Axioms of Adaptivity

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1. Introduction and Outline

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Rate Optimality of Adaptive Algorithms

The overwhelming practical success of adaptive mesh-refinement in computational sciences and engineering has recently obtained a mathematical foundation with a theory on optimal convergence rates. This article first explains an abstract adaptive algorithm and its marking strategy. Secondly, it elucidates the concept of optimality in nonlinear approximation theory for a general audience. It thirdly outlines an abstract framework with fairly general hypotheses (A1)—(A4), which imply such an optimality result. Various comments conclude this state of the art overview.

All details and precise references are found in the open access article [C. Carstensen, M. Feischl, P. Page, D. Praetorius, Comput. Math. Appl. 67 (2014)] at http://dx.doi.org/10.1016/j.camwa.2013.12.003.

The Algorithm

The geometry of the domain Ω in some boundary value problem (BVP) is often specified in numerical simulations in terms of a triangulation $\mathcal{T}$ (also called mesh or partition) which is a set of a large but finite number of cells (also called element-domains) $T_1, \dots, T_N$. Based on this mesh $\mathcal{T}$, some discrete model (e.g., finite element method (FEM)) leads to some discrete solution $U(\mathcal{T})$ which approximates an unknown exact solution u to the BVP. Usually, a posteriori error estimates motivate some computable error estimator

$$\eta(\mathcal{T})^2 = \sum_{i=1}^{N} \eta_{T_i}(\mathcal{T})^2.$$

The local contributions $\eta_{T_i}(\mathcal{T})$ serve as refinement-indicators in the adaptive mesh-refining algorithm, where the marking is the essential decision for refinement and written as a list of $\mathcal{M}$ cells (i.e. $\mathcal{M} \subseteq \mathcal{T}$) with some larger refinement-indicator. The refinement procedure then computes the smallest admissible refinement $\mathcal{T}'$ of the mesh $\mathcal{T}$ (see Section 3) such that at least the marked cells are refined.

The successive loops of those steps lead to the following adaptive algorithm, where the coarsest mesh $\mathcal{T}_0$ is an input data.

Adaptive Algorithm
Input: initial mesh $\mathcal{T}_0$
Loop: for $\ell = 0, 1, 2, \dots$ do steps 1-4:
1. Solve: Compute discrete approximation $U(\mathcal{T}_\ell)$.
2. Estimate: Compute refinement indicators $\eta_T(\mathcal{T}_\ell)$ for all $T \in \mathcal{T}_\ell$.
3. Mark: Choose set of cells to refine $\mathcal{M}_\ell \subseteq \mathcal{T}_\ell$ (see Section 4 for details).
4. Refine: Generate new mesh $\mathcal{T}_{\ell+1}$ by refinement of at least all cells in $\mathcal{M}_\ell$ (see Section 3 for details).
Output: Meshes $\mathcal{T}_\ell$, approximations $U(\mathcal{T}_\ell)$, and estimators $\eta(\mathcal{T}_\ell)$.

The Optimality

Figure 1 displays a typical mesh for some adaptive 3D mesh-refinement of some L-shaped cylinder into tetrahedra with some global refinement as well as some local mesh-refinement towards the vertical edge along the re-entrant corner. The question whether this is a good mesh or not is an important issue in the mesh-design with many partially heuristic answers and approaches. We merely mention the coarsening techniques as in [Binev et al., 2004] when applied to the adaptive hp-FEM with the crucial decision about h- or p-refinement.

For the optimality analysis of the adaptive algorithm of Section 1, the

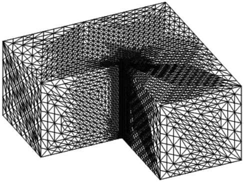


Figure 1: Strongly adaptively refined surface triangulation

Short History

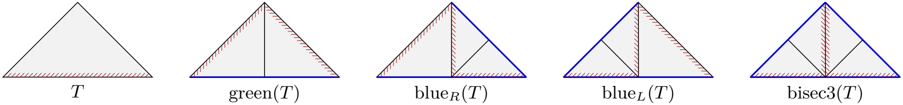
- Advert Johnson-Eriksson/Babuska; early 1D results Babuska et al.
- Dörfler marking [Dörfler, 1996]
- Convergence [Morin-Nochetto-Siebert 2000]
- Optimal rates for the Poisson problem [Binev-Dahmen-DeVore 2004]
- Optimal rates without coarsening [Stevenson 2007]
- Convergence for nonconforming/mixed FEM [Carstensen-Hoppe 2006]
- NVB included [Cascon-Kreuzer-Nochetto-Siebert 2008]
- Integral equations and BEMs [Feischl et al. 2013], [Gantumur 2013]
- Poisson with general boundary conditions [Aurada et al. 2013]
- Abstract framework [C-Feischl-Page-Praetorius 2014]
- Instance optimality [Diening-Kreuzer-Stevenson 2015]

Overview

- Introduction
- (A1) Stability
- (A2) Reduction
- (A12) and plain convergence
- (A3) Reliability
- Quasimonotonicity
- (A4) Quasiorthogonality
- R-linear convergence
- Comparison lemma
- Optimal convergence rates

Admissible Triangulations

- NVB refinement strategy and initial triangulation $\mathcal{T}_0$ specifies set $\mathbb{T}$ of all admissible triangulations



- NVB in any dimension [Stevensen (2008) Math.Comp.]
- Overlay control $|\mathcal{T}_\ell \oplus \mathcal{T}_{\text{ref}}| + |\mathcal{T}_0| \leq |\mathcal{T}_\ell| + |\mathcal{T}_{\text{ref}}|$
- Closure Overhead Control $|\mathcal{T}_\ell| - |\mathcal{T}_0| \leq C_{BDV} \sum_{j=0}^{\ell-1} |\mathcal{M}_j|$

Adaptive Algorithm

Input: Initial triangulation $\mathcal{T}_0$ with NVB refinement edges and $0 < \theta \ll 1$

$\forall \ell = 0, 1, 2, 3, \dots$ until termination do

- Given $\mathcal{T}_\ell$, solve discrete problem and compute error estimators

$$\eta_\ell(K) \text{ for all } K \in \mathcal{T}_\ell$$

- Determine (almost) minimal set of marked cells $\mathcal{M}_\ell \subset \mathcal{T}_\ell$ s.t.

$$\theta\, \eta_\ell^2 \leq \eta_\ell^2(\mathcal{M}_\ell) := \sum_{M \in \mathcal{M}_\ell} \eta_\ell^2(M)$$

- Design minimal refinement $\mathcal{T}_{\ell+1}$ of $\mathcal{T}_\ell$ with $\mathcal{M}_\ell \subseteq \mathcal{T}_\ell \setminus \mathcal{T}_{\ell+1}$

Output: Sequence of triangulations and estimators

Optimal Convergence Rates

Axioms (A1)–(A4) involve

estimators $0 \le \eta(\mathcal{T};K) < \infty$ for all $K \in \mathcal{T} \in \mathbb{T}$ and

distances $0 \le \delta(\mathcal{T},\hat{\mathcal{T}}) < \infty$ for all refinements $\hat{\mathcal{T}}$ of $\mathcal{T}$ in $\mathbb{T}$

