Classification of Graphs by Regularity

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We give a classification of graphs by two parameters p and k such that a graph is regular if p ≥ 0, edge-regular if p ≥ 1, and distance-regular if distance d = 1, 2, 3.

1. INTRODUCTION

Among the regular graphs, certain classes have received much attention in the past. Strongly regular graphs (see, e.g., Bose [3], Halin [9], Shult [14], edge-regular graphs (see, e.g., Bose and Kautz [1]), and distance-regular graphs (see Biggs [1], Delsarte [4]). The object of this paper is to show that all graphs can be classified in such a way that the above classes are contained in our classification. Also, certain properties of distance-regular graphs can be generalized to arbitrary graphs.

In view of applications in subsequent papers we state our theory in terms of Dijkstra's method of searching. A Dijkstra matrix is a nonzero symmetric matrix with zero row sums and off-diagonal elements e, i.e., For all indices i and j, the elements e are always positive semidefinite. The number x is a distinct positive eigenvector of E is called the geometric mean of $\mathbf{E}$. Another invariant $\mathbf{x}$ is defined by properties of the principal product of polynomials in $\mathbf{E}$ and measures the inner efficiency of $\mathbf{E}$.

Various calculations give insight into the algebraic structure of the $\mathbf{E}$ algebra, i.e., the algebra of polynomials in $\mathbf{E}$. This algebra can be described in terms of a special basis $\mathbf{E}_{0}, \mathbf{E}_{1}, \ldots, \mathbf{E}_{k}$, and a related family $\mathbf{p}_{0}(\mathbf{E})$ of orthogonal polynomials. If $\mathbf{E}$ is distance-regular, then $\mathbf{D}_{0}, \ldots, \mathbf{D}_{k}$ are $\mathbf{D}_{0}$-invariant, that is, $\mathbf{D}_{0}$ is the adjacency matrix of a distance-regular graph $\Gamma$ of diameter $d$. In this case, the $\mathbf{D}_{0}$-eigenvector of $\mathbf{E}$ for a graph with vertex set $\Gamma$, then we define a matrix $\mathbf{X} = (x_{ij})_{i,j}$, where $x_{ij}$ is the valency of $i$ if $x = y$, $x_{ij} = 0$ if $x$ and $y$ are adjacent, and

\begin{align*}
x_{ij} = 0 & \quad \text{if } x \text{ and } y \text{ are adjacent,} \\
x_{ij} = 1 & \quad \text{if } x = y.
\end{align*}
\textbf{Spectrum of a Graph}

Let \( X \) be a graph. A Fibonacci matrix on \( X \) is a primitive \( n \times n \)-matrix \( B = (b_{ij})_{n \times n} \) with zero new sums and off-diagonal entries \( 0 \) (cf. Fibonacci [1]). A Fibonacci matrix on \( X \) is called centered if it is possible to split \( X \) into two isomorphic subgraphs \( Y, Z \) such that \( B_{ij} = 0 \) for \( i \neq j \) and \( i, j \in Y \) or \( Z \).

1. **Theorem.** Every Fibonacci matrix \( B \) is positive semidefinite, and the all-zero vector \( \mathbf{0} \) is an eigenvector of the eigenvalue \( \lambda = 0 \). \( B \) is centered if \( \mathbf{0} \neq \mathbf{0} \) is a simple eigenvalue of \( B \).

2. **Proof.** (Sketch.) Let \( B \) be a positive definite matrix for the eigenvalue \( \lambda \). Then \( B \) has a non-zero entry, and we can normalize \( u \) such that the spectral radius of \( B \) is 1. Define \( Y = \{ i | \lambda \mathbf{x} = \mathbf{1} \} \). Then \( B \mathbf{1}_Y = \mathbf{0}_Y \), and for \( i, j \in Y, x_{ij} = \sum_{k \in Y} b_{ij} x_{jk} = 0 \) since \( b_{ij} \leq 0 \), \( \lambda < 1 \) for \( i \neq j \), and \( \mathbf{1}_Y = \mathbf{1}_Y \). Hence \( \lambda \mathbf{1}_Y = \mathbf{1}_Y \) and \( B \) is positive semidefinite.

In conclusion, if \( B \) is a positive definite matrix, we may split \( X \) into two isomorphic subgraphs \( Y, Z \) with \( B_{ij} = 0 \) for \( i \neq j \) and \( i, j \in Y \) or \( Z \). This proves the theorem.

