An Improved Interval Newton Operator

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1. Introduction

Recently, a number of authors (Adorno [1], Gay [1], Krawczyk [8]) considered the extension of various forms of the interval Newton method to sets of nonlinear systems of equations. Following [8] we consider here function systems, i.e., interval valued mappings \( G(x) \) defined for vectors in a subset \( D \) of \( R^n \). A vector \( x \in D \) is considered as a pair of \( G(x) = (G_1(x), \ldots, G_n(x)) \).

We study two related interval Newton operators \( N_{\text{N}}(x) \) and \( N_{\text{N}}(x) \) with the property that every \( x \in D \) and \( G(x) \) of a Lipschitz continuous function \( G \) satisfies \( G(x) \approx \frac{G(\tilde{x})}{G(\hat{x})} G(\tilde{x}) \). Interval operators with this property are frequently used to enclose iteratively the set of zeros of \( G \) in \( D \) by an iterative process of the type \( x_{k+1} = (N(x_k) \setminus \{x_k\}) \) or \( x_{k+1} = x_k \cdot (N(x_k) \setminus \{x_k\}) \). If the function \( G(x) \) is nonsingular, its level \( \tilde{z} \) is a functional of the Newton operator; hence it is useful to have results about existence and uniqueness of such functionals.

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2. Notation and Basic Concepts

Small letters denote real values, vectors, matrices, and real-valued vector functions (except \( \mathbf{v}, \mathbf{w}, \mathbf{m}, \mathbf{n} \) for index notation). Capital letters denote intervals, interval vectors, interval matrices, and interval functions.

\( \mathbb{D} \) denotes the set of positive integers, \( \mathbb{N}_{\geq 0} = \{0, 1, 2, \ldots\} \), \( \mathbb{R}^m \) the set of all \( m \times n \)-matrices with \( 0 \) as zero in \( \mathbb{R}^{m \times n} \). We consider the subinterval vector space \( \mathbb{R}^m \to \mathbb{R}^m \) and the set of real numbers \( \mathbb{R} \to \mathbb{R} \) as special cases.

Let the \( \mathbb{C}^n \)-notation be defined componentwise and let \( \mathbb{R}^{m \times n} \) denote the positive cone with respect to the \( \mathbb{C}^n \)-notation. The sign of a vector \( \mathbf{a} \in \mathbb{R}^m \) is defined as the vector sign \( \mathbf{a} \) whose each component is \( 1 \) if \( a_i > 0 \), \( 0 \) if \( a_i = 0 \), and \(-1\) if \( a_i < 0 \). If \( \mathbf{a} \in \mathbb{R}^{m \times n} \), then \( \mathbf{a} \) denotes the spatial radius of the matrix \( \mathbf{a} \).

Let an interval \( \Delta = [a_\Delta, b_\Delta] \) with \( a_\Delta, b_\Delta \in \mathbb{R}^m \) and \( \Delta d \) be defined in the usual manner; i.e., \( \Delta = [a_\Delta, b_\Delta] \), \( \Delta^{**} \) denotes the set of all intervals of \( \mathbb{R}^m \). If \( \mathbf{a} \in \mathbb{R}^{m \times n} \) and \( m > 1, n > 1 \), then we also call \( \mathbf{a} \) an interval matrix; we also call an interval vector \( \mathbf{x} \in \mathbb{R}^{m \times 1} \) and \( n \times 1 \). We call an interval matrix \( \Delta \in \mathbb{R}^{m \times n} \) regular, if all \( a_\Delta, b_\Delta \) are regular. If \( \mathbf{a} \in \mathbb{R}^m \) then \( \Delta = [a_\Delta, b_\Delta] \subseteq \mathbb{R}^m \). We identify a real value \( a_\Delta \in \mathbb{R}^{m \times n} \) and the degenerate interval \( \Delta = [a_\Delta, a_\Delta] \subseteq \mathbb{R}^{m \times n} \). Concerning interval arithmetic operations we refer to Alefeld and Herzberger [2].
Let \( a \in \mathbb{R}^n \). Then

\[
\text{mid} \ a = \frac{1}{2} (a - a_0) + a_0
\]

(2.1)

defines the midpoint of \( a \).

\[
\text{rad} \ a = \|a - a_0\|_1 \in \mathbb{R}_+^n
\]

(2.2)

defines the radius of \( a \), and

\[
|a| = \|a - a_0\|_1 \in \mathbb{R}_+^n
\]

(2.3)

defines the absolute value of \( a \).