Axioms (A1)–(A4) imply rate-optimality for $\theta \ll 1$ in the sense that

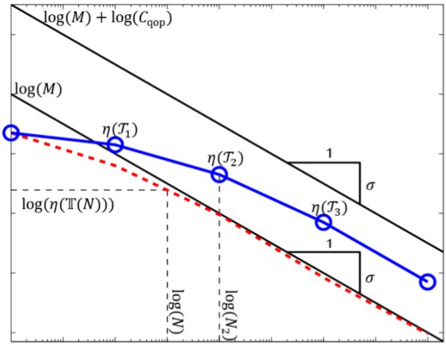
$$\sup_{N\in\mathbb{N}_0} (1+N)^s \min_{\mathcal{T}\in\mathbb{T}(N)} \eta(\mathcal{T})$$

$$\approx \sup_{\ell\in\mathbb{N}_0} (1+\underbrace{|\mathcal{T}_\ell|-|\mathcal{T}_0|}_{N_\ell})^s \eta_\ell$$

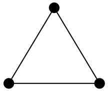
$$\eta_\ell^2 := \eta^2(\mathcal{T}_\ell)$$

$$\eta^2(\mathcal{T}) := \eta^2(\mathcal{T},\mathcal{T})$$

$$\eta^2(\mathcal{T},\mathcal{M}) := \sum_{K\in\mathcal{M}} \eta^2(\mathcal{T},K)$$

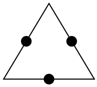


Poisson Model Problem (PMP) $f + \Delta u = 0$



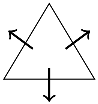
CFEM seeks $u_C \in P_1(\mathcal{T}) \cap C_0(\Omega)$ s.t.

$$\int_\Omega \underbrace{\nabla u_C}_{p_\mathcal{T}} \cdot \nabla v_C \, dx = \int_\Omega f v_C \, dx \quad \text{for all } v_C \in P_1(\mathcal{T}) \cap C_0(\Omega)$$



CR-NCFEM seeks $u_{CR} \in CR^1_0(\mathcal{T})$ s.t.

$$\int_\Omega \underbrace{\nabla_{NC} u_{CR}}_{p_\mathcal{T}} \cdot \nabla_{NC} v_{CR} \, dx = \int_\Omega f v_{CR} \, dx \quad \text{for all } v_{CR} \in CR^1_0(\mathcal{T})$$



RT-MFEM seeks $p_{\mathrm{RT}} \in \mathrm{RT}_0(\mathcal{T})$ and $u_{\mathrm{RT}} \in P_0(\mathcal{T})$ s.t.

$$\begin{aligned} \int_\Omega \underbrace{p_{\mathrm{RT}}}_{p_\mathcal{T}} \cdot q_{\mathrm{RT}} \, dx + \int_\Omega u_{\mathrm{RT}} \operatorname{div} q_{\mathrm{RT}} dx &= 0 \qquad \text{for all } q_{\mathrm{RT}} \in \mathrm{RT}_0(\mathcal{T}) \\ \Pi_0 f + \operatorname{div} p_{\mathrm{RT}} &= 0 \end{aligned}$$

Estimator in PMP $f + \Delta u = 0$

CFEM, CR-NCFEM, MFEM generate discrete flux

$$p_{\mathcal{T}} \equiv P \in P_k(\mathcal{T};\mathbb{R}^2) \quad \text{for } k = 0,1 \text{ and for } \mathcal{T} \in \mathbb{T}$$

in PMP with jumps (with unspecified sign)

$$[P]_E := (P|_K)|_E - (P|_{K'})|_E \in L^2(E;\mathbb{R}^2)$$

across an interior edge $E = \partial K \cap \partial K' \in \mathcal{E}(\Omega)$ of two neighboring triangles $K, K' \in \mathcal{T}$ and appropriate modifications on the exterior boundary with tangent unit vector τ_E along $E \in \mathcal{E}(\partial\Omega)$

$$[P]_E := P \cdot \tau_E$$

The (error) estimator for $K \in \mathcal{T}$ reads

$$\eta^2(K) \equiv \eta^2(\mathcal{T}, K) := |K|\,\|f\|^2_{L^2(K)} + |K|^{1/2} \sum_{E \in \mathcal{E}(K)} \|[P]_E\|^2_{L^2(E)}$$

($|K|\,\|f\|^2_{L^2(K)}$ possibly replaced by $|K|\,\|f - f_K\|^2_{L^2(K)}$ for RT-MFEM).

(A1)—(A4) at a Glance

$\exists 0 < \Lambda_1 < \infty \quad \forall \mathcal{T} \in \mathbb{T} \, \forall \hat{\mathcal{T}} \in \mathbb{T}(\mathcal{T})$

$$|\eta(\hat{\mathcal{T}}, \mathcal{T} \cap \hat{\mathcal{T}}) - \eta(\mathcal{T}, \mathcal{T} \cap \hat{\mathcal{T}})| \le \Lambda_1 \delta(\mathcal{T}, \hat{\mathcal{T}}) \tag{A1}$$

$\exists 0 < \rho_2 < 1 \, \exists 0 < \Lambda_2 < \infty \quad \forall \mathcal{T} \in \mathbb{T} \, \forall \hat{\mathcal{T}} \in \mathbb{T}(\mathcal{T})$

$$\eta(\hat{\mathcal{T}}, \hat{\mathcal{T}} \setminus \mathcal{T}) \le \rho_2 \eta(\mathcal{T}, \mathcal{T} \setminus \hat{\mathcal{T}}) + \Lambda_2 \delta(\mathcal{T}, \hat{\mathcal{T}}) \tag{A2}$$

$\exists 0 < \Lambda_3 < \infty \quad \forall \mathcal{T} \in \mathbb{T} \, \forall \hat{\mathcal{T}} \in \mathbb{T}(\mathcal{T})$

$$\delta(\mathcal{T}, \hat{\mathcal{T}}) \le \Lambda_3 \eta(\mathcal{T}, \mathcal{T} \setminus \hat{\mathcal{T}}) \tag{A3}$$

$\exists 0 < \Lambda_4 < \infty \quad \forall \ell \in \mathbb{N}_0$ (exclusively for the AFEM output)

$$\sum_{k=\ell}^{\infty} \delta^2(\mathcal{T}_k, \mathcal{T}_{k+1}) \le \Lambda_4 \eta_\ell^2 \tag{A4}$$

Outline of Optimality Analysis I

- (A12) Estimator reduction $\eta_{\ell+1}^2 \le \rho_{12}\eta_\ell^2 + \Lambda_{12}\delta_{\ell,\ell+1}^2$
- Convergence from

$$\sum_{k=\ell}^{\infty} \eta_k^2 \le \Lambda \eta_\ell^2 \quad \text{and then} \quad \sum_{k=0}^{\ell-1} \eta_k^{-1/s} \lesssim \eta_\ell^{-1/s}$$

- Quasimonotonicity $\quad \eta^2(\hat{\mathcal{T}}) \le \Lambda_7 \eta^2(\mathcal{T})$
- Comparison Lemma: Given $\mathcal{T}_\ell$, $0 < \kappa < 1$, and

$$M := \sup_{N\in\mathbb{N}_0} (1+N)^s \min_{\mathcal{T}\in\mathbb{T}(N)} \eta(\mathcal{T}),$$

there exist $\hat{\mathcal{T}}_\ell$ and $0 < \theta_0 < 1$ s.t.

