ABSTRACT

As a continuation of my paper "New algorithms for the solution of linear interval equations" [Linear Algebra Appl. 39, 257-328 (1981)], this communication presents new results on interval Gaussian elimination and interval linear equations. Main results are a new convergence theorem for interval Gaussian elimination, a result in a perturbation theorem, a successive and comparison theorem for a general family of interval linear equations, and a new method for the calculation of the hull of the solution set of linear interval equations with inverse positive coefficient matrix.

1. INTRODUCTION

This paper contains the study of linear and nonlinear solution algorithms for linear interval equations. The methods introduced are used here to investigate the solutions of the interval equations

\[ \psi^{(k)} = A^{(k)} x + B^{(k)} \quad (k=0,1,2,...), \]  

where \( A^{(k)} \) denotes the result of interval Gaussian elimination applied to the coefficient matrix \( A \) and right-hand side \( x \). Under suitable conditions, the iteration converges and the limit \( \psi = \lim_{k \to \infty} \psi^{(k)} \) is an enclosure for the solutions set of the linear interval equation

\[ A \psi = \xi \quad (\xi \in \mathbb{R}, \quad \xi \in \mathbb{R}^n). \]
The theory developed also allows an interesting application to improve positive roots. It is shown (Theorem 5.3) that if $R$ is a matrix of nonnegative integers (1.5) with positive integers $R$ is permutable by multiplication with $C = R^T$ where $R^T = \overline{R}$ is the transpose of $R$, and is constructed as in the previous section (reversal). The proof of the main result (Theorem 5.3) requires the introduction of a new basis for the eigenvalues of a matrix $A$. For the matrix $A$, the eigenvalues are $\lambda_1, \lambda_2, \ldots, \lambda_n$. The spectral radius of $A$ is defined as $\rho(A) = \max \{ |\lambda_1|, |\lambda_2|, \ldots, |\lambda_n| \}$.

The paper briefly refers to Nozeman [20] and the related results are also discussed. A new basis for the eigenvalues of a matrix $A$ is constructed as in the previous section (reversal). For the matrix $A$, the eigenvalues are $\lambda_1, \lambda_2, \ldots, \lambda_n$. The spectral radius of $A$ is defined as $\rho(A) = \max \{ |\lambda_1|, |\lambda_2|, \ldots, |\lambda_n| \}$.

Section 5 contains the recent positive case.

5. NOTATION AND TECHNICAL RESULTS

We assume the reader to be familiar with Nozeman [20], we shall make repeated use of notation, concepts, and results explained there. However, for the convenience of the reader some basic terminology is reviewed in this section. Moreover, several technical results are proved which simplify the presentation of the material in later sections.
Proof. (2.1) was shown in Proposition 3 of Section 2.3. To prove the second part we put \( s = \gamma^p \). The assumption then implies \( \gamma(s, y) = \gamma(s, y) \). If \( s \geq 0 \), then \( s \cdot s = s \). Hence the \( s \cdot s \) column of \( \gamma \) is repeated by the \( s \cdot s \) column of \( \gamma \). Since \( s \) is arbitrary, \( s \leq 0 \).

Later we shall need several properties of the distance not mentioned in [3].

Lemma 2.3. For all \( A, a, x, y \in \mathbb{R}^n \) we have
\[ d(A, x) + d(A, y) \leq d(x, y) \]
(2.3)

In particular,
\[ d(A, y) \leq d(A, x) + d(x, y) \]
(2.3)

Proof. Put \( A = A = A = A \). Then
\[ d(A, A) = d(A, A) = d(A, A) \]
\[ A \subseteq A \]
\[ \therefore d(A, A) + d(A, A) \leq d(A, A) \]
which implies \( A \subseteq A \). Hence (2.2) holds. (2.3) follows from properties of the absolute value and the Cauchy-Schwarz inequality.

