Distance Matrices and $m$-dimensional Designs

A. Neumaier

1. $M_d$-designs

All sets considered are finite. A design consists of a set $P$ of points and a collection $\mathcal{B}$ of subsets of $P$, called blocks. A $t$-design is a design, together with a set of subspaces $\mathcal{B}$ of $P$, called subblocks, such that the empty set, the non containing of a single point, the intersection of subspaces and the intersection of a subspace with a block are subspaces. In particular, this implies that the set of all subspaces contained in a given block, together with this block, is a $t$-design. A $t$-design is either a subspace, a block or the set $P$. Note that if any two blocks intersect in a subspace then the varieties form an atomic lattice. A subspace is said to extend if every subspace in the intersection of two blocks.

A quantization is defined as a sublattice of dimension $n$ if the following hold ($L_n$, $B_n$, $R_n$ and $R_n$ are satisfied):

1. If $n$ is a variety then all maximal chains $\mathcal{A} = x_1 < x_2 < \cdots < x_n$ of varieties have the same length $l$ (we call such a variety an $l$-variety, and we write $K_l$ for the set of all $l$-subsets $\mathcal{A}$).
2. The $n$-subvarieties are the blocks).
3. For $i < n$, every $i$-subvariety contains exactly $K_i$ points, $0 = K_0 < K_1 < \cdots < K_n$.
4. For $i < n$, every $i$-variety $\mathcal{A}$ has exactly $K_i$ blocks. Note that $K_n$ is the total number of blocks.

An $M_d$-design is a regular quantization of dimension $n$ satisfying:

(A1) If $n$ is an $l$-variety, a $i$ block containing $x$, and $p \in \mathcal{A}$ a point out to $x$ then there is an $(l-1)$-variety $y = p$ containing $x$ and $p$.

Note that by the intersection property, there is (for $i = n-2$) at least one $(i + 1)$-variety $\mathcal{A}$ containing $x$ and $y$ so that $x$ is in all blocks $x$ containing $x$.

It is easy to see that a regular quantization of dimension $n$ is an $M_d$-design if only if the lattice of subspaces of every block is a material (see e.g., Welsh [3] for a definition).

EXAMPLES:

1. Every $l$-design is an $M_1$-design; in fact the two concepts are the same.
2. The set of all proper subsets of an $(n + 1)$-set $P$ is an $M_1$-design with $K_1 = \{\emptyset, P\}$.
3. The set of $n$-element transversals of a partition of $n$ into $r$-classes of $n$ points each is an $M_r$-design with $K_r = \{\emptyset, P\}$ (see for complete transversals).
4. The set of all proper subspaces of a projective or affine space of dimension $n$ over a finite field $GF(q)$ is an $M_{n}$-design with $K_n = (q^n - 1)/(q - 1)$.
5. The set of all proper subspaces of an $n$-dimensional space over $GF(q)$ is an $M_{n}$-design with $K_q = (q^n - 1)/(q - 1)$. Several families of polar spaces are known (El. [7], Buekenhout and Sovel [1]).

M6. For polar spaces of type $\Omega_v(q)$ (Thm. [7]), the set of varieties belonging to the node $x$ in the diagram

\[
\begin{array}{c|c|c}
\text{points} & r + 2 - 2x & \text{blocks} \\
\hline
\text{C} & 1 & 2 \\
\text{C} & 2 & 1 \\
\hline
\end{array}
\]

(which we shall call a node polar space) is an $M_{p, q}$-design with $K_w = (q^p - 1)/(q - 1)$ for $1 \leq w \leq p - 1$, $K_w = (q^p - 1)/(q - 1)$ for $p - 1 < w < q - 1$.

M7. In the same way, the starred nodes (v) of the diagram for buildings of type $\Omega_v(q^p)$

\[
\begin{array}{c|c|c}
\text{points} & r + 2 - 2x & \text{blocks} \\
\hline
\text{C} & 1 & 2 \\
\text{C} & 2 & 1 \\
\hline
\end{array}
\]

and $R_v$ give rise to $M$-designs (for $\nu =$ number of stars). This follows from the transitivity properties of the automorphism group of the relevant building (Thm. [7]), together with the fact that the union of a block (= variety belonging to the rightmost starred node) is a truncated projective space, hence a $M$-design.