(a) $\eta(\hat{\mathcal{T}}_\ell) \le \kappa \eta(\mathcal{T}_\ell)$

(b) $\kappa \eta_\ell |\mathcal{T}_\ell \setminus \hat{\mathcal{T}}_\ell|^s \lesssim M$

(c) $\theta_0 \eta_\ell^2 \le \eta_\ell^2(\mathcal{T}_\ell \setminus \hat{\mathcal{T}}_\ell)$

Outline of Optimality Analysis II

- $\mathcal{T}_\ell \setminus \hat{\mathcal{T}}_\ell$ satisfies the bulk criterion for $\theta \le \theta_0$ by (c). This implies

$$|\mathcal{M}_\ell^\star| \le |\mathcal{T}_\ell \setminus \hat{\mathcal{T}}_\ell|$$

with the optimal set $\mathcal{M}_\ell^\star$ of marked cells in AFEM. The utilized set $\mathcal{M}_\ell$ of marked cells is almost minimal: $\exists 0 < \Lambda_{\mathrm{opt}} < \infty \, \forall \ell \in \mathbb{N}_0$,

$$|\mathcal{M}_\ell| \le \Lambda_{\mathrm{opt}} \, |\mathcal{M}_\ell^\star| \le \Lambda_{\mathrm{opt}} |\mathcal{T}_\ell \setminus \hat{\mathcal{T}}_\ell|$$

- Recall $M := \sup_{N \in \mathbb{N}_0} (1+N)^s \min_{\mathcal{T} \in \mathbb{T}(N)} \eta(\mathcal{T})$ and from (b) deduce

$$|\mathcal{T}_\ell \setminus \hat{\mathcal{T}}_\ell| \lesssim \left(\frac{M}{\kappa \eta_\ell} \right)^{1/s} \approx M^{1/s} \eta_\ell^{-1/s}$$

- Recall closure overhead control and combine with aforementioned estimates for

$$|\mathcal{T}_\ell| - |\mathcal{T}_0| \le C_{BDV} \sum_{j=0}^{\ell-1} |\mathcal{M}_j| \lesssim M^{1/s} \sum_{j=0}^{\ell-1} \eta_j^{-1/s} \lesssim M^{1/s} \eta_\ell^{-1/s} \quad \square$$

Conclusions

- Theory without efficiency and so includes adaptive BEM
- Abstract framework (A1)–(A4) almost covers existing literature (up to instance optimality)
- Rate optimality for AFEMs may be based on collective and separate marking
- Separate marking necessary for $H(\mathrm{div})$ Least-Squares FEM but leads to optimal convergence rates in [C-Park SINUM 2015].
- Possible generalizations: Higher-order problems, more complex PDEs, non-constant coefficients, more nonconforming FEMs, inhomogeneous Dirichlet data etc.
- Inexact solve possible for iterative solve. Proof of information-based optimal complexity is missing — hopefully realistic assumptions on the performance of the nonlinear solver guarantee optimal complexity
- List of open cases for linear problems e.g. for Taylor-Hood, dG, Kouhia-Stenberg and hard nonlinear problems e.g. in comp. calc. var.

2. Plain Convergence

(A1)-(A2) & Dörfler marking imply (A12)

This section concerns the output η_ℓ and $\mathcal{T}_\ell$ of AFEM.

$\exists 0 \le \varrho_{12} < 1 \; \exists 0 < \Lambda_{12} < \infty \quad \forall \ell \in \mathbb{N}_0$

$$\eta_{\ell+1}^2 \le \varrho_{12}\, \eta_\ell^2 + \Lambda_{12}\, \delta^2(\mathcal{T}_{\ell+1}, \mathcal{T}_\ell). \tag{A12}$$

The following identity is frequently used throughout the proofs. Any $a, b \ge 0$ satisfy

$$(a+b)^2 = \inf_{0<\lambda<\infty} ((1+\lambda)a^2 + (1+1/\lambda)b^2).$$

(For a proof, observe that $\lambda = b/a$ leads to the minimum if $a, b > 0$.)

Theorem (estimator reduction in AFEM)

For any $1-\theta(1-\varrho_2^2) < \varrho_{12} < 1$, there exists $\Lambda_{12} < \infty$ so that (A1)-(A2) & Dörfler marking with bulk parameter $0 < \theta \le 1$ imply (A12).

Proof of (A12)

Let $\lambda > 0$ satisfy $1-\theta(1-\varrho_2^2) = \varrho_{12}/(1+\lambda)$. (A1) leads to

$$\eta^2(\mathcal{T}_{\ell+1}, \mathcal{T}_{\ell+1}\cap\mathcal{T}_\ell) \le \big(\eta(\mathcal{T}_\ell, \mathcal{T}_\ell\cap\mathcal{T}_{\ell+1}) + \Lambda_1\delta(\mathcal{T}_{\ell+1},\mathcal{T}_\ell)\big)^2$$
$$\le (1+\lambda)\eta^2(\mathcal{T}_\ell, \mathcal{T}_\ell\cap\mathcal{T}_{\ell+1}) + (1+1/\lambda)\Lambda_1^2\delta^2(\mathcal{T}_{\ell+1},\mathcal{T}_\ell).$$

The same argument with (A2) leads to

$$\eta^2(\mathcal{T}_{\ell+1}, \mathcal{T}_{\ell+1}\backslash\mathcal{T}_\ell) \le \varrho_2^2(1+\lambda)\eta^2(\mathcal{T}_\ell, \mathcal{T}_\ell\backslash\mathcal{T}_{\ell+1}) + (1+1/\lambda)\Lambda_2^2\delta^2(\mathcal{T}_{\ell+1},\mathcal{T}_\ell).$$

Combine the previous estimates with the decomposition

$$\eta_{\ell+1}^2 = \eta^2(\mathcal{T}_{\ell+1}, \mathcal{T}_{\ell+1}\cap\mathcal{T}_\ell) + \eta^2(\mathcal{T}_{\ell+1}, \mathcal{T}_{\ell+1}\backslash\mathcal{T}_\ell)$$
$$\le (1+\lambda)\underbrace{\big(\eta^2(\mathcal{T}_\ell, \mathcal{T}_\ell\cap\mathcal{T}_{\ell+1}) + \varrho_2^2\eta^2(\mathcal{T}_\ell, \mathcal{T}_\ell\backslash\mathcal{T}_{\ell+1})\big)}_{(*):=\eta_\ell^2-(1-\varrho_2^2)\eta^2(\mathcal{T}_\ell,\mathcal{T}_\ell\backslash\mathcal{T}_{\ell+1})} + \Lambda_{12}\delta^2(\mathcal{T}_{\ell+1},\mathcal{T}_\ell)$$

