1. Introduction

Recently the problem of enclosing the zero set of a system of equations, depending on intervally known data, has been of considerable interest, e.g., in the following references: [1], [3, 4, 8, 9, 30, 31]. In [13], common features of zero finding methods for data independent systems were described using the notion of the “zero set” of an interval matrix. Thus, if \( F: D \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^m \) is a real function, \( D \subseteq \mathbb{R}^n \), and \( d^* \) denotes an interval of \( d \), regular interval Lyapunov matrix. A key finding in [13] is that the unique zero \( x^* \) of \( F \) \( (x^* \in X) \) satisfies

\[ x^* \in X - d^* I. \]

In the present paper we extend this approach to a function \( G(x) = \{ (x, x) \} \) representing an intervally known system of equations and show that under appropriate conditions the zero set \( X^* \) of \( G(x) \) satisfies

\[ X^* \in X - d^* I \] \( (x^* \in X^*) \).

Therefore, if \( X^* \subseteq X_0 \), the iteration

\[ x_{k+1} = x_k - d^* I (g_k), \quad x_0 \in X_0, \]

with a sequence of nested intervals \( X_k \) containing \( x^* \). Under suitable assumptions this sequence is shown to converge to a fixed interval \( X \), but further improvement is generally not possible by this method.
2. NOTATION AND BASIC CONCEPTS

The terminology of the paper follows Krasnosel' ski and Krasnosel'skii [10] and Krasnosel'skii [11]; but for convenience of the reader, some definitions are repeated. Lower case italic letters denote sets, intervals, and maps. We denote the set of n-dimensional Euclidean matrices by R^n and R^n x R^n, respectively, sur X, and a, for the minimum, maximum, and the vector a for the radius, and |X|, |a| for the absolute value of X and a in R^n, respectively. Moreover, we set N := [0,1,2,3,4]. The unit circle is written as S; and E, a, as E := [-1,1] [a]. The circle denotes the measure circle of a, and the ball of E the (or n × r) of a, which is defined by:

\[
q(X, Y) := \{ a(x, y) \mid 0 < a(x, y) \leq 1, x, y \in \mathbb{R}^n \}
\]

(2.1)

(2.2)

Moreover, we state the following

**Lemma 2.1.** \( a \in \mathbb{R}^n \) \( \Rightarrow \) \( \{ a \} \in \{ a \} + 2 \mathbb{R}^n \)  

(2.3)
Theorem 2.3. Let \( S' : B^* \to B^* \) be a surjective map, and define \( S : B^* \to B^* \) by
\[
S = S' + \epsilon(S)(E_{\text{rad}} X)
\]
Then \( S \) is a normal surjective map and the following holds:
(i) \( Sx = 0 \) if and only if \( x = 0 \).
(ii) \( |S| = |S'| + |\epsilon(S)| \).
(iii) If \( S \) is regular (complete) then \( S \) is also regular (complete).
(iv) In particular, if \( S = a' \) is an inverse of \( a \) in \( B^* \) then \( S = S' \) is again an inverse of \( a \).

Proof. Let \( a' \) exist. By (iii) and \( |S'| \) in (i) we have:
\[
Sx = Sx + (Sx)(E_{\text{rad}} X) = Sx + (Sx)(E_{\text{rad}} X)
\]
and in view of \( x = 0 \) we obtain
\[
Sx = Sx + a(Sx)(E_{\text{rad}} X) + a(Sx)(E_{\text{rad}} X)
\]
because the factor (the distribution \( \epsilon \) is valid).

From (iii) and Eq. for \( |S'| \) and (iv) we obtain:
\[
|Sx| = |Sx| + |\epsilon(S)(E_{\text{rad}} X)|
\]
in view of the subadditivity of \( |S| \) and \( |\epsilon(S)| \), (iv) implies
\[
|S| + |\epsilon(S)(E_{\text{rad}} X)| = |S' + |\epsilon(S)(E_{\text{rad}} X)|
\]
and in view of the subadditivity of \( |S'| \) and \( |\epsilon(S)| \), (iv) implies
\[
|S| + |\epsilon(S)(E_{\text{rad}} X)| = |S' + |\epsilon(S)(E_{\text{rad}} X)|
\]
Therefore, \( S = S' \) is an inverse of \( a \).