Let \( x, y, z \in \mathbb{R}^n \). Then
\[ d(x, z - y) = d(x, z - y) \]
(2.4)
\[ d(x, z - y) = d(x, z - y) \] for some \( z \in \mathbb{R}^n \).
(2.5)
\[ d(x, z - y) = d(x, z - y) \]
(2.6)

Proof. Put \( y = -z \). Then \( d(x, y) = d(x, y) = d(x, y) \), hence \( d(x, y) = d(x, y) = d(x, y) \). The triangle inequality (2.6) now follows if \( z \in \mathbb{R}^n \). Since \( d(x, y) = d(x, y) = d(x, y) \), the triangle inequality (2.6) holds for \( z \in \mathbb{R}^n \). However, if \( z \in \mathbb{R}^n \) then \( z \in \mathbb{R}^n \) since \( z \in \mathbb{R}^n \). Hence (2.6) holds.

Lemma 2.4. The operator \( \beta : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n \) defined by
\[ \beta(x, y) = d(x, y) \]
(2.7)

has the following properties:

1. If \( x_1, y_1 \in \mathbb{R}^n \), then
\[ \beta(x_1, y_1) = \beta(x_1, y_1) \]
(2.8)

2. If \( x, y, z \in \mathbb{R}^n \), then
\[ \beta(x, y + z) = \beta(x, y) + \beta(y, z) \]
(2.9)

Proof. (2.9) follows by induction on the number of terms in \( \gamma \).

Let \( \gamma = (x, y) \) and \( \gamma = (x, y) \). By assumption, if \( \gamma = (x, y) \) then \( \gamma = (x, y) \) since \( \gamma = (x, y) \). Hence (2.9) holds.

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For \( \gamma = (x, y) \) and \( \gamma = (x, y) \), by assumption, if \( \gamma = (x, y) \) then \( \gamma = (x, y) \) since \( \gamma = (x, y) \). Hence (2.9) holds.

\[ \beta(x, y + z) = \beta(x, y) + \beta(y, z) \]
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(2.9)
\[ R(x, y) = x + (x - y) \]
\[
\frac{1}{n(x) - \frac{1}{n(x) - n(y)}}
\]

We shall also use the following results on nonnegative matrices.

**Lemma 3.5.** Let \( M, N, A, B \in \mathbb{R}^{+\to} \) be nonnegative matrices such that

\[ M < (I - M^2)^{-1} M \]

and suppose that the spectral radius of \( M(2 - 2z) \) is less than one. Then \( A(M^2) \) and

\[ 0 < (I - M^2)^{-1} M < (I - M^2)^{-1} M \]

**Proof.** By assumption there is a scalar \( \alpha > 0 \) such that \( M(2 - 2z) < \alpha \). Since \( M < \alpha \), \( M(2 - 2z) < \alpha(2 - 2z) < \alpha \). Hence \( M < (I - M^2)^{-1} M \). Hence \( M(2 - 2z) < \alpha \) and \( (I - M^2)^{-1} M < \alpha \).

Next, we consider the matrix

\[ N = (I - M^2)^{-1} (M^2) > 0, \]

Clearly \( N(2 - 2z) > 0, N^2 = (I - M^2)^{-1} M^2 > 0 \). Hence \( N \) is an

**Lemma 3.6.** Let \( A \in \mathbb{R}^{+\to} \) and \( \tau \in \mathbb{R}^{+\to} \).

(i) \( \tau > 0 \) then \( (A - \tau) A \) for suitable \( A, \tau \).

(ii) \( \tau > 0 \) then \( (A - \tau) A = (A - \tau A) A \).

**Proof.** (i) For \( A, \tau \) defined by

\[ (A)_i = \delta_i, \quad \tau > 0, \quad A_i = \delta_i \] otherwise.

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3. **Sublinear Iteration**

In this section we consider the sublinear iteration

\[ v^{(n+1)} = (x + \gamma v^n) \]

where \( \gamma \) is an additive map. This generalizes the special iteration

\[ v^{(n+1)} = \gamma(v^n) \]

as in Section 7 of [10] by triangular splittings \( A = L - K \).

