Strongly regular graphs with smallest eigenvalue

By

A. Borel
Strongly regular graphs with smallest eigenvalue — IV

By

D. E. Norris

1. Definitions and well-known results. All our graphs are undirected, without loops and multiple edges. A graph \( G \) is strongly regular (or a SRG) if

2. every vertex is adjacent to \( k \) vertices,
3. the number of vertices adjacent to any two adjacent vertices is \( \lambda \).

We denote the number of vertices by \( v \), and assume always the nondegeneracy condition \( 2 \leq k \leq v - 2 \). The vertex set of \( G \) is denoted by \( V \). A counting argument gives

\[ \mu = \frac{v(v - 1) - \lambda v - \lambda - \mu}{2} \]

We call two vertices (first nearest) adjacent if they are distinct and adjacent (non-adjacent). Then the number of vertices in each of the associations to \( \mu \) and \( \lambda \) vertices adjacent to \( v \) and \( k \) vertices adjacent to \( \mu \) and \( \lambda \) vertices adjacent to \( v \) and \( k \); if \( x \) and \( y \) are both associate, then

\[ \mu = \frac{x(x - 1) - \lambda x - \lambda - \mu}{2} \]

\[ \lambda = \frac{y(y - 1) - \lambda y - \lambda - \mu}{2} \]

The complete graph \( G' \) with the same pairs, adjacent if they are distinct and nonadjacent in \( G \) is also strongly regular, with parameters

\[ x = 1, \quad y = v - 1, \quad \mu = v - 2 \delta, \quad \lambda = \frac{v(v - 1)}{2} - \delta \]

The adjacency matrix \( A = (a_{ij}) \) of \( G \) has \( a_{ij} = 1 \) if \( i \neq j \) and \( a_{ij} = 0 \) otherwise. \( A \) satisfies the equation

\[ \mu A^2 = \delta A + \mu I \]

and has the eigenvalues \( \mu, \mu - \delta, -\mu \) with multiplicities \( 1, 1, v - 1 \), where

\[ \mu = \delta(2k + 4 - \lambda) \quad \lambda > 0, \quad \delta = \frac{1}{v + \mu - 2}, \quad \mu = \frac{1}{v + \mu - 2}, \]

\[ \delta = \frac{2v - 1}{2v - 1} \]

* Part of this work was done during the author’s visit to Wadham College, Oxford.
1.1. Lemma. The parameters of a SNQ can be expressed on

\[ a = \mu + \nu + \alpha(m + 1) + 2n(n - 1), \]

\[ b = \mu + \alpha(n - \nu), \]

\[ c = \mu + \alpha(n - 1) + (n - 1)(2n - 2m + 1). \]

where \( m, n, \mu, \nu \) satisfy the restrictions

\[ \mu \geq 0, \nu \geq 0, \]

\[ \mu + \nu \leq \alpha m, \]

\[ \frac{\alpha m (m - 1)}{\mu + \nu} \leq \alpha n. \]

Proof. The expressions for the parameters, and the equations for \( \mu \) and \( \nu \) can be easily verified using (1) and (2), \( 1 \leq m \leq n \) follows from the fact that \( \mu \geq 0 \) and \( \nu \geq 0 \) are non-negative, and the inequalities for \( \mu \) and \( \nu \) come from the fact that the other \( \mu \) and \( \nu \) are non-negative.

A conference graph (or pseudosnake graph) is a graph with \( v = 2(p + 1), e = 2(p - 1), \) and hence \( x = \frac{px}{2p - 2}, y = 1 + \frac{(2b + 1)}{2} \). It is not difficult to prove from (1) and (2).

1.2. Lemma. The parameters \( v, n \) of a SNQ which is not a conference graph are improper.

A straightforward proof yields also

1.3. Lemma.

1. \( v = n \) \( \iff \) \( \mu = 0 \) \( \iff \) \( \Gamma \) is disconnected, if \( \Gamma \) is the union of at least 2 univalent digraph subgraphs.