M8. Perfect maximal designs (With [3]) of dimension $n$ are $M$-designs; in fact a maximal (normalized such that $K_v = 0$, $R_v = 1$) is a perfect maximal design if it is an $M$-design for the appropriate $n$. The only known perfect maximal designs are projective spaces, affine spaces, Steiner systems, affine plane systems and their truncations, see [3].

M9. The regular embeddings of Type II over $GF(q)$ defined in Delhali [2] are $M$-designs with $K_v = (q^p - 1)/(q - 1)$.

M10. For $1 \leq w \leq n$, the set of $w$-sections of an $M$-design is an $M$-design, and the set of $w$-sections, together with the blocks, is an $M_{p, q}$-design. Thus, the previous examples give rise to many others.

In 1.1.1.7, we assume that an $M_{p, q}$-design is given.

Lemma 1.1. Suppose that $0 < r < q$, $1 \leq t < n, 1 \leq j < k$, $\lambda = 1$. Then, for given $x \in X$, $y \in X$, $\lambda \not\equiv x \oplus y \mod q$, the number of $\lambda$-sections with $\gamma(x, y) = \lambda$ is

\[
N_{t, t}^{K_v} = (K_v - K_v) \cdots (K_v - K_v)
\]

(1)

Proof. Obviously, $N_{t, t}^{K_v} = 1$, and for $t = 1$ this product (1) is empty. Hence we may assume by induction that $r = 2$, and the formula holds for $N_{t, t}^{K_v}$. We count the number $N$ of pairs $p, q \in X$ with $\lambda = \gamma(x, y)$ and $x \not\equiv y \mod q$. For each of the $K_v$-points $p$ with $x$, there is a unique $(p + 1)$-plane containing $x$ and $p$, and we have $r + 1$ times $p, x, y$ by induction there are $N_{t, t}^{K_v}$ points $p$, whence $N = N_{t, t}^{K_v} K_v$. On the other hand, given $x, y$ can be chosen in $K_v - K_v$ ways, whence $N = N_{t, t}^{K_v} K_v$ and (1) follows.
Corollary 1.2. For given $s = x, t = x_1$ with $s < t$, the number of o-variety $y$ with
\[
\mu_s = \frac{\sum_{i=1}^{t} (K_i - K_i (K_i - K_i) \cdots (K_i - K_i))}{K_i (K_i - K_i) \cdots (K_i - K_i)}
\]
(2)

Corollary 1.3. The number of o-variety contained in a k-variety (if $s < k$) is
\[
\mu_k = \frac{\sum_{i=1}^{s} (K_i - K_i (K_i - K_i) \cdots (K_i - K_i))}{K_i (K_i - K_i) \cdots (K_i - K_i)}
\]
(3)

Remark. For the $M^*$-designs of Examples M2 and M3, $\mu_i$ is the ordinary binomial coefficient, and for the $M_2^*$-designs of Examples M4 and M5, $\mu_i$ is the Gaussian binomial coefficient. This explains the notation used.

Corollary 1.4. The set of subspaces of a block of an $M^*$-design is a perfect matroid design.

Proof. The set in question is a matroid, and by 1.2, (8, 10) are satisfied with $K^* = K$, $R^* = \mu_2$.

Lemma 1.5. For $i \leq s$, the number of o-variety contained in two given blocks $x$ and $y$ is
\[
\mu_n = \frac{\sum_{i=1}^{s} (K_i - K_i (K_i - K_i) \cdots (K_i - K_i))}{K_i (K_i - K_i) \cdots (K_i - K_i)}
\]
(4)

where $\mu_i$ is the number of points contained in $x$ and $y$.

Proof. Will be deferred (similar to the proof of 1.1).

