Distances, Graphs and Designs

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1. INTRODUCTION

This paper outlines the treatment of certain problems dealing with intersection matrices of $t$-designs, strongly regular graphs, finite metric spaces, low-dimensional sets on the Euclidean sphere, and $t$-designs in $Q$-polynomial association schemes.

Central to our theory is the concept of a distance matrix. Distance matrices are real, symmetric matrices closely related to finite metric spaces. We classify distance matrices according to two parameters, the degree $s$ and the strength $t$. The degree is the number of distinct off-diagonal entries, whereas the strength measures the linear regularity of the matrix. A distance matrix without repeated rows has box strength $t$ for all $t > 0$ is called a $D$-matrix.

$D_0$-matrices are $D_0$-matrices. For every $t$-design, or transversal $t$-design if there is an associated distance matrix $C$ of $t$-strength which is closely related to the intersection matrix of $B$, the degree of $C$ is the number of distinct intersection numbers of $B$. Using this, we are able to derive insights by Majumdar and others for the intersection numbers of $t$-designs, by Bannai and Ito, and by Cameron for $t$-designs with few intersection numbers.

Finally, spherical $t$-designs introduced by Delange, Gyllings, and Seidel [3], with the spherical metric, give rise to distance matrices of strength $t$ which explains the similarity of the theory in [1] and [5].

2. DISTANCE MATRICES

Let $X$ be a $n$-set. Since we use $X$ in labeling set for the rows (and columns) of symmetric $w$-matrices we call the elements of $X$ nodes.

A distance matrix (or $X$ is a non-zero real symmetric $w$-matrix $C = (c_{ij})_{i,j \in X}$ with non-negative entries, $c_{ii} = 0$, and zero diagonal, $c_{ii} = 0$, such that the distance function $d(x,y) = \sqrt{2s}$ satisfies the triangle inequality for all $x, y, z \in X$.

\[ d(x,z) = d(x,y) + d(y,z), \quad \text{for all } x, y, z \in X. \]  

(2.1)

If $c_{vv} = 0$ then (2.1) implies $c_{uv} = c_{vu}$ for all $u, v \in X$ and the converse holds since $c_{vv} = 0$. In particular, $C$ has no repeated rows $\forall c_{uv} = c_{vu}$ for all $u, v \in X$. In this case, $d(x,y)$ makes $X$ into a metric space. Conversely, if $X$ is a finite metric space with metric $d(x,y)$ then the matrix $C = (d(x,y))_{x,y \in X}$ is a distance matrix without repeated rows.

We say that two distance matrices $C = (c_{ij})_{i,j \in X}$ and $C' = (c'_{ij})_{i,j \in X'}$ are isomorphic if there is a bijection $\pi: X \to X'$ and a positive number $\lambda$ such that $c_{ij} = \lambda^t c'_{i\pi(j)}$ for all $i, j \in X$. Clearly, isomorphism is an equivalence relation.

We denote the identity of the set $w = \{x \in X : d(x,x) = 1\}$ and the all-one matrix of size $n \times n$ by $\mathbb{I}_n$. We call any matrix isomorphic to $\mathbb{I}_n$ an $I$-matrix. If $X$ is a low-dimensional set on the Euclidean sphere, the all-one matrix of size $n \times n$ by $J_n$ and the all-one matrix of size $n \times n$ by $\mathbb{I}_n$. If there is no doubt we simply write $I$ and $J$. We call any matrix isomorphic to $\mathbb{I}_n \otimes \mathbb{I}_m$ (where $\otimes$ denotes the Kronecker product $\times$ with multiple of $1$).
A symmetric matrix $C$ is a submatrix of a matrix without repeated rows if every row of $C$ is repeated exactly $r$ times.

A distance matrix $d$ is the off-diagonal entries are the same localized initial; all initial distance matrices are isometric to $J$.

We say that $x, y \in X$ is an antipodal pair of rows of $C$ if $d(x, y) = d$, for all $x \in X$. It is easy to see that the number $d(y, x)$ is independent of the antipodal pair, and that $x, y$ is an antipodal pair if and only if $d(y, x) = 0$. In particular, if $C$ has no repeated rows then every row $x$ has at most one antipodal row $x'$ such that $x, x'$ is an antipodal pair.

We say that $C$ is antipodal if it has no repeated rows, and every row has an antipodal pair. In this case we may split $X$ into two sets $Z, Y$ such that $Z$ contains no antipodal pairs of rows. Then the matrix $D = d - C$ has diagonal entries $d$, and no repeated rows, and $C$ can be written as

$$C = (d - D) \oplus (d + D),$$

(2.2)

Conversely, every such matrix is antipodal.

