23rd European Conference on Iteration Theory

ECIT 2022

Reichenau an der Rax

Book of Abstracts

ECIT 2022

23rd European Conference on Iteration Theory

Reichenau an der Rax

13th June 2022 – 17th June 2022

Welcome to the 23rd European Conference on Iteration Theory (ECIT 2022) in Reichenau an der Rax.

We hope you enjoy your stay in Reichenau an der Rax and have a fruitful scientific meeting.

Contents

General information

In Reichenau there are two supermarkets. These are Spar (Eurospar) at Hauptstraße 38 and Billa at Hauptstraße 34–36. Both are about 5 minutes by foot from Hotel Marienhof, and are opened 7.15–19.30 on working days. However, note they are **closed on Thursday, 16th June** because this is a holiday in Austria.

Rax-Seilbahn (the cable car to Rax) operates from 8.00–17.30. The regular price for ascent and descent is 31 € (children up to 15 years 15.50 € – note that there is a reduction if parents go with their children). Since Thursday, 16th June is a holiday in Austria the cable car could be more crowded. For the same reason the paths on the Rax plateau and the paths to Rax can be more crowded on this day. Also the buses could be more crowded.

Unfortunately Freibad Reichenau does no longer exist. There is an open-air swimming pool in the neighbouring town Payerbach (Freibad Payerbach) which is opened 9.00–19.00 (daily ticket 5 €, for children 2.60 €).

Figure 1. A chamois (Rupricapra rupricapra) on Rax.

Rax and Schneeberg have a very large population of chamois (it is among the regions of highest density of chamois). Since they are very cautious and try to avoid people it is obviously not guaranteed to see them. But the probability to see chamois on Rax (see Figure 1) or Schneeberg is higher than to see chamois for example in Tyrol!

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Programme

Monday, 13th June 2022

Arrival

19.00–20.00 Dinner

Tuesday, 14th June 2022

08.00–09.00 Breakfast

09.40 Opening
10.00–10.25 Ľubomír Snoha
10.30–10.55 Pasquale Commendatore
11.00–11.30 Break
11.30–11.55 Lyudmila Efremova
12.00–12.25 Marek Lampart

13.00–14.00 Lunch

15.00–15.25 Anastasiia Panchuk
15.30–15.55 Fabio Lamantia
16.00–16.25 Davide Radi
16.30–17.00 Break
17.00–17.25 Michaela Mihoková
17.30–17.55 Fedor Pakovich

19.00–20.00 Dinner

Wednesday, 15th June 2022

08.00–09.00 Breakfast

09.30–09.55 Wirot Tikjha
10.00–10.25 Alžběta Lampartová
10.30–10.55 Iryna Sushko
11.00–11.30 Break
11.30–11.55 Viktor Avrutin
12.00–12.25 Eddy Kwessi

13.00–14.00 Lunch

15.00–15.25 Peter Raith
15.30–15.55 Witold Jarczyk
16.00–16.25 t.b.a.

16.30–17.00 Break

17.00–17.25 Henrique Oliveira
17.30–17.55 t.b.a.

19.00–20.00 Dinner

Thursday, 16th June 2022

08.00–09.00 Breakfast

09.30–09.55 Roberto De Leo
10.00–10.25 Anastasiia Panchuk
10.30–10.55 t.b.a.

11.00–11.30 Break

11.30–11.55 Sara Perestrelo
12.00–12.25 Miroslav Výbošťok

13.00–14.00 Lunch

15.00–15.25 Janusz Morawiec
15.30–15.55 Víctor Mañosa
16.00–16.25 Francisco Balibrea

16.30–17.00 Break

17.00–17.25 Antonio Garijo
17.35 Closure

19.00–20.00 Dinner

Friday, 17th June 2022

08.00–09.00 Breakfast

Departure

Abstracts of talks

Remarks on Distributional Chaos

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B. Schweizer and J. Smítal introduced in 1994 in a celebrated paper, published in TAMS, the notion of *distributional chaos* for dynamical systems of the form $([0,1]=I,f)$ where $f\in C(\mathrm{I,I})$. They were inspired in ideas from the book: *Probabilistic metric spaces* from B. Schweizer and J. Sklar and the paper from Balibrea and Jaro: *A Chaotic continuous map generates all probability distributions*(1993, JMAA).

The complexity of the system (I,f) can be described in many different ways. In particular by two approaches. One, considering the behavior of distances $\{|f^n(x)-f^n(y)|\}$ between trajectories. The result is chaos in Li and Yorke sense when there is a pair of points $\{x,y\}\in I$ such that

$$\liminf_{n\to\infty}|f^n(x)-f^n(y)|=0,$$
$$\limsup_{n\to\infty}|f^n(x)-f^n(y)|>0.$$

Topological entropy in the sense of Bowen also deals with such distances.