Note that $\mu_n$ is a polynomial of degree $s$ in $\mu_i$ and
\[
\mu_n = \frac{\mu_1}{\mu_1 \frac{\mu_2}{\mu_1}}
\]
(5)

Lemma 1.6. The intersection of two blocks is either a subspace, or a union of $(n - 1)$-variety.

Proof. Let $x, y$ be blocks. Every point of $x \cap y$ is in a maximal subspace of $x \cap y$, whereas $x \cap y$ is the union of its maximal subspaces. Let $z$ be such a maximal subspace. If $x$ is an $i$-variety with $i = n - 3$, and $z$ a point of $x \cap y$ set as $x$ then $x$ and $y$ coincide this unique $(i - 1)$-variety containing $x$ and $y$, whereas $z$ is not contained. Hence there is no such $z$, and $x \cap y = \emptyset$.

Corollary 1.7. Two blocks with at least $K_{-1}$ common points intersect in a subspace.
We assume some of our results in matrix form. Denote by $f$ the identity matrix of any size, by $f_k$ (resp. $f$) any all one matrix (resp. vector), and by $f_k - A$ (where $A = [a_{ij}]$) the matrix $(f_k - a_{ij})$.

Let us define the incidence matrix $A_k = (a_k(x,y))_{x,y = 1,\ldots,n}$, where $a_k(x,y) = 1$ if $x = y$, and 0 otherwise; the incidence matrix $C = f_k A_k$, and the distance matrix $C + K$ is the number of points in the intersection of two blocks, called the incidence numbers of the design. The next result is an immediate consequence of the above results.

Lemma 1.8. For an $M_n$-design, the following is true.

(i) For all $k = 1,\ldots,n$, we have $A_k f_k = f_k A_k$.

(ii) $A_k f_k = f_k$, $C_k f_k = f_k$, $A_k A_k^T = f_k$.

(iii) $A_k f_k = f_k A_k$, $C_k f_k = f_k$, $A_k A_k^T = f_k$.

Theorem 1.9. A short regular lattice of dimension $n$ is an $M_n$-design if and only if $f_k - A_k$ is a polynomial $(f_k - a_{ij})$ of degree $k - 1$ in $(x - y)$ such that $f_k f_k = f_k$ for $i = 0,\ldots,n-1$.

Proof. By 1.8(i), every $M_n$-designs has incidence matrices that satisfy this property. Conversely, assume this property. Since $C_k f_k = f_k$, $C_k f_k = f_k$, $A_k A_k^T = f_k$, we refer to the existence of polynomials $f_k$ of degree $k - 1$ such that $C_k f_k = f_k$. If we compare the $(x,y)$-entries we find that the number of entries contained in two given blocks $x$ and $y$ is $g(x,y)$, where $g$ is the number of entries contained in two given blocks $x$ and $y$.

Let $x$ and $y$ be blocks whose incidence is $i$. Then $x = y$, and for $i = 0,\ldots,n-1$ there is exactly one. Hence $K_i$ is a zero of $g$ for $i = 0,\ldots,n-1$, and $g(x,y) = 0$ for $x = y$. Since $g(x,y) = 0$ for $i = 0,\ldots,n-1$, and $g(x,y) = 0$ for $x = y$, we have $g(x,y) = 0$ for $x = y$. Hence the $M_n$-designs is a polynomial $(f_k - a_{ij})$ of degree $k - 1$ in $(x - y)$ such that $f_k f_k = f_k$ for $i = 0,\ldots,n-1$.