2. \( \mu = n \) \( \iff \) \( \nu = 0 \) \( \iff \) \( \Gamma \) is a complete subgraph graph with clique of size \( v = n \).
1.6. Lemma. The complement of a SBG with parameters \( n, m, k, \mu \) for parameters
\[ n' = n + 1 - k, \quad m' = n, \quad \mu' = \mu. \]

1.7. Kneser and absolute bound. If \( \Gamma \) is a SBG such that \( \Gamma \) and its complement are connected (i.e., \( 1 < n < \infty \)) then there is another notation for the parameters. Holroyd [9] and Ekedahl [15] use

<table>
<thead>
<tr>
<th>self parameters</th>
<th>ray parameters</th>
</tr>
</thead>
<tbody>
<tr>
<td>( s )</td>
<td>( r )</td>
</tr>
<tr>
<td>( s = m )</td>
<td>( r = n )</td>
</tr>
<tr>
<td>( s = n )</td>
<td>( r = m )</td>
</tr>
<tr>
<td>( s = n - 1 )</td>
<td>( r = n - 1 )</td>
</tr>
<tr>
<td>( s = 1 )</td>
<td>( r = 1 )</td>
</tr>
</tbody>
</table>

Holroyd's paper contains constructions for most of the known SBGs, whereas Ekedahl's paper is a survey of theoretical results on SBGs. In particular, Ekedahl improves two necessary conditions for the parameters of a SBG which he calls the Kneser conditions and the absolute bound.

1.8. Lemma (Kneser conditions). If \( 1 < n < \infty \) then
\[ \mu(n - m(n - 1)) \leq (m - 1)(n - m + m(n - 1)) \]

1.9. Lemma (absolute bound). If \( 1 < n < \infty \) then
\[ r \leq \frac{i}{2} + \frac{1}{2} \]

Equality in this condition holds exactly when geometric properties.

2. The bound for \( \mu \).

2.1. Theorem. For a random SBG with typical random size \( m = -1 < n < \infty \), we have
\[ \mu \geq m + \frac{3}{2}(m - 3) \]
Equality implies \( m = n(1 - 2) < 1 \).

Remark: We call (8) the \( \mu \) bound.

Proof. Fix \( m < 1 < n < \infty \). Then \( \mu \geq 0 \).
\[ \mu = \mu - 0 \leq m + m - 1 + 1 \leq m \]
\[ \mu \leq m + 1 - m \leq 1 \]
\[ \mu \leq m - 2 m + 1 \]

Remark: We call (8) the \( \mu \) bound.

Proof. Fix \( m < 1 < n < \infty \). Then \( \mu \geq 0 \).
\[ \mu = \mu - 0 \leq m + m - 1 + 1 \leq m \]
\[ \mu \leq m - 2 m + 1 \]

Remark: We call (8) the \( \mu \) bound.
This expression is correct in a, hence has its maximum at either \( n = (m - 1) \) or \( n = n \), which implies:

\[ a > \frac{n}{a(n-a)} \]

(4) If \( n \leq m/3 \), then the expression is maximized at \( n = (m - 1) \), which implies:

\[ a > \frac{n}{a(n-a)} \]

(5) If \( n > m/2 \), then the expression is maximized at \( n = n \), which implies:

\[ a > \frac{n}{a(n-a)} \]

(6) If \( n > m/2 \), then the expression is maximized at \( n = n \), which implies:

\[ a > \frac{n}{a(n-a)} \]

(7) If \( n > m/2 \), then the expression is maximized at \( n = n \), which implies:

\[ a > \frac{n}{a(n-a)} \]

(8) If \( n > m/2 \), then the expression is maximized at \( n = n \), which implies:

\[ a > \frac{n}{a(n-a)} \]

(9) If \( n > m/2 \), then the expression is maximized at \( n = n \), which implies:

\[ a > \frac{n}{a(n-a)} \]

(10) If \( n > m/2 \), then the expression is maximized at \( n = n \), which implies:

\[ a > \frac{n}{a(n-a)} \]

(11) If \( n > m/2 \), then the expression is maximized at \( n = n \), which implies:

\[ a > \frac{n}{a(n-a)} \]

(12) If \( n > m/2 \), then the expression is maximized at \( n = n \), which implies:

\[ a > \frac{n}{a(n-a)} \]