Lemmas 2.4. Suppose that
\[
A : (S' \to B^*) \to (S \to B^*)
\]
where the paritioning is such that \( D \subseteq E_2 \), and let
\[
S(A) = S - SD^{-1} E_2
\]
be the Sohr complement of \( A \). If \( S \) is regular, then \( S \) can be expressed in terms of
\[
Y = S(A)(E_2 - SD^{-1} E_2)
\]
as

\[ A^T Y = \begin{bmatrix} U \end{bmatrix} X Y \begin{bmatrix} Z \end{bmatrix} \begin{bmatrix} V \end{bmatrix} \]

Proof. By the discussion in Section 5 of [11], the triangular decomposition of \( A \) has the form \((L, R)\) with

\[
L = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ \vdots & \vdots \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}, \quad R = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ \vdots & \vdots \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix},
\]

where \((L, R)\) is the triangular decomposition of \( \Delta(A) \). Now it is easy to see that

\[
L^T \left( \begin{bmatrix} U \\ \vdots \\ V \end{bmatrix} \right) = \begin{bmatrix} U \\ \vdots \\ V \end{bmatrix}, \quad R^T \left( \begin{bmatrix} U \\ \vdots \\ V \end{bmatrix} \right) = \begin{bmatrix} V \\ \vdots \\ U \end{bmatrix},
\]

where

\[
X = \begin{bmatrix} L \end{bmatrix} X = (X - \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}) = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}, \quad R = \begin{bmatrix} R \end{bmatrix} \begin{bmatrix} Z \end{bmatrix} = (Z - \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}) = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}.
\]

Therefore

\[
A^T Y = R^T \left( \begin{bmatrix} U \\ \vdots \\ V \end{bmatrix} \right) = \begin{bmatrix} V \\ \vdots \\ U \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}.
\]

Proposition 2.3. Let \( A \in \mathbb{R}^{n \times n} \) be an elementary matrix with \( n = n \). Then \( A \) has a triangular decomposition \((L, R)\) with

\[
\text{and the relation}
\]

\[
A^T X = X + E(A) - e \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix},
\]

valid for all \( X \in \mathbb{R}^{n \times n} \).

Proof. By multiplying \((2.9)\) by \( A^T \), the triangular decomposition \((L, R)\) of \( A \) exists. We prove (2.10) and (2.11) by induction on \( n \).

First, suppose that \( n = 1 \). Since \( A = e \), we may write \( \Delta = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \) with a number \( \epsilon, 0 < \epsilon < 1 \). We have \( L = 1, \ R = e \), hence (2.10) holds trivially. To prove (2.11) we observe that

\[
(\Delta) - 1 - e = \epsilon \begin{bmatrix} 0 \\ 1 \end{bmatrix} - e \begin{bmatrix} 0 \\ 1 \end{bmatrix}.
\]
Now, by the above argument for \( n = 1 \), we have
\[
(U \rightarrow Y) = (U \rightarrow Y) \cap (U \rightarrow Y) = (U \rightarrow Y) \cap (U \rightarrow Y)
\]
and by the induction hypothesis we have
\[
P = (U \rightarrow Y) \cap (U \rightarrow Y)
\]
and
\[
P = (U \rightarrow Y) \cap (U \rightarrow Y)
\]
Therefore, by Lemma 2.6 and (2.13)
\[
s^a \in \left( (U \rightarrow Y) \cap (U \rightarrow Y) \right) \cap \left( (U \rightarrow Y) \cap (U \rightarrow Y) \right)
\]
which implies (2.11). 

Corollary 2.6. Let \( A \in \mathbb{R}^{n \times n} \) be an \( n \)-matrix with \( \lambda = 0 \). Then
\[
\sigma(a^T) = \{ -a^T \} \cup \{ -a^T \}
\]
which implies (2.11).