The Dörfler marking guarantees $\theta\eta_\ell^2 \le \eta^2(\mathcal{T}_\ell, \mathcal{T}_\ell\backslash\mathcal{T}_{\ell+1})$ and so

$$(*) \le \big(1-\theta(1-\varrho_2^2)\big)\,\eta_\ell^2 = \varrho_{12}\eta_\ell^2/(1+\lambda). \quad \square$$

Convergence $\eta_k \to 0$ as $k \to \infty$ from AFEM

Theorem (plain convergence)

(A12) and (A4) imply that $\Lambda := (1 + \Lambda_{12}\Lambda_3^2)/(1 - \varrho_{12}) < \infty$ *satisfies*

$$\sum_{k=\ell}^{\infty} \eta_k^2 \le \Lambda \eta_\ell^2 \quad \text{for all } \ell \in \mathbb{N}_0.$$

Proof. Recall (A12) in the notation $\eta_{k+1}^2 \le \varrho_{12}\eta_k^2 + \Lambda_{12}\delta_{k,k+1}^2$ and deduce

$$\sum_{k=\ell}^{\ell+m} \eta_k^2 \le \sum_{k=\ell}^{\ell+m+1} \eta_k^2 \le \eta_\ell^2 + \varrho_{12} \sum_{k=\ell}^{\ell+m} \eta_k^2 + \Lambda_{12} \sum_{k=\ell}^{\ell+m} \delta_{k,k+1}^2$$

Utilize $\varrho_{12} < 1$ and (A4) for the last sum to prove

$$(1 - \varrho_{12}) \sum_{k=\ell}^{\ell+m} \eta_k^2 \le (1 + \Lambda_{12}\Lambda_3^2)\,\eta_\ell^2 \quad \square$$

R-Linear Convergence on Each Level

Theorem

(A12), (A4) and $\Lambda < \infty$ *from above lead to* $q := 1 - 1/\Lambda < 1$ *with*

$$\eta_{\ell+m}^2 \leq q^m \Lambda\, \eta_\ell^2 \quad \textit{for all } \ell,\, m \in \mathbb{N}_0.$$

Proof. Rewrite plain convergence theorem as

$$\sigma_\ell^2 := \sum_{k=\ell}^{\infty} \eta_k^2 \leq \Lambda\, \eta_\ell^2$$

Then

$$\Lambda^{-1}\sigma_\ell^2 + \sigma_{\ell+1}^2 \leq \eta_\ell^2 + \sum_{k=\ell+1}^{\infty} \eta_k^2 = \sigma_\ell^2$$

This is $\sigma_{\ell+1}^2 \leq q\sigma_\ell^2$ and, successively, $\sigma_{\ell+m}^2 \leq q^m \sigma_\ell^2$ for all $m \in \mathbb{N}_0$

Consequently

$$\eta_{\ell+m}^2 \leq \sigma_{\ell+m}^2 \leq q^m \sigma_\ell^2 \leq q^m \Lambda \eta_\ell^2. \quad \square$$

3. Quasimonotonicity and Comparison

Estimator Quasimonotonicity

Theorem

(A1)—(A3) imply that $\Lambda_{mon} := 1 + \sqrt{\Lambda_1^2 + \Lambda_2^2}\Lambda_3$ *and any refinement* $\widehat{\mathcal{T}}$ *of any* $\mathcal{T}$ *in* $\mathbb{T}$ *satisfy*

$$\eta(\widehat{\mathcal{T}}) \leq \Lambda_{mon}\eta(\mathcal{T}).$$

Proof. For any $0 < \lambda < \infty$, utilize (A1)-(A2) in the decomposition

$$\begin{aligned}\eta^2(\widehat{\mathcal{T}}) &= \eta^2(\widehat{\mathcal{T}}, \widehat{\mathcal{T}} \cap \mathcal{T}) + \eta^2(\widehat{\mathcal{T}}, \widehat{\mathcal{T}} \backslash \mathcal{T}) \\ &\leq (1+\lambda)\Big(\underbrace{\eta^2(\mathcal{T}, \widehat{\mathcal{T}} \cap \mathcal{T}) + \eta^2(\mathcal{T}, \mathcal{T} \backslash \widehat{\mathcal{T}})}_{\eta^2(\mathcal{T})}\Big) \\ &\quad + (1 + 1/\lambda)(\Lambda_1^2 + \Lambda_2^2)\delta^2(\mathcal{T}, \widehat{\mathcal{T}})\end{aligned}$$

(A3) reads $\delta^2(\mathcal{T}, \widehat{\mathcal{T}}) \leq \Lambda_3^2\eta^2(\mathcal{T})$ and leads to

$$\eta^2(\widehat{\mathcal{T}}) \leq \underbrace{(1 + \lambda + (1 + 1/\lambda)(\Lambda_1^2 + \Lambda_2^2)\Lambda_3^2)}_{\Lambda_{\text{mon}}^2}\eta^2(\mathcal{T}) \quad \square$$

Comparison Lemma

Given $0 < \varkappa < 1$ and $s > 0$ with $M := \sup_{N \in \mathbb{N}_0} (N+1)^s \min_{\mathcal{T} \in \mathbb{T}(N)} \eta(\mathcal{T}) < \infty$,

there exists $0 < \theta_0 < 1$ such that for all $\mathcal{T}_\ell$ there exist $\hat{\mathcal{T}}_\ell \in \mathbb{T}(\mathcal{T}_\ell)$ s.t.

(a) $\eta(\hat{\mathcal{T}}_\ell) \le \varkappa \eta(\mathcal{T}_\ell)$, (b) $\varkappa \eta_\ell |\mathcal{T}_\ell \setminus \hat{\mathcal{T}}_\ell|^s \le \Lambda_{\mathsf{mon}} M$, (c) $\theta_0 \eta_\ell^2 \le \eta_\ell^2(\mathcal{T}_\ell \setminus \hat{\mathcal{T}}_\ell)$.

Proof. (1) W.l.o.g. $\eta_\ell \equiv \eta(\mathcal{T}_\ell) > 0$. By quasimonotonicity, $0 < \eta_0 \le M$

(2) Choose minimal $N_\ell \in \mathbb{N}_0$ s.t.

$$(N_\ell + 1)^{-s} \le \frac{\varkappa \eta_\ell}{\Lambda_{\mathsf{mon}} M} < N_\ell^{-s} \le 1$$

($N_\ell \ge 1$ because $\eta_\ell \Lambda_{\mathsf{mon}}^{-1}/M \le \eta_0/M \le 1$)

(3) Set $\widehat{\mathcal{T}}_\ell := \mathcal{T}_\ell \otimes \mathcal{T}'$ for $\mathcal{T}'$ with $\mathcal{T}' \in \mathbb{T}(N_\ell)$ s.t. $(N_\ell + 1)^s \eta(\mathcal{T}') \le M$
Quasimonotonicity and overlay control lead e.g. to (a),

$$\eta(\widehat{\mathcal{T}}_\ell) \le \Lambda_{\mathsf{mon}} M (N_\ell + 1)^{-s} \le \varkappa \eta_\ell \quad \text{and} \quad |\widehat{\mathcal{T}}_\ell| \le |\mathcal{T}_\ell| + N_\ell$$