(13) If \( n > m/2 \), then the expression is maximized at \( n = n \), which implies:

\[ a > \frac{n}{a(n-a)} \]

(14) If \( n > m/2 \), then the expression is maximized at \( n = n \), which implies:

\[ a > \frac{n}{a(n-a)} \]

(15) If \( n > m/2 \), then the expression is maximized at \( n = n \), which implies:

\[ a > \frac{n}{a(n-a)} \]

(16) If \( n > m/2 \), then the expression is maximized at \( n = n \), which implies:

\[ a > \frac{n}{a(n-a)} \]

(17) If \( n > m/2 \), then the expression is maximized at \( n = n \), which implies:

\[ a > \frac{n}{a(n-a)} \]

(18) If \( n > m/2 \), then the expression is maximized at \( n = n \), which implies:

\[ a > \frac{n}{a(n-a)} \]

(19) If \( n > m/2 \), then the expression is maximized at \( n = n \), which implies:

\[ a > \frac{n}{a(n-a)} \]

(20) If \( n > m/2 \), then the expression is maximized at \( n = n \), which implies:

\[ a > \frac{n}{a(n-a)} \]

(21) If \( n > m/2 \), then the expression is maximized at \( n = n \), which implies:

\[ a > \frac{n}{a(n-a)} \]

(22) If \( n > m/2 \), then the expression is maximized at \( n = n \), which implies:

\[ a > \frac{n}{a(n-a)} \]

(23) If \( n > m/2 \), then the expression is maximized at \( n = n \), which implies:

\[ a > \frac{n}{a(n-a)} \]

(24) If \( n > m/2 \), then the expression is maximized at \( n = n \), which implies:

\[ a > \frac{n}{a(n-a)} \]

(25) If \( n > m/2 \), then the expression is maximized at \( n = n \), which implies:

\[ a > \frac{n}{a(n-a)} \]

(26) If \( n > m/2 \), then the expression is maximized at \( n = n \), which implies:

\[ a > \frac{n}{a(n-a)} \]

(27) If \( n > m/2 \), then the expression is maximized at \( n = n \), which implies:

\[ a > \frac{n}{a(n-a)} \]

(28) If \( n > m/2 \), then the expression is maximized at \( n = n \), which implies:

\[ a > \frac{n}{a(n-a)} \]

(29) If \( n > m/2 \), then the expression is maximized at \( n = n \), which implies:

\[ a > \frac{n}{a(n-a)} \]

(30) If \( n > m/2 \), then the expression is maximized at \( n = n \), which implies:

\[ a > \frac{n}{a(n-a)} \]

(31) If \( n > m/2 \), then the expression is maximized at \( n = n \), which implies:

\[ a > \frac{n}{a(n-a)} \]

(32) If \( n > m/2 \), then the expression is maximized at \( n = n \), which implies:

\[ a > \frac{n}{a(n-a)} \]

(33) If \( n > m/2 \), then the expression is maximized at \( n = n \), which implies:

\[ a > \frac{n}{a(n-a)} \]

(34) If \( n > m/2 \), then the expression is maximized at \( n = n \), which implies:

\[ a > \frac{n}{a(n-a)} \]

(35) If \( n > m/2 \), then the expression is maximized at \( n = n \), which implies:

\[ a > \frac{n}{a(n-a)} \]

(36) If \( n > m/2 \), then the expression is maximized at \( n = n \), which implies:

\[ a > \frac{n}{a(n-a)} \]

(37) If \( n > m/2 \), then the expression is maximized at \( n = n \), which implies:

\[ a > \frac{n}{a(n-a)} \]

(38) If \( n > m/2 \), then the expression is maximized at \( n = n \), which implies:

\[ a > \frac{n}{a(n-a)} \]

(39) If \( n > m/2 \), then the expression is maximized at \( n = n \), which implies:

\[ a > \frac{n}{a(n-a)} \]

(40) If \( n > m/2 \), then the expression is maximized at \( n = n \), which implies:

\[ a > \frac{n}{a(n-a)} \]

(41) If \( n > m/2 \), then the expression is maximized at \( n = n \), which implies:

\[ a > \frac{n}{a(n-a)} \]