If $Y$ is a subset of $X$, and $Z = (X \setminus Y) \setminus Y$, then we can split $X$ into $P = (Y - Y) \cup P_0$, $Z = (Z - Z) \cup Z_0$; the corresponding matrix $B = d - C|_{Y_1, Y_2}$ is obtained from $D$ by multiplying the rows and columns corresponding to $Z_0$ by $-1$. This operation is well-known under the name of switching (Heldt [13]). Switching with respect to arbitrary subsets is an equivalence relation; and any two switching equivalent matrices $D, B$ give rise to isometric matrices $C$ via (2.2).

2. Lemmas. Let $C = (c_{xy})$ be a non-zero real symmetric matrix (on $X$) with zero diagonal. If, for some $y \in Y$, $\sum c_{xy}^2$ $= 0$, then $C$ is positive semi-definite. If $C$ is a distance matrix. $C$ is positive semi-definite then $C$ is a distance matrix.

Moreover, $c_{xy} = 0$ for all $x, y \in X$, and $c_{xy} = 2y$ implies that $x, y$ is an antipodal pair.

Proof. The principal submatrices of dimension 2 and 3 of $d - C$ are the matrices

$$P_0 = \begin{pmatrix} d & c_{xy} \\ c_{xy} & d \end{pmatrix}, \quad P_0 = \begin{pmatrix} d & c_{xy} \\ c_{xy} & d \end{pmatrix}, \quad P_0 = \begin{pmatrix} d & c_{xy} \\ c_{xy} & d \end{pmatrix}, \quad P_0 = \begin{pmatrix} d & c_{xy} \\ c_{xy} & d \end{pmatrix}, \quad P_0 = \begin{pmatrix} d & c_{xy} \\ c_{xy} & d \end{pmatrix}.

$$

Hence $\sum c_{xy}^2 = 2c_{xy}^2$, whereas $\sum c_{xy}^2 = 2d$, and $\sum c_{xy}^2 = 2P_0$, whence $\sum c_{xy}^2 = 2P_0$, which can easily be transformed into the triangle inequality for $d(x, y) = c_{xy}$. If $c_{xy} = 2y$, then $d_{xy} = c_{xy} - 2y$, which is non-negative only if $d_{xy} = c_{xy} = 2y$.

Hence $x, y$ is an antipodal pair.

We now present the examples which relate combinatorial and geometric structures to distance matrices.

A design (or incidence structure) is a triple $(P, B, I)$ (where written: $I$) consisting of a set $P$ of points, a set $B$ of blocks, and a relation $I \subseteq P \times B$ called incidence. We write $x \in B$ if $x$ is incident with $B$. Then, as $B, B'$ is a pair of repeated blocks if $B \neq B'$ and $B$ and $B'$ are incident with exactly the same points. The block size is the number of points incident with $B$.

The incidence matrix of a design $(P, B, I)$ is the $b$-matrix $A$, whose rows are labelled by the points, whose columns are labelled by the blocks, and whose entry in cell $(p, B)$ is $k_{pb}$ if $p \in B$ and $I_{pb} = 1$ or 0 according as $p$ and $B$ are incident or not. The matrix $A$ is called the incidence matrix.
of B, and mention in call \((B, B')\) the number of pairs involved with \(B\) and \(B'\). Hence the diagonal elements count just the block sizes of \(B\). The off-diagonal elements are called the intersection numbers of \(B\). From Lemma 2.1 we derive the following lemma.

**Lemma 2.2.** Let \(A\) be the incidence matrix of a design \(B\) with crossings block size \(k\). Then \(C = A^T - A\) is a distance matrix, or \(C = f(r)\) (We call \(C\) the distance matrix of \(B\)).

A graph \((G, E)\) (where \(G\) is a vertex set \(V\) with an edge set \(E\)) consists of a finite set of vertices \(V\) (also known as nodes, or points, or graph points) and a set \(E\) of unordered pairs of points, called edges. The adjacency matrix of a graph \(G\) is the symmetric matrix \(M\) whose rows and columns are labelled by the vertices, and whose entry in cell \((i, j)\) is \(1\) if \((i, j)\) is an edge, and \(0\) otherwise (some authors use other types of adjacency matrices). The eigenvalues of a graph are eigenvalues of its adjacency matrix.

Again from Lemma 2.1, we have the following lemma.

**Lemma 2.3.** Let \(M\) be the adjacency matrix of a graph \(G\) with smallest eigenvalue \(-\lambda\). Then \(C = \det(J - I)\) is a distance matrix, or \(C = 0\) (We call \(C\) the distance matrix of \(G\)).

In the \(d\)-dimensional real vector space \(R^d\), we define the standard inner product \((x, y) = x_1 y_1 + \cdots + x_d y_d\). The set \(S\) is the set of points \(x \in \mathbb{R}^d\) such that \((x, x) = 1\). In other words, \(S\) is the unit sphere. The Gram matrix of \(X\) is the matrix \(G_X = (x_i, x_j)\). It is well-known that this matrix is positive semi-definite. Hence we obtain the following lemma from Lemma 2.1.