We write \( S \) for the set of nonzero eigenvalues of \( B \), and \( S' = \mathbb{R} \setminus S \). The multiplicity of the eigenvalue \( 0 \) of \( B \) is determined by \( S' \), since \( B \mathbf{1}_Y = \mathbf{0}_Y \) is centered. The number \( n - |S| \) of distinct nonzero eigenvalues of \( B \) is called
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the geometric rank of B. We also introduce the annihilator polynomial
\[ \text{Ann}_A(x) \] of A by
\[ \text{Ann}_A(x) = \sum_{i \in I} \left( 1 - \frac{x}{A_i^2} \right) \]
(1.1)

1.2. Lemma. The minimal polynomial of a Faddeev matrix B is a scalar multiple of \( \text{Ann}_A(x) \). Moreover, B is connected if
\[ \text{Ann}_A(B) = 1. \]
(1.2)

Proof. The minimal polynomial of a symmetric matrix \( B \) has as zeros just the eigenvalues of \( B \), all simple. Hence the first result follows from the definition of \( \text{Ann}_A(x) \). If \( B \) is not connected, and \( X, Y \) are as above, then the

tori of \( X \) is a plane, \( \langle x, x \rangle \in X \times X \) is zero, for all \( i \) (induction). In

particular, \( \text{Ann}_A(B) \) contains zero entries and (1.2) does not hold. If \( B \) is

closed, then \( H = \text{Ann}_A(B) \) is symmetric and satisfies \( HB = 0 \). Hence the
closure of \( H \) is a multiple of \( f \) whereas \( f \) is a multiple of \( 2 \). Hence \( 2f = f \)

implies \( H = f \).

For a Faddeev matrix \( B \), we denote the F-algebra by \( F \). By Lemma 1.2, \( F \)

has dimension \( 1 \). Let us define the polynomials
\[ A_0(x) = 1 \]
(1.3)

Then for \( a, b \in F \),
\[ A_a(b) = \text{Ann}_A(x) \]
(1.4a)

\[ A_a(b) = A_b(a) \quad \text{for all } a, b \] \quad (1.4b)

\[ A_a(B) = A_b(A_0) \]
(1.4c)

and we have

1.3. Theorem. The matrices \( A_a(B) \), \( a \in S \), form a basis of the

F-algebra \( F \), and satisfy for \( a, b \in S \),
\[ A_a(B) = A_b(a) \quad \text{for all } a, b \] \quad (1.4a)

\[ A_a(B) = A_b(A_0) \]
(1.4c)

\[ A_a(B) = A_b(B) \quad \text{for polynomial } p(x) \]
(1.4d)

\[ \sum_{a \in S} A_a = 1. \]
(1.4e)
Moreover, if $\mathcal{B}$ is connected, then

$$\Lambda = \Lambda' = \Lambda''.$$  

(1.10)

(1.11)

Proof. Equation (1.50) implies (1.6). From (1.7), $\Lambda'$ has the eigenvalue $A_{\alpha}(\alpha) = x$ with multiplicity $L_{\alpha}$ and other eigenvalues only. The trace is the sum of all eigenvalues weighted with their multiplicities, whence (1.7) holds.

(1.12)

Equation (1.50) implies (1.6). A symmetric matrix whose eigenvalues are all zero, whereas $d = 0$, and (1.2) follows. Equation (1.50) is the special case $f(\Lambda) = 0$ of (1.2). Equation (1.90) follows from (1.4) and Lemmas 1.3, and (1.11) from (1.90), (1.8), and (1.4).

2. Regularity

Let $\mathcal{B}$ be a fixed connected Fickler matrix of generic rank $s$, and $V$ be the corresponding $s$-algebra. For $i = 0, \ldots, r$, $V_i$ denotes the subspace of $V$ consisting of all polynomials in $\mathcal{B}$ of degree at most $i$. Obviously, $V_i$ has dimension $i + 1$, and $V_i \subset V_{i+1} \subset \cdots \subset V_r = V$.

2.1. Lemma. There are unique matrices $B_0, \ldots, B_n$ satisfying

$$V_i = \langle \epsilon_{0}, \ldots, \epsilon_{i} \rangle$$  

for $i = 0, \ldots, r$.  

(2.1)

$$f_i(B) = B_i$$  

for $i = 0, \ldots, r$.  

