The extremal case of some matrix inequalities

by

A.P. Nikulin

Introduction. The paper discusses the problem of characterizing the matrices satisfying one of the inequalities \( A \preceq \lambda I + A^*B + \sum_{i=1}^{n} A_i^*B_i \) for all \( \lambda \geq 0 \), where \( A \) is a Hermitian matrix and \( \{ A_i \} \) with \( A_i \preceq B_i \) are Hermitian matrices for \( i = 1, \ldots, n \). The matrix inequalities are treated for Hermitian matrices. The problem is solved by applying the following result.

1. The equation \( xA + \lambda I \preceq B \) Determine the set of all matrices \( A \) which satisfy the inequality \( xA + \lambda I \preceq B \) for all \( \lambda \geq 0 \). The equation is solved by applying the following result.

Theorem 1. Let \( A \in \mathbb{C}^{n \times n} \) and \( B \in \mathbb{C}^{n \times n} \) satisfy the equation

\[ \lambda B = A \]

Then \( A \) is a Hadamard factor of \( B \).
Then one of the following holds:

(i) Every row of \( A \) contains some zero entry.

(ii) Every column of \( B \) contains some zero entry.

(iii) There are nontrivial diagonal matrices \( D, D' \) such that \( DAB \) and \( D'AB^{-1}B' \) are reducible and nonseparable.

**Proof.** The assumptions imply that

\[
\left[ \begin{array}{c}
A_i \cdot B_i \\
... \\
A_n \cdot B_n
\end{array} \right] = (AB)_i = (AB)_n \in \mathbb{R},
\]

for \( i = 1, \ldots, n \) and \( k = 1, \ldots, n \). With

\[
\alpha = \text{diag}(A), \quad \beta = \text{diag}(B),
\]

and the lemma (apply the existence of nontrivial matrices \( \alpha, \beta \in \mathbb{R} \)), we have

\[
\alpha_i \beta_j = \delta_{ij} \quad \text{for all } i, j,
\]

Now suppose that neither (ii) nor (iii) holds. Then there are indices \( i, j \) such that \( (AB)_k = 0 \) for \( k = 1, \ldots, n \) and by (ii),

\[
\alpha_i \beta_j = \delta_{ij} \neq 0.
\]

Therefore, the diagonal matrices

\[
D = \text{diag}(\alpha), \quad D' = \text{diag}(\beta), \quad D' = \text{diag}(\beta)
\]

are nonsingular, and by (ii) and (iii) we have

\[
(AB) = \alpha_i \beta_j = (\alpha) (\beta) = (\alpha) (\beta),
\]

\[
(AB) = \beta_i \alpha_j = (\beta) (\alpha) = (\beta) (\alpha).
\]

Hence (ii) holds.

Conversely, it is obvious that \( (AB) = (\alpha) (\beta) \) for all pairs \( (A, B) \) satisfying (i). On the other hand, the characterization of those pairs \( (A, B) \) with \( (AB) = (\alpha) (\beta) \) satisfying (i) or (ii) depends on the zero structure of \( A \) and \( B \) and seems to be a nontrivial conditional problem. Simple examples which may occur here are the matrix pairs of type

\[
A = (C, 0), \quad B = \left( \begin{array}{c}
0 \\
... \\
0
\end{array} \right), \quad A = (C, 0), \quad B = \left( \begin{array}{c}
0 \\
... \\
0
\end{array} \right).
\]

with arbitrary \( C, D \) of the proper size. Moreover, with \( (A, B) \), the pairs \( (M \cdot A, M^{-1} \cdot B') \) occur, where \( M, M', M' \) are nonsingular matrices, i.e., have precisely one nonzero entry in each row and column. A particular case, used in the next section, is

(iii) \( (AB) = (\alpha) (\beta) \) if \( A \) or \( B \) is diagonal.
Theorem 1. If \( T \) is a continuously invertible operator, then for any operator \( S \) in \( K \), there exists an operator \( T^{-1} \) such that:

\[ T^{-1}ST = S \]

Proof: Since \( T \) is invertible, there exists an operator \( T^{-1} \) such that:

\[ TT^{-1} = I \]

where \( I \) is the identity operator. Then, for any operator \( S \) in \( K \), we have:

\[ T^{-1}ST = S \]

This completes the proof of the theorem.
Corollary 1. There is \( i \neq 0 \) such that \( A_i = 0 \) for some \( i \neq 0 \). Fix one such \( i \neq 0 \) and put \( a_i = A_i \). Then for \( i \neq 0 \), we have

\[ a_i^* A_i a_i - a_i^* A_i = a_i^* A_i a_i - a_i^* A_i \geq 0. \]

Moreover, since \( A_i \) is irreducible, there are indices \( j_1, \ldots, j_k \) such that \( a_{j_1} \ldots a_{j_k} a_i = 0 \) for any \( i \neq 0 \) due to this.

\[ a_i^* A_i a_i - a_i^* A_i = a_i^* A_i a_i - a_i^* A_i \geq 0. \]

Together with (i) for \( a_i = 0 \), this contradicts the minimality of \( E \).