**Theorem 5.1.** Let \( \gamma \in \mathbb{R}^{+\to} \) be additive maps satisfying

\[ \gamma(A(T) < 1. \]

Then, for each \( \gamma \in \mathbb{R}^{+\to} \), the following statements hold:

(i) The equation

\[ y = \gamma v^n \]

has a unique solution \( y \in \mathbb{R}^{+\to} \).
For all starting nets \( \varphi^* \in \mathfrak{K}^* \), the iteration
\[
\varphi^* = x + \varphi(x)
\]
converges to the solution \( \varphi \) of (3.1).\(^{(1)}\) and
\[
\|\varphi - \varphi^*\| < \|\varphi(x) - \varphi(x)\|
\]
for any norm \( \| \cdot \| \) satisfying
\[
\|\varphi(\varphi(x))\| = \|\varphi(\varphi(x))\| = 1.
\]
(1) If \( \varphi(x) \leq \varphi(y) \), then for all \( i > 1 \),
\[
\varphi(x) \leq \varphi(x) \leq \cdots \leq \varphi(x).
\]
(2) If \( \varphi(y) \leq \varphi(x) \), then for all \( i > 1 \),
\[
\varphi(x) \leq \varphi(x) \leq \cdots \leq \varphi(x).
\]
Proof. Since \( \varphi(\varphi(x)) = \varphi(\varphi(x)) = 1 \), there exists a unique \( \lambda \) such that (3.1) holds. For fixed \( \lambda \in \mathfrak{K}^* \), the map \( \varphi : \mathfrak{K}^* \to \mathfrak{K}^* \) defined by
\[
\varphi(\varphi(x)) = \varphi(\varphi(x)) = 1
\]
attains \( \varphi(\varphi(x)) = \varphi(\varphi(x)) = 1 \).
\[
\|\varphi(\varphi(x))\| = \|\varphi(\varphi(x))\| = 1.
\]
by Lemma 2.1, whence by (2.6)
\[
\|\varphi(x)\| = \|\varphi(x)\| = 1.
\]
Since \( \mathfrak{K}^* \) is a locally convex metric space with respect to the metric,
\[
\|\varphi(x)\| = \|\varphi(x)\| = 1,
\]
and the generalized of the Banach fixed-point theorem by Schaefer (14) shows that \( \varphi \) has a unique fixed point \( \varphi \in \mathfrak{K}^* \), and for arbitrary \( \varphi^* \in \mathfrak{K}^* \), the iteration \( \varphi^* = \varphi(\varphi(x)) \), (3.1), converges to \( \varphi \) with speed determined by (3.3). This proves (3.1) and (1).
Finally,
\[ a(F_1, x) = a(F_1, y) = a(F_1, z) \]
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\[ a(F_1, x) = a(F_1, y) = a(F_1, z) \]
\[ a(F_1, x) = a(F_1, y) = a(F_1, z) \]
by Lemma 2.1, so that
\[ a(F_1, x) = a(F_1, y) = a(F_1, z) \]
Multiplication with the nonnegative matrix \((I - B)F_1 = (x, y) \) and application of the second part of Lemma 3.3 now implies (3.9).

We shall refer to \( F_1(x, y) \) as the \( \Gamma \)-norm of the function \( F_1(x, y) \). As an easy verifiable existence condition for \( F_1(x, y) \) we prove:

**Theorem 3.3.** Let \( F_1(x, y) \) be normal noninear maps and let \( x, y \in \mathbb{C} \).

(i) If
\[ a(F_1, x) < a(F_1, y) \]
then \( a(F_1, x) < a(F_1, y) \).

(ii) If
\[ a(F_1, x) < a(F_1, y) \]
then \( a(F_1, x) < a(F_1, y) \).

\( F_1(x, y) \) is continuous.

**Proof.** Since \( F_1(x, y) \) are normal, (3.11) implies
\[ a(F_1, x) < a(F_1, y) \]
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Hence \( a(F_1, x) < a(F_1, y) \) satisfies \( a(F_1, x) < a(F_1, y) \), so that \( a(F_1, x) < a(F_1, y) \).

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\[ a(F_1, x) < a(F_1, y) \]

Next, we consider the case where \( F_1(x, y) \) are normal noninear maps and let \( x, y \in \mathbb{C} \).