(2.2)

Proof. Define on $V$ an inner product $\langle A, B \rangle = \sum_{n=0}^{r} a_n b_n$. The $s$-dimensional inner product on an $s$-algebra considered as $s$-dimensional vector space is linear in $A$ and in $B$. Hence, the Gram-Schmidt algorithm yields a unique basis $B_0, \ldots, B_n$ of $V_i$ orthonormal with respect to $\langle \cdot, \cdot \rangle$ and satisfying (2.1).

2.2. Theorem. There are nonnegative numbers $d_i$ and polynomials $p_i(x)$ of degree $i$ such that the matrices

$$D_i = p_i(x) x^{i-1} \sum_{n=0}^{i} p_n(x) B_n$$  

for $i = 0, \ldots, r$ satisfy the relations

$$f(D_i + D) = D_i f(D)/D_i$$  

for $i = 0, \ldots, r$.  

(2.3)

$$D_0 f = f, \quad p_0(x) = 1, \quad d_0 = 1.$$  

(2.4)

(2.5)

(2.6)
Lemma.

(1) \( \sum_{i \leq m} D_i = \sum_{i \leq m} A_i = 0 \Rightarrow \sum_{i \leq m} A_i = 0 \).

(2) \( A_i = 0 \Rightarrow D_i = 0 \).

(2) \( C^2 = A^2 \).

Proof.

(1) By the definition of \( C \), we have \( A_i = 0 \) for all \( i \leq m \).

(2) \( D_i = 0 \) for all \( i \leq m \).

(3) \( A_i = 0 \) for all \( i \leq m \).

(4) \( C^2 = A^2 \).

Theorem.

The following conditions are equivalent for any \( \in \mathbb{C} \):

(1) \( B \in \mathbb{C} \), where \( B = \sum_{k \leq m} B_k \).

(2) \( \sum_{k \leq m} B_k = 0 \).

(3) \( \sum_{k \leq m} B_k = 0 \).

(4) \( \sum_{k \leq m} B_k = 0 \).

(5) \( \sum_{k \leq m} B_k = 0 \).

(6) \( \sum_{k \leq m} B_k = 0 \).

(7) \( \sum_{k \leq m} B_k = 0 \).

(8) \( \sum_{k \leq m} B_k = 0 \).

(9) \( \sum_{k \leq m} B_k = 0 \).

(10) \( \sum_{k \leq m} B_k = 0 \).

Proof:

(1) \( \Rightarrow \) (2): By the definition of \( B \), we have \( B_k = 0 \) for all \( k \leq m \).

(2) \( \Rightarrow \) (3): By (2), \( \sum_{k \leq m} B_k = 0 \).

(3) \( \Rightarrow \) (4): By (3), \( \sum_{k \leq m} B_k = 0 \).

(4) \( \Rightarrow \) (5): By (4), \( \sum_{k \leq m} B_k = 0 \).

(5) \( \Rightarrow \) (6): By (5), \( \sum_{k \leq m} B_k = 0 \).

(6) \( \Rightarrow \) (7): By (6), \( \sum_{k \leq m} B_k = 0 \).

(7) \( \Rightarrow \) (8): By (7), \( \sum_{k \leq m} B_k = 0 \).

(8) \( \Rightarrow \) (9): By (8), \( \sum_{k \leq m} B_k = 0 \).

(9) \( \Rightarrow \) (10): By (9), \( \sum_{k \leq m} B_k = 0 \).

(10) \( \Rightarrow \) (1): By (10), \( \sum_{k \leq m} B_k = 0 \).
We say that $B$ is regular if the conditions of Theorem 2.6 are satisfied.

2.6. Theorem. A connected regular matrix $B$ is always diagonalizable. It is 1-
regular iff the diagonal entries of $B$ are constant, i.e., all diagonal entries are $a$ for some $a \neq 0$. (Thus we obtain a regular graph on $X$ by coloring $x, y \in X$ adjacent iff $b_{xy} = 1$.)

Proof. Irregularity requires $i - f = (a, a, 0)$, which is always true. Ir-
regularity requires $i - f = (0, 0, 1)$, which is always true. Irregularity requires in addition that $B' = f' = 0$, i.e., $B'$ has constant diagonal. Irregularity requires in addition that $B' = f' = 0$, i.e., $B'$ has constant diagonal. Feat.

2.7. Theorem. Let $B$ be a connected $i$-regular Fiedler matrix of

where $\sum_{x \in X} b_{xy} = 0$ when $i(t, u) = 0$.

Proof. (i) follows from Theorems 2.6 and 2.5.

