$ and $t \in [\underline{z}_1 - 2, -\Delta)$ then $t \in (-1, -\Delta)$, $t + \delta \in (-1, 0)$, and thus $q(t) = 0$ and $p(t+\delta) < 0$.

The cases $\Delta \in (1, 2]$ and $\Delta > 2$ are similar and much simpler. This completes the proof of the Claim. $\square$

For any $s \in [-1, 0]$ one has

$$p(z_1 + s) = p(z_1 + s) - p(z_1) = -\int_{z_1+s}^{z_1} \dot{p}(u)\, du = -\int_{s-1}^{-1} f_\alpha(p(z_1 + u))\, du.$$

Combining the above equality, Claim 1 and the monotone increasing property of $-f_\alpha$, it follows that

$$p(z_1 + s) \leq -\int_{s-1}^{-1} f_\alpha(q(\underline{z}_1 + u))\, du \quad \text{for all } s \in [-1, 0].$$

Now the new bounding function $\hat{y}^1_+ \in B$ can be defined as follows. For all $s \in [-1,0]$,

$$\hat{y}^1_+(s) = \min\left\{ y^1_+(s), q(\underline{z}_1 + s), -\int_{s-1}^{-1} f_\alpha(q(\underline{z}_1 + u))\,du \right\}. \tag{5.11}$$

Analogously, we can construct step functions $\underset{\sim}{P},\ \tilde{P} : [z_1, \infty) \to [-m, M]$ so that $\underset{\sim}{P}(t) \le p(t) \le \tilde{P}(t)$ for all $t \ge z_1$. Two mesh points of the numerical integration yield a verified enclosing interval $[\underline{z}_2, \overline{z}_2]$ for z_2. Similarly to q we define a step function $r : [z_1 - 1, \underline{z}_2] \to \mathbb{R}$ such that $r(t) \le p(t)$ for all $t \in [z_1 - 1, \underline{z}_2]$. Analogously to Claim 1, it is easy to verify that $r(\underline{z}_2 + s) \le p(z_2 + s)$ for all $s \in [-2, 0]$. Then the bounding function $\hat{y}^3_- \in B$ is defined by

$$\hat{y}^3_-(s) = \max\left\{ y^3_-(s), r(\underline{z}_2 + s), -\int_{s-1}^{-1} f_\alpha(r(\underline{z}_2 + u))\,du \right\} \tag{5.12}$$

for $s \in [-1, 0]$.

We remark that the inequalities (1.4), (1.5), (1.6) of Wright [37] can be obtained in a straightforward way by applying two iterations of the above process.

6. Algorithms utilizing the bounding scheme. In this section we describe the algorithms based on the bounding schemes defined in Section 5 so that the algorithms can be directly adopted to computers and are capable to reject intervals of M and m, that is to prove that slowly oscillating solutions cannot exist for those M and m values. The computational part of the proof of Theorem 1.1 will prove the following

THEOREM 6.1. *If $\alpha \in [1.5, 1.5706]$ and $y : \mathbb{R} \to \mathbb{R}$ is a slowly oscillating periodic solution of (1.2), then* $\max_{t\in\mathbb{R}} |y(t)| \le 1 - \frac{2\alpha}{\pi}$.

Now, a combination of Theorem 3.1, Corollary 4.2, and Theorem 6.1 proves Theorem 1.1.

Since general elements of the set B defined in Section 5 cannot be represented on a computer, we reformulate the construction (5.5), (5.6), (5.8), (5.9), (5.11), (5.12) of the sequence of bounding functions for step functions. For a fixed positive integer l divide the intervals $[-1, 0]$ and $[-2, 0]$ into 2^l and 2^{l+1} equal parts, respectively. Let $t_k = -k2^{-l}$ for $k \in \{0, 1, \ldots, 2^{l+1}\}$, and let τ_k denote the interval $[t_k, t_{k-1})$ for $k \in \{2, 3, \ldots, 2^{l+1}\}$, and let $\tau_1 = [t_1, t_0]$. Define the sets S_1 and S_2 as the sets of maps from the sets of intervals $\{\tau_1, \tau_2, \ldots, \tau_{2^l}\}$ and $\{\tau_1, \tau_2, \ldots, \tau_{2^{l+1}}\}$ into $\mathbb{R}$, respectively. Then $Y \in S_1$ can be given by the finite sequence $(Y(\tau_j))_{j=1}^{2^l}$, and similarly for $Y \in S_2$ An element $Y \in S_1$ can be obviously identified with the step function from $[-1, 0]$ into $\mathbb{R}$ taking the value $Y(\tau_k)$ on the interval τ_k, $k \in \{1, 2, \ldots, 2^l\}$. With this identification we may consider S_1 as a subset of B. Replacing y by Y in (5.1), formula (5.1) defines the bounding functions $Y^0_\pm$, Y^1_+, $Y^2_\pm$, Y^3_- as elements of S_1 for the slowly oscillating periodic solution p. The capital letter Y is used to emphasize that these bounding functions are in S_1.