**Lemma 2.4.** Let \(X\) be a finite set of points on the unit sphere in a finite-dimensional real vector space. Then \(C = J - G_X\) is a distance matrix with repeated zeros (We call \(C\) the distance matrix of \(X\)).

We say that the metric corresponding to \(C\) is (up to a scale factor) the metric induced on \(X\) by the euclidean metric.

Let \(X\) be a finite set of points. An (\(r\)-current) scheme on \(X\) is a partition of the set \(\binom{X}{r}\) of all \(r\)-subsets of \(X\) into \(\geq 2\) non-empty classes. Two points, \(a, b\), are called associates if \(a \neq b\) and \((a, b)\) is in the class with label \(0\). We define \(A_r\) as the number of such associates of \(a\) and \(B_r(b, a)\) as the number of such associates of \(b\). If we write \(D_r = (D_{r,j})\) with \(D_{r,j} = 1\) according to \(r\) and \(j\) are such associates or not, then

\[
J = f = \sum_j D_{r,j}
\]

\[
D_{r,j} = 1_{A_r(a, b)}
\]

\[
D_{r,0} = 1_{A_r(a, b)}
\]

\[
\rho = \sum_{x \in X} \rho(x, x)
\]

where \(\rho = 1\) if \(x = x\), and \(\rho = 1\) in the Kronecker symbol. The sum \(\sum_j D_{r,j}\) is the Kronecker symbol.

A regular scheme is a scheme with \(A_r(a, b) = 1\), for all points \(a, b\), and no associates scheme is a regular scheme with \(A_r(a, b) = 1\), for all points \(a, b\), and no associates scheme is a regular scheme with \(A_r(a, b) = 1\), for all points \(a, b\), and no associates scheme is a regular scheme with \(A_r(a, b) = 1\). For the case, we define some notions of distance matrices.

Let \(C\) be a distance matrix on a set \(X\). We call \(S = \{x : x \neq y\}\) the set of distance numbers, and \(s = |x|\) the degree of \(C\). In particular, \(C\) is trivial if \(s = 1\).
A. Namore

If \( C \) has no repeated rows then \( \tilde{C} \), and we define the annihilator polynomial of \( C \) to be the polynomial

\[
A_{\tilde{C}}(\alpha) = \prod_{i=1}^{\alpha} \left( 1 - \alpha \right)
\]

Then \( A_{\tilde{C}}(0) = \alpha \), and the distance numbers are just the roots of \( A_{\tilde{C}}(\alpha) \).

The distribution scheme of \( C \) is the \( x \)-the distribution scheme on \( X \) defined by using \( x, y \)-th associated if \( x \neq y \) and \( x, y \neq \infty \). The corresponding \( \tilde{G} \)-matrices \( D_\alpha \) are called the distribution matrices of \( C \), and we can express \( \alpha \) as

\[
C = \sum_{\alpha} D_\alpha
\]

(2.4)

3. STRENGTH. DELAUNAY MATRICES

If \( A = (a_{ij}) \) is a matrix, \( f \) is a non-negative integer, and \( f^{(n)} \) is a neighborhood function, then we define the matrices \( A^{f^{(n)}} = (b_{ij}) \), \( A = (b_{ij}) \), and \( f^{(n)} \). Here \( f^{(n)} = f \), so that \( A^{f^{(n)}} = A^{f^{(n)} - A} \).

Let \( f \) be a non-negative integer. We say that a distance matrix \( C \) has strength \( f \) if for all non-negative integers \( i, j \) with \( i \neq j \) there is a polynomial \( f_{i,j}(\alpha) \) of degree \( \leq \min(i, j) \) such that

\[
f_{i,j}(\alpha) = f_{i,j}(C), \quad i, j \neq \infty
\]

(3.1)

Isomorphic distance matrices have the same strength. If \( C \) has strength \( f \) then \( C \) has strength \( f \) for all \( f \neq \infty \). Besides the degree, the maximum strength is the most important characteristic of a distance matrix.

A Delaunay matrix is a distance matrix with no repeated rows which has strength \( f \) for all non-negative integers \( f \). It is easy to see that the trivial distance matrices are Delaunay matrices. Examples of non-trivial Delaunay matrices arise, e.g., from certain \( t \)-designs, strongly regular graphs, and spherical codes.

Immediately from the definitions, we have the following lemma.

**Lemma 3.3.**

(i) Every distance matrix has strength 0.

(ii) A distance matrix has unique row sums iff it has strength 1.

Distance matrices of strength 2 can be characterized by algebraic equations.