Remark. Instead of \( \text{mid} a \) we also write \( a \).

We shall make use of the following rules about the composition of midpoint, radius, and absolute value.

Let \( X, Y \in \mathbb{R}^n \). Then

\[
\text{mid} (X + Y) = \frac{1}{2} (X + Y) \quad \text{and} \quad \text{mid} (X) = \frac{1}{2} (X).
\]

(2.4)

\[
\text{mid} (a - a) = 0.
\]

(2.5)

\[
\text{mid} \ 0 = 0 = \text{mid} (0) = 0.
\]

(2.6)

\[
\text{rad} (X + Y) = \text{rad} (X) + \text{rad} (Y).
\]

(2.7)

\[
\text{rad} (a) = \text{rad} (a) = 0.
\]

(2.8)

\[
\text{mid} (a + b) = \text{rad} (X) + \text{rad} (Y).
\]

(2.9)

\[
|a| = |a| = |a|.
\]

(2.10)

\[
|X + Y| = |X| + |Y|.
\]

(2.11)

\[
|X - Y| = |X| - |Y|.
\]

(2.12)

Furthermore, we need the following propositions:

Proposition 2.1. If \( A \in \mathbb{R}^{n \times n} \) is a diagonal matrix and \( X \in \mathbb{R}^n \) then

\[
\text{mid} (AX) = A \text{mid} (X) \quad \text{and} \quad \text{rad} (AX) = |A| \text{rad} (X),
\]

(2.13)

\[
|AX| = |A| \text{mid} (X) \quad \text{and} \quad |AX| = |A| \text{mid} (X),
\]

(2.14)

Proof. Equations (2.13) and (2.14) are consequences from (2.8) in [17], applied to each component of \( AX \).
PROPOSITION 2.5. If \( A \in \mathbb{R}^{n \times n} \) is a diagonal matrix \( L \in \mathbb{R}^{n} \) then

\[
\text{rank}(A) = 0 = \text{null } A \text{ and } x = 0.
\]  

(2.15)

Proof. From (2.13) follow \( \text{rank}(\text{det}(A)) = \text{rank}(A) \) and \( x = 0 \), because the two terms of (2.13) have the same sign, so (2.15) is true. \( \square \)

3. The Functions \( G \) and Associated Lyapunov Operators

Let \( G : \mathbb{R}^n \to \mathbb{R}^n \) be a map which associates with each \( x \in D \) an interval

\[
G(x) := [x_1, b_1(x)]
\]

(3.1)

We denote a map \( G(x) \) a function \( y(x) \).

We assume that both end functions \( g(x) < b(x) < 0 \) satisfy a common interval Lipchitz condition

\[
g(x_1) - g(x_2) \leq L(x)(x_1 - x_2)
\]

for all \( x_1, x_2 \in D \). (3.2)

where the Lipchitz norm \( L(x) \) is regular for all \( x \in D \) and the operator \( L \) is continuous and bounded, i.e., \( X \subseteq Y \) implies \( L(X) \subseteq L(Y) \).

Furthermore we define

\[
\phi(x) := (\text{mult} L(x))^{-1}
\]

(3.3)

and

\[
\phi(K) := (\phi(x)) \setminus \text{mult} L(x)
\]

(3.4)

and we assume that

\[
\phi(x) < 1 \quad \text{for all } x \in D.
\]

(3.5)

Remark. It is sufficient to assume that \( \text{mult} L(x) \) is regular and \( \phi(0) = 1 \). For by Theorem 4 in \([5]\) and Proposition 6 in \([16]\) these simple assumptions imply the regularity of \( L(x) \) and \( \phi(x) < 1 \) for all \( x \in D \).

In view of (3.5),

\[
\phi(x) := (\phi(x) - (\phi(x)))^{-1}
\]

(3.6)

where \( \phi(x) \neq 0 \).
The nonnegative matrix \( R \) defined by (2.4) can be interpreted as a pseudoe寻常 action on \( \mathcal{L}(X) \) for the functions \( x \mapsto \text{mod}(\mathcal{L}(X), \chi(x)) \) and \( \text{mod}(x, \chi(x)) \). They read:

\[
(\chi(x) - \text{mod}(x, \chi(x))) = (\chi(x) - \text{mod}(\mathcal{L}(X), \chi(x)))
\]

and

\[
\text{mod}(\mathcal{L}(X), \chi(x)) = \text{mod}(\mathcal{L}(X), \chi(x))
\]

for all \( \chi(x) \in \mathcal{X} \). (3.7)

The proof of both Lipschitz conditions (3.7) and (3.8) is analogous to (1.15) in \( \mathcal{X} \) and can be omitted here.