(iii) $\sum_{x \in X} b_{xy} = 0$ when $i(t, u) = 0$.

(iv) $\sum_{x \in X} b_{xy} = 0$ when $i(t, u) = 0$.

(v) $\sum_{x \in X} b_{xy} = 0$ when $i(t, u) = 0$.

2.8. Lemma. The following conditions are equivalent:

(i) $D + D = B$.

(ii) $D$ is a constant matrix.

(iii) $D$ is an integral matrix.

(iii) $D = B$.

(iii) $D = B$.

Proof. (i) $(i) \Rightarrow (ii)$ follows from (2.5) for $t \in X$.

(iii) $D_{i} + D_{j} = 0$ when $i(t, u) = 0$.

(iii) $D_{i} + D_{j} = 0$ when $i(t, u) = 0$.

(iii) $D_{i} + D_{j} = 0$ when $i(t, u) = 0$.

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2.9. Theorem. Let $B$ be a connected $i$-regular Fiedler matrix of

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Proof. (i) follows from Theorems 2.6 and 2.5.

(ii) $\sum_{x \in X} b_{xy} = 0$ when $i(t, u) = 0$.

(iii) $\sum_{x \in X} b_{xy} = 0$ when $i(t, u) = 0$.

(iii) $\sum_{x \in X} b_{xy} = 0$ when $i(t, u) = 0$.

(iii) $\sum_{x \in X} b_{xy} = 0$ when $i(t, u) = 0$.
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is closed under positive multiplication. By (i) and Lemma 3.3(i), $D_1 \cdots D_r$ are matrices whose sum is $I$, and by (ii), $D_1 \cdots D_r = I$ implies $1 = \det(I)$. Hence $D_i$ is a $(S,\sigma)$-matrix, too, and $D_i \cdots D_r = D_1 \cdots D_r$. By (ii) and Lemma 3.3(ii), all $D_i$ are nonzero, so that $D_1 \cdots D_r$ implies Theorem 3.3(ii)

with $\tau = 2n$, i.e., $3r$ regularity.

Remark: Theorem 3.3(ii) implies that $F$ is the adjacency algebra of a $P$-polynomial association scheme in the sense of Delsarte [4].

3. The Characteristic Matrix

In the following, $\theta$ is a fixed ordered vector of error correction codes. We show that the $p_i(\theta)$ form a family of orthogonal polynomials, and derive some formulas which allow us to calculate the $\theta$-spectrum. These results are relevant, e.g., for the investigation of perfect error correcting codes in $\theta$-regular graphs.

This will be done somewhere else. (For the case $\theta = 2n - 2$ see, e.g., Delsarte [4].)

We shall always assume that $\theta_i > 0$ for $i = 1, \ldots, r$. By (2.1) and the proof of Theorem 2.3, this is equivalent to the assumption that $D_1 \cdots D_r$ generate $F$.

3.1. Lemma

(i) $D_i D_j = \sum_{k=1}^r p_i(\theta_k) p_j(\theta_k) I_k$,

(ii) $\theta D_i D_j = p_i(\theta) D_j + \theta p_j(\theta) D_i$,

(iii) $D_i^2 = \sum_{k=1}^r p_i(\theta_k) p_i(\theta_k) I_k$,

(iv) $D_i^3 = \sum_{k=1}^r p_i(\theta_k) p_i(\theta_k) p_i(\theta_k) I_k$,

where

\[ p_i(\theta) = \sum_{\theta_k \in \Theta} \frac{\theta_k}{\theta} p_i(\theta_k) p_i(\theta_k) p_i(\theta_k). \]  

(3.1)

Proof. (i) follows from (3.1) and (1.6), and (ii) from (2.4) and (2.5). By the above remark, $I_k = \sum_{\theta_k \in \Theta} \delta(\theta_k) I_k$ for certain numbers $\delta(\theta_k)$. Hence

\[ \theta D_i D_j = \sum_{\theta_k \in \Theta} \theta \delta(\theta_k) p_i(\theta_k) p_j(\theta_k) I_k, \]

on the other hand, $\theta D_i D_j = p_i(\theta) D_j + \theta p_j(\theta) D_i$ by (2.1) and (1.6). Hence $\delta(\theta_k) = \delta(\theta_k) p_i(\theta_k) p_j(\theta_k) I_k = I_k$ (i.e., (3.1) holds. (iv) follows by iterating (ii) twice.)