(42) If \( n > m/2 \), then the expression is maximized at \( n = n \), which implies:

\[ a > \frac{n}{a(n-a)} \]

(43) If \( n > m/2 \), then the expression is maximized at \( n = n \), which implies:

\[ a > \frac{n}{a(n-a)} \]

(44) If \( n > m/2 \), then the expression is maximized at \( n = n \), which implies:

\[ a > \frac{n}{a(n-a)} \]

(45) If \( n > m/2 \), then the expression is maximized at \( n = n \), which implies:

\[ a > \frac{n}{a(n-a)} \]

(46) If \( n > m/2 \), then the expression is maximized at \( n = n \), which implies:

\[ a > \frac{n}{a(n-a)} \]

(47) If \( n > m/2 \), then the expression is maximized at \( n = n \), which implies:

\[ a > \frac{n}{a(n-a)} \]

(48) If \( n > m/2 \), then the expression is maximized at \( n = n \), which implies:

\[ a > \frac{n}{a(n-a)} \]

(49) If \( n > m/2 \), then the expression is maximized at \( n = n \), which implies:

\[ a > \frac{n}{a(n-a)} \]

(50) If \( n > m/2 \), then the expression is maximized at \( n = n \), which implies:

\[ a > \frac{n}{a(n-a)} \]
The dual of a geometric 1-design is obtained by interchanging the role of points and lines, and is again a geometric 1-design (with \( \cal{B} \) and \( \cal{L} \) interchanged). Two points (lines) are called adjacent if they are on a common line (contains a common point). This defines a graph on the set of points (lines), and we call it the point graph (line graph) of the design. Point graph and line graph are dual concepts.

A set of \( \cal{E} \) is a geometric 1-design on \( \cal{P} \) if the property that any two distinct points are on a unique line. Thus \( \lambda = \frac{r}{K - 1} \), and the number of lines is \( \lambda = \frac{r}{K - 1} \). The line graph of a \( 2-(n \times + m \times + 1) \), m, 1-design with \( n \geq m + 1 \) is called a Steiner graph \( S(G) \). \( S(G) \) is also called a triangular graph \( G(n, 3, 2) \). By Steiner theorem of Wilson [3] and Hanzo [3], if \( n \geq m + 1 \), then \( K \)-design exist for all \( n \geq m'(n) \) with \( K = 1 \). \( K \) is unique for \( n \geq m'(n) \). We have \( n'(n) = n + \lambda \) and \( n' = n \), for \( K = 0 \).

4.1. Lemma. Steiner graphs \( S(G) \) exist for all \( n \geq m'(n) \) with \( n(n + 1) + 1 \), and \( n'(n) = n + \lambda \) for \( n \leq m \).

A \( 2-(m, 1, \lambda) \)-design is a geometric 1-design on \( n \) points which can be partitioned into \( \lambda \) classes of \( m \) points each such that two distinct points are on a line if they are in distinct classes. The line graph of a \( 2-(m, 1, \lambda) \)-design with \( m \geq n + 1 \) is called a Latin square graph \( L(S) \) where \( n \geq m + 1 \). A Latin square graph is equivalent to a complete graph \( G(n, 4) \). By Steiner theorem of D. E. Hanzo [3], if \( n \geq m + 1 \), then \( K \)-design exist for \( n \geq m'(n) \), and \( n' = n - 1 \) for \( n \leq 4 \).

4.2. Lemma. Latin square graphs \( L(S) \) exist for all \( n > n(n) \), and \( n'(n) = n + 1 \) for \( n \leq 4 \).

A partial geometry \( P(G, \cal{E}, \cal{K}) \) is a geometric 1-design with the property that for any nonadjacent points \( p \) and \( q \), there are exactly \( n \geq 1 \) points on \( q \) adjacent to \( p \). The dual of a \( P(G, \cal{E}, \cal{K}) \) is a partial geometry \( P(G, \cal{E}, \cal{K}) \). A partial geometry with \( n = 1 \) is called a projective geometry. The following become natural and straightforward.