Proof of (b)-(c) in Comparison Lemma

(4) Proof of (b). Count triangles to verify

$$|\mathcal{T}_\ell \setminus \widehat{\mathcal{T}}_\ell| \leq |\widehat{\mathcal{T}}_\ell| - |\mathcal{T}_\ell| \underset{\text{from } \otimes}{\leq} N_\ell \underset{\text{from } (2)}{<} \varkappa^{-1/s} \eta_\ell^{-1/s} \Lambda_{\text{mon}}^{1/s} M^{1/s} \quad \square$$

(5) Any $\widehat{\mathcal{T}}_\ell \in \mathbb{T}(\mathcal{T}_\ell)$ with (a) allows for (c). Given any $0 < \mu < \varkappa^{-2} - 1$, (A1) followed by (a) and (A3) imply

$$\begin{aligned} \eta_\ell^2(\mathcal{T}_\ell \cap \widehat{\mathcal{T}}_\ell) &\leq (1+\mu)\eta^2(\widehat{\mathcal{T}}_\ell, \mathcal{T}_\ell \cap \widehat{\mathcal{T}}_\ell) + (1+1/\mu)\Lambda_1^2 \delta^2(\mathcal{T}_\ell, \widehat{\mathcal{T}}_\ell) \\ &\leq (1+\mu)\varkappa^2 \eta_\ell^2 + (1+1/\mu)\Lambda_1^2 \Lambda_3^2 \eta_\ell^2(\mathcal{T}_\ell \setminus \widehat{\mathcal{T}}_\ell) \end{aligned}$$

This and the decomposition

$$\eta_\ell^2 = \eta_\ell^2(\mathcal{T}_\ell \cap \widehat{\mathcal{T}}_\ell) + \eta_\ell^2(\mathcal{T}_\ell \setminus \widehat{\mathcal{T}}_\ell)$$

lead to

$$\left(1 - (1+\mu)\varkappa^2\right) \eta_\ell^2 \leq \left(1 + (1+1/\mu)\Lambda_1^2 \Lambda_3^2\right) \eta_\ell^2(\mathcal{T}_\ell \setminus \widehat{\mathcal{T}}_\ell) \quad \square$$

4. (A1)-(A2) for Courant FEM

Recall trace inequality, inverse estimate, discrete trace inequality and compute their constants in terms of a lower bound of the minimal angle in the triangulation, recall the Euclid norm in ℓ^2.

Λ_1 Comes from Discrete Jump Control

Given $g \in P_k(\mathcal{T})$ for $\mathcal{T} \in \mathbb{T}$, set

$$[g]_E = \begin{cases} (g|_{T_+})|_E - (g|_{T_-})|_E & \text{for } E \in \mathcal{E}(\Omega) \text{ with } E = \partial T_+ \cap \partial T_-, \\ g|_E & \text{for } E \in \mathcal{E}(\partial\Omega) \cap \mathcal{E}(K). \end{cases}$$

Lemma (discrete jump control)

For all $k \in \mathbb{N}_0$ there exists $0 < \Lambda_1 < \infty$ s.t., for all $g \in P_k(\mathcal{T})$ and $\mathcal{T} \in \mathbb{T}$,

$$\sqrt{\sum_{K \in \mathcal{T}} |K|^{1/2} \sum_{E \in \mathcal{E}(K)} ||[g]_E||^2_{L^2(E)}} \leq \Lambda_1 ||g||_{L^2(\Omega)}.$$

Proof with discrete trace inequality on $E \in \mathcal{E}(K)$ for $K \in \mathcal{T}$

$$|K|^{1/4}\, ||g|_K||_{L^2(E)} \leq C_{\mathsf{dtr}}\, ||g||_{L^2(K)}.$$

Compute Λ_1 in Proof of Discrete Jump Control

The contributions to LHS of interior edge $E = \partial T_+ \cap \partial T_-$ with edge-patch $\omega_E := \mathrm{int}(T_+ \cup T_-)$ read

$$
\begin{aligned}
&(|T_+|^{1/2} + |T_-|^{1/2})||[g]_E||^2_{L^2(E)} \\
&\le (|T_+|^{1/2} + |T_-|^{1/2}) \left(||g|_{T_+}||_{L^2(E)} + ||g|_{T_-}||_{L^2(E)}\right)^2 \\
&\le C^2_{\mathsf{dtr}} (|T_+|^{1/2} + |T_-|^{1/2}) \left(|T_+|^{-1/4}||g||_{L^2(T_+)} + |T_-|^{-1/4}||g||_{L^2(T_-)}\right)^2 \\
&\le C^2_{\mathsf{dtr}} \underbrace{(|T_+|^{1/2} + |T_-|^{1/2})(|T_+|^{-1/2} + |T_-|^{-1/2})}_{\le C^2_{\mathsf{sr}}} ||g||^2_{L^2(\omega_E)} \\
&\le C^2_{\mathsf{dtr}} C^2_{\mathsf{sr}} ||g||^2_{L^2(\omega_E)}.
\end{aligned}
$$

The same final result holds for boundary edge $E = \partial T_+ \cap \partial\Omega$ with $\omega_E := \mathrm{int}(T_+)$. The sum of all those edges proves the discrete jump control lemma with

$$\Lambda_1 := \sqrt{3}\, C_{\mathsf{dtr}} C_{\mathsf{sr}}. \quad \square$$

Proof of (A1) with $\delta(\mathcal{T},\hat{\mathcal{T}}) = ||\widehat{P} - P||_{L^2(\Omega)}$

Recall that $\hat{\mathcal{T}}$ is an admissible refinement of $\mathcal{T}$ with respective discrete solutions $\widehat{P} := p_{\hat{\mathcal{T}}} \in P_1(\hat{\mathcal{T}};\mathbb{R}^2)$ and $P := p_{\mathcal{T}} \in P_1(\mathcal{T};\mathbb{R}^2)$. Given any $T \in \mathcal{T} \cap \hat{\mathcal{T}}$, set

$$\eta(T) := \sqrt{\alpha_T^2 + \beta_T^2} \quad \text{and} \quad \widehat{\eta}(T) := \sqrt{\alpha_T^2 + \widehat{\beta_T}^2}$$

for $\alpha_T := |T|^{1/2}\,||f||_{L^2(T)}$,

$$\beta_T^2 := |T|^{1/2} \sum_{E\in\mathcal{E}(T)} ||[P]_E||^2_{L^2(E)} \quad \text{and} \quad \widehat{\beta_T}^2 := |T|^{1/2} \sum_{E\in\mathcal{E}(T)} ||[\widehat{P}]_E||^2_{L^2(E)}$$

Then, $\eta(\mathcal{T}\cap\hat{\mathcal{T}}) := \sqrt{\sum_{T\in\mathcal{T}\cap\hat{\mathcal{T}}} \eta^2(T)}$ and $\widehat{\eta}(\mathcal{T}\cap\hat{\mathcal{T}}) := \sqrt{\sum_{T\in\mathcal{T}\cap\hat{\mathcal{T}}} \widehat{\eta^2}(T)}$ are Euclid norms of vectors in $\mathbb{R}^J$ for $J := 2\,|\mathcal{T}\cap\hat{\mathcal{T}}|$.