Let us recall the following

Lemma 3.1. If \( \text{mod}(\mathcal{L}(X), \chi(x)) \) is differentiable, then

\[
(\mathcal{L}(X))^{-1} \ni (\chi(x) - \text{mod}(\mathcal{L}(X), \chi(x)))
\]

is differentiable.

Proof. With the abbreviations \( \mathcal{L}(X) = (\mathcal{L}(X))^{*} \) and \( \chi(x) = (\chi(x))^{*} \), we have

\[
\chi(x) - \text{mod}(\mathcal{L}(X), \chi(x)) = \chi(x) - \text{mod}(\mathcal{L}(X), \chi(x))
\]

in \( \mathcal{X} \). It follows that

\[
(\chi(x) - \text{mod}(\mathcal{L}(X), \chi(x))) = (\chi(x) - \text{mod}(\mathcal{L}(X), \chi(x))
\]

in \( \mathcal{X} \). Since \( \mathcal{L}(X) \) is differentiable, we obtain the inequality

\[
(\chi(x) - \text{mod}(\mathcal{L}(X), \chi(x))) = (\chi(x) - \text{mod}(\mathcal{L}(X), \chi(x))
\]

in \( \mathcal{X} \).

4. Internal Newton Operators for the Function Group \( \Gamma \)

Now we define an internal Newton operator

\[
N(\gamma) = \gamma - \text{mod}(\mathcal{L}(X), \gamma)
\]

where \( \mathcal{L}(X) \) is a positive operator on \( \mathcal{X} \) satisfying (2.5) and (2.6), with \( \gamma \) defined by (2.5) satisfying the condition (2.6). (4.4)

\[
N(\gamma) \in \text{cont}(\mathcal{X}, \text{cont}(\mathcal{X})) \quad \text{for all} \quad \gamma \in \mathcal{X}
\]

and \( N(\gamma) \) is continuous.
From (4.3) follow by (2.4)-(2.10) the relations

\[ \text{rad NN}(\delta) = \delta - \text{rad}(\delta) G(\delta) \]  \hspace{1cm} (4.2) \\

and

\[ \text{rad NN}(\delta) = \text{rad}(\text{rad}(\delta) G(\delta)) + q(\text{rad}(\delta) G(\delta)). \]  \hspace{1cm} (4.3) \\

Remark. For the case that \( G \) degenerates to a function \( g \) the operator (4.4) was introduced in (6). It was also dealt with in [5] for the case that \( L(\delta) = L \) is a constant matrix, so that \( a, e, q \) are constant matrices, too.

We obtain an improved internal Newton operator by the definition

\[ \text{NN}(\delta) = \delta - \text{NN}(\delta) G(\delta), \]  \hspace{1cm} (4.4) \\

where

\[ \text{NN}(\delta) = \left[ \delta - q(\delta), \delta + q(\delta) \right]. \]  \hspace{1cm} (4.5) \\

Because of the subadditivity the inclusion

\[ \text{NN}(\delta) \subseteq \text{NN}(\delta) \]  \hspace{1cm} (4.6) \\

is true.

For the midpoint and radius of \( \text{NN}(\delta) \) we have the following

Lemma 4.4. With the abbreviations \( H = \text{NN}(\delta) G(\delta) \) and \( S_\delta \) on \( (\text{NN}_\delta, \text{NN}_\delta) \) the following relations hold:

\[ \text{mid NN}(\delta) = \delta - \text{mid}(S_\delta H). \]  \hspace{1cm} (4.7) \\

and

\[ \text{rad NN}(\delta) = \text{rad} H - \min(\text{rad} S_\delta, \text{rad} H). \]  \hspace{1cm} (4.8) \\

Proof. With the diagonal part \( S_\delta \) of \( S_\delta \) and the off-diagonal part \( S \) on \( (\text{NN}_\delta, \text{NN}_\delta) \) we have the direct splitting \( \text{NN}(\delta) = S_\delta \oplus S \) and

\[ \text{rad} S_\delta = e, \quad \text{rad} S = 0. \]  \hspace{1cm} (4.9) \\

By (4.4) and formula (27) of [10] we get

\[ \text{NN}(\delta) = \delta - (S_\delta \oplus S) H = \delta - S_\delta H - S H. \]  \hspace{1cm} (4.7) \\

By (4.8) and (2.8) \( \text{rad} S H = 0 \) so that (4.7) follows from (2.4).