**Theorem 3.2.** A non-zero square \( n \times n \) matrix \( C \) with zero diagonal satisfies the equations

\[
C_{ij} = 0, \quad C_{ij}^{1} + C_{ij}^{2} = \alpha_{ij} + \delta_{ij},
\]

(2.2)

for some positive real numbers \( \alpha_{ij} \) if \( C \) is a distance matrix of strength 2. In this case,

(i) \( f(C) = \alpha \) is an integer.

(ii) \( f(C) \) is a non-negative integer.

(iii) If \( C \) is a distance matrix of strength 2, then \( \alpha = n \), \( \alpha \leq n \), and \( \delta_{ij} = 0 \).

Remark. We call \( C \) the complement of \( C \). The complement of the complement of \( C \) is again \( C \).

**Proof.** If \( C \) is a distance matrix of strength 2 then (2.1) for \( i, j \neq 0, \infty, i, j \neq 1 \) imply the existence of numbers \( a, b, c \) with \( C_{ij} = a, C_{ij}^{1} = a \), \( C_{ij}^{2} = -a \), and multiplication of the second equation by \( 2 \) gives \( e = a - e \). Hence (2.2) holds. Moreover, \( a \neq 0 \).
positive since $C$ has only non-negative entries, and $C > 0$. Conversely, suppose that $C$ is a real symmetric, non-negative matrix with two diagonal entries $a$ and $b$, suppose that $a$ is an eigenvalue of $D$ and $b$ is an eigenvalue of $F$. Then $a + b = 2$, and $0 < a < b$, where $a > 0$, and $b > 0$. $C$ is positive.

In particular, $D$ is positive semi-definite, and by Lemma 2.1, $C$ is a distance matrix. Now, for $i 
eq j$, if $i$ follows directly from (3.2), and (3.3) for $i = j$, we have $C_{ij} > 0$, and $C$ is a s-distance matrix. $C$ follows from looking at the diagonal entries of $C = w (D - I)$. This proves the equivalence, and (ii).

Now it is easy to show that $C = (D - I) - C$ satisfies equations like (3.2) with the stated parameters.

**Lemma 3.3.** If $C$ is a distance matrix of order $n$, with the notation of Theorem 3.2,

(i) $0 < c_{ij} = c_{ji}$ for all $i \neq j$.

(ii) $c_{ij} > 0$ for all $i \neq j$.

(iii) $D_{jj} = w - 2a - 2b < 2w$ for all $j$, with $a \neq b$.

(iv) $c_{ij} = 2a - w (1 - 2a)$ for $i \neq j$, $i$ is an isolated point, and

(v) $c_{ii} = 2 - w (1 - 2a)$ for $i, j, z$ is an isolated point.

**Proof.** From (3.2), we find

\[ \sum_i c_{ij} = a, \]  
\[ \sum_i c_{ij} = b, \]  
\[ \sum_i c_{ij} = c_{ij}, \]  
\[ \sum_i c_{ij} = 3a - 2b, \]  
\[ \sum_i c_{ij} = 3a - 2b. \]

Hence

\[ 2a \sum_i c_{ij} = \left( a - c_{ii} \right)^2 = 2a c_{ii}, \]

which gives (i) and (ii). Also, for $a \neq b$,

\[ \sum_i \left( c_{ij} - \frac{c_{ij}}{w} \right)^2 = \sum_i \left( c_{ij} - \frac{2a - \frac{c_{ij}}{w}}{w} \right)^2. \]

This gives (iii). Finally,

\[ \sum_i \left( c_{ij} - \frac{c_{ij}}{w} \right)^2 = \sum_i \left( c_{ij} - \frac{2a - \frac{c_{ij}}{w}}{w} \right)^2, \]

which gives (iv) since by (iii), $c_{ij} > 0$. (v) is the complement of (iii).

The next two theorems give sufficient conditions for a distance matrix to be a Euclidean matrix. We also obtain some information on the distribution scheme.
Theorem 3.4. Let C be a distance matrix of degree s and length r.

(i) If r = s = 1, and C has no repeated rows, then C is a multiple of a distance matrix of degree 1 and length 1, which has no repeated rows.

(ii) If r > 2s - 2, then the distribution scheme is an association scheme, and C is either a distance matrix of degree s or a multiple of a distance matrix of degree r - 1.

Proof. First we remark that $C^{(i)}$ can be expressed by the distribution matrices as

$$C^{(i)} = I_s - \sum_{j=0}^{r-1} A_{ij} + A_{ii}.$$  \hspace{1cm} (3.4)

(i) For i = 1, we have $C^{(1)} = f_1C = \sum_{j=0}^{r-1} f_{1j}A_{ij} + A_{ii}$, wherein, for some constant $c_1$

$$\sum_{j=0}^{r-1} f_{1j} = 1$$

for i = 0, 1, ..., r - 1.