4.3. Lemma. For a partial geometry \( P(G, \cal{E}, \cal{K}) \).

4.4. Theorem (Chow [2]). The point graph of a partial geometry \( P(G, \cal{E}, \cal{K}) \) with \( n < K \) is a \( (K, \lambda) \)-graph with

\[\lambda = \begin{cases} 
\lambda = 1 & \text{for } K = 1, \\
\lambda = 0 & \text{for } K > 1.
\end{cases}\]
In particular, a Steiner graph $\gamma_n$ is strongly regular with parameters $m, n = m^2$, and a Latin square graph $L(n)$ is strongly regular with parameters $m, n = m^2 - m + 1$.

By (23) and (24), the point graph of a partial geometry has $m, n = m^2$. Any SRG with $m, n = m^2$ is called a pseudo-resolvability, and geometric if it is the point graph of a partial geometry. If $G(n)$ is called a pseudo Steiner graph $G(n, m)$ if $m = m^2 - m + 1$, and a pseudo Latin square graph $G(n)$ if $m = m^2 - m + 1$.

The former conditions for SRG allow to generalize an inequality of Rawson [5] for pseudo-resolvability:

4.6. Theorem. For a partial geometry $PG(n, K, s)$ with $s < K - 1$,

$$n - 1 \leq (K - s)(K - s + 1)$$

and equality implies $s = 1$ or $K = 2s + 1$.

Proof. The complement of the point graph has parameters $m = K - s - 1, n = K - s - 1$,

$$\rho = \frac{1}{m} (K - s)(K - s + 1).$$

For $K \geq 2s + 1$, we employ the trivial condition which yields

$$\rho \geq 0.$$ 

Since $K - 2s - 1 \geq 2s + 1 - 1 \geq 0,$

$$\rho \leq \frac{1}{m} (K - s)(K - s + 1).$$

4.7. Theorem. A SRG with smallest eigenvalue $\rho$, $\rho > 1$ integral, is geometric if

$$\rho = \cos \pi \theta = \cos \pi \left(\frac{m + 1}{m} x - 1\right)$$

This result has been proved by Rawson [7]. Under the additional assumption that the graph is pseudo geometric (which is essential for his proof), it turns out that inequality (23) can be satisfied only if one of pseudo Latin square graphs and pseudo Steiner graphs.

4.8. Theorem. Let $F$ be a SRG with smallest eigenvalue $\rho$, $\rho > 1$ integral.

(1) If $\rho = m^2 - m$, i.e., $F$ is a $L(n)$, and $m > 1$ then $n = m^2 - m + 2$.

(2) If $\rho = m^2$ then $F$ is a $G(n)$, and $n = m^2 + 1$.

(3) If $\rho = m - 1$, then $F$ is a Steiner graph $\gamma_n$.

(4) If $\rho = m^2 - m$, then $m = m^2 - m + 2$.
Remark. We will consider (2) the base case.
Proof. Suppose first that (2) holds. Then theorem 4.2 implies that there is a
partial geometry with \( R = \eta, R = \eta + n \geq 1 - m, n = \frac{1}{3}. \) Now
(17) 
\[ K = 1 \geq (R - x) R (n - 1) \geq (R - x) \]
store otherwise
\[ a = R + x - 1 \leq 2m - a(m - 1) \leq 0 \leq 2m - 1 \leq (a - 1) + 1 = 2m - 1 \leq m + 1 - a. \]
If \( a + m(m - 1), m \geq n \) then \( R = 1 \) since \( a \) is a integer. For \( R \leq 2m, (1) \) contradicts the dual of (15), and for \( R \geq 3a + 1, (1) \) contradicts the dual of (13). If \( R = m \) then the dual of (12) says \( R = \leq (R - 1), n = \leq m - 1, \)
contradicting (10). States \( n = m - 1, \) or \( m = 0, \) and we get (5) and (6).
Now suppose that (15) and (16) fail. Then
(19) 
\[ (m - 1)(a + 1) = (m - 1)(a + 1) \leq m + 1 - a. \]
where \( 0 < 2m - 1 \leq 1 \leq (a - 1)(a + 1), \) or \( m < 4. \) For \( m = 5 \)
and \( m = 3, \) a somewhat tedious calculation shows that (19) contradicts the absolute bound (this part of the argument is due to Grohe [10]). Hence (10) holds.
5. The characterization theorem. In this section we give a new proof of the follow-
ing theorem by Sivas [6, Ray-Chaudhuri (V)].
6.3. Theorem (Hoff). The SEGs with smallest possible \( m = n, n \geq 2 \) integral,
are the following:
(a) Complete multigraphs with a closure of size \( n. \)
(b) Line square graphs \( L(G_2). \)
(c) Wheel graphs \( K_2, K_n. \)
(d) Finally many other graphs.
Proof. Suppose first that \( n \geq m(m - 1). \) Hence by 6.1, \( n = \leq m(m - 1). \)
we have \( a = \leq m(m - 1) \leq m + 1 - a. \) Hence the dual implies that
\[ a = (n - 1) \leq m + 1. \] But then theorem 4.2 implies that we have case (1)
or (9). Suppose now that \( m \leq n. \) Then \( n = m, \) and by 6.3 (3), we have case (6).
Finally suppose that \( m = n \leq \sqrt{n}. \) By 6.1, \( 1 \leq \sqrt{n}(m - 2) \) whenever
there are only finitely many possible triplets \( e, m, n \) with the given \( n. \) For each
of these there are only finitely many graphs, so that (6) holds.
6.3. Theorem. The SEGs not named in theorem 6.1 are the following:
(a) Complete graphs.
(b) The union of a complete bipartite graph.
Proof. By 1.2 and 1.3, we look more closely at the six possibilities:

1. 
The graphs have $n = m, \mu = m - 1$, and exist for all $m \geq 2$.
2. 
The graphs have $n = m, \mu = m - 2$, and exist for all sufficiently large $n$ by Chaves, Ehrlich, and Fawrie [4].
3. 
The graphs have $\mu = m$. By 1.2, they can exist only for $n = m + 1$. By Watan [10], they exist for all sufficiently large $n$ with $n = m + 1$.
4. 
The graphs fail into three classes:
   a. 
      Pandio Latin square graph $L_{(m)}(n)$ which are not Latin square graphs. They have $\mu = n(n - 1)$, and by theorem 4.1, $n \geq (m - 1)(m - 2) + m - 2n = 2n - 2m = 2m - 2n$.
   b. 
      Pandio hypercube graph $L_{(m)}(n)$ which are not hypercube graphs. They have $\mu = m$, and by theorem 5.7, and 1.1 [9], $m(n(n - 1))(n - 2) + m = 2n$.
   c. 
      Graphs with exceptional parameter sets, see below.

The graphs have $\mu = 2, n = 0, \mu = m, \mu = m - 1$, and exist for many values of $\mu$ to be a sum of two integer squares (see e.g., [5]).

The graphs have $n = 1, \mu = 0$, and exist for all positive integers $n$ and $m$.

A parameter set $\mathbf{P} = \{n, \mu, \rho, j, k, l, \nu\}$. Integers satisfying the conditions (4) and (5) of section 1. A parameter set $\mathbf{P}$ is admissible if it satisfies the vertex condition (lemma 2.1), the absolute bound (lemma 2.2), the $\mu$-bound (theorem 2.3), and the size bound (theorem 2.4) (2). An admissible parameter set $\mathbf{P}$ with $\mu = m, \rho = m, \nu = m - 1$ is called exceptional. By theorem 2.1 (b), there are only finitely many exceptional parameter sets for each integer $m \geq 2$.

In table 1 we state the three exceptional parameter sets with $n = 2$.

<table>
<thead>
<tr>
<th>No.</th>
<th>$n$</th>
<th>$\mu$</th>
<th>$j$</th>
<th>$k$</th>
<th>$l$</th>
<th>$\nu$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>3</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>4</td>
</tr>
</tbody>
</table>

The uniqueness of the three exceptional graphs has been shown by Schott [11]. Brittke [12] showed that there is a unique graph with $m = 2$ in class [12].

Seymour (to be continued, to be published).
References


A. Neumann
Technische Universität Berlin
D-100 18 Berlin
5600 Berlin

Academy of Sciences
A. Neumann
Technische Universität Berlin
D-100 18 Berlin
5600 Berlin