By (2.7)-(2.10),

\[ \text{rad NN}(\delta) = \text{rad}(S_\delta H) + \text{rad} S \oplus H \]
\[ = \text{rad}(S_\delta H) + \text{rad} S \oplus \text{rad} S_\delta H. \]
On the other hand, (4.3) implies
\[ \text{rad} N(X) = \text{rad} H + (\text{rad} S_0; H) \]
so that
\[ \text{rad} N(X) = \text{rad} NP(X) = \text{rad} H + (\text{rad} S_0; H) = (\text{rad} S_0; H) \]
If we apply Proposition 2.1 with \( A = S_0 \), \( X = H \) and observe (4.9) we obtain
\[ \text{rad} N(X) = \text{rad} NP(X) = \text{rad} H + (\text{rad} S_0; H) \]
\[ = \text{rad} H + (\text{rad} S_0; H) \]
\[ = \text{rad} \{ (\text{rad} H + \text{rad} S_0; H), \text{rad} H + \text{rad} S_0; H, \text{rad} S_0; \} \]
\[ = \text{rad} H + (\text{rad} S_0; H) = (\text{rad} H, \text{rad} S_0; H) \]
\[ = \text{rad} H + \text{rad} S_0; H \]
\[ = \text{rad} H \cup \text{rad} S_0; H \]
\[ = \text{rad} (H \cup S_0; H) \]
\[ = \text{rad} H \cup \text{rad} S_0; H \]
By (2.11), this simplifies to (4.8).

Remark 1. The inference in (4.8) is a measure for the improvement of the operator \( N(X) \) over \( X \).

2. If either \( \text{rad} H = 0 \) or \( \text{rad} H = 0 \) then (4.8) implies \( \text{rad} N(X) = \text{rad} N(X) \) whereas \( \text{rad} N(X) = \text{rad} N(X) \) by (4.9).

If \( H \) is diagonalizable to a function \( g \) and if \( x^* \) is a zero of \( g \) then \( x^* \in X \) and \( x^* = x^* \). An analogous property is also valid for the improved operator (4.1).

Theorem 4.1. \( H \in \text{Op}(X) \) and \( x^* \in X \) show \( x^* \in \text{N}(X) \).

Proof. By assumption (3.2) there exist two matrices \( I, \in \text{L}(E) \) and \( S_0 \in \text{L}(E) \) so that
\[ g(I) = g(x^*) = I (I - x^*) \]
\[ g(I) = g(x^*) = I (I - x^*) \]
Because of (4.6) \( g(x^*) \) there exists a non-negative diagonal matrix \( d \) with \( d \in I \), so that the equation
\[ dg(x^*) + (d - I) g(x^*) = 0 \]
holds.
From (4.11) and (4.12) we obtain

\( \xi := \mathcal{D}(\xi) = (c - d) \mathcal{D}(1) = (d, \gamma) \mathcal{S}(x, x^* = \sigma_0(x)) \),

where \( I := [d, \gamma] \mathcal{S}(1) = [d, \gamma] \mathcal{L}(x, x^* = \sigma_0(x)) \) (since \( d > 0 \) and \( x^* = \sigma_0(x) \), the distributive law is valid), and we have

\( \sigma^* = x^* = x^* \).

To include \( I^* \gamma \) we define

\[ 0 = (d - \xi) = c - \sigma \] (because \( d = \sigma \)).

Then:

\[ |I| \leq |a| |I| - |a| |I| = |x| = x. \]  

From (4.14) follows \( I^* = (c - x)^{-1} \), so

\( I^* \gamma = (c - x)^{-1} \gamma = (c - x)^{-1} \gamma \).

