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NEW INEQUALITIES FOR THE PARAMETERS
OF AN ASSOCIATION SCHEME

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New inequalities for the parameters of an association scheme

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Let X be a finite set with v objects. An s-class scheme
on X is a partition of the set of all 2-subsets of X into
s nonempty classes. Two objects x and y are i-th
associated if x \neq y and x,y is in the i-th class of
the partition. We define k_i(x) as the number of i-th
associates of x, and p_{ij}(x,y) as the number of z \in X
which are i-th associates of x, and j-th associates of y. If we
write D_i = \{d_{xy}^i\} with d_{xy}^1 = 1 or 0 according as x and y
are i-th associates or not then

\[ D_i D_j = \{p_{ij}(x,y)\}, \]

(1)

\[ p_{ij}(x,z) = k_i(x) \delta_{ij}, \]

(1')

where \(\delta_{ij}\) ( = 1 or 0 according as i = j or not) is the
Kronecker symbol. An association scheme is a scheme with
k_i(x) = k_i for all x \in X, and p_{ij}(x,y) = p_{ij} whenever
x and y are i-th associates.

For the following well-known results see e.g. Cameron, et al [1].

The Bose-Wenner-algebra of an association scheme is the
algebra generated by the matrices I and D_1,...,D_s. There
is a basis \(E_1,...,E_s\) of \(V\) satisfying for i,j = 1,...,s

\[ E_i E_j = \delta_{ij} E_j. \]

(2)

Also, the Bose-Wenner-algebra is closed under pointwise
multiplication \((e_{xy})^2 = (e_{xy})^2\), whence
\[ E_{j}^{j} = \sum_{\ell=0}^{n} q_{j}^{\ell} \xi_{j}^{\ell} \]  

(3)

For appropriate numbers \( q_{j}^{\ell} \), which can be calculated from the parameters, the

**Rokhlin condition**

\[ q_{j}^{\ell} > 0 \text{ for } i, j, \ell = 0, \ldots, n \]  

(4)

gives a restriction on the parameters.

The ranks \( E_{j}^{j} \) can also be calculated from the parameters; they satisfy

\[ f_{0}^{*} \ldots f_{n}^{*} = v, \]  

and the fact that all \( f_{i} \) must be integers places a severe restriction on the possible parameter sets. We prove here a new inequality for the ranks:

**Theorem 1**

The following inequalities hold:

\[ \sum_{\{\xi_{j}^{\ell} \neq 0\}} f_{\ell} \leq \begin{cases} \frac{f_{i}f_{j}}{2f_{i}} & \text{for } i \neq j, \\ \frac{f_{i}^{*}}{2f_{i}}(f_{i}^{*} + 1) & \text{for } i = j. \end{cases} \]  

(5)

**Proof.** By the following lemmas, the rank of \( E_{j}^{j} \) is at most

\[ \min f_{i} f_{j} \text{ if } i \neq j, \]  

and

\[ \frac{f_{i}^{*}}{2f_{i}}(f_{i}^{*} + 1) \text{ if } i = j. \]  

On the other hand, since the \( E_{j}^{j} \) are mutually orthogonal, (3) implies that the rank of \( E_{j}^{j} \) is given by the left hand side of (5).

**Lemma**

(i) Let \( A \) be a matrix of rank \( r \). Then \( A^2 \) has rank \( \leq \frac{1}{2}(r+1) \).

(ii) Let \( A \) and \( B \) be matrices of the same size of rank \( f \) resp. \( g \). Then \( AB \) has rank \( \leq fg \).
Proof. Write $A = (a_{xy})$, and let $x_1, \ldots, x_f$ be the labels of $f$ independent rows. Then each $a_{xy}$ is a linear combination of $a_{x_jy}$, $j = 1, \ldots, f$. Hence each entry $a_{xy}$ of $AA$ is a linear combination of $a_{x_jy}$, $j = 1, \ldots, f$, of which there are $f + \binom{f}{2} = \frac{f(f+1)}{2}$ terms. This proves (i), and the proof of (ii) is completely analogously.

A 2-class association scheme is essentially the same as a strongly regular graph (see e.g. Seidel [2] for a definition). A strongly regular graph with $d_2^2 = 0$ is called a Smith graph. Cameron et al. [1] show that $d_2^0$ and $d_2^1$ are non-zero. Hence theorem 1 gives

Theorem 2

(i) The parameters of a strongly regular graph which is not a Smith graph satisfy
\[ v \leq \frac{1}{2} \left( f_2^4 + 1 \right). \]  

(ii) The parameters of a Smith graph satisfy
\[ v \leq \frac{1}{2} \left( f_2^2 + 1 \right). \]

Proof. Apply theorem 1 with $i = j = 2$, and observe that $f_0 f_1 f_2 = v$.

Example

The following parameter set for a strongly regular graph satisfies all previously known conditions for strongly regular graphs (as stated e.g. in Seidel [2]) but fails (6):
\[ v = 881, k = 300, \lambda = 97, \mu = 35, r_2 = 10. \]
Problem. Characterize those graphs for which (6) is satisfied with equality.

Remarks. 1. More inequalities can be obtained similarly by looking at $E_{\lambda, \mu}(\mathbf{X})$, etc., but it is not known whether they are really more restrictive than those of theorems 1 and 2.

2. The special case of theorem 2, where the graph has a rank 2 automorphism group, has been proved already in Cameron, et al [1].

3. Theorem 2 improves the absolute bound (see e.g. Seidel [3]) for strongly regular graphs; it is not known how theorem 1 relates to the more general absolute bound mentioned in [1], proposition 6.1.

References