Proof. By (2.11), we have
\[
\sigma(a^T) = \sigma(a^T) \cap \sigma(a^T) \cap \sigma(a^T) = \{ -a^T \} \cup \{ -a^T \}
\]
Moreover, using Theorem 4 of (11) and (2.15), we find
\[
\sigma(a^T) \cap \sigma(a^T) \cap \sigma(a^T) = \{ -a^T \} \cup \{ -a^T \}
\]
which implies (2.11).
Theorem 4.2. Let $x \in D$ and $\tilde{x} = A^T(x) \in \tilde{D}$. Then
\[
\tilde{x}_n^* < [1 - t_n^{-1}t_n^{-1}]x_n + \tilde{y}_n^{-1}(\tilde{y}_n^{-1})^{\top}\tilde{y}_n^{-1} \tilde{y}_n^{-1}n.
\]
Moreover, if $\tilde{x}_n$ is regular and $\tilde{x}_n \in D$ then
\[
x_n = [1 - t_n^{-1}t_n^{-1}]x_n + \tilde{y}_n^{-1}(\tilde{y}_n^{-1})^{\top}\tilde{y}_n^{-1} n.
\]

Proof: By Proposition 3.1(i), each $\tilde{x}_n = \tilde{x}$ satisfies $g_\tilde{x} = 0$ for some $\tilde{x} \in [0, x]$. Hence, by Proposition 3.1(ii), $g(x) = g_0(x) = \tilde{y}_n^{-1}(\tilde{y}_n^{-1})^{\top}\tilde{y}_n^{-1} n$ for some $x, \tilde{x} \in D$ by Proposition 3.1. This implies (4.3). To prove (4.4) we define
\[
\tilde{z} = [1 - t_n^{-1}t_n^{-1}]x_n + \tilde{y}_n^{-1}(\tilde{y}_n^{-1})^{\top}\tilde{y}_n^{-1} n.
\]

Since $x_n \in D$, define the diagonal matrices $A^\top D$ and $A^\top D$ in Proposition 3.1(i) by
\[
A^\top D = A^\top D = \tilde{y}_n^{-1}(\tilde{y}_n^{-1})^{\top}\tilde{y}_n^{-1} n < [1 - t_n^{-1}t_n^{-1}]x_n + \tilde{y}_n^{-1}(\tilde{y}_n^{-1})^{\top}\tilde{y}_n^{-1} n.
\]

In practice it is difficult to compute $A^\top D$, and one can instead the convex hull by $A^\top D$, where $A^\top D$ is a separable convex set.

Theorem 4.3. Let $\tilde{x}_n$ be a regular inverse of $A$, and suppose that $x_n \in D$ is such that
\[
\tilde{x}_n = \tilde{x}_n - A^\top D \tilde{x}_n.
\]
If \( A \) is regular then, for every \( x \), we have
\[
0 \in \text{int} X' = \text{int} X - x \text{rad } G(1)
\]
(2) \( \text{rad } A' = \text{rad } A \) and \( \text{rad } G(1) = \text{rad } G(1) \)

Proof. In view of (11), Theorem 4, we have
\[
\text{rad } X' = \text{rad } A' G(1)
\]
\[
= \text{rad } A' (G(1) + x \text{rad } G(1))
\]
\[
= \text{rad } A' (G(1) + x \text{rad } G(1)) \text{rad } G(1)
\]

Now for every \( x \), we have \( A \) and \( A' = \text{rad } A \) are as shown in (11), and \( A' G(1) = \text{rad } A \text{rad } G(1) \), and we get
\[
\text{rad } X' = \text{rad } A' G(1) = \text{rad } A' G(1) \text{rad } G(1)
\]
\[
= \text{rad } A' (G(1) + x \text{rad } G(1)) \text{rad } G(1)
\]

since \( x \text{rad } G(1) \text{rad } G(1) = \text{rad } G(1) \text{rad } G(1) \). Now apply Theorem 4.2.

Theorem 5.1 Let \( G \) be a linear space. Let \( G \) be an integral space and let \( \mathcal{N} \) be a null set. Then
\[
\mathcal{N} = \text{int} X
\]

Thus, if \( G \) is a null set then, the integral space and let \( \mathcal{N} \) be a null set. Then
\[
\mathcal{N} = \text{int} X
\]

5. INTEGRAL NEWTON OPERATORS AND THEIR FIXED INTERVALS

Let \( \mathcal{N} \) be a function space defined by (5.1) which satisfies the Lipschitz condition (5.2) with a regular matrix \( A \), and let \( A' \) be a null set. Then we call the operator \( N' \) defined by
\[
N'(x) = A' G(1), \quad \text{where } x = \text{int } X
\]

an integral Newton space operator for \( G \). An integral \( \mathcal{N} = \text{int } X \) is called

Now apply Theorem 4.2. 