Now we can include \( (p - b) - c \). By (4.13) we have \( d(p, x, c) < 1 \) and therefore

\[ (p - b) - c = \sum_{i=1}^{n} \left( d(p, x, c) - 1 \right) \gamma = \sum_{i=1}^{n} \left( d(p, x, c) - 1 \right) = n \] by (3.6).

This implies

\[ (c - x)^{-1} \gamma = (c - x)^{-1} \gamma = s. \]  

Inserting the relation (4.17) in (4.16) and using \( d(p, x, c) < 1 \) we obtain

\[ I^* \gamma = \mathcal{S}(\gamma) \mathcal{G}(c). \]

By (4.4) and (4.13) the inclusions (4.10) follows.

Remarks:

1. The application of the associative law to (4.16) is important, because otherwise we must use the inclusion:

\[ (c - x)^{-1} \gamma \subseteq \mathcal{S}(\gamma) \mathcal{G}(c) \]

which yields

\[ (c - x)^{-1} \mathcal{S}(\gamma) \mathcal{G}(c). \]

For \( c > 1 \) this is worse than (4.8) since then \( \mathcal{S}(\gamma) \mathcal{G}(c) \subseteq \mathcal{S}(\gamma) \mathcal{G}(c) \) and the notation is generally proper, even if \( 0 \) degenerates to a function (cf. p. 134).
of Adolph and Handersperger [2]. In particular, the interval Newton operator for functions described in Adolph and Handersperger [3] which is based on (4.19) is inferior to the present one.

2. An operator which possesses the property (4.12) is called inclusion operator in [9].

3. Comparing with an Optimal Inclusion

From the proof of Theorem 4.4 we learn that

\[
\left( \tau - y, \tau \right) = \left( L(x), y \right) \triangleq G(x, y) = S\left\{ G(x) \right\}.
\]

(5.1)

In order to assess the quality of the inclusion (5.1) we need to compare (5.1) with the optimal inclusion interval. This is possible if \( G(x) \neq 0 \) is open and \( L \) is inverse increasing (i.e., if all \( L \) are inverse increasing).

Let \( \Delta = (0, 1) \) and let \( \Delta \) be inverse increasing. Then

\[
\left( \tau - y, \tau \right) = \left( L(x), y \right) \triangleq G(x, y) = \left[ \tau - \left( \frac{\tau - y}{1 - \tau} \right) \Delta \right]
\]

(5.2)

is an optimal inclusion interval (Breit [1]). On the other hand \( G(x) \neq 0 \) and \( G(x) \neq 0 \) that

\[\text{sup} \left\{ G(x) \right\} = \left( \epsilon + q \right) \left \{ \left( 1 - \epsilon \right) \Delta \right \} \] from where we obtain

\[
\left( \tau - y, \tau \right) \triangleq G(x, y) = \left( \epsilon + q \right) \left \{ \left( 1 - \epsilon \right) \Delta \right \}.
\]

(5.3)

Inserting \( y = \left( \epsilon + q \right) \left \{ \left( 1 - \epsilon \right) \Delta \right \} \), it becomes identical to (5.1), so \( \left( \epsilon + q \right) \left \{ \left( 1 - \epsilon \right) \Delta \right \} \). Since \( \left( \epsilon + q \right) \left \{ \left( 1 - \epsilon \right) \Delta \right \} \) is identical to (5.1), it becomes identical to (5.1) because of (5.1).

\[\text{sup} \left\{ G(x) \right\} = \left( \epsilon + q \right) \left \{ \left( 1 - \epsilon \right) \Delta \right \} = \left( \epsilon + q \right) \left \{ \left( 1 - \epsilon \right) \Delta \right \} = \left( \epsilon + q \right) \left \{ \left( 1 - \epsilon \right) \Delta \right \}.
\]

(5.4)

Thus, the upper bound of the optimal inclusion (5.2) coincides with the upper bound of the inclusion (5.1). Similarly if \( G(x) \neq 0 \) and \( L \) is inverse increasing then

\[
\text{inf} \left\{ G(x) \right\} = \left( \epsilon + q \right) \left \{ \left( 1 - \epsilon \right) \Delta \right \} = \left( \epsilon + q \right) \left \{ \left( 1 - \epsilon \right) \Delta \right \} = \left( \epsilon + q \right) \left \{ \left( 1 - \epsilon \right) \Delta \right \}.
\]

(5.5)

In this special case the lower bounds are identical.

