Some Triple System Constructions with Applications to Resolvable Designs

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The doublet construction and the Kneser product construction for Hadamard matrices are generalized to recursive constructions for homogeneous triple systems. These generalizations lead to new recursive constructions for affine 2-designs and several other related designs.

Introduction

This paper is one of a number (e.g., Neumaier [13,15]) whose aim is to present various design-theoretical results within the framework of a uniform and transparent theory. The new theory uses the language of triple systems, which enables many results (both old and new) to be stated, and proved more briefly and elegantly. For example, affine 2-designs can be interpreted quite simply as triple systems with certain homogeneity and maximality properties.

The key result of the paper is Theorem 3.2, which relates homogeneous triple systems and resolvable 1-designs. Using this theorem it is a simple matter to translate the triple system constructions derived in Section 2 into constructions for design. As applications we obtain generalizations of affine designs of the Sheingold and Kneser product constructions for Hadamard matrices (see Wallis et al., [20]), simple proofs of some results of Maenhaut [11,12] and Mullen [15], and some new constructions for affine designs of the types $D(p_t, m)$ and $D(p, m)$ (in the notation of Maenhaut [11]).
1. Homogeneous Tuple Systems

Let $K$ and $E$ be finite sets. An $E$-tuple over a set $K$ is a map $f: E \to K$. As usual, we write $x = x(y)$ for $x = x_1(y), x_2(y), \ldots, x_n(y)$ if $E = \{1, 2, \ldots, n\}$. Elements of $K$ are called points, and those of $E$ places. A $K$-tuple space over $E$ is a valuation $Q$ of (non-necessarily distinct) $E$-tuples over $K$. For $i \in E$, if $Q \in Q^i$, we can extract $x_i = x_i(Q) \in K$, and then $Q = (x_1, x_2, \ldots, x_n)_{Q^i}$ is the restriction of $Q$ to $i$.

Let $a, b, x, y, z$ be positive integers with $a > b$. A $1-\{a, b, 1\}$-homogeneous tuple space $(Q, \{a, b, 1\})$ HTS (or simply HTS) over $K$ is a tuple system $Q$ over $K$ with $Q^a = (a, a, \ldots, 1, 1)$ and such that for all elements $x, y, z \in K$ and all $x, y, z$ triplets $x, y, z \in Q$, there are exactly $a$ tuples of $Q$ with $x_i = x$, $y_i = y$, and $z_i = z$. Such an HTS is $a$-reachable (HTS for short) if it can be partitioned into $1-\{a, b, 1\}$ HTSs $Q_1, Q_2, \ldots, Q_k$. Then $|Q|_{\{a, b, 1\}}$ is the resolution change of $Q$ and $Q_1$ are the resolution classes of $Q$. Clearly, $Q^a = (a, a, \ldots, 1, 1)$ and $Q^b = (b, b, \ldots, 1, 1)$. Two resolutions $a^r = (a, a, \ldots, 1, 1)$ and $b^r = (b, b, \ldots, 1, 1)$ of $Q$ are orthogonal iff $|Q^a| < |Q^b|$ for all $a, b \in E$. The following examples will be useful in the sequel.

1. EXAMPLE. For $a = b = 1, b = 2, 3, 4, \ldots$ HTS over $K$ and $Q_1 = (1, 1, 2, 2, \ldots, 1, 1)$ with $x_i = y_i = 1$, $z_i = 2$ HTSs with orthogonal resolutions $(x, x, 1, 1)$, $(y, y, 1, 1)$, and $Q_2 = (1, 1, 2, 2, \ldots, 1, 1)$ $1-\{2, 3, 1\}$ HTSs where $x_1 = y_1 = 1$, $z_1 = 2$.

We are mainly interested in HTSs with $a = 2$. For such HTSs, we can find a sharp upper bound for $a$ in terms of $a$ and $b$.

2. THEOREM. Let $Q$ be a $1-\{a, b, 1\}$ HTS over $K$. Then $a < b^2 - (b - 1)(a - 1)$, with equality if and only if the following conditions hold for some positive integer $r$:

$$a, b \in Q, \text{ for } a \neq b \text{ or } \exists i \in E: (\beta)$$

If $|Q|$ satisfies (1), then $a < b^2 - (b - 1)(a - 1)$ and $(a - 1)(b - 1) = 0$ is no longer valid.

The inequality in Theorem 1.2 is established in Ref. [3] in terms of orthogonal arrays. It is easily seen that a $1-\{a, b, 1\}$ HTS is equivalent to an orthogonal array $\alpha$ ($a^b$) in HTS language, a result simpler proof can be given. Fix $x \in Q$, and for $y \in Q, x \neq y$, define $\lambda_y(x) = \min \{i \in E : x_i \neq y_i\}$. Then $\lambda_{y}(x)$ are mutually distinct, and the result follows from the inequality $\lambda_{y}(x) < b^2$.

A $1-\{a, b, 1\}$ HTS with $b^2 - (b - 1)(a - 1)$ will be called maximal.

Simple quadratic covering arguments also yield the following results, the first of which is a well-known theorem in Ref. [1].