The second approach deals with *limit distributions of trajectories*, if they exist. For any x and t of I, let

$$\Phi_x^l(t) = \liminf_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n} \chi[0,t)(f^n(x))$$
$$\Phi_x^u(t) = \limsup_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n} \chi[0,t)(f^n(x))$$

$$\mathcal{D}(f) = \left\{g : \Phi_x^l = \Phi_x^u \text{ for some } x \in I\right\}$$

It is well known that $\mathcal{D}(f)$ is just the class of distributions of all probability invariant measures of f. This follows from Birkhoff's Ergodic Theorem.

Substituting in the former formulas $f^n(x)$ by $|f^n(x) - f^n(y)|$ where $y \in I$ we have the well known formulas of distributional chaos $\Phi_{xy}^l(t)$ and $\Phi_{xy}^u(t)$.

Denote now

$$\mathcal{F}(f) = \left\{g : \Phi_{xy}^l = \Phi_{xy}^u \text{ for some } x \in I\right\}$$

Our main goal is prove that there are universal generators of probability distributions, that is, functions f such that $\mathcal{F}(f)$ equals the class of non-decreasing functions of I into itself. This follows from the result:

Let f be such that f^2 be topologically transitive. If the set $\{x - y : x \text{ and } y \text{ are fixed points of } f\}$ contains I then $\mathcal{F}(f)$ is the class of all non-decreasing functions from I into itself.

One additional remark is that the class $\mathcal{F}(f)$ is always bigger than $\mathcal{D}(f)$ and $\mathcal{F}(f)$ is *convex and closed.* To prove such properties we can not apply Birkhoff's Ergodic Theorem to each of the corresponding two trajectories separately.

Big or small? A new economic geography model with an endogenous switch in the market structure

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(joint work with **Ingrid Kubin** and **Iryna Sushko**)

We present a new economic geography (NEG) model with a linear demand function where firms may change the perception of their relative dimension with respect to the local market. If they perceive themselves as big, they behave as Cournot oligopolists, otherwise they behave as monopolistic competitors. We first compare the pure cases in which only one market form prevails in the two-region economy. Comparing these pure cases of monopolistic and oligopolistic competition only a quantitative difference emerges. Subsequently, we assume that firms switch behaviour and start to interact strategically when the number of local firms is below a threshold; in that case, the market form evolves endogenously. Results change substantially. 'Break' and 'sustain' points are separated as in standard NEG models with an isoelastic demand function leading to a co-existence of equilibria. Stable partial agglomeration and oscillations with small amplitude are possible. The dynamics are described by a 1D piecewise smooth map, which can be continuous (in the pure monopolistic or oligopolistic cases) or discontinuous (when the market structure evolves endogenously). We analyse the bifurcation structure of the parameter space of the map comparing these cases. We show that the continuous maps have rather standard dynamics, while the discontinuous map is characterised by border collision bifurcations of fixed points and cycles, which lead to rich and complex bifurcation structures.

Backward Dynamics in S-unimodal maps

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While the forward trajectory of a point in a discrete dynamical system is always unique, in general a point can have infinitely many backward trajectories (or none at all). The union of the limit points of all backward trajectories through x was called by M. Hero the "special α-limit" ($s\alpha$-limit for short) of x.

This concept plays a fundamental role in the construction of the graph of a dynamical system given by C. Conley [1], extending a seminal idea by S. Smale [5]: the nodes of the graph are the equivalence classes of all chain-recurrent points of the system and there is an edge from node A to node B if there is a point that asymptotes backward to A and forward to B. In a recent work with Jim Yorke [2], we studied the graph of the logistic map (more generally, of any S-unimodal map) and proved that there is a linear hierarchy between all nodes: nodes can be sorted as $N_0, N_1, \dots, N_p$, where N_p is the unique attractor and p is possibly infinite, so that arbitrarily close to each node N_i there are points that asymptote to N_j for each $j > i$. This behavior is not specific of S-unimodal mamps but appears also in higher-dimensional systems [3].

In this talk we show that, correspondingly to the hierarchy above, there is a hierarchy of $s\alpha$-limits of a S-unimodal map [4]. The $s\alpha$-limit of any point of the attractor is the whole non-wandering set. The $s\alpha$-limit of points x "farther and farther away" from the attractor gets smaller and smaller until it is a single point (if x is close to the boundary fixed point) or the empty set (if x is close to the other boundary point).