2. Diameter Matrices (Redundant Elements)

We start with a summary of some definitions and results of Maximum [2]. A distance matrix $C$, with symmetric matrix $C = (c_{ij})$ with the properties $c_{ii} = 0$, $c_{ij} = 0$ if $i = j$, and $c_{ij} = c_{ji}$ for all $i, j, k$, is a distance matrix for $n$ points such that $c_{ij} = c_{kl}$ for all $i, j, k, l$. A distance matrix with diagonal elements equal to 0 is called a distance matrix without off-diagonal zeros such that $C$ has strength $s$ for all $i$. A distance matrix with diagonal elements equal to 0 is called a distance matrix without off-diagonal zeros such that $C$ has strength $s$ for all $i$.
Lemma 2.3. A distance matrix \( C \) without off-diagonal zeros contains a distance matrix \( C' \) in a Delaunay matrix \( D \) if

\[
\sum_{i,j} C_{ij} \quad \text{for the } (i,k) \text{-th entry, there are polynomials } p_i(x) \text{ of degree } n \text{ such that (5) holds, here } n \text{ is the number of distinct non-zero entries of } C.
\]

If \( A \) is the incidence matrix of a 1-design with block size \( k \), then \( C = M - A \) is a distance matrix of strength 1 if the design is a 1-design, i.e. satisfies the following assertion:

1. If \( x \) is a point and \( y \) is a block then the number of pairs \((x, y)\) consisting of a point \( x \) and a block \( y \) with \( x \neq y \) depends only on whether \( x \) is on \( y \) or not.

We now give sufficient conditions for the distance matrix of a 1-design to be a Delaunay matrix. For \( x \in X_1, x \neq X_2 \), we define \( a_{ij}(x) \) to be the number of points \( (x, y) \) in \( X_1 \times X_2 \) with \( y \neq x \) and \( a_{ij}(x) = a_{ji}(x) \). We consider the following generalization of (v):

(v') If \( x \in X_1, y \in X_2 \), then \( a_{ij}(x, y) = a_{ij}(x) \), for all \( n \times n \) matrices \( A, B \) with \( x \neq y \), \( i, j, k < n \). An \( M_n \)-design with property (v) is called an \( A_n \)-design.

Lemma 2.2. (ii) implies the extra sum \( \mu_n \), such that for \( i < n, \)

\[
\mu_n = \sum_{i=1}^{n} \mu_n A_{m,n}.
\]

Proof. The \((a, b)\)-term of the matrix on the left of (10) is just the number counted in (10), whenever it is \( a = b \) if \( a \neq b \) is a \( b \)-variety. On the other hand, the \((a, b)\)-entry of \( \mu_n A_{m,n} \) is the number of \( b \)-varieties contained in \( b \) if \( a \) which is \( a \). Hence (8) holds if

\[
\mu_n = \sum_{i=1}^{n} \mu_n A_{m,n}.
\]
Let $C$ be the distance matrix of an $S$-design. Then $C$ has strength $s$, and if there are no repeated blocks and at least $s$ distinct intersection numbers then $C$ is even a Delaunay matrix.

Proof. For $i < n$, $C_i$ is a polynomial of degree $i$ in $C$, whereas each $C_i$ is a linear combination of $C_0, \ldots, C_n$. Hence if $C_i$ implies the existence of a polynomial of degree $i$ with (7) for $i < n$, $C$ is even a Delaunay matrix. Now $C_{i+1} = C_i + 1$ has an $(i+1)$-entry for $C_i = C_{i-1}$, which is the diagonal entry of $C_{i-1}$. Therefore $C_0 = C_0 + 1$. Hence $C$ has strength $s$. Since the entries of $C$ are $K_n$, every intersection number, the second part follows from Lemma 2.1.

Corollary 2.3. The blocks of an $S_n$-design without repeated blocks and with $s = o$ distinct intersection numbers form an association scheme with $s$ classes.

Proof. In this case, the intersection numbers are even in $\{K_n, K_{2n}, \ldots, K_{sn}\},$ where $s = o$.

For $s = 2$, there are well known results implying the existence of a strongly regular graph on the blocks for quasi-symmetric $S_n$-designs, partial geometries and Bose systems $S(n, k, \lambda)$. For $s = 3$, the corollary is related to Theorem 7 of Delaunay [3]. For a block $B$ is an $S_n$-design then it is a regular association. In fact, Delaunay's $3(K_n, K_{2n}, K_{3n})$ equals $K_3$, if $s < 3$, and $\{K_3, K_4\}$ if $s = 3$. Hence for Theorem 7 implies 2.3.