**Theorem 3.3.** Let \( F_1(x, y) \) be normal noninear maps and let \( x, y \in \mathbb{C} \).

(i) If
\[ a(F_1, x) < a(F_1, y) \]
then \( a(F_1, x) < a(F_1, y) \).

(ii) If
\[ a(F_1, x) < a(F_1, y) \]
then \( a(F_1, x) < a(F_1, y) \).

\( F_1(x, y) \) is continuous.

**Proof.** Since \( F_1(x, y) \) are normal, (3.11) implies
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Hence \( a(F_1, x) < a(F_1, y) \) satisfies \( a(F_1, x) < a(F_1, y) \), so that \( a(F_1, x) < a(F_1, y) \).

\[ a(F_1, x) < a(F_1, y) \]
4. FINITIZED ITERATION AND GAUSS ELIMINATION

If \( A = \text{diag}(a_{11}, \ldots, a_{nn}) \) has a triangular decomposition \((L_r, R_r)\), then, as in [10], we denote by \( \mathbf{d}^* \) the result of Gauss elimination applied to the coefficient matrix \( A \) and the right-hand side \( \mathbf{s} \); i.e., \( \mathbf{s} \) is the right-hand side of \((A, \mathbf{s})\) for the right-hand side \( \mathbf{s} \) of \((A, \mathbf{s})\).

**Theorem 4.3** Let \( A \in \mathbb{R}^{n \times n} \), and suppose that \( A \) has a triangular decomposition \((L_r, R_r)\) and \( B = A - E \). If

\[
\mathbf{s}[A^T][\mathbf{b}] < 1,
\]

then \( P = (B^T)^{-1}(A^T) \) is a matrix such that

\[
(P) := (B^T)^{-1}(A^T) \mathbf{b} < 1.
\]

**Proof.** We begin by observing that (4.5) holds, since by continuity of the spectral radius, there is a matrix \( \mathbf{A} \) such that for some \( \mathbf{A} \) there exists a matrix \( \mathbf{A} > 0 \) such that

\[
\mathbf{A}[A^T][\mathbf{b}] = 1,
\]

and therefore, \( \mathbf{A}^T \mathbf{b} = 1 \). This implies that for some \( \mathbf{A} > 0 \), there exists a matrix \( \mathbf{A} > 0 \) such that

\[
\mathbf{A}[A^T][\mathbf{b}] = 1,
\]

and therefore, \( \mathbf{A}^T \mathbf{b} = 1 \). This implies that for some \( \mathbf{A} > 0 \), there exists a matrix \( \mathbf{A} > 0 \) such that

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\]

and therefore, \( \mathbf{A}^T \mathbf{b} = 1 \). This implies that for some \( \mathbf{A} > 0 \), there exists a matrix \( \mathbf{A} > 0 \) such that

\[
\mathbf{A}[A^T][\mathbf{b}] = 1,
\]

and therefore, \( \mathbf{A}^T \mathbf{b} = 1 \). This implies that for some \( \mathbf{A} > 0 \), there exists a matrix \( \mathbf{A} > 0 \) such that
Proposition 3.4 implies that

\[ [P] = (1 - (\alpha_i^2)(\beta_i^2)^{-1})^{-1} = (1 - (\beta_i^2)^{-1})(\alpha_i^2)^{-1} \]

\[ = (1 - (\alpha_i^2)(\beta_i^2)^{-1})^{-1} = (\alpha_i^2)^{-1} \]

Remark. If \( A = L \) is lower triangular, then the inversion \( y = \frac{1}{\alpha_i}(1 - \frac{\beta_i}{\alpha_i}) \) reduces to the inversion \( y = \frac{1}{\alpha_i}x \) considered in Section 7 of [40]. Indeed, it is not difficult to show that if \( A \) is regular and lower triangular, then \( y_i \) exists and agrees with the forward substitution map \( A^T \). This is trivial if \( A \) has a subdiagonal, in which case the triangular decomposition is given by \( R_i = \text{Diag}(\alpha_{i+1}, \ldots, \alpha_n) \).