6. Fractional Theorems

Let \( G \) be a function which defined in \( B \) (see (3.1)) and let \( N \) map \( N \) denote the interval Newton operator of \( G \) with the domain \( D \) defined by (4.1) map (4.4).
We say that \( F \) is a subinterval of \( N \) if \( F \subset N \), i.e., \( F \subset N \cap \mathbb{K} \).

In the section we make some assertions about the uniqueness and existence of subintervals of \( N \) resp. \( N \).

**Proposition 6.1.** If \( F \) is a subinterval of \( N \) if \( F \) is a subinterval of \( N \).

**Proof.**
1. From \( F \subset N \) and \( F \subset N \) follows that \( \rho \subset \rho \subset \rho \subset \rho \). By Lemma 3.1, this implies \( \rho \subset \rho \).

2. If \( F \subset N \), then \( F \subset N \) and \( F \subset N \) follows from \( \rho \subset \rho \) and \( \rho \subset \rho \). The application of Proposition 6.1 with \( \rho \subset \rho \) is valid for those of the operator \( N \).

For the following we confine ourselves to the operator \( N \) since by Proposition 6.1 the assertions about subintervals of the operator \( N \) are also valid for those of the operator \( N \).

**Proposition 6.2.** All subintervals of \( N \) have the same subinterval \( F \).

**Proof.**
Let \( \mathbf{D} \subset \mathbb{D} \) and \( \mathbf{D} \subset \mathbb{D} \) be two different subintervals of \( N \). Then \( \rho \subset \rho \) and \( \rho \subset \rho \) follows as in Proposition 6.1 because \( \rho \) is regular.

Applying the hypothesis (3.1), we obtain
\[
|\lambda_0 - \lambda_1| < \rho(\rho) \subset \rho.
\]

Because of \( \rho(\rho) < 1 \) this inequality can be true only if \( \lambda_0 = \lambda_1 \).

If \( F \) degenerates to a subinterval \( F \) then this implies that these exist at worst one subinterval of \( N \). But in the general case it is possible that \( N \) has more than one subinterval.

**Example 6.1.**
Let \( \lambda_0 = 1.0 \times 10^6 \), \( \lambda_1 = 3 \times 10^6 \). Then \( \rho_0 = 1.0 \times 10^6 \), \( \rho_1 = 3 \times 10^6 \), \( \rho_2 = 3 \times 10^6 \), \( \rho_3 = 3 \times 10^6 \).

\[
\mathbf{A} = \begin{bmatrix}
1 & 2 \\
3 & 4
\end{bmatrix}, \quad \mathbf{T} = \begin{bmatrix}
2 & 3 \\
4 & 5
\end{bmatrix}, \quad \mathbf{N} = \begin{bmatrix}
6 & 7 \\
8 & 9
\end{bmatrix}
\]

For each subinterval \( \mathbf{D} \) the condition (3.1) is fulfilled. Now suppose that \( \lambda_0 = 1 \). Then \( \rho(\rho) = \rho(1) \) and \( \rho(\rho) = \rho(1) \) follows by (3.1), that \( \rho(\rho) = \rho(1) \) and \( \rho(\rho) = \rho(1) \). The application of Proposition 6.1 with \( \rho \subset \rho \) is valid.

If \( \mathbf{D} = \{1, 2, 3, 4\} \), then \( \rho(\rho) = \rho(1) \) and \( \rho(\rho) = \rho(1) \).

If \( F = \{1, 2\} \), then \( \rho(\rho) = \rho(1) \) and \( \rho(\rho) = \rho(1) \).
So $x^2$ and $x^3$ are two different fixed points of $N$.

A consequence of Proposition 6.2 is the following

**Theorem 6.3.** Let $N$ be defined by (7.1) with $L(x) = L(x) = -x$. Then the operator $N$ has at most one fixed point in $D$.

**Proof.** By Proposition 6.2 all fixed points have the same modulus $k$. If $k = 1$ and $k > 1$, then $N$ and $N^{-1}$ depend only on $k$, therefore all fixed points have the same radius too, which means that they are identical.

For the operator defined by (4.1) we write in the following $N_{0}$ if $L(x) = L(x)$ for all $x > 0$, and $N_{0}$ otherwise, (i.e., if $L(x)$ is varying).

Generally fixed points of $N_{0}$ (if they exist) need not coincide with fixed points of $N$ for the same $G(x)$.

