Some sporadic geometries related to $PG(3,2)$

By A. P. Dempster

Introduction. The motivation for writing the present paper has been twofold. Firstly, in 1973, I discovered two sporadic geometries whose diagrams to the same of Jaffke (5) in a chapter designed, I wrote about them in Perspec. Perspect., and since he didn’t answer I judged them to be of little relevance. But last year (1982) he asked me to publish my work about the new approach to building of the first sporadic groups related to such geometries (10), (11), (12), (13) which I did not (11) now wish to present which are now called building (CAB). Secondly, John Cohens and myself are working on a collection and survey of the known distance regular graphs. The present work will provide a convenient reference for one such graph, a regular thin near octagon on 196 vertices induced to one of the CAB’s, whose existence was claimed before only indirectly from results of Cohen and Wales (14) in the Hoffman-Singleton graph.

1. The projective space $PG(3, 2)$. We denote by $PG(n, q)$ the projective space of (projected) dimension $n$ over a finite field with $q$ elements (cf. McWilliams and Sloane (12)). The projective space $PG(3, 2)$ has 15 points, 35 lines and 10 planes. Each plane contains 7 points and 7 lines, each line contains 3 points and 3 lines. Any two lines in a plane intersect, and if two non-intersecting planes they have a unique common point and lie in a unique plane. $PG(3, 2)$ has an automorphism group the group $A_6$, isomorphic to the alternating group $A_6$ on a set of 6 letters. This implies the existence of a partition of the 15 lines of $PG(3, 2)$ into the 10 partitions of type 4 of a standard partition into two sets of size 4 of the 6 letters. Specifically we have

Preposition 1. There is a bijection $I = 1$ between the 30 lines $l$ of $PG(3, 2)$ and the 30 partitions $I$ of type 4 of $X \times X$, such that

(a) two lines $l_1$, $l_2$ intersect if and only if the partitions $I_1$ and $I_2$ intersect in a partition of type 2; the three lines determined by such a partition are concurrent and coplanar.

(b) for any three distinct letters $a, b, c \in X$, and each point $x \in PG(3, 2)$, there is a unique line $l(x)$ such that one of the two 4-sets of $I$ contains $a, b, c$ and $x$. 

Proof. The bijection is easy to see inside the Steiner system $S = S(2,4,8)$. By Cameron [6], whose terminology, we use the group of automorphisms of $S$ fixing an orbit $A$ and a point $p$. Let $A_0$ be an orbit of $A$ and $(x, y, z)$ a point of $A_0$. Then, $A_0$ on $x$, $y$, $z$ and $(x, y, z)$ a point of $A_0$.

With lines being the sets $P_{1}, P_{2}$, where $T$ is an orbit intersecting $S$ in a point, $P_{1}$ is a projective space $P(2,4)$, and there is a one-to-one correspondence between the 35 partitions of type $1_{4}$ of $S$, the 35 spaces defined by them, and the 35 lines induced on $S$ by such a partition of $S$. Now to determine $S_{1}$ of $S$, define three axioms, corresponding to those mentioned in Kaplansky's case, and conversely, two intersecting lines of $P_{1}$ define two distinct intersecting lines in a partition of type $1_{3}$ of $S$, since they induce on $S$ a partition of type $0_{4}$.

To show (i), let $A_{2}$ be the set of $4$-sets contained in some partition $I_{2}$, where $I_{2}$ varies over the lines $a_{S}$, $b_{S}$, and $c_{S}$. Then $A_{2}$ contains 14 quadruples, and by (i), any two of these have 0 or 2 common letters. Since a triple $a$, $b$, $c$ of distinct letters is in at most one quadruple of $A_{2}$, but the total number of triples is a quadruple of $A_{2}$ is 14, 4 = $\binom{4}{0}$. Hence each triple occurs precisely once.