3. THEOREM. Every $1-\{a, b, 1\}$ HTS satisfying condition (2),

$$a < b^2 - (b - 1)(a - 1)$$

is a maximum. Thus $a = b^2 - (b - 1)(a - 1)$.

4. THEOREM. Let $Q$ be a $1-\{a, b, 1\}$ HTS with resolution $r$. Then $r = \min \{\lambda_y(x) \in Q : x, y \in Q, x \neq y\}$. The resolution classes of $r$ are the resolution classes of $Q$. Clearly, $Q^a = (a, a, \ldots, 1, 1)$ and $Q^b = (b, b, \ldots, 1, 1)$. Two resolutions $a^r = (a, a, \ldots, 1, 1)$ and $b^r = (b, b, \ldots, 1, 1)$ of $Q$ are orthogonal iff $|Q^a| < |Q^b|$ for all $a, b \in E$.

5. THEOREM. Let $Q$ be a $1-\{a, b, 1\}$ HTS that satisfies (1), then $a < b^2 - (b - 1)(a - 1)$.

$Q$ (or $\alpha$) is satisfied, then $a = b^2 - (b - 1)(a - 1)$.

6. THEOREM. Let $a, b \in Q, x \neq y$ be in $\alpha$ with resolution $r$ and fix $x \in \alpha$. Then $r = \min \{\lambda_y(x) \in Q : x, y \in Q, x \neq y\}$. Thus $|Q|_{\{a, b, 1\}}$ is the resolution of $Q$.

2. Constructions for HTSs

We give here some fairly general recursive constructions for HTSs. Most of the theorems of the proofs are omitted, as the construction of HTS language rules...
2.1 Lemma. If there exists a $\alpha - (\alpha, \beta, \gamma)\text{HTS}$, then there exists a $\alpha - (\alpha, \beta, \gamma, \lambda)\text{HTS}$.

2.2 Theorem. Let $Q$ be a $\alpha - (\alpha, \beta, \gamma, \lambda)\text{HTS}$ over $K/L$, and $Q'$ be a $\alpha - (\alpha', \beta', \gamma', \lambda')\text{HTS}$ over $K'/L$, with resolutions $(\Gamma)^{\phi}(x) \in Q)$. Then $\alpha^\phi \alpha'$. Let $(\theta)^{\phi} \phi(x) \in \Gamma(x)$, where $(\theta)^{\phi} \phi(x) \in \Gamma(x)$. Define $\Gamma' := \Gamma(x) \cup \Gamma(x)$, and let $\alpha^\phi \alpha'$. Let $\alpha^\phi \alpha'$ be a base, then $\alpha^\phi \alpha'$ is a $\alpha - (\alpha, \beta, \gamma, \lambda)\text{HTS}$ over $K'$. Moreover, if $\alpha^\phi \alpha'$ is both maximal, then so is $Q'$.

Our first construction uses a $\alpha - (\alpha, \beta, \gamma, \lambda)\text{HTS}$ with two orthogonal resolutions. Example 1 gives a sense of obtaining such an HTS.

2.3 Theorem. Let $Q$ be a $\alpha - (\alpha, \beta, \gamma, \lambda)\text{HTS}$ over $K/L$, and $Q_1$ a $\alpha - (\alpha_1, \beta_1, \gamma_1, \lambda_1)\text{HTS}$ over $K/L$, with two orthogonal resolutions. Then $\alpha$, $\alpha_1$ and $\alpha^\phi \alpha'$ are both maximal, and $\alpha^\phi \alpha'$ is a $\alpha - (\alpha, \beta, \gamma, \lambda)\text{HTS}$ over $K'$. Then $\alpha^\phi \alpha'$ is both maximal, and so is $Q'$.

Proof. Suppose the resolutions of $Q_1$ are resolved $T_{ij}(x) \in 2 \times K$ with $T_{ij} \in 2 \times K$. Define $Q'$ over $K' \times 2 \times K'_L$ by

$$Q' := \{(x, y) \in 2 \times K \mid (x, y) \in Q \cup T_{ij}(x) \in 2 \times K\}.$$

Let $\alpha^\phi \alpha'$ be a base, then $\alpha^\phi \alpha'$ is a $\alpha - (\alpha, \beta, \gamma, \lambda)\text{HTS}$ over $K'$. Moreover, if $\alpha^\phi \alpha'$ is both maximal, then so is $Q'$.

A similar construction for 1-HTS's gives

2.4 Theorem. Suppose $Q$ is a $\alpha - (\alpha, \beta, \gamma)\text{HTS}$ over $K/L$, and $Q_1$ is a $\alpha - (\alpha_1, \beta_1, \gamma_1)\text{HTS}$ over $K/L$, with two orthogonal resolutions. Then $\alpha$, and $\alpha_1$ are both maximal, and $\alpha^\phi \alpha'$ is a $\alpha - (\alpha, \beta, \gamma)\text{HTS}$ over $K'$. Then $\alpha^\phi \alpha'$ is a $\alpha - (\alpha, \beta, \gamma)\text{HTS}$ over $K'$. Hence, $\alpha^\phi \alpha'$ is both maximal, and so is $Q'$.