Remark. One might hope to get
\[
\text{rad } X = \text{int } X
\]

(4.6) so that the theorem, \( A' \), or the example
\[
A = \begin{bmatrix}
-1 & 2 \\
3 & -1
\end{bmatrix}, \quad X = \begin{bmatrix}
-1 & 1 \\
1 & -1
\end{bmatrix}, \quad A' = \begin{bmatrix}
-1 & 2 \\
3 & -1
\end{bmatrix},
\]

shows that \( A' \) must be regular.

Moreover, the bound in Theorem 4.3 is not optimal since for \( A' = A' \) it does not reduce to that of Theorem 4.2.
\[ |a| |r|^{-1} = 2 \text{ rad}(a) \tag{5.10} \]

Thus
\[ 0 < \text{ rad } X = \text{ rad } \mathbb{R}^n < \text{ rad } (a^{n-1}) = \text{ rad } (a^n) \text{ rad } (a) \tag{5.11} \]

Proof: (i) follows from Theorem 4.3(i) with \( X = Y \) and \( \mathbb{R}^n = \mathbb{R}^n \cdot \mathbb{R}^n \).

(ii) follows from Theorem 4.3.1 and \( \mathbb{R}^n = 0 \) by (5.3).

(iii) follows directly from (5.9) and (5.30).

Remarks: (1) If \( X = Y \) and \( A = B \) then (5.2) means merely multiplication, as Example 1 in Sect. 3 of [11] shows.

(ii) If \( X = Y \) and \( A = B \) then (5.3) means order \( \Theta(\eta) \) in (5.11) is of order \( \Theta(\eta) \). This means quadrilateral convergence, if \( \Theta(\eta) = 0 \).

(iv) In [7] the internal operator \( N \) was called inessential preserving if (5.7) holds, and essential if (5.8) holds.

6. A CONVERGENCE THEOREM FOR INTERNAL ITERATION

In this section we use an internal iteration operator \( X \) to determine an internal sequence \( \{X_n\} \), and we consider the question under which assumptions this sequence converges to a fixed interval of \( N \). For this purpose we use the following proposition.

**Proposition 6.1.** Let \( X, Y \in D \) and \( N \) be defined by (5.10).

Then
\[ \text{ rad } (X^N) \cdot \text{ rad } N = \text{ rad } (X \cdot Y \cdot N) \tag{5.11} \]

Proof: By Proposition 3.2 and Proposition 1 in [12], we have
\[ \text{ rad } (X^N) = \text{ rad } (X^N Y^n (X^{n-1}) Y^{n-1} (X^{n-2}) Y^{n-2} \ldots Y) \leq \text{ rad } (X^N) = \text{ rad } (X \cdot Y \cdot N) \]

Moreover, if for some \( i, k \),
\[ |a| |r|^{-1} = 2 \text{ rad}(a) \tag{5.10} \]

then
\[ 0 < \text{ rad } X = \text{ rad } \mathbb{R}^n < \text{ rad } (a^{n-1}) = \text{ rad } (a^n) \text{ rad } (a) \tag{5.11} \]

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hence \( \text{ rad } X = \text{ rad } (a^{n-1}) = \text{ rad } (a^n) \cdot \text{ rad } (a) \) by interchanging \( X \) and \( Y \) and we also get \( \text{ rad } (X^N) = \text{ rad } (X \cdot Y \cdot N) \text{ rad } (a) \), which implies (6.1).

**Theorem 6.2.** Let \( N \) be defined by (5.2) if there is a fixed interval \( \overline{X} \) of \( \mathbb{R} \) such that
\[ \text{ rad } (X^N) \cdot \text{ rad } N = \text{ rad } (X \cdot Y \cdot N) \tag{5.11} \]

then
\[ \text{ rad } X = \text{ rad } Y = \text{ rad } N = \text{ rad } (X \cdot Y \cdot N) \tag{5.11} \]

Proof: Let \( N^2 \geq 0 \leq \overline{X} \) Then it follows by (5.2) and Proposition 6.1 that
\[ \text{ rad } X = \text{ rad } (X^N) = \text{ rad } Y = \text{ rad } (Y \cdot X) \]

hence
\[ (\text{ rad } X)^2 = (\text{ rad } Y)^2 = (\text{ rad } N)^2 = 0 \]

Since \( (\text{ rad } X)^2 (\text{ rad } Y)^2 > 0 \) because of (5.2), this implies that
\[ \text{ rad } X = \text{ rad } Y = \text{ rad } N = 0 \text{ and therefore } X = \overline{X}. \]

An interval vector \( X \) is called stable with respect to \( N \), if
\[ X \in \mathbb{R}^n \] implies \[ N \in \mathbb{R}^n \] by (5.4).