Lemma 2.6. Let $\mathcal{B}$ be an $S_n$-design with $s = 3$ distinct intersection numbers. $\mathcal{B}$ is an $S_n$-design if the following axioms hold:

(P) The number of $x \in X_1, y \in X_2$ is a constant $\gamma(x, y)$, for every $x \in X_1$, $y \in X_2$, with $x \neq y$, every non-negative number $\mu$, and all integers $s$ with $0 < s < n$.

Proof. Suppose first that (P) holds. Then the pairs admissible in axiom (8), there are $\gamma(x, y)$, for every $x \neq y$, and then $\gamma(x, y)$, for $x = y$, trivially. Hence

$$\gamma(x, y) = \sum_{i=1}^{n} \gamma_{i}(x, y),$$

independent of $x$ and $y$. Hence (8) holds. Conversely, if (8) holds then we have (11) with $\gamma_{i}(x, y)$ possibly depending on $x$ and $y$. Since (11) holds for all $i$, and since $\gamma_i$ is a linear combination of $[0, \ldots, 0, 1, 0, \ldots, 0]$, we have, for $i = 0, \ldots, s - 1$, $\sum_j \gamma_{i}(x, y) = \gamma_{i}(x, y)$ independent of $x$ and $y$. Since these equations have a Vandermonde matrix, they have a unique solution, whereas $\gamma_i(x, y) = \gamma_{i}(x, y)$ independent of $x$ and $y$.

Remark. (P) implies (8) for every $S_n$-design.
Distance matrices and subcombinatorial ideals

Lemma 2.7. If a distance matrix has minors $q_i(x)$ of degree $i$ (for all $i < n$) such that the matrix $A = q_0$ is a $C$-matrix, for $i < n$, $C_i = RA_iA_i$, $i = 1, \ldots, k$, $k < n$,

\[ J_i = RA_iA_i \]

(12)

By (9), the vector space $V_i$ generated by $C_1, \ldots, C_i$ is an ideal. Let $J$ be a transverse matrix in the orthogonal complement of $V_i$ in $V$, so that $J_i = V_i$ for $i < n$. Then $J_i$ is the set of all $J_i$ and we may normalize $J_i$ so that $J_i = R_i A_i$. Then (12) holds and $V_i$ is generated by $J_i$. Let $C_i = J_i A_i$, $i = 1, \ldots, k$, for appropriate $a_{j,i}$, $i = 0, 1, \ldots, k$. By (12) and (9), $C_i = RA_iA_i = C_i$, and since $J_i = V_i$, the $J_i$ are also generated by $C_1, \ldots, C_i$, whence $J_i = \cdots = C_i$, with a polynomial $q_i(x)$ of degree $i$.

Remark. If $J_i$ is the inverse of the triangular matrix $\rho_{ij}$, with $\rho_{ij} = 0$ if $i < j$, then $J_i = \frac{1}{\rho_{ii}}R_i A_i C_i$, hence $\rho_{ij} = \frac{1}{\rho_{ii}}R_i A_i C_i$.

Lemma 2.8. If there are a distinct intersection number $\rho_{ij}$, for $i < j$, then

\[ \rho_{ij} = \frac{1}{\rho_{jj}}R_i A_i C_i \frac{1}{\rho_{ii}}R_j A_j C_j, \quad \rho_{ij} = \frac{1}{\rho_{jj}}R_i A_i C_i \frac{1}{\rho_{ii}}R_j A_j C_j. \]

(14)

Proof. By (9), the vector space $V_i$ generated by $C_1, \ldots, C_i$ is an ideal. Hence they commute with each other and, by comparing the coefficients of $J_i$ in $C_i$ and $C_i$, the result follows.

Lemma 2.9. Let $C_i$ be the rank of $J_i$, then $C_i$ and $A_i$ are such that $J_i = \cdots = J_i$.