Proof of (A1) with $\delta(\mathcal{T},\hat{\mathcal{T}}) = \|\widehat{P} - P\|_{L^2(\Omega)}$

The reversed triangle inequality in $\mathbb{R}^J$ bounds the LHS in (A1), namely $|\eta(\hat{\mathcal{T}},\mathcal{T}\cap\hat{\mathcal{T}}) - \eta(\mathcal{T},\mathcal{T}\cap\hat{\mathcal{T}})| = |\widehat{\eta}(\mathcal{T}\cap\hat{\mathcal{T}}) - \eta(\mathcal{T}\cap\hat{\mathcal{T}})|$ from above by

$$\sqrt{\sum_{T\in\mathcal{T}\cap\hat{\mathcal{T}}} |\widehat{\eta}(T) - \eta(T)|^2} = \sqrt{\sum_{T\in\mathcal{T}\cap\hat{\mathcal{T}}} |\underbrace{\sqrt{\alpha_T^2 + \widehat{\beta_T}^2} - \sqrt{\alpha_T^2 + \beta_T^2}}_{\leq|\widehat{\beta_T}-\beta_T| \text{ (triangle inequality in } \mathbb{R}^2)}|^2}$$

The reversed triangle inequality in $\mathbb{R}^3$ shows

$$|\widehat{\beta_T} - \beta_T| = |T|^{1/4}\left|\sqrt{\sum_{E\in\mathcal{E}(T)} \|[\widehat{P}]_E\|_{L^2(E)}^2} - \sqrt{\sum_{E\in\mathcal{E}(T)} \|[P]_E\|_{L^2(E)}^2}\right|$$

$$\leq |T|^{1/4}\sqrt{\sum_{E\in\mathcal{E}(T)} \|[\widehat{P} - P]_E\|_{L^2(E)}^2}.$$ Altogether implies

$$|\eta(\hat{\mathcal{T}},\mathcal{T}\cap\hat{\mathcal{T}}) - \eta(\mathcal{T},\mathcal{T}\cap\hat{\mathcal{T}})| \leq \sqrt{\sum_{T\in\mathcal{T}\cap\hat{\mathcal{T}}} |T|^{1/2} \sum_{E\in\mathcal{E}(T)} \|[\widehat{P} - P]_E\|_{L^2(E)}^2}$$

The discrete jump control lemma for $\widehat{P} - P \in P_1(\hat{\mathcal{T}};\mathbb{R}^2)$ yields (A1). □

Proof of (A2) with $\varrho_2 = 1/\sqrt{2}$ and $\Lambda_2 = \Lambda_1$

Recall that $\hat{\mathcal{T}}$ is an admissible refinement of $\mathcal{T}$ with respective discrete solutions $\widehat{P} := p_{\hat{\mathcal{T}}} \in P_1(\hat{\mathcal{T}};\mathbb{R}^2)$ and $P := p_{\mathcal{T}} \in P_1(\mathcal{T};\mathbb{R}^2)$. Given any refined triangle $T \in \hat{\mathcal{T}}(K) := \{T \in \hat{\mathcal{T}} : T \subset K\}$ for $K \in \mathcal{T} \setminus \hat{\mathcal{T}}$, recall $\alpha_T := |T|^{1/2}\,||f||_{L^2(T)}$,

$$\beta_T^2 := |T|^{1/2} \sum_{F\in\mathcal{E}(T)} ||[P]_F||^2_{L^2(F)} \quad \text{and} \quad \widehat{\beta_T}^2 := |T|^{1/2} \sum_{F\in\mathcal{E}(T)} ||[\widehat{P}]_F||^2_{L^2(F)}.$$

The left-hand side in (A2) reads

$$\widehat{\eta}(\hat{\mathcal{T}} \setminus \mathcal{T}) = \sqrt{\sum_{K\in\mathcal{T}\setminus\hat{\mathcal{T}}} \sum_{T\in\hat{\mathcal{T}}(K)} (\alpha_T^2 + \widehat{\beta}_T^2)} \quad \text{(by a triangle inequality)}$$

$$\leq \sqrt{\underbrace{\sum_{K\in\mathcal{T}\setminus\hat{\mathcal{T}}} \sum_{T\in\hat{\mathcal{T}}(K)} (\alpha_T^2 + \beta_T^2)}_{(i)}} + \sqrt{\underbrace{\sum_{K\in\mathcal{T}\setminus\hat{\mathcal{T}}} \sum_{T\in\hat{\mathcal{T}}(K)} (\widehat{\beta}_T - \beta_T)^2}_{(ii)}}.$$

Observe $[P]_F = 0$ for $F \in \hat{\mathcal{E}}(\text{int}(K))$ and $|T| \leq |K|/2$ for $T \in \hat{\mathcal{T}}(K)$.

Proof of (A2) with $\varrho_2 = 1/\sqrt{2}$ and $\Lambda_2 = \Lambda_1$

Since $[P]_F = 0$ for $F \in \hat{\mathcal{E}}(\text{int}(K))$ and $|T| \le |K|/2$ for $T \in \hat{\mathcal{T}}(K)$,

$$(i) := \sum_{K \in \mathcal{T} \setminus \hat{\mathcal{T}}} \sum_{T \in \hat{\mathcal{T}}(K)} (\alpha_T^2 + \beta_T^2) \le \frac{|K|}{2} ||f||^2_{L^2(K)} + \frac{|K|^{1/2}}{\sqrt{2}} \sum_{E \in \mathcal{E}(K)} ||[P]_E||^2_{L^2(E)}.$$

Reversed triangle inequalities in the second term prove

$$|\widehat{\beta}_T - \beta_T| = |T|^{1/4} \left| \sqrt{\sum_{F \in \mathcal{E}(T)} ||[\widehat{P}]_F||^2_{L^2(F)}} - \sqrt{\sum_{F \in \mathcal{E}(T)} ||[P]_F||^2_{L^2(F)}} \right|$$

$$\le |T|^{1/4} \sqrt{\sum_{F \in \mathcal{E}(T)} ||[\widehat{P} - P]_F||^2_{L^2(F)}} \quad \text{and so lead to}$$

$$(ii) := \sum_{K \in \mathcal{T} \setminus \hat{\mathcal{T}}} \sum_{T \in \hat{\mathcal{T}}(K)} (\beta_T - \widehat{\beta}_T)^2 \le \sum_{K \in \mathcal{T} \setminus \hat{\mathcal{T}}} \sum_{T \in \hat{\mathcal{T}}(K)} |T|^{1/2} \sum_{F \in \mathcal{E}(T)} ||[\widehat{P} - P]_F||^2_{L^2(F)}.$$

The combination of the above with the discrete jump control lemma conclude the proof of (A2). □

5. (A3)-(A4) for Courant FEM

Recall Poincare and Friedrichs inequalities and write $|||\bullet||| := \|\nabla\bullet\|_{L^2(\Omega)}$ for the H^1 semi-norm which plays a dominant role as the energy norm in $H^1_0(\Omega)$.