The monotone property of function f_α can be used to show that if the elements $Y^0_\pm$, Y^1_+, $Y^2_\pm$, Y^3_- of S_1 are bounding functions of a slowly oscillating periodic solution $p(\alpha, M, m)$ according to definition (5.1), then the elements $\hat{Y}^0_\pm$, $\hat{Y}^2_\pm \in S_1$ defined analogously to (5.5), (5.6), (5.8) and (5.9) by

$$\hat{Y}^0_+(\tau_i) = \min\left\{ Y^0_+(\tau_i),\ -\alpha \sum_{j=i}^{2^l} \left(e^{Y^3_-(\tau_j)} - 1 \right) / 2^l \right\}, \tag{6.1}$$

Algorithm 1 Determination of the enclosure of the distance of subsequent zeros

Input:	–	s: M ($k=1$) or $-m$ ($k=2$) as extremal values of the periodic solution,
	–	α: the parameter in (1.2),
	–	l resolution parameter, the step size is 2^{-l},
	–	L and U: $Y^0_\pm$ ($k=1$) or $Y^2_\pm$ ($k=2$) lower and upper bound functions.
Output:	–	An enclosure $[\underline{d}, \overline{d}]$ of the distance of subsequent zeros: $z_1 - 0$ ($k=1$) or $z_2 - z_1$ ($k=2$).

Step 1. Compute $Y(T_i)$ $(i = 1, \ldots, 2^l)$ as the enclosures of the periodic solution on subintervals of the unit length time period by using the functions $L(\tau_{2^l+1-i})$ and $U(\tau_{2^l+1-i})$.
Step 2. Set $j = (2^l + 1)$ and $Y_{last} = [s, s]$.
Step 3. Enclose $Y(T_j)$ with the expression $\left(Y_{last} + \left(-\alpha\left(e^{Y(T_{j-2^l})} - 1\right)\right) \cdot [0, 1/2^l]\right)$.
Step 4. Set $Y_{last} = Y_{last} + \left(-\alpha\left(e^{Y(T_{j-2^l})} - 1\right)\right)/2^l$.
Step 5. If $0 \notin Y(T_{j-1})$ and $0 \in Y(T_j)$, then calculate the lower bound for z_i: $\underline{d} = (j-1)/2^l$.
Step 6. If $0 \in Y(T_{j-1})$ and $0 \notin Y(T_j)$, then calculate the upper bound for z_i: $\overline{d} = (j-1)/2^l$ and STOP.
Step 7. Set $j = j + 1$.
Step 8. If $j < 2^{l+2}$, then continue with Step 3, otherwise STOP.

$$\hat{Y}^2_-(\tau_i) = \max\left\{Y^2_-(\tau_i),\ -\alpha\sum_{j=i}^{2^l}\left(e^{Y^1_+(\tau_j)} - 1\right)/2^l\right\}, \tag{6.2}$$

$$\hat{Y}^0_-(\tau_i) = \max\left\{Y^0_-(\tau_i),\ M + \alpha\sum_{j=1}^{i}\left(e^{Y^3_-(\tau_j)} - 1\right)/2^l\right\}, \tag{6.3}$$

$$\hat{Y}^2_+(\tau_i) = \min\left\{Y^2_+(\tau_i),\ -m + \alpha\sum_{j=1}^{i}\left(e^{Y^1_+(\tau_j)} - 1\right)/2^l\right\} \tag{6.4}$$

will be also bounding functions.