Proof. Denote by $\dim C_i$ the row space of $A_i$. Then $\dim C_i$ is also the row space of $C_i = A_i A_i$, and contains the row of all matrices $C_i = A_i A_i$, with $i < k$. Therefore $\dim C_i$ contains the row of all $0$, $i < k$, and hence its dimension is at least $J_i = \cdots = J_i$. On the other hand, $C_i$ is a linear combination of $J_1, \ldots, J_i$, whence we have equality.

Remarks.

1. By Lemma 2.9, we have

\[ f_k = \cdots = f_k \leq \sqrt{k}. \]

(15)

2. There are already two-dimensional examples (partial geometries) with rank $A_2 < 1|K|$, hence inequality (15) is possible. On the other hand, we have equality for Examples 1 and 2 (see Cameron [2]).

3. By (15), the rank of $C_i$ equals the sum of the $f_k$ with $\rho_{kk} = 0$. Hence we have

\[ \rho_{kk} = 0 \quad \text{for all } i < k. \]

(16)

Theorem 2.10. The distance matrix of an $M$-design with exactly a distinct intersection number is a $C$-matrix if it satisfies the following axioms:

1. $\mu_k(x) = 0$, $x \neq x$.

2. $\mu_k(x) = 0$ for all $k < n$, $x \neq x$.

Here $\mu_k(x) = 1$ if $x = x$, $\mu_0 = 0$ otherwise.
Theorem 2.11. The distance matrix of a three regular lattice $\Omega$ is a Delaunay matrix if $\Omega$ is $M_\lambda$-design satisfying $(\mathbb{S})$.

Proof. Since $(\mathbb{S})$ is equivalent to either of the following two statements:

(i) $A_0 C = (b_0)$ is a linear combination of $A_{0,k} C_0, \ldots, A_{0,n} C_n$, for all $0 < k < n$;

(ii) $C_0 C$ is a linear combination of $C_0 C_0, \ldots, C_0 C_n$, for all $0 < k < n$.

In fact, $(\mathbb{S})$ and $(\mathbb{S})$ are equivalent since $a_{0,k}(x)$ is the $(x,y)$-entry of $A_{0,k}$, and $(\mathbb{S})$ follows from $a_{0,k}$ by left-multiplication with $A_{0,k}$ if $(\mathbb{S})$ holds, see $C_0 C = \sum_{i=0}^n x_i C_i C_i y_i$.

Then define $X = A_0 C - (b_0) = \mu C - (p_0)$. Then $A_0 X = 0$, whereas $X' X = 0$, so that $X = 0$ which implies $(\mathbb{S})$.

Now suppose that the distance matrix $C$ of $\Omega$ is a Delaunay matrix. Then $(\mathbb{S})$ implies that $C_0 C$ is a linear combination of $C_0 C_0, \ldots, C_0 C_n$, for all $0 < k < n$; conclude $a_{0,k}(x) = 0$ for $0 < x < n$.

We have with matrix entries $c_{ik}$ since the proofs of $(2)$ and $(3)$ depend only on $(4)$ they still hold. Hence

$$C C = \sum_{i=0}^n \delta_{i0} C_i C_i$$

and induction on $i$ proves that for $0 < k, C_0 C_0, \ldots, C_0 C_n$ generate $V_i$ (see $(10)$). Hence $C_0 C_0, \ldots, C_0 C_n$ generate $V_i$, and $(17)$ implies $(\mathbb{S})$.

Finally, suppose that $(\mathbb{S})$ holds, and assume that we know already that, for all $0 < k < n$ and all $i$, $C_i C_i$ is a linear combination of $C_i C_i, \ldots, C_i C_n$; $(10)$ is true for $x = i$. Hence $C_i$ and $C_i$ generate as polynomials in $C_i$ (11) implies by our assumption that $C_i C_i$ is a linear combination of $C_i C_0, \ldots, C_i C_n$. Hence this holds for all $i$, and $C$ is shown to be a Delaunay matrix exactly as in (2.3).

References