Discrete Quasiinterpolation

Theorem (approximation and stability). $\exists 0 < C = C(\min \angle \mathbb{T}) < \infty$ $\forall \mathcal{T} \in \mathbb{T}\, \forall \hat{\mathcal{T}} \in \mathbb{T}(\mathcal{T}) \quad \forall \hat{V} \in S^1_0(\hat{\mathcal{T}})\, \exists V \in S^1_0(\mathcal{T})$

$$V = \hat{V} \text{ on } \hat{\mathcal{T}} \cap \mathcal{T} \quad \text{and} \quad ||h_{\mathcal{T}}^{-1}(\hat{V} - V)||_{L^2(\Omega)} + |||V||| \le C\, |||\hat{V}|||.$$

Proof. Define $V \in S^1_0(\mathcal{T})$ by linear interpolation of nodal values

$$V(z) := \begin{cases} \hat{V}(z) & \text{if } z \in \mathcal{N}(\Omega) \cap \mathcal{N}(T) \text{ for some } T \in \mathcal{T} \cap \hat{\mathcal{T}} \\ \int_{\omega_z} \hat{V}\, dx / |\omega_z| & \text{if } z \in \mathcal{N}(\Omega) \text{ and } \mathcal{T}(z) \cap \hat{\mathcal{T}}(z) = \emptyset \\ 0 & \text{if } z \in \mathcal{N}(\partial\Omega) \end{cases}$$

Since V and $\hat{V}$ are continuous at any vertex of any $T \in \mathcal{T} \cap \hat{\mathcal{T}}$, the first case applies in the definition of $V(z) = \hat{V}(z)$ for all $z \in \mathcal{N}(T)$.

This proves $V = \hat{V}$ on $T \in \mathcal{T} \cap \hat{\mathcal{T}}$. □

Given any node $z \in \mathcal{N}$ in the coarse triangulation, let $\omega_z = \text{int}(\cup \mathcal{T}(z))$ denotes its patch of all triangles $\mathcal{T}(z)$ in $\mathcal{T}$ with vertex z.

Lemma A. There exists $C(z) \approx \text{diam}(\omega_z)$ with

$$||\hat{V} - V(z)||_{L^2(\omega_z)} \leq C(z)\, ||\nabla \hat{V}||_{L^2(\omega_z)}.$$

Proof4Case II: $z \in \mathcal{N}(\Omega)$ and $\mathcal{T}(z) \cap \hat{\mathcal{T}}(z) = \emptyset$ with $V(z) = \int_{\omega_z} \hat{V}\, dx / |\omega_z|$.
Then, the assertion is a Poincare inequality with $C(z) = C_P(\omega_z)$. □
Proof4Case III: $z \in \mathcal{N}(\partial\Omega)$ and $V(z) = 0$. Since $\hat{V} - V$ vanishes along the two edges along $\partial\Omega$ of the open boundary patch ω_z with vertex z. Hence the assertion is indeed a Friedrichs inequality with $C(z) = C_F(\omega_z)$. □
Proof4Case I: $\exists T \in \mathcal{T}(z) \cap \hat{\mathcal{T}}(z)$ for $z \in \mathcal{N}(\Omega)$ and $V = \hat{V}$ on T. This leads to homogenous Dirichlet boundary conditions on the two edges of the open patch $\omega_z \setminus T$ with vertex z and $\hat{V} - V$ allows for a Friedrichs inequality (on the open patch as in Case III for a patch on the boundary)

$$||\hat{V} - V||_{L^2(\omega_z)} \leq C_F(\omega_z \setminus T)\, ||\nabla(\hat{V} - V)||_{L^2(\omega_z)}$$

However, this is not the claim! The idea is to realize that LHS$=||w||_{L^2(\omega_z)}$ for $w := \hat{V} - \hat{V}(z)$, which is affine on T and vanishes at vertex z. Hence (as an other inverse estimate or discrete Friedrichs inequality)

$$||w||^2_{L^2(T)} \leq C_{dF}(T)^2 \, ||\nabla w||^2_{L^2(T)} \leq C_{dF}(T)^2 \, ||\nabla w||^2_{L^2(\omega_z)}$$

E.g. the integral mean $w_T := \int_T w \, dx/|T|$ of $w := \hat{V} - \hat{V}(z)$ on T satisfies

$$|w_T|^2 \, |T| \leq C_{dF}(T)^2 \, ||\nabla w||^2_{L^2(\omega_z)}$$

Compare with integral mean $\overline{w} := \displaystyle\int_{\omega_z} w \, dx/|\omega_z|$ and compute

$$\begin{aligned}|\overline{w} - w_T|^2 \, |T| &= |T|^{-1} \, |\int_T (\overline{w} - w) dx|^2 \leq ||w - \overline{w}||^2_{L^2(T)} \\ &\leq ||w - \overline{w}||^2_{L^2(\omega_z)} \leq C_P(\omega_z)^2 \, ||\nabla w||^2_{L^2(\omega_z)}\end{aligned}$$

Consequently, $$|\overline{w} - w_T|^2 \, |\omega_z| \leq \underbrace{|\omega_z|/|T|}_{\leq C_{sr}} \, C_P(\omega_z)^2 \, ||\nabla w||^2_{L^2(\omega_z)}$$

The orthogonality of 1 and $w - \overline{w}$ in $L^2(\omega_z)$ is followed by Poincare's and geometric-arithmetic mean inequality to verify

$$\begin{aligned} ||w||^2_{L^2(\omega_z)} &= |\overline{w}|^2\,|\omega_z| + ||w-\overline{w}||^2_{L^2(\omega_z)} \\ &\le 2|\overline{w}-w_T|^2\,|\omega_z| + 2|w_T|^2\,|\omega_z| + C_P(\omega_z)^2\,||\nabla w||^2_{L^2(\omega_z)} \end{aligned}$$

The above estimates for $|w_T|^2\,|T|$ and $|\overline{w}-w_T|^2\,|T|$ lead to

$$||w||^2_{L^2(\omega_z)} \le \underbrace{\left(2|\omega_z|/|T|\,(C_{dF}(T)+C_P(\omega_z)^2) + C_P(\omega_z)^2\right)}_{=:C(z)^2}\,||\nabla w||^2_{L^2(\omega_z)} \qquad \square$$

W.r.t. triangulation $\mathcal{T}$ and nodal basis functions φ_1, φ_2, φ_3 in $S^1(\mathcal{T})$, let $T = \mathrm{conv}\{P_1, P_2, P_3\} \in \mathcal{T}$ and $\Omega_T := \omega_1 \cup \omega_2 \cup \omega_3$ for $\omega_j := \{\varphi_j > 0\}$