Now, that for \( \mathbb{R}^n \), if \( \text{ rad } \mathbb{R}^n = 0 \).

Now we will improve the theorem for \( \mathbb{R}^n \) and accordingly.

**Theorem 6.3.** Let \( G : D \times \overline{X} \rightarrow \overline{X} \) be a function that satisfies the Lipschitz condition (5.2) with regular \( \delta \), and let \( X^k \) be the sequence of \( G \)-defined by (5.3). Let \( N \) be defined by (5.2) with an inverse \( a^m \) of \( A \). Suppose \( \{X^k\} \) is an interval sequence with \( X^k = 10^k \) (\( X_{k+1} = a^m X_k \) \( X_{k+1} = a^{m+1} X_k \) \( a = 1, \ldots, \)) such that \( X_{k+1} = \overline{X} \), \( X_k = 0 \), and otherwise
\[ \text{ rad } X_{k+1} = \text{ rad } (a^m X_k) \]  

Then
\[ X_{k+1}, \text{ to } X_k \text{ is stable.} \]  

(5.4)

(5.5)

(5.6)
By Theorem 2.2 we can improve $A^*$ by considering instead the inverse $A^*$, we then have $A^* = \sigma(A^*)$. However, we do not know how to compare the corresponding convergence rates $\tau(A^*)$ and $\tau(A^*)$. The arguments leading to (6) only give

$$\alpha^*(\mu)^{-1} \approx \alpha(A)^{-1}$$

so that both the lower and upper bound for $\tau(A^*)$ are at least as poor as the corresponding bounds for $\tau(A^*)$.

Unless $\eta > 2$ or $A$ has special properties, the normal Gauss algorithm very likely breaks down or yields a perverse solution for $A^*(1)$. The usually recommended remedy is preconditioning with the midpoint inverse, see [11] and the citations there. For the remaining examples we therefore assume the following situation:

$$A^* = \sigma(A)^{-1}, \quad e := (\mu, \lambda)^{-1}$$

It is well known that under this assumption $A$ is regular. We shall call $A$ strongly regular if (11) holds.

Example 2: If $A$ is strongly regular then by Eq. (4.4) in [12], the map $A^*$ defined by

$$A^* \rightarrow (e^{A^*} / \beta)$$

is an inverse of $A$. The corresponding Newton operator is

$$N_A(x) = A^{-1} - (e^{A^*} / \beta)$$

and was introduced in [8].

Properties of the inverse $A^*$:

(i) $A^*$ is normal and consistent.

(ii) $\tau(A^*) = 2\alpha(A)^{-1}.$

(iii) if $\rho(A^*) = 1$ then $\sigma(A^*) = \sigma(A)^{-1}$.