In the terminology of designs of M.H. Hall and E. Troida [23], the 15 sets $A_0$, and classify the 35 sets $S_{1}$, excluding of the four in the partition $I_{1}$ for some line $l_{1}$ in a plane of $P(2,4)$, and $S_1$ - $\{a, b, c\}$. The group $S_{1}$ is isomorphic to the affine group $A(2,4)$. With $S_{1}$ in our Group $P(2,4)$, there are precisely 30 and 2 - $\{a, b, c\}$, $S_{1}$ design falling into two orbits of 15 under $S_{1}$ - $\{a, b, c\}$, then specifying the points and blocks of $P(2,4)$. We remark that this description can be used to give an elementary proof of Proposition 3, the Steiner system being used only for brevity.

As is apparent from the previous remarks, the odd permutations of $S_{1}$ induce relabels of $P(2,4)$. In particular, the partitions of $P(2,4)$ are induced by odd permutations of $S_{1}$. As a result, the group $S_{1}$ (by the group $S_{1}$) has precisely the three lines defined by the permutations $(a, b, c, a, b, c)$ and the corresponding intersecting point and extending plane, since it defines an algebraic geometry. A translation $\tau$ leaves the 13 lines corresponding to the permutations $\{a,b,c,a,b,c\}$ and the 10 points-planar flags corresponding to the partition of the triple $(a, b, c, a, b, c)$ and their fixed points and line form a projective quadrangle of order two.

We now fix a point $\eta$ of $T$, and write $T = X(\eta)$. Then we may complete a triple (invariant) $D_{2}$ of $T$ with the partition $(\eta, \eta, \eta)$, $\eta$ is a point of $X(\eta)$. In this way we obtain from Proposition 1:

**Proposition 2.** There is a bijection $I_{2} \mapsto I_{1}$ between the 35 lines $I_{1}$ of $P(2,4)$, and the 35 hyperplanes $I_{2}$ of $P(2,4)$ such that

(i) two lines $l_{1}$ and $l_{2}$ intersect if and only if the triples $l_{1}$ and $l_{2}$ have precisely one common point,

(ii) for any two distinct letters $a_{S}$, $b_{S}$, and for each point $p_{S}$ in $P(2,4)$, there is a unique line $l_{1}$ such that $a_{S}$, $b_{S}$, $c_{S}$, $d_{S}$, $e_{S}$.
This second representation is invariant under $\mathcal{A}$ only (the stabilizer of $\sigma$). The points and planes may now be viewed as the 30 [five plane] PG(2, 2) definable over $\mathcal{K}$, which again give rise to $10 \times 10$ under $\mathcal{A}$. The transpositions of $\mathcal{K}$ (i.e., these fixing $\sigma$) again form symplectic polarities.

2. Three linked partial 4-geometries. A special feature of the representation of $PG(2, 2)$ as in Proposition 2 is that pairs of skew lines may be classified into two types, namely depending on whether the associated triples are disjoint or not. Moreover, this distinction is preserved by the symplectic polarity induced by transpositions of $\mathcal{T}$.

This fact has been used by Hammaker [9] to construct the following graph $\Gamma$. Vertices (or points) of $\Gamma$ are the symbols $p_0$ and $p_1$, where $a \in \mathcal{A}$ point and $b \in \mathcal{B}$ is a line of $PG(2, 2)$; the neighbors of $p_a$ are the $7$ vertices $p_b$ with $a \neq b$, and the neighbors of $p_b$ are the $7$ vertices $p_a$ with $a \neq b$. It is immediate to see that $\Gamma$ is regular of valency $\Delta = 7$, and since the number of vertices is $2 \cdot 10 \cdot 10 = 300$, it is the Hoffman-Designs graph (Hoffman and Singleton [12]). The Hoffman-Designs graph $\Gamma$ contains no triangles or quadrangles, and two nonadjacent points have a unique neighbor.