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It is important to understand dynamics of skew products on n-dimensional ($n \geq 2$) cells, cylinders and tori

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J. Smítal's survey [2] played an important role in my works on dynamics of skew products. In this talk new results on the structure of the nonwandering set of simplest skew products on n-dimensional cells, cylinders and tori are presented [1]. The following cases are considered:

(1) the set of periodic points is not empty (if the phase space is an n-dimensional cylinder or torus), and this set is closed for all maps under consideration;

(2) the set of periodic points is empty for self-maps cylinders and tori, and the nonwandering set is minimal.

The topological entropy of maps under consideration equals 0, and for trajectories of points two types of the returnability are realized: periodicity in the case 1 (under some additional conditions) and uniform recurrence, but not periodicity, in the case 2.

Examples are given, unsolved problems are formulated.

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A Bifurcation analysis of the Microscopic Markov Chain Approach to contact-based epidemic spreading

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(joint work with **Alex Arenas**, **Sergio Gómez** and **Jordi Villadelprat**)

The problem of modeling the spread of a disease among individuals has been studied in deep over many years. The development of compartmental models, models that divide the individuals among a set of possible states, has given rise to a new collection of techniques that enables, for instance, the analysis of the onset of epidemics, the study of epidemics in structured networks, or the study of the impact of a vaccination campaign. All previous works heavily rely on the mathematical approach to the study of epidemic spreading.

We consider a connected undirected network $\mathcal{N}_n$ made up of n nodes, whose probability connections are represented by the entries $r_{ij} \in [0,1]$ of an $n \times n$ symmetric matrix R. We now define a discrete dynamical system based on the infection process on the network. We introduce two parameters in order to control the epidemic spreading model, the first one is $\beta \in (0,1)$, that controls the probability to transmit the disease to its neighbors, and the second one is $\mu \in (0,1)$, that controls the probability of re-infection. The evolution of this infection process is governed by the iteration of a map $F : \mathbb{R}^n \to \mathbb{R}^n$.

Numerical simulations show that these kind of systems, governed by the map F, converge to an asymptotic distribution

$$\lim_{k \to \infty} F^k(p_1^0, \ldots, p_n^0) = (p_1^\infty, \ldots, p_n^\infty),$$

independently on the initial condition $(p_1^0, \ldots, p_n^0)$. Our goal in to investigate the bifurcation process occurring from an analytic point of view.

Gaussian iterative algorithm for various notions of means

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(joint work with **Justyna Jarczyk**)

We recall a celebrated Gaussian algorithm for the iterations of the pair (A, G) of bivariable arithmetic and geometric means, and then its generalization. Next we define three modifications of the concept of mean and formulate suitable versions of the Gausssian algorithm.

Strong Allee Effect-type plasticity rule in unsupervised learning environment

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The Allee effect was introduced by Allee (1949) and characterizes a phenomenon in population dynamics where there is a positive correlation between a population density or size and its per capita growth rate.The strong Allee effect occurs when a population has a "critical density" below which it declines to extinction while the weak Allee effect occurs when a population lacks such a "critical density", but at lower densities, the population growth rate arises with increasing densities. Brain plasticity can be thought of as the ability of the brain to adapt to external activities by modifying some of its synaptic structure. Synapses play an important role in the brain because they constitute junctions between nerve cells and therefore facilitate diffusion of chemical substances called neurotransmitters from the brain to other parts of the body. To understand these modifications at the functional and behavioral levels, one must understand how experience and training modify synapses and how these modifications change patterns of neuronal firings to affect behavior. In this talk, I will propose a discrete time model of brain plasticity based on the strong Allee effect that captures synaptic modifications at the functional level. Stability analysis of the model will be discussed and simulations will be given.

A heterogeneous beliefs asset-pricing model based on predictor accuracy

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Starting from the work [1], we propose an alternative version of their asset pricing model with heterogeneous agents and asynchronous updating of beliefs. In particular, we assume that the predictors are selected based on their accuracy in predicting the market price of the risky asset and not on the net profits made by fundamentalists and trend followers. From a mathematical point of view, the deterministic skeleton of the present model is a two-dimensional piecewise-smooth map. We present an analytical study of the fundamental equilibrium and the coexisting non-fundamental equilibria and propose a comparison with the results in [1]. The model is then extended by introducing noise through stochastic dividends and noisy traders.

This is a joint work with M. Anufriev (University of Technology Sydney, Business School, Economics Discipline Group, Australia and VŠB - Technical University of Ostrava, Ostrava, Czech Republic), T. Tichy (VŠB - Technical University of Ostrava, Ostrava, Czech Republic) and D. Radi (University Cattolica di Milano and VŠB - Technical University of Ostrava, Ostrava, Czech Republic)

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Hidden and self-excited attractors in a heterogeneous Cournot oligopoly model

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(joint work with **Marius-F. Danca**)

The sudden appearance of some hidden chaotic attractor [4, 5], could represent a major disadvantage for the underlying system.