This system of equations for $f_{1j}(t)$ has a Vandermonde determinant, hence a unique solution; therefore, $C^{(1)}$ is independent of $C$, nor the distribution scheme is regular.

(ii) If C has repeated rows then it is a distance matrix. (ii) implies that $f_{1j}(t) = \delta_j(t) \delta_j(t)$ for all $j$, hence every row is repeated exactly $m = \lambda_0 - 1$ times. Therefore, C is a regular multiple of another matrix $C_0$ without repeated rows. As a principal submatrix of $C$, this is a distance matrix of degree $s - 1$ which is a distance matrix of degree $s - 1$ (since 0 is not a distance number of $C_0$, and from $f_1(C_0) = f_1(C_0)$, $C_0^{(s-1)}$). The distribution scheme is an association scheme.

Theorem 3.5. Let C be a distance matrix, and suppose that C and its complement have non-empty 3. Then either C is a distance matrix of degree 2, or a multiple of a trivial matrix.

Proof. Under the hypothesis, $C^{(i)} = \nu C^{(i)}$ for some real numbers $\nu, \lambda$, and we find

$$\sum_{i=0}^{r-1} \nu = 1.$$  \hspace{1cm} (3.5)

Similarly, we obtain from the complement,

$$\sum_{i=0}^{r-1} \nu = \nu \lambda_1.$$  \hspace{1cm} (3.6)
for appropriate $x, y$. If we add (3.5) and (3.6), then straightforward calculations, using
Equations (3.3a), (3.3b), show that, for $x \neq y$, $\gamma(x)$ satisfies a quadratic equation with
coefficients independent of $x$, and $\gamma(x) = x$ can take at most two values, and $C$
the degree $x \leq 2$. Now the result follows from Theorem 3.4 (iv).
Another consequence of Theorem 3.4 is the following lemma.

Lemma 3.6. Let $C$ be a distance matrix of degree 1, and strength $x > \max(2, x - 1)$. If $C$
has no repeated rows and contains an antipodal pair then $C$ is antipodal.

Proof. By Lemma 3.3 (iv), $x = 2x/2$ is a distance number, and by 3.4 (iii), $\nu(x) = x - 1$ for all $x$. Hence, again by 3.3 (iv), every point has an antipodal mate, whence $C$
is antipodal.

4. The Distribution Algebra

Let $C$ be a distance matrix of degree 1 without repeated rows, and $S$ be the set of
distance numbers of $C$. From now on we shall make use of the conventions
$S = S\setminus\{0\}$, $D(C) : I$.

(4.1)

Using the Hadamard product $(\alpha_1 \cdot \alpha_2)_{ij} = \alpha_1(i) \alpha_2(j)$ for matrices, we have for the dis-
tribution matrices

$D(C)D(C) = S\otimes D(C)$ for $\alpha, \beta \in S$.

(4.2)

Hence the (real) vector space $V$ generated by the $D(C)$, $\alpha \in S$, is an algebra of dimension
$s = 1$ over the Hadamard product. We call $V$ the distribution algebra of $C$. The
distribution algebra reflects many properties of $C$. For example, we have the following theorem.

Theorem 4.1. The distribution algebra of $C$ is a subalgebra of $S$.

Proof. The distribution algebra is closed under matrix multiplication if $D(C)D(C) = S\otimes D(C)$ for all $\alpha, \beta \in S$. By (4.2) and (4.3), this is equivalent to the fact that the
distribution algebra is a subalgebra of $S$.

We now construct a special basis $E_0, \ldots, E_s$ for the distribution algebra.

Lemma 4.2. The distribution algebra $V$ of a distance matrix $C$ contains a complex.

$V_0 \subseteq V_1 \subseteq \cdots \subseteq V_s = V$

(4.3)

of vector spaces $V_i$ of dimension $i + 1$ defined by

$V_i = \{x \mid x = \text{polynomial of degree } i\}$.

(4.4)

There are unique matrices $E_0, \ldots, E_s$ satisfying

$V_i = \mathbb{C}E_0 \cdots E_i$,

(4.5)

and the usual orthogonality relations

$E_i E_j = \delta_{ij}$, $i, k \in \{0, \ldots, s\}$.

(4.6)
Theorem 4.3. C has strength 1 if

\[ E_{k+1} = E_k (E_{k+1})^T \quad \text{for} \quad k = 0, 1, \ldots, n. \]  

Proof. (4.7) implies that for \( i, k \leq n \), \( C_i C^T \in \mathbb{R}^{m \times m} \) where \( E_{k+1} \) has strong 1. Conversely, by the above remark, (4.7) holds for \( r = 0 \). Hence, assume by induction that (4.7) holds for \( r = k \) - 1.Define \( E_{k+1} = (E_{k+1})^T \mathcal{E}_{E_{k+1}} \) for certain numbers \( a_i \). If \( i < k \), then \( a_i = a_i - a_{i+1} \) for all \( a_i \). Hence, the eigenvalues of \( E_{k+1} \) are \( a_1, \ldots, a_{k+1} \) and the multiplicity of \( a_i \) equals the rank of \( E_{k+1} \). Hence, this is an extension of Theorem 3.2.