3.2. Lemma (Orthogonality relations)

(i) $\sum_{\theta_k \in \Theta} \theta_k p_i(\theta_k) p_j(\theta_k) = \delta_{ij} \theta_i$.

(ii) $\sum_{\theta_k \in \Theta} \theta_k^2 p_i(\theta_k) p_j(\theta_k) I_k = \delta_{ij} \theta_i$.

(iii) $\sum_{\theta_k \in \Theta} \theta_k^3 p_i(\theta_k) p_j(\theta_k) I_k = \delta_{ij} \theta_i$.
(ii) \( \frac{1}{x} \sum_{k=1}^{x} f(k)x(k) = \frac{1}{x} \sum_{k=1}^{x} a_k \).

(iii) \( p_k = a_k \).

(iv) \( p_k = 0 \quad \forall i-j-1 \leq i-j \leq j \).

(v) \( p_j = 0 \) if \( j \leq 0 \) or \( j > L \).

Proof: (i) and (ii) follow by substituting (3.3) and Lemma 3.1(iii) into each other and comparing coefficients. (iii) follows from (1), (2.8), and (5). Lemma 3.1(iv) implies that \( p_k = 0 \) for \( j \leq k \), and each for \( k \leq j \), i.e., \( \forall i-j \leq j \). By (3.3), \( a_j \) is dynamic in \( L_j \), and 1, this implies (iv) and (v). For \( j = \sum_{i=0}^{L-1} a_i \), we have \( T = \sum_{i=0}^{L-1} a_i \).

\[ \frac{1}{x} \sum_{k=1}^{x} x(k) = \frac{1}{x} \sum_{k=1}^{x} a_k \] (3.2)

This is important and is called the characteristic series of \( \mu \). We also define \( \tau_{k+1} = 0 \).

3.3. Theorem

(i) The characteristic series \( T = \sum_{i=0}^{L-1} a_i \) is defined and satisfies

\[ \tau_i = 0 \quad \forall i \leq L, \] (3.2a)

\[ \tau_i = 0 \quad \forall i \geq L, \] (3.2b)

\[ \tau_i = \tau_i \tau_i \quad \forall i \geq L, \] (3.2c)

\[ \tau_i = 0 \quad \forall i \geq L. \] (3.2d)

(ii) \( a_i \) can be recursively defined by

\[ a_i = 1, \] (3.3a)

\[ a_i = a_{i-1} \tau_i a_{i-1} \quad \forall i = 1, \ldots, L. \] (3.3b)

(iii) \( p(x) \) can be recursively defined by

\[ p(0) = 0, \quad p(x) = 1, \] (3.4a)

\[ t_i = 0, \quad t_i = a_i a_{i+1} \quad \forall i = 1, \ldots, L. \] (3.4b)
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(9) $D_t$ can be recursively defined by

\[ D_0 = 0, \quad D_1 = 1. \quad (3.8a) \]

\[ t_{ij} = 0 \quad \text{for} \quad i > j + 1, \quad \text{and} \quad t_{ii+1} = 0. \quad (3.8b) \]

Proof. By (3.2) and Lemma 3.2.4, $\sum_{t=\lambda} t_{\lambda t} p(\tau)$ simplifies to $\rho(\tau)$. Hence

\[ \rho(\tau) = \sum_{t=\lambda} t_{\lambda t} p(\tau) \quad \text{and} \quad \lambda \geq t_{\lambda t} \lambda t_p(\lambda). \quad (3.7) \]

A comparison of the degree shows that $\tau_t = 0$ if $i > j + 1$, and $t_{ij+1} = 0$. By (3.5),

\[ t_{ij+1} = t_{ij}, \quad (3.8) \]

which also shows $\tau_t = 0$ if $i > j + 1$, and $t_{ij+1} = 0$. Now $t_{ij+1} + t_{ij} = \sum_{\lambda} t_{\lambda t} \sum_{\mu} \mu_{\lambda t} p(\mu) = 0$ by Lemma 3.2.6 since $\lambda \geq t_{\lambda t} \rho(\lambda) = 0$ for $\lambda \geq 0$. Therefore, (3.2a) and (3.5a) hold now from (3.3), (3.5) from (3.7), and (4.3) from (3.5) and (3.7).

Remarks. (1) Results Lemmas 3.2.4 and Theorem 3.3(2) show that the $p(\tau)$ are a family of orthogonal polynomials, cf. St defended in [1].