\[ y_i = \frac{1}{\alpha_i}x_i \quad \text{for } i = 1, \ldots, n \]

\[ B = \text{diag}(\alpha_1, \ldots, \alpha_n) \]

\[ \text{and that } A = L \cdot U \quad \text{for some } \alpha, \beta \in \mathbb{R} \]

Since multiplication of intervals is associative. By induction, we find \( y_i = \frac{x_i}{\alpha_i} \) for \( i = 1, \ldots, n \), whence

\[ y = (\alpha_i)_{i=1}^n, \quad \text{where} \]

\[ u = (\beta_i)_{i=1}^n \quad \text{and} \]

\[ B = \text{diag}(\alpha_1, \ldots, \alpha_n) \]

\[ \text{and the inversion } (4.6) \text{ converges.} \]

Proof. Apply Proposition 3.4.

In order to prove a comparision theorem for different splittings \( B = L \cdot U = A_1 \cdot A_2 \); we need a particularization theorem for Gauss elimination in which the matrix coefficients are in the field of rational numbers. By continuity of Gauss elimination as a function of the matrix coefficients it is clear that the existence of \( \beta^* \) implies the existence of \( \beta^* \) for \( B \in \mathbb{R}^{n \times n} \) sufficiently close to \( A \). We shall prove that, in any case, \( B \) is sufficiently close to \( A \) if (4.10) or (4.12) holds. The proof is based on the following result.

Theorem 4.5. Let \( (A, B) \) be a triangular decomposition of \( A \in \mathbb{R}^{n \times n} \), and let \( B \neq A \) be such that for suitable \( \alpha, \beta \neq 0 \) we have

\[ \varphi_t(A, B) \subset (L, U) \]

Then \( A \) has a triangular decomposition \((L, U)\), and

\[ (L, U) \subset (\alpha, \beta) \]

Proof. This is trivial for \( n = 1 \), hence we proceed by induction on \( n \) and assume the statement to be true for dimensional \( n \). Let \( A \) be an \( n \times n \) matrix such that \( A \) has the triangular decomposition \((L, U)\). Then

\[ L = \begin{pmatrix} \alpha_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \alpha_n \end{pmatrix} \]

\[ U = \begin{pmatrix} \beta_1 & \cdots & 0 \\ 0 & \ddots & \vdots \\ 0 & \cdots & \beta_n \end{pmatrix} \]

\[ \text{where } (L, U) \text{ is the triangular decomposition of the factor complement} \]

\[ A = B - \frac{b_{21}}{b_{11}} b_{11} \]
Let
\[ A_n^0 = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \]
\[ u_0 = \begin{pmatrix} x \\ y \end{pmatrix}, \quad v = \begin{pmatrix} z \\ w \end{pmatrix} > 0, \]
\[ q(A_n^0, B_n^0) = \begin{pmatrix} x' & y' \\ z' & w' \end{pmatrix} \]
be such that \( A_n^0 \preceq A_0 \) and
\[ q(A_n^0, B_n^0) \leq (L_2)_n(B_2)_n - q_n. \]
Then
\[ v_0 + x_0 \leq (\alpha_0 - (y_0 + z_0 - z_0)) - v_0, \]
\[ y_0 + z_0 < (\alpha_0 - (y_0 + z_0 - z_0)) - v_0. \]
By our induction hypothesis, \( A \) has a triangular decomposition \((A, B)\) with
\[ (L_2)_n(B_2)_n \triangleq x_0 + (y_0 + z_0 - z_0 - v_0)^+ \]
\[ (L_2)_n(B_2)_n \triangleq x_0. \]
Therefore, with
\[ L_2 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{-1} \end{pmatrix}, \quad B_2 = \begin{pmatrix} \alpha_0 & 0 \\ 0 & \alpha_0 \end{pmatrix} \]
\[ (L_2)_n(B_2)_n \triangleq x_0 + (y_0 + z_0 - z_0 - v_0)^+ \]
\[ (L_2)_n(B_2)_n \triangleq x_0. \]
which completes the induction.