4.3 may be regarded as a special case of Theorem 3.3.

Theorem 4.4. Let \( C \) be a Delta matrix. Then

(i) The Delta matrix \( C \) is a closed under matrix multiplication.

Proof. (4.2) and (4.3) follow from (4.3) and (4.5). To prove (4.2), let \( H \) be a \( n \times n \)-matrix such that the columns form a set of \( \delta \)-orthonormal eigenvectors of \( C \). For the eigenvalue 1 of \( E \), by standard linear algebra, (4.3) and (4.9) hold, and the equations for \( H \) are as easy consequence of (4.8) and (4.9).

(ii) The Delta matrix \( C \) is \( \Omega \)-polynomial association schemes in the sense of Dehmer [4]. Conversely, it is easy to see from his definitions that the adjacency algebra of a \( \Omega \)-polynomial association scheme contains a distinguished matrix which is a Delta matrix. Therefore, Delta matrix and \( \Omega \)-polynomial association schemes are equivalent concepts. This fact led me to choose Dehmer's name for these matrices since he was the first who considered \( \Omega \)-polynomial association schemes.
Let $C$ be a Delaunay matrix of degree $k$. Let $B$ be a non-empty collection of rows of $C$ by also we mean a set $B$ of labels such that each label denotes some row, distinct labels may denote the same row. We define the matrices $C(B) = (c_{ij})_{|B|}$ and $N(B) = (n_{ij})_{|B|}$ where $c_{ij} = n_{ij}$. The number of labels in $B$ will be denoted by $|B|$. If $B$ is called a $k$-design of $C$ if $n_{ij} = k$ for $(i,j) \in E_k$. By (4.10), the set of all rows of $C$ is an $|B|$-design. Also, a $k$-design is an $l$-design for all $l < k$, and by (4.3) a $k$-design on a Delaunay matrix is a $k$-design in the corresponding $O$-polynomial association scheme. 

Theorem 4.5. Let $B$ be a $k$-design of a Delaunay matrix $C$. Then $C(B)$ is a distance matrix of strength $k$. 

Proof. The triangle inequality for $C(B)$ holds since it holds for $C$. By (4.10), $N(B) = C(B) - C$, and for some polynomial $(f_i)$ of degree $i$. Hence also $N(B) N(B)^T = C(B) - C = (|B|) - C(B)$, eq. Now (4.10) implies $|B| C(B) = A(B) B$ for $c_{ij} = k$, $1 \leq i,j \leq |B|$, and in the proof of Theorem 4.3 it follows that $C(B)$ has strength $k$. 

5. Combinatorial Examples

In this section, we apply the results of Section 4 to $k$-designs, graphs, and spherical designs. Among others, we obtain familiar results by Behzad, Brouwer, Cayley, Delaunay, Hajnal, Goethals, and Seidel.

Theorem 5.1. Let $C = A - A^T$ be the distance matrix of a $k$-design with constant block size $k$ and incidence matrix $A$. Then $A = A^T$, and

(i) $C$ has strength $k$. 
(ii) $A^T A = A A^T = k A$ for constant $k$.

(iii) The tropical $A^T A = m A$ for constant $m$.

In these formulas, $A$ has the same meaning as in 3.2.

Proof. Obviously, $A = A^T$.

(i) $C$ has strength $k$ iff $C = 0$ for some $a = j_0 \in t A$, $b = j_0 \in t A$.

(ii) $A^T A = k A$ implies $A A^T = k A$.

(iii) $A^T A = m A$ implies $A A^T = m A$.

The dual of a $k$-design is obtained by interchanging the roles of points and blocks, and reversing incidence. The dual of a $k$-design with constant block size satisfying the conditions of (i), (ii), or (iii) or Theorem 5.1 is called a weak $k$-design, weak 2-design, or weak $k$-design, respectively. A weak 2-design is the same as an $(O, A)$-design, and for weak $k$-designs see [15].

A $(v, k, \lambda)$-design is a design on $v$ points with constant block size $k$ such that every $k$ distinct points are in exactly $\lambda$ blocks. A $(v, k, \lambda)$-design is also a $k$-design for all $v' < v$. A transversal is a set of $k$ points in a design on $v$ points. A transversal contains exactly one point from every class. A transversal in a $k$-design is also a $k$-transversal, and the $k$-transversals of a design are $k$-points. A transversal in a $k$-design is also a $k$-design. A $(v, k, \lambda)$-design (also called a $k$-design or partial geometric design) is a design whose incidence matrix $A$ satisfies $A^T A = 0, A A^T = k A$ for certain integers $v, k, \lambda$. 