Thus, the consequences could be dramatic such as in the case of pilot-induced oscillations that caused the YF-22 crash in April 1992 and Gripen crash in August 1993 [1].

It is understandable that identifying unwanted hidden chaotic behaviour is desirable. There exists the risk of the sudden jump from a desirable attractor to the possible undesired behaviour of some hidden attractor. Recently, it has been shown that multistability is connected with the occurrence of hidden attractors. If there are unstable fixed points, the basins of attraction of the hidden attractors do not touch them. Note that if the system exhibits chaotic or regular behaviour and the system's equilibria are stable, then the chaotic or regular underlying attractors are implicitly hidden. Therefore, the stability of equilibria is important.

For a hidden attractor, its attraction basin is not connected with unstable equilibria. For example, hidden attractors can be found in e. g. systems without equilibria or with stable equilibria [1].

Also, as in the case of the studied discrete-time system in this paper, systems with an infinite number of equilibria (also called a line of equilibria), can admit hidden attractors. Systems with a line of equilibria are very few (see e. g. [6, 7]). Hidden attractors into an impulsive discrete dynamical system have been found in [2], where the case of a supply and demand economical system is studied.

The aim is focussed on the *heterogeneous Cournot oligopoly model* with q_1 (called *gradient* player) and q_2 (called *imitator* player) variables:

$$\left\{\begin{array}{lcl} q_1^{n+1} & = & q_1^n + \gamma q_1^n (a - b((N(1-\omega)+1)q_1^n + \omega N q_2^n) - c), \\ q_2^{n+1} & = & \dfrac{\pi_2^n}{\pi_2^n + \pi_1^n} q_2^n + \dfrac{\pi_1^n}{\pi_2^n + \pi_1^n} q_1^n, \end{array}\right.$$

where

$$\pi_1^n = (a - c - bN((1-\omega)q_1^n + \omega q_2^n))q_1^n,$$
$$\pi_2^n = (a - c - bN((1-\omega)q_1^n + \omega q_2^n))q_2^n,$$

the meaning of remaining parameters can be found in [3].

The present research outputs focussing on the oligopoly model follow the usual path in research development: revealing at the first step the existence of hidden chaos in the model. Our second step to explain the essence of the new finding of hidden chaos will be carried out, hopefully in the near future, since there is no precise rigorous theory about hidden attractors yet in the current literature where researchers are still intensively working on the topic. Besides, this important but difficult issue is beyond the scope of the present paper of the first attempt.

More precisely, it is numerically shown that the dynamics of a heterogeneous Cournot oligopoly model depending on two bifurcation parameters can exhibit hidden and self-excited attractors. The system has a single equilibrium and a line of equilibria. The bifurcation diagrams show that the system admits several attractor coexistence windows, where the hidden attractors can be found. Depending on the parameters ranges, the coexistence windows present combinations of periodic, quasiperiodic and chaotic attractors.

The talk will be based on results published in [3].

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On extensive dynamics of a Cournot heterogeneous model with optimal response

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(joint work with **Marek Lampart** and **Giuseppe Orlando**)

The theme of this talk is the study of the dynamical properties analysis of an original specification of the classical Cournot heterogeneous model with optimal response; specifically, a new approach that considers ordinal utility instead of cardinal monetary amounts is proposed where the classical decision of quantity is disentangled from the decision on imitation.

The analysis is performed by means of bifurcation diagrams, the 0-1 test for chaos, Power Spectral Density, histograms, and trajectory analysis. For this purpose, a new perturbation parameter ϵ of the initial condition is introduced, and together with the intensity of choice parameter β determining the share of responders vs imitators, the system is researched. Depending on the ϵ and β, extreme reach dynamics, and coexisting attractors, periodic and chaotic trajectories are investigated through massive simulations. Those dynamics represent alternation between stability, cycles and chaos in the market. As the dynamics are completely endogenous, it means that swings in economy are intrinsic to the system and that they may persist unless controlled.

The talk is based on the paper [Lampart, M., Lampartová, A., Orlando, G. Chaos 32(2), 2022].