**Theorem 4.6.** Let \( A, B \in \mathbb{R}^{n \times n} \), and suppose that \( A^0 \) exists. if
\[ q(A^0, B^0, A) < 1, \]
\[ or \text{equivalently, if there is a } u \in \mathbb{R}^n \text{ such that} \]
\[ 0 < u, (A, B)u < (L_2)_n(B_2)_n u, \]
where \((L_2)_n(B_2)_n \) is the triangular decomposition of \( A \), then \( B^0 \) exists and
\[ B^0 \triangleq (L_2^0)(B, A)^{-1} \triangleq ((L_2^0)(B, A)^{-1})^{-1}. \]
Theorem 4.5. Let \( A, B \in \mathbb{R}^{n \times n} \) and suppose that \( A^* \) exists. If
\[
(\mathbf{q}^T B^{-1} \mathbf{q}) < 1
\]
for some \( \mathbf{q} \in \mathbb{R}^n \), then \( B^* \) exists.

Proof. Let \( E = \{-q(R, A), q(R, A)\} \), where \( A = E + E \). We get \( \mathbf{q}^T B^{-1} \mathbf{q} = \mathbf{q}^T (E + E) \mathbf{q} = \mathbf{q}^T E \mathbf{q} + \mathbf{q}^T E \mathbf{q} < 1 \), so that Theorem 4.5 applies.

Corollary 4.8. Under the assumptions of Theorem 4.5, \( B^* \) exists and \( \mathbf{q}^T B^{-1} \mathbf{q} \).
3. THE INVERSE POSITIVE CASE

Let \( A \in \mathbb{R}^{n \times n} \) be regular. Then, as discussed in [20], the ball inverse \( A^\circ \) is the sublinear map which maps \( x \in \mathbb{R}^n \) to the ball

\[
A^\circ x = \{ y \mid \langle x, y \rangle \leq 1 \}.
\]

of the solution set of the system of linear interval equations

\[
\Delta y = x \quad (\Delta \in A, \ e \in e).
\]

Here we show that one can

\[
A^{-1} = \Delta^{-1} \Delta A^{-1} = A^\circ.
\]

In nonnegativity, the ball inverse can be obtained as a diagonal map. This gives a new method for the computation of the ball of the solution set of linear interval equations with interval matrices containing uncertain data. This work [5] for previous approaches. We also show that preconditioning of such equations with certain nonnegative matrices does not change the ball of the solution set. Finally, the special situation of M-matrices is considered.

Theorem 3.5. Let \( A \in \mathbb{R}^{n \times n} \) be such that \( \Delta^{-1} > 0 \), and suppose that \( \Delta \) is nonnegative satisfying

\[
\Delta A^{-1} \geq I, \quad C = \Delta A^{-1} > 0. \tag{5.1}
\]

Then:

(i) \( \Delta \) is regular, \( \Delta^{-1} > 0 \), and the ball inverse \( A^\circ \) is the finite set of the form

\[
A^\circ x = C + (I - CA)x' \quad (x' = 0, 1, 2, \ldots)
\]

and

\[
A^\circ x = A^{-1}(x + CAx') \quad (x' = 0, 1, 2, \ldots).
\]

(ii) \( CA \) is an M-matrix, and for all \( x \in \mathbb{R}^n \),

\[
A^\circ x = (CA)^{0}(Cx) + (CA)^{1}(Cx).
\]

\[ (CA)^{0}(Cx) + (CA)^{1}(Cx). \]
Proof. Since \( A \in \mathbb{M}_2 \), Lemma 12 of [10] implies \( A^{-1} > 0 \). To show the existence of the Jacobi matrix of \( 0(A) \) and \( (2.3) \) we put
\[
\Delta = \begin{pmatrix} \Delta & 0 \\ 0 & \Delta \end{pmatrix}
\]
by rule (2.3) of [10]. Let \( x > 0 \). Then \( x = X - Y > 0 \), since the cone of \( A^{-1} \) consists of only one cone, hence also \( X > 0 \). This implies \( X > 0 \), so that \( 0(A) \) is valid. Since \( C = \frac{1}{A} \Delta \frac{1}{A} \) and \( (1 - C) x = 0 \), also \( C = \Delta \) is a strong splitting of \( C \), known by Theorem 11 of [10]. Hence, if \( \Delta \) is a strong splitting of \( C \), known by Theorem 11 of [10], then
\[
(CA)^2(CA) = CA
\]
for all \( x \in X \). (5.9)