On the basis of the bounding functions $Y^0_\pm$ we can get lower and upper bounds $\underline{P}$ and $\overline{P}$ for $p(\alpha, M, m)$ (see Section 5). Then a verified enclosing interval $[\underline{z}_1, \overline{z}_1]$ can be obtained as given in Algorithm 1. By using $\overline{P}$ and $\underline{z}_1$, a step function q is given in Section 5 with the property stated in Claim 1. Define

$$Q : S_2 \ni \tau_k \mapsto q(\underline{z}_1 + t_k) \in \mathbb{R}$$

and

$$R : S_2 \ni \tau_k \mapsto r(\underline{z}_2 + t_k) \in \mathbb{R},$$

where r is the step function defined analogously to q in Section 5. With this definition

Algorithm 2 Check the existence of a periodic solution

Input:
- M and $-m$: the extreme values of the periodic solution,
- α: the parameter of the studied delay differential equation (1.2),
- l resolution parameter, the step size is 2^{-l},
- *cycle*: the maximal number of iterations.

Output:
- a statement whether a periodic solution can exist with the given extreme values.

Step 0. Check the conditions (1.4)-(1.6) and that of Corollary 4.2 for the given m and M values. If any of these is false then the answer is that the given periodic solution does not exists, and STOP.
Step 1. Set $c = 1$ and for all $i = 1, \ldots, 2^l$:
$Y_+^0(\tau_i) = M$, $Y_-^2(\tau_i) = -m$,
$Y_-^0(\tau_i) = 0$, and $Y_+^2(\tau_i) = 0$,
$Y_+^1(\tau_i) = M$, and $Y_-^3(\tau_i) = -m$.
Step 2. Calculate sharper bounding functions for Y_+^0 and Y_-^0 from Y_-^3 by the expressions (6.1) and (6.3).
Step 3. Calculate sharper bounding functions for Y_-^2 and Y_+^2 from Y_+^1 by the expressions (6.2) and (6.4).
Step 4. If $Y_+^0(\tau_1) < M$ or $-m < Y_-^2(\tau_1)$ holds then the answer is that the given periodic solution does not exists, and STOP.
Step 5. Apply Algorithm 1 to calculate the lower and upper bounds of z_1 and a new bounding function for the periodic solution on $[1, z_1]$.
Step 6. Based on the new bounding functions of Step 5 construct the bounding function Q.
Step 7. Apply (6.5) to calculate a sharper bounding function Y_+^1.
Step 8. Apply Algorithm 1 to calculate the lower and upper bounds of $z_2 - z_1$ and a new bounding function for the solution on $[z_1 + 1, z_2]$.
Step 9. Based on the new bounding functions of Step 8 construct the bounding function R.
Step 10. Apply (6.6) to calculate a sharper bounding function Y_-^3.
Step 11. If $c \geq cycle$, then answer that the existence of a periodic solution could not be excluded, and STOP.
Step 12. Set $c = c + 1$, and continue to Step 2.

the step function version of (5.11) and (5.12) is as follows:

$$\hat{Y}_+^1(\tau_i) = \min\left\{ Y_+^1(\tau_i),\ Q(\tau_i),\ \alpha \sum_{j=2^l+1}^{2^l+i} \left(e^{Q(\tau_j)} - 1\right)/2^l \right\}, \tag{6.5}$$

and

$$\hat{Y}_-^3(\tau_i) = \max\left\{ Y_-^3(\tau_i),\ R(\tau_i),\ \alpha \sum_{j=2^l+1}^{2^l+i} \left(e^{R(\tau_j)} - 1\right)/2^l \right\}. \tag{6.6}$$

This completes the description of the iterative procedure to improve bounding functions on the slowly oscillating periodic solutions of equation (1.2). Since $p(1) = M$

and $p(z_1 + 1) = -m$ hold for $p = p(\alpha, M, m)$, the bounding functions Y_+^0 and Y_-^2 satisfy $Y_+^0(\tau_1) \geq M$ and $Y_-^2(\tau_1) \leq -m$ by definition (5.1). If

$$Y_+^0(\tau_1) < M \quad \text{or} \quad Y_-^2(\tau_1) > -m$$

is obtained at a certain step of the iteration, then it is a contradiction. This means that there is no slowly oscillating periodic solution with M and m.