(2) The characteristic matrix contains a lot of parallel information about $\mathcal{B}$: We can compute the $t_{ij}$ by (3.4), the $p(\tau)$ by (3.2) and $\Lambda_{\alpha}(\lambda)$ by Lemma 3.2.6, if as the set of zeros of $\Lambda_{\alpha}(\lambda)$, and the $t_{ij}$ by Lemma 3.3(3).

3.4. Theorem

(i) The algebra $\mathcal{U}$ of polynomials in $\mathcal{T}$ is generated by the pairwise orthogonal polynomials

\[ T_\alpha = \left( \sum_{\tau=\lambda} p(\tau) \rho(\tau) \right), \quad \alpha \in \mathcal{B}. \quad (3.8) \]

(ii) $\rho(\tau) = \sum_{\tau=\lambda} p(\tau) T_\alpha$ for all polynomials $p(\tau)$.

(iii) $\rho(\tau)$ is an eigenvalue of $\mathcal{T}$ for the eigenvector $\alpha \in \mathcal{B}$.

(iv) The correspondence $\mathcal{T} \rightarrow \mathcal{B}$ is an isomorphism between the algebra $\mathcal{U}$ and $\mathcal{V}$. Moreover, $p(\tau) \rightarrow T_\alpha$. 

(\text{continued on next page...})
By induction from (3.10)-(3.12), \( T = \sum_{n=1}^{k} nT_n \), and (6) follows. By (3.12), the \( T_n \) are linearly independent, so (6) implies (5). By (6) and (6b), \( T \) is associated with the algebra generated by the \( T_n \) and has the same dimension \( k+1 \), whereas (3.1) holds. By (3.7), \( T_1 = u \), which implies (9).

(9) follows from (3.12) and (3.6).

3.5. Proposition. If \( B \) is \( 2 \)-regular, \( p \geq 1 \), then

\[
\rho_{i} < 0 \quad \text{for } \pi_{i} \in E_{1}, \ldots, e_{k}, \quad (3.13)
\]

\[
t_{i} = 0, \quad t_{i+1} = 0 \quad \text{for } i = 1, \ldots, n. \quad (3.14)
\]

Proof. If \( B \) is \( 2 \)-regular, then \( C_2 = 0 \) are the (1) equations whose for \( \lambda_{j} \in 1, \ldots, e \). Let \( c_{i}(1)(D_{i}) = 0 \) be a \( i \)-equation, then for \( i, j = 1, \ldots, e \). By (3.1), (3.5), (3.7), and (3.1) this implies (3.13). By (3.2), \( b \in (D_{i}, D_{j}) \) with \( D_{i} \), and \( D_{j} \) a \( 0 \)-equation, so that \( (3.9) \) and (3.10) imply that \( B = c_{i}(1)(D_{i}) = 0 \), i.e., \( x = (1) a_{i} - 2p_{i}(1) \), whereas \( c_{i}(1)(D_{i}) = -c_{i}(1) a_{i} - 2p_{i}(1) \) for \( p_{i}(1), (1.9), (1.13) \), and (3.9). Equation (3.14)

follows from (3.13).

Remarks. (1) By (3.1), (3.9), and Theorem 3.4(1), \( p_{i}(1) \) is the \( i \)-coordinate of \( \rho_{i}(1) \).

(2) If \( B \) is \( 3 \)-regular, then, in the notation of Theorem 2.6, \( \rho_{i}(1)(a) = c_{i}(1) a_{i}, \delta_{i}(1) = c_{i}(1) a_{i} \).

In the next section we shall need the following results which hold without the assumption that all \( \delta_{i} \) are positive.

3.6. Lemma. Let \( B \) be a \( 3 \)-regular Poisson algebra, and \( e = \emptyset(1) \). Then

\[
B_{i} = B_{j} = 0, \quad B_{i} = 0, \quad \text{for all } i, j = 1, \ldots, e \text{ satisfying } i + j = i + e. \quad \text{Moreover,}
\]

(3.15) and (3.16) hold for \( i = 1, \ldots, e. \)

Proof. Define the polynomial \( f_{i}(1) \) by \( E_{i} = f_{i}(1) \), so that \( D_{i} = c_{i}(1) e_{i}, \rho_{i}(1) = e_{i}, \delta_{i}(1) = a_{i}, \rho_{i}(1) = e_{i}, \delta_{i}(1) = a_{i} \), and by Theorem 2.6(1), \( \delta_{i}(1) \neq 0 \) for \( i = 1, \ldots, e. \)
Essentially the same proofs as for Lemmas 2.1 and 5.2 and Theorem 3.3 show that
\[ E_{ij} = \sum_{k} E_{ik} E_{kj} \]

where \( E_{ij} = 0 \) implies \( i \neq j \).