Corollary 2. There is \( i \neq 0 \) such that \( A_i = 0 \) for some \( i \neq 0 \). This leads to a contradiction by the usual argument.

Therefore, \( E = \{0, 1, \ldots, n\} \), and with \( D = \text{Diag}(a) \) we have \( (D - M)^{-1} A_i D - \text{tr}^{-1} A_i \geq 0 \) for all \( i \neq 0 \) and that (ii) holds. Finally, (iii) holds as the nonnegative matrix \( D = D^{-1} a \) is similar to \( a \), hence \( a_{\text{tr}} = 1 \), \( (D - M)^{-1} \geq 0 \), and \( D - M^{-1} \geq 0 \). Therefore (ii) holds.

Corollary. If \( x \in \mathbb{K}^n \) is an irreducible matrix, with \( x_i < 0 \) for satisfying (ii) then

\[ (x - D^{-1} x) \leq 0. \]

If \( A \) is reducible then these theorems are no longer equivalent; we only have the implication (B) \( \Rightarrow\) (A). Counterexamples to (iii) are the matrices of shape \( \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \) which have

\[ a_{22}^* A_{22} a_{22} = 0, \quad \text{and} \quad (D - M)^{-1} \geq 0. \]

and counterexamples to (ii) are the strictly upper triangular matrices \( A \), where \( a_{22}^* A_{22} a_{22} = 0 \) and (D) agree on and below the diagonal. The demonstration of all reducible \( A \) satisfying (ii) or (iii) again seems to be a numerical problem.

Theorem 3. Let \( A \) be an irreducible. Then \( A \) and \( (A)^{-1} \) are nonsingular, and

\[ |A|^* \leq |(A^{-1})|^*. \]

Moreover, if \( A \) is irreducible then one of the following holds:

(i) \( A \) satisfies the semipositive inequality \((A^*)^* \leq (A)^*\).

(ii) \( A \) is semipositive and \( (A^*)^* \leq (A)^*\).

(iii) \( A \) is semipositive and \( (A^*)^* \leq (A)^*\).

Proof of (ii). By Theorem 1, let \( A \) be an irreducible. Then \( A \) is an irreducible and there is a real vector \( a > 0 \) such that \( A a = 0 \). Since \( A \) is an irreducible, there is a real vector \( b > 0 \) such that \( a^* b = 0 \). Now \( a^* a \leq a^* b \leq a^* (A^{-1} a) \leq a^* a \) by (ii) and since...
Mani inequalities

$[M]^{-1} [N] = \frac{N}{\text{det}(N)} [M]^{-1} [N] \quad \text{for } C = [M]^{-1} [N]$.

Thus $[A] = [M]^{-1} [N]$ and $(A) = [N]^{-1} [M]$ are nonsingular.

By (6) and Theorem 2, we have:

$$(A) = [M]^{-1} [N] = \frac{N}{\text{det}(N)} [M]^{-1} [N] = \frac{N}{\text{det}(N)} [M]^{-1} [N] \frac{N}{\text{det}(N)} [M]^{-1} [N]$$

$= [M]^{-1} [N] [M]^{-1} [N] = (A)^{-1}.$

So that (6) holds. Moreover, $[C] = [C]^{-1}$ and $(A) = (A)^{1},$ have a common root, as do

$[M]^{-1} [N]^{-1}$ and $[M]^{-1} [N]^{-1}.$ Now if $A$ is not of the form $M^{-1} N$, it is nonsingular, and by Lemma 2, the matrix $[M]^{-1} [N]^{-1}$ of such that $C = [M]^{-1} [N]^{-1} N$ is real and nonsingular. The vector $v = [M]^{-1} [N]^{-1} N$ is positive and satisfies

$$v = [M]^{-1} [N]^{-1} N \quad \text{for } v = (C) \text{ and } (C) = [M]^{-1} [N]^{-1} N.$$ 

Hence the matrix $B = B_{C}$ - constitutes $(B) = [M]^{-1} [N]^{-1} N < (D)^{-1} = v$. But $B = [M]^{-1} [N]^{-1} N$ is therefore as Moraitis. But $B = [M]^{-1} [N]^{-1} N \neq [M]^{-1} [N]^{-1} N$ with the nonsingular diagonal matrix $B = [M]^{-1} [N]^{-1} N$.

Conversely if $B$, $C$ are nonsingular diagonal matrices such that $B = [M]^{-1} [N]^{-1} N$ is an $\text{invertible}$ (see $\text{invertible}$) $B_{C} = B_{C}$ - constitutes $(B) = [M]^{-1} [N]^{-1} N$ and

$$(A) = [M]^{-1} [N]^{-1} N = (A)^{-1}.$$

References


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