The checking algorithm (Algorithm 2) is also able to decide on these conditions when the M and m values are given as intervals. To exclude such possible intervals of M we apply the above conditions for the upper bounds of the Y, and for the lower bound of the M intervals:

$$Y_+^0(\tau_1) < \underline{M}. \tag{6.7}$$

7. Implementation and verified results. In this section we provide the implementation details that are necessary to obtain the rigorous numerical results, and we discuss the achieved results.

We composed a computer program, a verified numerical algorithm that is able to check reliably whether given values of α, m, and M in intervals allow a slowly oscillating periodic solution. To check condition (6.7) we used an interval version of it in an adaptive branch-and-bound technique (see [7] with its correctness proof). Recall that according to (1.7) it is enough to consider $(m, M) \in [0, 6] \times [0, 6]$. The branch-and-bound procedure generates a subdivision of the interval $[0, 6] \times [0, 6]$ such that for all subintervals either

1. one of the conditions (1.4), (1.5), and (1.6) fails to be satisfied, or
2. $M < 1 - 2\alpha/\pi$, or
3. Algorithm 2 verifies that no slowly oscillating periodic solution exists for the investigated intervals of α, m, and M, or
4. none of the above conditions hold, and the subinterval is small (its size Mm_{eps} is set by the user).

The last possibility is necessary to have the finiteness of the algorithm. Obviously, we have to run the checking algorithm with decreasing Mm_{eps} values to be able to have only subintervals fulfilling conditions 1, 2, or 3, in the above list. The efficiency of our checking algorithm is crucial regarding the computational complexity of the problem. Hence we have applied a sophisticated adaptive framework algorithm to tune the algorithm parameters in such a way that subintervals of the search domain are proven with minimal amount of computation. Illustrations on how well this adaptive subdivision technique works can be found in [7].

To achieve the reliability of numerical calculations necessary for computer aided proofs, we applied interval arithmetic based verified algorithms [1] as also in the solution of other mathematical problems [4, 6, 7, 26]. The computational environment for the computer aided proof was C-XSC [15] and PROFIL/BIAS [16]. These provide support for the interval arithmetic, for the outward rounding, and for the interval versions of the standard functions. The runs were executed on a 2 processor, 4 core SUN Fire V490 workstation and on a 64 core HP ProLiant DL980 Generation 7 computer. The parallelization of the branch-and-bound algorithm was described in the paper [30]. The source code of the algorithm is available at the internet address

`http://www.inf.u-szeged.hu/~csendes/Wright/WrightNM.cpp`

Parameters Mm_{eps}	resolution	proven interval	elapsed time (s)	CPU time (s)	acceleration rate
	1,024	[1.500, 1.557]	428	1,504	3.51
10^{-4}	2,048	[1.500, 1.558]	1,150	4,040	3.51
	4,096	[1.500, 1.558]	2,222	7,817	3.51
	1,024	[1.500, 1.565]	2,755	10,310	3.74
10^{-5}	2,048	[1.500, 1.565]	4,046	15,099	3.73
	4,096	[1.500, 1.565]	7,243	26,730	3.69
	1,024	[1.500, 1.566]	7,009	26,422	3.76
10^{-6}	2,048	[1.500, 1.568]	23,956	90,442	3.77
	4,096	[1.500, 1.568]	34,000	130,156	3.82
	1,024	[1.500, 1.566]	6,972	26,449	3.79
10^{-7}	2,048	[1.500, 1.568]	82,366	319,629	3.88
	4,096	[1.500, 1.568]	34,041	130,233	3.82

TABLE 7.1

Numerical results for $\alpha \in [1.5, 1.568]$. Mm_{eps} is the stopping criterion parameter: when subintervals of this width are reached by the branch-and-bound algorithm, then it is terminated. Resolution gives the number of subintervals per time unit. Elapsed time denotes the length of time interval between the start and the stop of the algorithm, while CPU time provides the total amount of CPU time used by the four processors.

The reliable numerical computations were successful, we were able to prove Theorem 6.1, and with it we completed the proof of Theorem 1.1.

Now we give the implementation details of the computational procedure. The choice of the algorithm parameters is crucial because of the enormous computational complexity of the problem. We started the computations on the SUN Fire V490 workstation, and it worked up to $\alpha = 1.5705$. Beyond this value, the computation time became too long, and hence we changed to a the more powerful HP ProLiant computer. The computation times of the latter computer were converted for the 4 core Sun Fire V490 workstation.