5.1. Combinatorial Examples

In this section, we apply the results of Section 4 to $k$-designs, graphs, and spherical designs. Among others, we obtain familiar results by Behzad, Brouwer, Cayley, Delaunay, Hajnal, Goethals, and Seidel.

Theorem 5.1. Let $C = A - A^T$ be the distance matrix of a $k$-design with constant block size $k$ and incidence matrix $A$. Then $A = A^T$, and

(i) $C$ has strength $k$. 
(ii) $A^T A = A A^T = k A$ for constant $k$.

(iii) The tropical $A^T A = m A$ for constant $m$.

In these formulas, $A$ has the same meaning as in 3.2.

Proof. Obviously, $A = A^T$.

(i) $C$ has strength $k$ iff $C = 0$ for some $a = j_0 \in t A$, $b = j_0 \in t A$.

(ii) $A^T A = k A$ implies $A A^T = k A$.

(iii) $A^T A = m A$ implies $A A^T = m A$.

The dual of a $k$-design is obtained by interchanging the roles of points and blocks, and reversing incidence. The dual of a $k$-design with constant block size satisfying the conditions of (i), (ii), or (iii) or Theorem 5.1 is called a weak $k$-design, weak 2-design, or weak $k$-design, respectively. A weak 2-design is the same as an $(O, A)$-design, and for weak $k$-designs see [15].

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5.1. Combinatorial Examples

In this section, we apply the results of Section 4 to $k$-designs, graphs, and spherical designs. Among others, we obtain familiar results by Behzad, Brouwer, Cayley, Delaunay, Hajnal, Goethals, and Seidel.
Every transversal 1-design is a 1-design, 2-designs, transversal 2-designs, and the duals of 1-designs are 1-designs. Moreover, every 1-design (1-design, 2-design) is a weak 1-design (1-design, 2-design). Much is known about 1-designs and 2-designs; for a summary, see, e.g., Neumaier [37].

The complete designs $F(k, 1)$ have $n$ points and all $n$ sets of points as blocks. The complete transversal designs $F(k, q)$ have $n = kn$ points, partitioned into $k$ classes of $q$ points each, and all transversals of the partition as blocks. The distribution schemes of $F(k, 1)$ and $F(k, q)$ are usually called the Distance maps. Merging schemes.

Theorem 5.3. (Dinitz [4]).
(i) The distance matrix $C_0$ of a complete design $F(k, 1)$ is a Distance matrix of degree $k + m - (k, k - 1, \ldots, 1)$. For $n = m$, a complete $k$-design of order $n$ is a $k$-design of $C_0$. If $k$ is a $k$-design, for some integer $k$.
(ii) The distance matrix $C_0$ of a complete transversal design $F(k, q)$ is a Distance matrix of degree $k$. $q$-classes of order $n$ is a $q$-design of $C_0$. If $n$ is a transversal $k$-design, for some integer $k$.

Theorem 5.4. (Dinitz [3]). The blocks of a $(2^1 - 1)$-design with a transversal number carry a natural association scheme.

Theorem 5.5. The blocks of a transversal $(2 - 2)$-design with a transversal number carry a natural association scheme.

Proof. By 5.2 and 5.5.

Corollary 5.4. (Dinitz [3]). The blocks of a $(2^1 - 1)$-design with a transversal number carry a natural association scheme.

Corollary 5.5. The blocks of a transversal $(2^1 - 2)$-design with a transversal number carry a natural association scheme.

Proofs. By 5.3 and 5.4 (ii).

Note that by Theorem 5.3, the distance matrix of a $(k - 1)$-design has strength 2; and since there are many $(k - 1)$-designs which are neither 2-designs nor transversal 2-designs, the converse of Theorem 5.3 is not valid.

Corollary 5.6. (Malajczyk [9], Baker and Pisansky [1]). Two blocks of a $(k - 1)$-design all contain an element $k - 1 + a$, for some $a$. Then, the solution is the blocks defined by $AB = AB$ or $AB = k + a$. An equivalence relation.

Proof. By 5.5, the distance matrix $C$ of $AB$ has strength 2, with $a^2 - 1, a^2 - k$.

Theorem 5.7. Let $C = (i(j - 1)) = B$ be the distance matrix of a graph with smallest eigenvalue $-\mu$, and adjacency matrix $M$. Then

1. $C$ has strength 2 if $|M| = 2$ for some $k$.
2. $C$ has strength 2 if $|M| = (i(j + 1)) = \mu$ for some $h, a, b, k$.
3. If $C$ is a multiple of a Chevalley distance matrix $C_0$, the distance matrix of complete graphs of the same size.
Dissertations, graphs and designs

PROOF. Similar to the proof of 5.1. Observe that the diagonal of $M^2$ contains the entries of $M^2$ since $M$ is a 0-1 matrix.