**Example 6.2.** $N_{0}$, $D = [1, 10]$, $g(x) = x^2 + 4x - 200$, $g(x) = x^2 + (5.5 - 200)$. $L(x) = 3x^2 + 18.5x + 15.5$ is a Lipschitz interval, and we have $N_{0}(x^2) = x^2$. $N_{0}(x^2) = x^2 < 1$ for all $x < D$.

$$\text{rad} \lambda(x) = 1 - (-1)^x \sqrt{a(x)} \mid \mid 0 \mid 1 = \frac{21}{2} = 10.5$$

Since $\text{rad} \lambda(x) = 1 - (-1)^x \sqrt{a(x)} \mid \mid 0 \mid 1$ is in $D$, the unique zero $x < 5$, this is the midpoint of all fixed points. With $L(x) = [23.5, 48.5]$ we obtain $\text{rad} \lambda(x) = 1 - (-1)^x \sqrt{a(x)} \mid \mid 0 \mid 1 = 3$. Therefore $x^2 = 2.5$ is a fixed point of $N$. But $x^*$ is a variable, $x^* = 2.5$ is not a fixed point of $N$, because $\text{rad} \lambda(x) = 1 - (-1)^x \sqrt{a(x)} \mid \mid 0 \mid 1$. The example shows that $x^* = x^2$.

**Theorem 6.4.** If $x^2 \in D$ is a fixed point of $N$, then the relation $x^2 = x^3$ holds for each fixed point $x^*$ of $N$.

**Proof.** From (4.3) follows because of $\text{rad} \lambda(x) = 0$, (2.11), and $x^* = (e - 1)^{-1}$ that

$$\text{rad} \lambda(x) = (e - 1)^{-1} a(x) \text{rad} \lambda(x) = (e - 1)^{-1} a(x) \text{rad} \lambda(x)$$

$$< (e - 1)^{-1} a(x) \text{rad} \lambda(x) \text{rad} \lambda(x)$$

$$= \text{rad} \lambda(x) \text{rad} \lambda(x)$$

The last inequality is a consequence of Lemma 3.3 \text{rad} \lambda(x) = x^2.$
It is even possible that \( N \) contains one or more discontinuities, but \( N \) has no finiterval in \( D \).

**EXAMPLE 6.5.** In Example 6.1 we have seen that \( N \) has two discontinuities \( X = [3, 4] \) and \( X = [1, \infty) \) with the common midpoint \( X^* = \infty \). If \( D = [0, 1] \) and \( \text{rad}X = 0 \) and \( \text{rad} X = 2 \) is independent of \( \text{rad}X \). Supposing that \( \text{rad}X \) is a finiterval \( X^* \) in \( D \), then \( \text{rad}X = 2 \) and \( \text{rad} X = 0 \) which is a contradiction to \( X^* \in D \).

But the converse situation is not possible:

**THEOREM 6.5.** If \( X \subseteq D \) is a finiterval of \( N \), then \( N \) has at most one finiterval.

**Proof.** Let \( \text{rad}X = X^* \) and \( \text{rad}X = \text{rad} X \). Consider the constant function \( f(X) = (\text{rad} X)^r - (\text{rad} X)^s \) and \( \text{rad} X \) for all \( X \) with \( \text{rad} X = X^* \) constant and \( 0 = \text{rad} X < \text{rad} X < \text{rad} X \). By Lemma 3.3, \( f \) is increasing in the interval \([ \text{rad} X, \text{rad} X^*] \). Moreover \( f(0) = (\text{rad} X)^r - (\text{rad} X)^s \geq 0 \) and \( f(\text{rad} X) = (\text{rad} X)^r - (\text{rad} X)^s \). Therefore \( f(\text{rad} X) = 0 \) if \( \text{rad} X \) is in \( [0, \text{rad} X] \), and by Brouwer's fixed point theorem there exists a \( x \) such that \( \text{rad} x^* = f(\text{rad} x^*) = 0 \). Since the midpoint \( X^* \) is also the midpoint of \( X^* \) it follows that \( X^* = X^* \).

The following existence theorem for a finiterval of the operator \( N \) is at the same time the converse theorem for a finiterval of \( N \) (which must not be unique).