For the parameter interval $\alpha = [1.500, 1.568]$ we summarize the results in Table 7.1. The table also contains preliminary results for some algorithm parameters. The bounding propagation cycle in Algorithm 2 was applied at most $c = 5$ times – four rounds were not always enough. The step size for the bounding step functions was 2^{-10}, 2^{-11}, and 2^{-12}. The minimum interval size (to enclose the values of m and M, denoted by Mm_{eps}) used in the branch-and-bound technique was set to 10^{-4}, 10^{-5}, 10^{-6}, and 10^{-7}. The shortest calculation required about 6.6 hours. The parallelization was successful according to the acceleration rates close to 4. This fact shows that more demanding problems could be solved by similar architecture computers with more cores and threads.

Our numerical results are summarized in Figure 7.1. For the interval $[1.500, 1.542]$ we used 13 seconds CPU time (and 47.01 seconds cumulative time regarding the four cores), we needed 4 iteration cycles, $l = 7$, and $Mm_{eps} = 10^{-5}$ was the stopping criterion parameter in the branch-and-bound algorithm. We remark that for $\alpha \leq 37/24 = 1,541666...$ Wright stated in [36] to have a proof, but did not give the details.

For $\alpha \in [1.500, 1.568]$ the algorithm parameters and the CPU times used are given in the last two rows of Table 7.1. Wright also hoped the extendability of his result to $\alpha < 1.567...$, but he had not completed it due to technical difficulties of the last step.

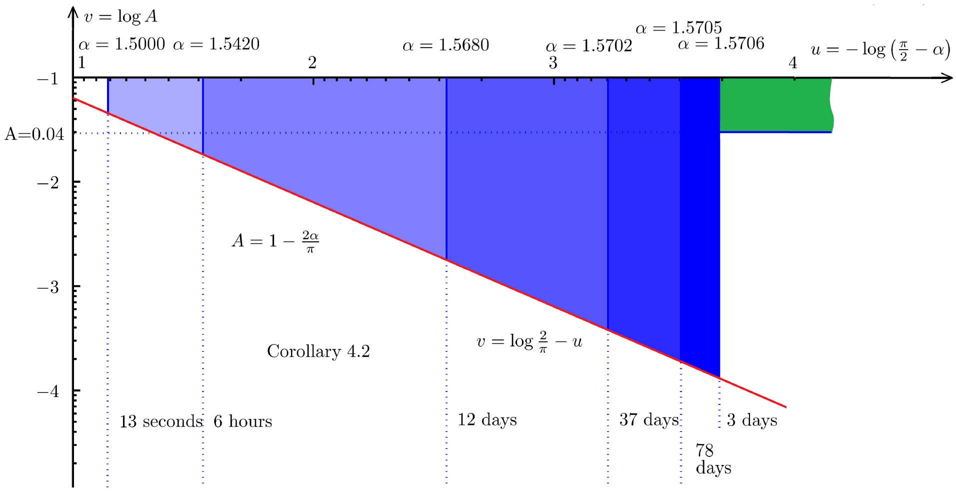

FIG. 7.1. *Illustration of the obtained results on a log-log scale: $u = -log(\frac{\pi}{2} - \alpha)$ is the horizontal axis, $v = logA$ is the vertical axis, $1.5 \le \alpha \le \pi/2$, $A > 0$, 10 is the base of the logarithm. A stands for the amplitude of a possible slowly oscillating periodic solution. Corollary 4.2 guarantees the nonexistence of slowly oscillating periodic orbits below the red line. The computer-assisted proof of Theorem 6.1 gives it above the red line for $\alpha \in [1.5, 1.5706]$. The green region illustrates Theorem 1.2. The required computation times are given for the corresponding intervals of α.*

Theorem 6.1 was also proven for the α intervals $[1.5000, 1.5702]$, $[1.5000, 1.5705]$, and $[1.5000, 1.5706]$ with the same technique, we needed 12 days, 37 days, and 78 days of CPU time, respectively.

The reliable computational technique was successful to prove Theorem 1.2. This is illustrated in Figure 7.1 by the green area in the top right corner. In order to prove Wright's conjecture for $\alpha \in [1.5706, \pi/2]$, obviously a new analytic technique is necessary near the critical $\alpha = \pi/2$. The work [19] under progress is a possible approach. It is expected that also the computational part should be strengthened to fill the gap with the combination of these two tools.

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