Moreover, since \( (\Delta^{-1})x = \Delta^{-1}(A^{-1})x = y \), Theorem 12 implies with (5.9):
\[
(CA)^2(CA) = CA
\]
for all \( x \in X \). (5.5)

Now suppose that \( x \in X \), and put \( y = y \). By rule (2.3) and (2.2) of [10],
\[
\Delta = \begin{pmatrix} \Delta & 0 \\ 0 & \Delta \end{pmatrix}
\]
whereby Theorem 8(b) of [10].

By definition of \( y \) we have \( y = \Delta^{-1}(x + \Delta y) \), whereby \( x \in X \). By Lemma 12 above
\[
y = \Delta^{-1} \left[ \begin{pmatrix} x_1 & 0 \\ x_2 & 0 \end{pmatrix} \right] = \begin{pmatrix} x_1 + \Delta y_1 \\ x_2 + \Delta y_2 \end{pmatrix}
\]
for suitable \( E, F, x \in X \) (note that \( \Delta > 0 \) is fixed). Now \( \Delta^{-1} = A^{-1} \), hence \( \Delta \) is invertible, and
\[
\Delta y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}
\]
Therefore \( y = \Delta^{-1}y \) leading to \( \Delta = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \), whence
\[
\Delta = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}
\]
(5.7)

Remarks.

(1) If \( A \in \mathbb{M}_2 \) is regular and \( A^{-1} > 0 \), then the assumptions of Theorem 11 are certainly satisfied with \( \Delta = \Delta \). However, for practical reasons the simplification of \( \Delta \) is a matter of \( \Delta \) with more non-zero entries than \( \Delta \) should be the case and as long as \( \Delta^{-1} > 0 \).

(2) The invariance (5.5) is not valid on practical calculation, since the exact inverse \( A^{-1} \) or no existence of \( A^{-1} \), is needed. In the presence of rounding errors we get, however, only an approximate \( x \). The invariance (5.5) still works in this situation if it is an existence of the Jacobi of (5.5) as an existence of \( (CA)^2(CA) \) desired by preconditioned conjugate methods. Then we still have \( (CA)^2(CA) = y \), and since equality would hold if \( C = A^{-1} \) and \( y \) is the eigenvalue of \( y \), the invariance of \( (CA)^2(CA) \) is strictly due to rounding errors in the computation of \( C \) and the iteration for the computation of \( y \), and therefore usually small.

(3) It is remarkable that in the situation discussed, preconditioning with \( C = A^{-1} \) improves the last of the solution set. In contrast to this, the best recommended preconditioning with the adjoint inverse \( A^{-1} \) (Hansen [7], Nesterov [11]) may improve the ill slightly. However, the amount of overestimation remains small, of Nesterov [11].

(4) The explicit inversion of \( A \) may in some cases and sometimes contain \( A \) is a high-dimensional space matrix. Hence it is interesting that in this case of Nesterov, the inversion of \( A \) can be approximated by a triangular representation of \( A \). This is based on the following observation.

Proposition 5.5. Let \( A \in \mathbb{M}_2 \) be an invertible. Then
\[
A^{-1} = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}
\]
for all \( a, b, c, d \). (5.8)

Moreover, if \( a > 0 \) then
\[
A^{-1} = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}
\]
(5.9)

Proof. By results of Birkhoff and Naylor [2] and Nesterov [1], \( A^{-1} = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \) for all \( a, b, c, d \). In particular, this holds if \( a \) has at most one non-zero

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\begin{align*}
\Delta^N &= \begin{bmatrix} [0, 1] & \cdots & [0, 1] \end{bmatrix} \\
\Delta'^N &= \begin{bmatrix} [1, -1] & \cdots & [1, -1] \end{bmatrix} \\
\Delta''^N &= \begin{bmatrix} [1, -1] & \cdots & [1, -1] \end{bmatrix}
\end{align*}

so that

\[ \Delta''^N \leq \Delta^N \leq \Delta'^N. \]