We now turn $\Gamma$ into a design $\mathcal{D}$ by defining $10 \times 10 = 100$ blocks $$(p_0 \times \mathcal{A}) \cup (p_1 \times \mathcal{B})$$

$R_0 = \{p_a : a \in \mathcal{A}\} \cup \{p_b : b \in \mathcal{B}\}$ for those $\alpha$ of $PG(2, 2)$.

The design $\mathcal{D}$ is point-line, for every symplectic polarity induced by a transposition of $\mathcal{T}$, the map $y \mapsto -y$, $y_0 = x_0$, is a polarity of $\mathcal{D}$. Since every block contains $10$ points, $\mathcal{D}$ is a symmetric $1$-design.

Now, in the terminology of Bieleck [10], a partial $\mathcal{A}$-geometries with block size $\mathcal{B}$ and some $c$ is a symmetric $1$-design $\mathcal{D}$ whose blocks contain $c$ points, such that two distinct points are in at most $k$ of $c$ blocks; two distinct blocks are in at most $t$ points, and $c \in c(k, t, B)$ is a nonnegative point-block pair then there are $\Delta$ constant point-block pairs $(0, a)$ with $a \in \mathcal{A}$.

\begin{center}
\begin{tikzpicture}
  \node (A) at (0,0) {$A$};
  \node (B) at (0,1) {$B$};

  \draw (A) -- (B);
\end{tikzpicture}
\end{center}

A geometric study (Seemeyer [14]) shows that there are precisely two feasible parameter sets with $c^2 = 8$, namely

- $K = 10$, $A = 8$, $r = 12$ (with 60 points),
- $K = 11$, $A = 9$, $r = 12$ (with 80 points).

We show here that our design realizes the first possibility.
Lemma. Two distinct points of $\mathcal{F}$ are 0,1, or 2 blocks of $\mathcal{F}$, depending on whether they are adjacent or not.

Proof. Let $a, b$ be distinct nonadjacent points of $\mathcal{F}$. Up to symmetry in $a$ and $b$, we have to consider four different situations. In each case we consider lines in $\mathcal{F}$ and planes in $\mathcal{F}$ which define 0, 1, or 2 blocks $\mathcal{F}_a$, $\mathcal{F}_b$.

(i) $a \neq b$, $a \neq b$. Then there are lines $l \perp a$, $\ell \perp b$, and planes $\mathcal{P} \perp a$, $\mathcal{Q} \perp b$.

(ii) $a \neq b$, $a = b$. Put $l = a, \ell = b$. Then there are lines $l \perp a$, $\ell \perp b$, and planes $\mathcal{P} \perp a$, $\mathcal{Q} \perp b$.

(iii) $a = b$, $a = b$. Then there are lines $l \perp a$, $\ell \perp b$, and planes $\mathcal{P} \perp a$, $\mathcal{Q} \perp b$.

(iv) $a = b$, $a = b$. Then there are lines $l \perp a$, $\ell \perp b$, and planes $\mathcal{P} \perp a$, $\mathcal{Q} \perp b$.

Therefore, two nonadjacent points are in at least 2 blocks. This gives a total of $45 \times 3 = 135$ blocks, each of which is a $\langle a, b, c \rangle$ triple with $a, b, c \in \mathcal{F}$. Hence we have accounted for all such triples, and the assertions of the Lemma follow.

Proposition 3. The design $\mathcal{F}$ is a partial geometry with block size $k = 15$ and order $n = 19$.