A graph $G$ is called regular, resp. strongly regular if the condition of (i), resp. (ii) of Theorem 5.7 is satisfied. (i) means that every vertex is adjacent to $k$ other vertices, and (ii) means that in addition, the number of vertices adjacent to two distinct vertices $a$ and $b$ is $\lambda$ according to a and $b$ are adjacent or not. A lot of results about strongly regular graphs is contained in Faust [7], Goodman [12], and Higgin [13].

Complementars strongly regular graphs give rise to complementary diameter matrices.

If $G$ is a graph on $n$ vertices, then there is a connected scheme of degree 2 on $X$ which has adjacent points as free associates, and non-adjacent points as second associates. This scheme is essentially the distribution scheme of the corresponding distance matrix, and is regular, resp. an association scheme of the graph is regular, resp. strongly regular. Of course, every scheme of degree 2 comes from a graph in this way.

Theorem 5.8. The distance matrix of a strongly regular graph which is connected and not complete is a Delsarte matrix of degree 2. Consequently, every distance matrix of degree 2 is isomorphic to the distance matrix of a strongly regular graph which is connected and not complete.

PROOF. The distance matrix $C$ of a strongly regular graph $G$ has strength 2 and degree 2. If $C$ is not a Delsarte matrix then by Theorem 5.4 (iii), $C$ is a multiple of a triagonal distance matrix, whence, by 5.7 (ii), $W$ is complete or disconnected.

Conversely, if $C$ is a Delsarte matrix of degree 2 with distance numbers $a_1 < a_2$ then the graph $G$ whose vertices are the rows of $C$, and whose edges are the pairs $(i, j)$ with $c_{ij} = a_1$ has adjacency matrix $A = (a_1 - a_2) I + (a_2 - a_1) J$, hence smallest eigenvector $\lambda = (a_1 - a_2) \lambda I + (a_2 - a_1) \lambda J$. Therefore, $M = a_1 - a_2, C$ is strongly regular, connected, and not complete.

In many cases, interesting graphs can be found from designs. The block graph of a $12$-design $G$ with two interaction numbers $\mu_1 > \mu_2$ is the graph whose vertices are the blocks of $G$, adjacent if there intersect in $\mu_1$ points. A $12$-design with interaction numbers 0 and 1 in a partial geometry [1], and in this case blocks are called lines, and the block graph is called the line graph. A $2$-design with two interaction numbers is called quasi symmetric [3]. From Theorem 5.4 (i) and Theorem 5.7 (i) we now obtain without difficulty the following well-known results.

Corollary 5.9. (ii), (iii), (iv), (v) (vi).

The block graph of a 12-design with two interaction numbers is strongly regular. This holds in particular for the block graph of a quasi symmetric 2-design, for the line graph of a partial geometry, and for the line graph of a 2-(v, k, \lambda)-design.

If a 2-design $G$ has three interaction numbers $\mu_1, \mu_2$, and $\mu_3 = k - \mu_1 - \mu_2$ then we may from the equivalence class of $G$ and $\mu_3 = k - \mu_1 - \mu_2$, where $G$ is a strongly regular graph on the classes of $G$, called the class graph of $G$ (see the following corollary).

Corollary 5.10. (Baker and Hooley [3]). The class graph of a 2-design with three interaction numbers, one of which is $k - \mu_1 - \mu_2$ is strongly regular, and all classes have the same size.
The results analogous to 5.9, 5.6, and 5.10 for transversal designs are summarized in the following corollary.

**Corollary 5.12.**

(i) The block graph of a transversal 2-design with two interaction numbers is strongly regular.

(ii) Two blocks of a transversal 2-(v, k, 1)-design meet in at least \( k - r \) points, where \( r \leq k \). If \( B \) has three interaction numbers, one of which is \( k \), then the blocks of \( B \) can be partitioned into classes of the same size such that the class graph of \( B \) is strongly regular.

**Proof.** Similar to the proofs of 5.9, 5.6 and 5.10.

Finally, we mention some results on the spherical case. The details follow easily from Deza, Godsil, and Sédel [5].

**Theorem 5.13.**

(i) The distance matrix of a spherical set \( X \) of points has strength 1 if the centre of mass of \( X \) is the origin.

(ii) The distance matrix of a spherical 1-design has strength 1.

(iii) Every distance matrix of strength 2 without repeated rows is isomorphic to the distance matrix of a spherical 2-design.

Again, many of the results of [5] can now be obtained as corollaries from 5.12 and the above results.

Note that there is no analogue of (ii) for the case \( r > 7 \). Distance matrices of strength \( r > 2 \) yield, in general, only spherical 2-designs. It is not known what happens in the case \( r = 1 \).

**References**