(iv) if $\rho(A^*) > 1$ then $\sigma(A^*)$ is regular.
where (iv) holds. Since
\[ A[x] = [A; z = x \leftrightarrow (x \in \mathbb{R})] = \left( x \in \mathbb{R}, x + g(x) \right) \]
and
\[ A[x] = \exists x \in \mathbb{R} \, (x + g(x) = x + q(x)) \]
then
\[ A[x] = \exists x \in \mathbb{R} \, (x + g(x) = x + q(x)) \]
and (i) holds as well. Now recall (iii) holds since
\[ \text{rad}(A)[x] \subseteq \text{rad}(A)[y] \subseteq \text{rad}(A)[z] \]
and (iii) holds since by regularity of \( x \in \text{rad}(A)[x] \subseteq \text{rad}(A)[y] \subseteq \text{rad}(A)[z] \) and \( x \in \mathbb{R} \). Hence (i) holds. Since \( \text{cd}(A) = \left( x \in \mathbb{R} \right) \) (see (4.1) in (7)), (iii) follows from
\[ \text{rad}(A)[x] = \text{rad}(A)[y] \subseteq \text{rad}(A)[z] = \text{rad}(A)[z], \quad x \in \mathbb{R}. \]
and (iii) holds since by regularity of \( x \in \text{rad}(A)[x] \subseteq \text{rad}(A)[y] \subseteq \text{rad}(A)[z] \) and \( x \in \mathbb{R} \). Hence (i) holds. Now (ii) and (iii) imply \( [A; x] = \left( x \in \mathbb{R}, x + g(x) = x + q(x) \right) \) and \( [A; x] = \left( x \in \mathbb{R}, x + g(x) = x + q(x) \right) \) and hence (ii) holds. Finally, if \( \exists x \in \mathbb{R} \) then (see (ii)) then (ii) follows
\[ \text{rad}(A)[x] \subseteq \text{rad}(A)[y] \subseteq \text{rad}(A)[z] = \text{rad}(A)[z], \quad x \in \mathbb{R}. \]
and (iii) holds since by regularity of \( x \in \text{rad}(A)[x] \subseteq \text{rad}(A)[y] \subseteq \text{rad}(A)[z] \) and \( x \in \mathbb{R} \). Hence (i) holds. Now (ii) follows from
\[ \text{rad}(A)[x] \subseteq \text{rad}(A)[y] \subseteq \text{rad}(A)[z] = \text{rad}(A)[z], \quad x \in \mathbb{R}. \]
and the lower bound in (ii) follows from (i) and (ii) since \( \text{rad}(A)[x] \subseteq \text{rad}(A)[y] \subseteq \text{rad}(A)[z] \) and \( x \in \mathbb{R} \) if and only if \( x \in \mathbb{R} \). Hence (ii) follows from
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\[ \text{rad}(A)[x] \subseteq \text{rad}(A)[y] \subseteq \text{rad}(A)[z] = \text{rad}(A)[z], \quad x \in \mathbb{R}. \]
In this context, we have

\[ A^T \mathbf{e}^T (d_\mathbf{e}) d \mathbf{e} = (d_\mathbf{e}) d \mathbf{e} \mathbf{e}^T \]

so that

\[ A^T \mathbf{e}^T (d_\mathbf{e}) d \mathbf{e} = A^T (d_\mathbf{e}) (d_\mathbf{e})^T \]

for all \( 0 \leq \mathbf{e}^T \mathbf{e} \leq 1 \).

(3.1)

However, in contrast to (3.4), already simple examples with \( n = 1 \) show that equality need not hold for \( n = 0 \). It is easy to see that \( A^T \mathbf{e}^T (d_\mathbf{e}) d \mathbf{e} \) for \( n = 0 \). It would be interesting to know whether this also holds for higher dimensions.

**Proposition of the Inverse \( A^T \):**

(i) \( A^T \) is normal and controllable.

(ii) \( A^T \mathbf{e}^T (d_\mathbf{e}) d \mathbf{e} \leq 2p \).

(iii) \( A^T \mathbf{e}^T (d_\mathbf{e}) d \mathbf{e} \leq \mathbf{e}^T (d_\mathbf{e}) \mathbf{e} \).

(iv) \( A^T \mathbf{e}^T (d_\mathbf{e}) d \mathbf{e} \geq \mathbf{e}^T (d_\mathbf{e}) \mathbf{e} \).

(v) \( A^T \) is regular.

Proof. By Proposition 3 and Proposition 2 of (3.1), \( A^T \) is normal and by Proposition 2.3, \( A^T \) is incurred since \( c \) is solvable. Hence (i) holds.

The lower bound in (ii) holds since by (3.4), controllability 3.6 and 3.3 we have

\[ (d_\mathbf{e}) d \mathbf{e} \leq c \mathbf{e}^T (d_\mathbf{e}) d \mathbf{e} \leq (d_\mathbf{e}) d \mathbf{e} \mathbf{e}^T \leq 2p \mathbf{e}^T (d_\mathbf{e}) \mathbf{e} \mathbf{e}^T \]

The upper bound in (iv) holds since it follows immediately from (3.7) and the properties of \( A^T \). Finally (vi) holds since

\[ (d_\mathbf{e}) d \mathbf{e} \leq c \mathbf{e}^T (d_\mathbf{e}) d \mathbf{e} \leq (d_\mathbf{e}) d \mathbf{e} \mathbf{e}^T \leq 2p \mathbf{e}^T (d_\mathbf{e}) \mathbf{e} \mathbf{e}^T \]

by rule (88) of (11).

**Remark:** The hypothesis (3.10) in Theorem 3.10 holds for all \( A^T \) and \( A^T \), as long as (3.11) is valid for three reasons. We did not manage to decide whether (3.10) also holds for all \( A^T \).