Proof. By the lemma, and the self-duality of $\mathcal{F}$, only the equation on the nonmaximal points need to be verified. Let $a$ be a block, and denote by $l_a$ the number of points of $\mathcal{F}$ intersecting with $a$. Then, since two nonadjacent points have a unique plane, we have

\[ \sum_i l_a = 30 - 15, \quad \sum_i l_a = 30 - 15, \quad \text{hence} \quad \sum_i l_a = 15. \]

Therefore, $l_a = 3$ for all $a \in \mathcal{F}$, and there are 13 points $b \not\in \mathcal{F}$ nonadjacent with a fixed $a \in \mathcal{F}$. By the lemma, this yields $9 - 15 = 15$ (indeed) point-block pairs $(a, b)$ with $a \perp b$. \hfill $\square$

By Demb [3], the incidence graph of a (proper) partial geometry is an primitive distance-regular graph of diameter four, or, in the terminology of Shult and Tanaka [36], a thin regular rerouter. In the present case, given by the following diagram.

\[
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\end{array}
\]

In fact, this graph $G^*$ is an even distance transitive, with automorphism group $P_3\Gamma\mu(2,3,8,3,9)$. We show this by exhibiting the graph as a set of vectors inside the Levi lattice. We use the description of the Levi lattice $A_2$ given by Gropp [8] and Curtis [7]. In accordance with our previous notation for the distance system $D(34,8,3)$.
we arrange coordinates such that the b last coordinates form an octad X. We write x for the first coordinate, y for the first coordinate of X, and z for the complement of X\x (y). and y for z = X - x).

The set S of vectors of ∏y of norm 9 orthogonal to the axes 46 vector e = (0, 1, 0) consists of 220 vectors of shapes
\[ \pm (2,7,1,0), \pm (220 vectors).\]
\[ a \pm (1,3,2,0). \]

They have mutual inner product \[ \pm 16 \] and define 784 equiangular lines (cf. Taylor [17], p. 6.6. for the resulting regular two-graph). The three vectors
\[ a = (0,0,0,0), \quad \mathbf{b} = (1,1,0,0), \quad \mathbf{c} = (0,0,0,1). \]

To le 248; they have none 2D and mutual inner product \[ = 8. \] Since by Curtis (1), p. 500, the group of automorphisms of the 2D lattice fixing the set \([a, b, c, d]\) is \( G = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \), where the subgroup \( \mathbb{Z}/2\mathbb{Z} \) fixes each of a, b, c, d, the vectors c and d and the vector e in multiplication by \[ = 1 \]. The subgroup of \( \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \) fixing a, b, c, d, respectively have index \( = 8 \) in \( G \), hence an isomorphism to \( C_4 \times C_4 \) (4 = 2\( \times 2 \)).

Thus \( a = 1 \) is a point of \( S \), the group \( G \) acts on \( S \). One consists of 220 vectors of shape
\[ \pm (0,1,3,2) \pm (0,1,3,2) \pm (0,1,3,2) \] (180 consists 2D intersect Y in two points). orthogonal to \( a \), \( b \), \( c \); they again define a regular two-graph (cf. Taylor [19], p. 4.4). The fibre with consists of \( 5, 9, 20 \) vectors falling into 16 types depending on the value of the class products \([a, b, c, d], [b, c, d, e], [a, b, c, d, e] \).

type \( \pm (0,1,3,2) \pm (0,1,3,2) \pm (0,1,3,2) \). type \( \pm (0,1,3,2) \pm (0,1,3,2) \pm (0,1,3,2) \). type \( \pm (0,1,3,2) \pm (0,1,3,2) \pm (0,1,3,2) \).

The 50 type I vectors are of shapes
\[ (0,1,3,2), \quad (0,1,3,2), \quad (0,1,3,2) \] (180 consists 2D intersect X in 4 points).

The 50 type II vectors are of shapes
\[ (0,1,3,2), \quad (0,1,3,2), \quad (0,1,3,2) \] (180 consists 2D intersect X in 4 points).

The 50 type III vectors are of shapes
\[ (0,1,3,2), \quad (0,1,3,2), \quad (0,1,3,2) \] (180 consists 2D intersect X in 4 points).

The 100 vectors are of shapes
\[ (0,1,3,2), \quad (0,1,3,2), \quad (0,1,3,2) \] (180 consists 2D intersect X in 4 points).