Theorem 3.5: For \( i, j \) in the required range, \( E_{ij} + E_{ji} = \sum_{k=1}^{n} E_{ik} E_{kj} \), where \( D_{ij} = \sum_{k} E_{ik} E_{kj} \).\( \Box \)

Theorem 3.6: A graph is regular if and only if it is strongly regular.

4. 1-regular Graphs

All graphs considered are necessary, undirected, without loops or multiple edges. The diameter of a vertex \( x \) is the number of vertex adjacent to \( x \). The diameter of two vertices is \( i \) and \( j \) denote the number of vertices such that \( d(i, j) = 1 \) and \( d(i, j) = 2 \). The diameter of a graph is the largest covering distance.

We consider the following conditions:

(i) Every vertex is adjacent to exactly 4 other vertices.

(ii) The number of vertices adjacent to any two nonadjacent vertices is 2.

(iii) The number of vertices adjacent to any two nonadjacent vertices is 4.

(iv) If \( d(i, j) = 1 \), then \( d(i, 1) = d(2, j) = 1 \).

A graph \( G \) is called regular, edge-regular, or strongly regular if (i), (ii), and (iii) hold, and diameter \( d \) or \( G \) is connected, and (iv) holds for all \( x, y (\neq x), z, \ldots \) in \( G \). It is easy to see that a distance regular graph is edge-regular, and the distance regular graph of diameter 2 is just the connected strongly regular graphs. In the next, Fig. 1 deals essentially with regular and distance regular graphs and proves many special.
cases of the results of Sections 1-3, interpreted in terms of graphs.

For a graph $G$ of diameter $d$, with vertices on $X$ of size $n$, we define symmetric $n \times n$-matrices $A_d = (a_{ij})_{i,j=0,1,\ldots,d}$ and $B = (b_{ij})_{i,j=0,1,\ldots,d}$ by

- $a_{ij} = 1$ if $d(i,j) = 0$,
- $a_{ij} = 0$ otherwise,
- $b_{ij} = \text{traces of } x^i$ if $x \neq 0$,
- $b_{ij} = 0$ otherwise.

Then $A_d = \lambda_1 A_1 + \cdots + \lambda_d A_d$, and $B$ is a simple matrix. We call $A_d$ the adjacency matrix of $G$, and $B$ the Fiedler matrix of $G$ (cf. Fiedler [8]).

We call $G$ regular of rank $r$ if $B$ is a regular of $G$ is regular and has generic rank $r$.

4.1. Theorem. If $G$ is a connected graph with diameter $d$ and rank $r = 1$, then $r = 0$.

Proof. Let $x_0, x_1, \ldots, x_d$ be vertices of diameter $i, i = 0, 1, \ldots, d$. Then $a_{i,j}$ is a $d$-entry of $B$ if $i < j$, and all if $i = j$. Hence if $s_0 B^0 + \cdots + s_d B^d = 0$, then considering the $(s_0, B)$-entry for $i = 0, 1, \ldots, d$, we find $s_i = c_0, c_1, \ldots, c_d = 0$. Hence the matrix polynomial $s A(B) G$ has degree $i + 1 > d$. Therefore $s = 0$.

Problem. Characterize the case $r = 0$.

4.2. Proposition. If $G$ is $1$ regular if $G$ is $2$ regular if $G$ is regular. In this case,

$$D = A_d - I = B.$$  \hfill (4.1)

Proof. By Theorem 2.5 and Remark (1) after Lemma 3.5.

4.3. Theorem. Let $G$ be a regular graph, and $x = [(2)].$ Then

\begin{enumerate}
  \item $D_0 = A_0 \neq 0$,
  \item $D_0 x = 0$ for all $x \neq 0$.
\end{enumerate}

Proof. Of course, $D_0 = A_0$. Hence $D_0 x = 0$, and $D_0 \neq 0$. For $i = 0, 1, \ldots, r$, we have $D_i x = 0$, where $x \in (0, x - 1)$. By Theorem 2.5, $D_i x = 0$, is a 0 if.
and by Theorem 2.4.2 and our assumption, \( D_{n+1} \cdot A_i = 0 \) for \( i \leq n \). Now by Lemma 2.5 and Proposition 4.3:

\[
A_{n+1} = D_{n+1} \cdot A_i + (\tau_{n+1} A_n) + \tau_{n+1} A_{n+1}
\]