Proof. Suppose $Q$, and $Q_1$ are both maximal, and $\alpha^\phi \alpha'$ is a $\alpha - (\alpha, \beta, \gamma)\text{HTS}$ over $K/L$. Then $\alpha$, and $\alpha_1$ are both maximal, and $\alpha^\phi \alpha'$ is a $\alpha - (\alpha, \beta, \gamma)\text{HTS}$ over $K'$. Then $\alpha^\phi \alpha'$ is both maximal, and so is $Q'$.

3. HTS's and Resolvable Designs

The following constructions shows how resolvable 3-designs and 1-HTS's are equivalent concepts. The notation and terminology for resolvability and affine designs is as in Vanstone [11]. The dual notions of "parasite" and "resolution" will be called "point paralleling" and "point resolution", respectively.

3.1 Construction. Let $Q$ be a $\alpha - (\alpha, \beta, \gamma)\text{HTS}$ over $K/L$. Define an incidence structure $Q$ with points set $Q$, and lines set $L$, such that if $L$ is a $\alpha - (\alpha, \beta, \gamma, \lambda)\text{HTS}$, then $Q$ is a resolvable $\alpha - (\alpha, \beta, \gamma)\text{HTS}$. Conversely, when given a resolvable design, we can recover the construction and generate a 1-HTS.
2.3 Theorem. 1 Conjecture 3.1.

(i) If there exist 1 – (v, r, k, λ)-designs for Q, then 2 – (v, k, λ)-designs.

(ii) If a 2-(v, k, λ)-design with Q, then there exist 2 – (v, k, λ)-designs.

(iii) If a 2-(v, k, λ)-design G with Q, then there exist 2 – (v, k, λ)-designs.

Proof: (i) and (ii) are immediate from Theorem 4.1 for (v, k, λ) and assume that there is a point partition of G into two sets, 

P = \{P_1, P_2, \ldots, P_m\}, G = \bigcup_{i=1}^{m} P_i.

Thus, there are two 2-(v, k, λ)-designs for Q, each containing G, and hence there exist 2 – (v, k, λ)-designs.

2.3 Theorem. 2 Conjecture 3.2.

(i) If there exist 2 – (v, k, λ)-designs for Q, then there exist 2 – (v, k, λ)-designs.

(ii) If there exist 2 – (v, k, λ)-designs with Q, then there exist 2 – (v, k, λ)-designs.

Proof: (i) and (ii) are immediate from Theorem 4.1 for (v, k, λ) and assume that there is a point partition of G into two sets, 

P = \{P_1, P_2, \ldots, P_m\}, G = \bigcup_{i=1}^{m} P_i.

Thus, there are two 2-(v, k, λ)-designs for Q, each containing G, and hence there exist 2 – (v, k, λ)-designs.

2.4 Theorem. 3 Conjecture 3.3.

(i) If there exist 3 – (v, k, λ)-designs for Q, then there exist 3 – (v, k, λ)-designs.

(ii) If there exist 3 – (v, k, λ)-designs with Q, then there exist 3 – (v, k, λ)-designs.

Proof: (i) and (ii) are immediate from Theorem 4.1 for (v, k, λ) and assume that there is a point partition of G into two sets, 

P = \{P_1, P_2, \ldots, P_m\}, G = \bigcup_{i=1}^{m} P_i.

Thus, there are two 3-(v, k, λ)-designs for Q, each containing G, and hence there exist 3 – (v, k, λ)-designs.

3. Conjecture. 1 Conjecture 4.1.

(i) If there exist 1 – (v, k, λ)-designs for Q, then there exist 1 – (v, k, λ)-designs.

(ii) If there exist 1 – (v, k, λ)-designs with Q, then there exist 1 – (v, k, λ)-designs.

Proof: (i) and (ii) are immediate from Theorem 4.1 for (v, k, λ) and assume that there is a point partition of G into two sets, 

P = \{P_1, P_2, \ldots, P_m\}, G = \bigcup_{i=1}^{m} P_i.

Thus, there are two 1-(v, k, λ)-designs for Q, each containing G, and hence there exist 1 – (v, k, λ)-designs.

4. Conjectures and Problems

We list some known results about existence and nonexistence of designs 2-(v, k, λ)-designs and 3-(v, k, λ)-designs and some unsolved conjectures.
A1. Theorem 6.6: An additive, m,n, can only exist if \( m - 1 \leq n \leq m - 1 \).

A2. Theorem 6.2: An additive, m,n, exists if and only if \( m - 1 \leq n \leq m - 1 \).

A3. Theorem 6.3: An additive, m,n, exists if and only if \( m - 1 \leq n \leq m - 1 \).

A4. Theorem 6.4: An additive, m,n, exists if and only if \( m - 1 \leq n \leq m - 1 \).

A5. Theorem 6.5: An additive, m,n, exists if and only if \( m - 1 \leq n \leq m - 1 \).

References

2. R. C. Bunn, "On the existence of additive, m,n, for fixed m and n, "J. Res. Nat. Bur. Std. 41, 1940, 331-335.