**THEOREM 6.6.** Let \( X \) be defined by (41) with \( L(X) = L(D) \) and \( r = 0, \infty \), and let the condition

\[
\text{rad} D > (r - 1) \left( \text{rad} L(D) \right) + (r - 1) \left( \text{rad} L(D) \right)
\]

be satisfied. Then \( X \) has exactly one finiterval in \( D \).

**Proof.** Let \( X \subseteq D \) with \( N X = X \) and

\[
X_{n+1} = N X_n, \quad X_0 = 0, 1, 2, \ldots
\]

Then from (41) follows \( X_{n+1} = X_n \). Applying (57) we obtain

\[
|N X_0| < \text{rad} L(D), \quad |X_0| < \text{rad} L(D)
\]

(63)
and

\[ |x_2 - x_1| \leq |x_2 - x_n| + |x_{n+1} - x_1| + \cdots + |x_{n+k} - x_1| \]

\[ = (\max |x_2 - x_n|) + (\max |x_{n+1} - x_1|) + \cdots + (\max |x_{n+k} - x_1|) \]

\[ \leq (n-k) \cdot \max |x_2 - x_1| + \cdots + \max |x_{n+k} - x_1|, \]

so

\[ |x_2 - x_1| \leq (n-k) \cdot \max |x_2 - x_1| \quad \text{for all } k \in \mathbb{N}. \quad (6.5) \]

If \( x_2 \notin B \) then from (4.3) and (6.4) it follows in view of (2.11) and (3.6)

\[ \text{and } \text{rad } N_{\delta}(x_2) = (n-k)^{-1} \cdot \text{rad} \text{dist}(x_2, B) \cdot (\max |x_2 - x_1|). \]

Inverting (3.6), (4.4), and (6.5) we get

\[ \text{rad } N_{\delta}(x_2), \leq (n-k)^{-1} \cdot (\text{rad} \text{dist}(x_2, B)) \]

\[ \leq (n-k)^{-1} \cdot (\text{rad} \text{dist}(x_2, B)) \cdot (\max |x_2 - x_1|) \]

\[ \leq (n-k)^{-1} \cdot (\text{rad} \text{dist}(x_2, B)) \cdot (\max |x_2 - x_1|) \]

\[ \leq (n-k)^{-1} \cdot (\text{rad} \text{dist}(x_2, B)) \]

by (6.2) and because \( (n-k)^{-1} + (n-k)^{-1} \leq (n-k)^{-1} + (n-k)^{-1} \). Considering (6.5) we obtain

\[ \text{rad } N_{\delta}(x_2), \leq (n-k)^{-1} \cdot (\text{rad} \text{dist}(x_2, B)) \leq (n-k)^{-1} \cdot \text{rad } N_{\delta}(x_2). \]

Because of (2.2) the last inequality is equivalent to \( x_2 \notin B \). Therefore the iteration (5.5) can be carried out for all \( k \geq 0 \). By (6.4) we have

\[ \text{rad } N_{k+1}(x_2) \leq (n-k)^{-1} \cdot \text{rad} \text{dist}(x_2, B). \]

So, since \( n-k \leq 1 \),

\[ \lim_{k \to \infty} \text{rad } N_{k}(x_2) = 0, \quad \lim_{k \to \infty} \text{rad} \text{dist}(x_2, B) = \text{rad} \text{dist}(x_2, B). \]

The triangle inequality \( x_{k+1} - x_k \leq x^* \), and since \( x^* \) is a continuous function, \( x^* = 0 \) holds. By Theorem 4.3, \( x \) is the unique fixedpoint in \( \mathcal{U} \). \( \square \)

Conclusion: If the condition (6.2) is fulfilled then \( W \) has at least one

\[ \text{fixedpoint in } \mathcal{U}. \]
REFERENCES

1. E. Amore, The use of solutions of nonlinear systems in RF, in "Electrical
1964.


3. G. Alefeld and J. Herzberger, Über das Newton-Verfahren bei nichtlinearen

4. H. Kaj, "Einige Anwendungen der L"osungsalgorithmen bei linear- und


6. H. Benker, "Einige Anwendungen der L"osungsalgorithmen bei linear-

7. R. B. Darwen, "Piecewise linear functional operators: boundedness and

8. H. Benker, "Einige Anwendungen der L"osungsalgorithmen bei linear-

9. R. B. Darwen, "Piecewise linear functional operators: boundedness and

10. L. B. Rice, "Representation of intervals and optimal error bounds," Math. Comp. 48