(6.2)

\( A_0 \) has nonzero entries just in places \( (i, j) \) with \( 1 \leq i \leq n, n+1 \leq j \leq n+2 \), so \( \tau_{n+1} A_n \) implies that \( D_{n+1} \cdot A_i \) has nonzero entries just at places \( (i, j) \) with \( n+2 \leq j \leq n+1 \). Hence \( \tau_{n+1} A_{n+1} \) and (6) follow by induction. To prove (5), we simply observe that the \((m, m)\) entry of \( A_i \) is \( a_i(n+1, n+1) \), and by (6) and Lemma 2.6, we have \( a_i(n+1, n+1) \cdot a_i(n, n) \) for \( i \geq j \) in the required range.

6.4 Example. Let \( G \) be a graph of diameter \( d \), and \( a \leq d - 1 \).

(1) If \( A_i \cdot \mathbf{1} \) holds for \( i = 0, \ldots, a-1 \) and \( i = a+1, \ldots, d-1 \), then \( G \) is \( a \)-regular.

(2) If \( G \) is \( d \)-regular and satisfies \( A_0 \cdot \mathbf{1} = \mathbf{1} \), then \( G \) is \((d-1)\)-regular.

Proof. Note first that \( A_0 \cdot \mathbf{1} \) is trivially satisfied with \( b_0 = 0 \) unless \( i = a+1, \ldots, d-1 \). Hence \( A_i \cdot \mathbf{1} = (a_i+1)\mathbf{1} \) implies, together with the assertions of (1), that for \( i = 0, \ldots, a-1 \):

\[
A_i \cdot \mathbf{1} = (a_i+1)\mathbf{1} \quad \text{if } i < a,
\]

\[
A_i \cdot \mathbf{1} = 0 \quad \text{if } i = a,
\]

(6.3)

(6.4)

\[
A_i \cdot \mathbf{1} = (a_i+1)\mathbf{1} \quad \text{if } i > a-1,
\]

and \( A_i \cdot \mathbf{1} = 0 \) implies that \( G \) is regular, whereas by Proposition 4.3.1, \( A_i \) is a polynomial of degree \( i \) in \( B \) (iteration for \( i = 0, \ldots, a-1 \)) and hence for \( i = a \).

\[
V_i = (A_n \cdot \mathbf{1}),
\]

(6.5)

\[
V_{a+1} = (A_{n+1} \cdot \mathbf{1}),
\]

(6.6)

The \( A_i \) are \((d, 1)\)-matrices whenever \( a_i \neq 1 \).

(6.7)

Note (4.4)-(6.8) imply \( V_i \cdot \mathbf{1} \leq V_i \) for \( i-j \leq 2a, i, j \) (and hence \( i \leq a \)), whereas by Theorem 2.4.2, \( G \) is \( a \)-regular.

The addition of assumption (6) implies similarly

\[
A_i \cdot \mathbf{1} = (a_i+1)\mathbf{1} \quad \text{for } i = 0, \ldots, a-1,
\]

(6.8)

\[
A_i \cdot \mathbf{1} = 0 \quad \text{for } i = a+1, \ldots, d-1\]
4.3. COROLLARY. A graph is 3-regular if and only if it is a graph where each vertex has degree exactly 3.

4.4. COROLLARY. A graph is 3-regular if and only if it is a graph where each vertex has degree exactly 3.

5.1. THEOREM. Let $G$ be a connected graph with diameter 3 and rank $r = 1$. Then the following conditions are equivalent:

(i) $G$ is $(f - 3)$-regular, and $r = 1$,

(ii) $G$ is $(f - 3)$-regular,

(iii) $G$ is $(f - 3)$-regular,

Proof. (i) $G$ is $(f - 3)$-regular, and $r = 1$. Hence, by Theorem 2.7(iii) and Corollary 4.5, $G$ is 3-regular.

(ii) $G$ is $(f - 3)$-regular, and $r = 1$. Hence, by Theorem 2.7(iii) and Corollary 4.5, $G$ is 3-regular.

(iii) $G$ is $(f - 3)$-regular, and $r = 1$. Hence, by Theorem 2.7(iii) and Corollary 4.5, $G$ is 3-regular.

5.2. COROLLARY. The strong regular graphs are just the 3-regular (or 3-connected) graphs of rank 3.

REFERENCES