Lemma B. There exists $C(T) \approx h_T$ with

$$||\hat{V} - V||_{L^2(T)} \le C(T)\,||\nabla \hat{V}||_{L^2(\Omega_T)}.$$

Proof of Lemma B. N.B. $V = \sum_{j=1}^{3} V(P_j)\,\varphi_j$ and $1 = \sum_{j=1}^{3} \varphi_j$ on T
Hence

$$
\begin{aligned}
\|\hat{V} - V\|^2_{L^2(T)} &= \int_T |\sum_{j=1}^{3} (\hat{V} - V(P_j))\,\varphi_j|^2\,dx \\
&\le \int_T (\sum_{j=1}^{3} |\hat{V} - V(P_j)|^2)(\underbrace{\sum_{k=1}^{3} \varphi_k^2}_{\le 1})\,dx \quad \text{(CS in } \mathbb{R}^3\text{)} \\
&\le \sum_{j=1}^{3} \|\hat{V} - V(P_j)\|^2_{L^2(T)} \\
&\le \sum_{j=1}^{3} C(P_j)^2 \|\nabla \hat{V}\|^2_{L^2(\omega_j)} \quad \text{(Lemma A)} \\
&\le \underbrace{(\sum_{j=1}^{3} C(P_j)^2)}_{C^2(T)}\ \|\nabla \hat{V}\|^2_{L^2(\Omega_T)} \qquad \square
\end{aligned}
$$

Lemma C. There exists $C > 0$ (which solely depends on $\min \angle \mathbb{T}$) with

$$||\nabla V||_{L^2(T)} \leq C\,||\nabla \hat{V}||_{L^2(\Omega_T)}.$$

Proof. N.B. $\nabla V = \sum_{j=1}^{3} V(P_j)\,\nabla\varphi_j$ and $0 = \sum_{j=1}^{3} \nabla\varphi_j$ on T

Hence

$$\begin{aligned}
||\nabla V||^2_{L^2(T)} &= \int_T |\sum_{j=1}^{3}(\hat{V} - V(P_j))\nabla\varphi_j|^2\,dx \\
&\leq \int_T (\sum_{j=1}^{3}|\hat{V} - V(P_j)|^2)(\underbrace{\sum_{k=1}^{3}|\nabla\varphi_k|^2}_{\leq C(\min\angle T)^2/h_T^2})\,dx \quad \text{(CS in } \mathbb{R}^6\text{)} \\
&\leq C(\min\angle T)^2\,h_T^{-2}\sum_{j=1}^{3}\int_T |\hat{V} - V(P_j)|^2\,dx \\
&\leq \dots \text{(as before)} \dots \\
&\leq \underbrace{C(\min\angle T)^2\,h_T^{-2}C^2(T)}_{=:C^2}\,||\nabla\hat{V}||^2_{L^2(\Omega_T)} \qquad \square
\end{aligned}$$

Finish of proof of theorem: $||h_{\mathcal{T}}^{-1}(\hat{V}-V)||_{L^2(\Omega)} + |||V||| \le C\,|||\hat{V}|||$.

Lemma B and C show for some generic constant $C>0$ and any $T\in\mathcal{T}$ that

$$||h_T^{-1}(\hat{V}-V)||^2_{L^2(T)} + ||\nabla V||^2_{L^2(T)} \le C\,||\nabla\hat{V}||^2_{L^2(\Omega_T)}$$

The sum over all those inequalities for $T\in\mathcal{T}$ concludes the proof because the overlap of $(\Omega_T)_{T\in\mathcal{T}}$ is bounded by generic constant $C(\min\angle\mathbb{T})$. □

Proof of (A3)

Given discrete solution U (resp. $\hat{U}$) of CFEM in PMP w.r.t. $\mathcal{T}$ (resp. refinement $\hat{\mathcal{T}}$), set $\hat{e} := \hat{U} - U \in S_0^1(\hat{\mathcal{T}})$ with quasiinterpolant $e \in S_0^1(\mathcal{T})$ as above. Then, $v := \hat{e} - e$ satisfies

$$\delta^2(\mathcal{T}, \hat{\mathcal{T}}) = |||\hat{e}|||^2 = a(\hat{e}, v) = \underbrace{F(v) - a(U, v)}_{\mathsf{Res}(v)}$$

A piecewise integration by parts with a careful algebra with the jump terms for appropriate signs shows

$$\begin{aligned} -a(U, v) &= -\sum_{E \in \mathcal{E}(\Omega)} \int_E v\, [\partial U / \partial \nu_E]_E \, ds \\ &\leq \sqrt{\sum_{E \in \mathcal{E}(\Omega)} |E|^{-1} ||v||^2_{L^2(E)}} \sqrt{\sum_{E \in \mathcal{E}(\Omega)} |E| \, ||[\partial U / \partial \nu_E]_E||^2_{L^2(E)}} \end{aligned}$$

Recall trace inequality

$$|E|^{-1} ||v||^2_{L^2(E)} \leq C_{tr} (h_{\omega_E}^{-2} ||v||^2_{L^2(\omega_E)} + ||\nabla v||^2_{L^2(\omega_E)})$$

Finish of Proof of (A3)

to estimate

$$\sum_{E\in\mathcal{E}(\Omega)} |E|^{-1}||v||^2_{L^2(E)} \lesssim \sum_{E\in\mathcal{E}(\Omega)} (h^{-2}_{\omega_E}||v||^2_{L^2(\omega_E)} + ||\nabla v||^2_{L^2(\omega_E)})$$

$$\lesssim ||h^{-1}_{\mathcal{T}} v||^2_{L^2(\Omega)} + |||v|||^2 \lesssim |||\hat{e}|||^2$$

with the approximation and stability of the quasiinterpolation.

A weighted Cauchy inequality followed by approximation property of quasi-interpolation show

$$F(v) \leq ||h_{\mathcal{T}} f||_{L^2(\Omega)}\, ||h^{-1}_{\mathcal{T}} v||_{L^2(\Omega)} \leq C||h_{\mathcal{T}} f||_{L^2(\Omega)}\, |||\hat{e}|||$$

All this plus shape-regularity (e.g. $|T| \approx h_T^2 \approx h_E^2$) lead to reliability

$$\delta^2(\mathcal{T}, \hat{\mathcal{T}}) = |||\hat{e}|||^2 \leq \Lambda_3\, \eta(\mathcal{T})|||\hat{e}|||$$

The extra fact $v = 0$ on $\mathcal{T} \cap \hat{\mathcal{T}}$ and a careful inspection on disappearing integrals in the revisited analysis prove the asserted upper bound in (A3),

$$\delta(\mathcal{T}, \hat{\mathcal{T}}) \leq \Lambda_3\, \eta(\mathcal{T}, \mathcal{T} \setminus \hat{\mathcal{T}}) \qquad \square$$

(A4) follows from (A3) for CFEM with $\Lambda_4 = \Lambda_3^2$

The pairwise Galerkin orthogonality in the CFEM allows for the (modified) LHS in (A4) the representation

$$\sum_{k=\ell}^{\ell+m} \delta^2(\mathcal{T}_k, \mathcal{T}_{k+1}) = \delta^2(\mathcal{T}_\ell, \mathcal{T}_{\ell+m+1})$$

for $m \in \mathbb{N}_0$. (A3) shows that this is bounded from above by $\Lambda_3^2 \eta_\ell^2$. Since $m \in \mathbb{N}_0$ is arbitrary, this implies

$$\sum_{k=\ell}^{\infty} \delta^2(\mathcal{T}_k, \mathcal{T}_{k+1}) = \lim_{m\to\infty} \sum_{k=\ell}^{\ell+m} \delta^2(\mathcal{T}_k, \mathcal{T}_{k+1}) \le \Lambda_3^2 \eta_\ell^2. \quad \square$$