The negative of these vectors are of type \( -1, -II, \) and \( -III \), respectively.
Proc. This it is clear that the $d_6$ is given $X$, and so split the type I and type II vertices on $10 + 20$, and the type III vertex on $1 + 7 + 4$. Indeed, the type III vertex

$$v = (1, 2, 3, 4, 5, 6)$$

has inner product $16$ with $10$ type I vertices, $10$ type II vertices, and $42$ type II vertices, and inner product $-16$ with $10$ type I vertices, $20$ type II vertices, and $7$ type III vertices. In particular, $G_n = P^2_1(16)$ acts as a cyclic $5$ group on the $100$ vertices of type I and II, with subgroups $1$, $5$, $10$, $15$, and $20$. Therefore, $G_n = P^2_1(16)$ also acts as a cyclic $5$ group on the $100$ vertices of type I and II, with the same sub-

10. But the graph on these vertices obtained by joining a vertex $x$ of type I with a vertex $y$ of type II whenever $(x, y) = 10$ is easily seen to be isomorphic to the incidence graph $\Gamma^*$ of the partial 2-geometry $G$ described above. Therefore, $\Gamma^*$ is distance transitive.

Remarks. 1. The vertices of type I, II, and III form a system of (Steiner) partial 2-geometries, related by the outer automorphism of $P^{-1/2}(16)$ of order three. Thus the situation is similar to that of the linked partial 2-geometries constructed by Coxeter and Moser ([2]). In [2], this is the only known example of a distance transitive graph $\Gamma^*$ that is not isomorphic to the incidence graph $\Gamma$ of a distance transitive group $G$ described in [2].

2. Perhaps the most natural way of drawing the graph $\Gamma^*$ is as follows: vertices of $\Gamma^*$ are the $100$ points of two $10$ in the Hoffman-Singleton graph $\Gamma$, two points are at distance $1, 2, 3, 4$ if they intersect in $0, 1, 2,$ or $3$ points, respectively. This can be deduced either directly from the above, or from a different representation of the Hoffman-Singleton graph described in [2].

3. A symmetric geometry for $d_6$ and $d_6$ is given by $G^*(6, 6)$. Using the description of $P^2_1(3, 2)$. In Section 1, we define a rank 4 geometry in the sense of Bader et al. ([2]). We assume the reader to be familiar with Bader et al.'s paper on the rank 4 geometry.

The 5-varieties of our geometry are the 5 v-irregulars from $X$ and $Y$-irregulars are the $20$-transpositions on $X$. If $X$-irregulars, we take the $10$ lines of $P$ and $Y$-irregulars are the $10$ points of $P$. We define incidence as follows: a symbol $a$ is incident with the transpositions moving $a$; with all lines,

$$A_5 \quad \text{and} \quad \text{Bader et al.'s diagram}$$

The verification of the outer and the diagram is a straightforward consequence of the results of Section 1. The intersection property does not hold here, e.g., every transposition is incident with every point.
Two notions of this geometry (both without the intersection property) are also interesting: namely the notion of a point, with diagram:

\[ 2 \times 1 (2) \]

and the notion of a symbol, with diagram:

\[ \text{Diagram} \]

This latter geometry is particularly interesting since it has no diagram a Quatem diamond; thus, in the terminology of Kantor [1] it is a GGL geometry which is about a buildings. It has been discovered independently by Shankdzee and Schen [2].

A second GGL arises from the partial 3-geometry of Schen in Section 2. Indeed, take as vertices the symbols \( e, e', e'' \), and in a block of \( 3 \). Addition is defined by:

\[ e + e = e, \quad e + e' = e', \quad e + e'' = e'' \]

From the results of Section 2 it is straightforward to show that the geometry has the diagram:

\[ \text{Diagram} \]

with rank 3 mathe isomorphic to the A-GLG. This geometry is also described in Kantor [10] in group-theoretic terms.

Bibliography

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A. Neusturev
Center for Applied Mathematics
Universität Tübingen 1 Dr.
53010 Tübingen