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Dynamics of a family of piecewise linear maps

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(joint work with **Anna Cima**, **Armengol Gasull** and **Francesc Mañosas**)

We consider the family of piecewise linear maps

$$F(x,y)=\begin{pmatrix} \cos(\alpha) & \sin(\alpha) \\ -\sin(\alpha) & \cos(\alpha) \end{pmatrix}\begin{pmatrix} x-\operatorname{sign}(y) \\ y \end{pmatrix}. \tag{1}$$

For the special cases $\alpha \in \mathcal{A} := \{\pi/3, \pi/2, 2\pi/3, 4\pi/3, 3\pi/2, 5\pi/3\}$, it is known that the maps are pointwise periodic but not globally periodic, hence their sets of periods are unbounded. For each of these cases, in [1], we prove the existence of a first integral whose energy levels are discrete and, furthermore, whose level sets are bounded sets whose interior is a necklace formed by a finite number of open tiles of a certain regular or uniform tessellation.

In [1], we also describe the dynamics of the maps on each invariant set of tiles in geometric terms that depend explicitly on the values of the first integral. More precisely, consider a map in (1) with first integral V, then: (a) We prove that F induces a dynamics between the M tiles of the necklace forming the level set $\{V(x,y)=c\}$, which is conjugate to the one generated by an affine map $h:\mathbb{Z}_M\to\mathbb{Z}_M$, which is k-periodic with $k\in\{M, M/2\}\cap\mathbb{N}$. Notice that, geometrically, F acts as a rotation among the tiles of the necklace. (b) We prove that each tile is invariant by F^k, which is a rotation of order p around the center of the tile. As a consequence, on each tile there is a k-periodic point (the center) and the rest of the points are kp-periodic. The map h and all the values of M, k and p depend explicitly on the energy level c.

The general properties of the maps F with $\alpha \in [0, 2\pi) \setminus \mathcal{A}$, being a rational multiple of π, are still not completely known. For instance, in [2] it is proved that for such cases there exists a sequence of open invariant nested necklaces that tend to infinity, whose beads are polygons, and where the dynamics of F is given by a product of two rotations. Remarkably, although the adherence of the union of all these invariant necklaces does not fill the full plane, it allows to prove that all orbits of F are bounded.

For these cases, we will present some sparse results. Consider the critical set $\mathcal{F} = \bigcup_{i \in \mathbb{N}} LC_{-i}$ formed by all the preimages of the critical line LC_0 where the discontinuity is located. Our simulations indicate that, in these cases, the critical set seems to fractalize. We also consider the set $\mathcal{U} = \mathbb{R}^2 \setminus \overline{\mathcal{F}}$. Among the results that we present, we prove that any connected component of $\mathcal{U}$ is open, bounded and periodic. Moreover, any element of $\mathcal{U}$ is periodic. Furthermore, it has been claimed -but not proved- that for some non-regular cases there exists non-periodic orbits in $\mathcal{F}$. We prove that if $\overline{\mathcal{F}} \setminus \mathcal{F} \neq \emptyset$, then the elements of $\overline{\mathcal{F}} \setminus \mathcal{F}$ are aperiodic. This last part of the talk is a work in progress.

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Correlation sums and recurrence determinism for interval maps

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Recurrence quantification analysis is a method for measuring the complexity of dynamical systems. Recurrence determinism is a fundamental characteristic of it, closely related to correlation sum. In this talk, I will speak about asymptotic behavior of these quantities for interval maps. I will show for which cases the asymptotic correlation sum exists. An example of an interval map with zero entropy and a point with the finite ω-limit set for which the asymptotic correlation sum does not exist will be given. Moreover, I will present formulas for computation of the asymptotic correlation sum with respect to the cardinality of the ω-limit set or to the configuration of the intervals forming it. I will also show that for a Li-Yorke non-chaotic (and hence zero entropy) interval map, the limit of recurrence determinism as distance threshold converges to zero can be strictly smaller than 1.

Invariant measures of random interval homeomorphisms

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(joint work with **Thomas Zürcher**)

Fix an integer $N \geq 2$, a probability vector $p = (p_1, \ldots, p_N)$, i. e.

$p_1, \ldots, p_N > 0$ with $\sum_{n=1}^N p_n = 1$, and a family $\mathcal{F} = \{f_1, \ldots, f_N\}$ of increasing homeomorphisms of $[0,1]$ into itself. The following result for the iterated function system with probabilities $(\mathcal{F}, p)$ was obtained by Tomasz Szarek and Anna Zdunik [4, Theorem 4]; see also [1, Lemma 1] and [2, Theorem 1].

Theorem. *Assume that $(\mathcal{F}, p)$ is such that for any $x \in (0,1)$ there exist $n, m \in \{1, \ldots, N\}$ with $f_n(x) < x < f_m(x)$. If $f_1, \ldots, f_N$ are continuously differentiable in neighbourhoods of 0 and 1, and additionally $\sum_{n=1}^N p_n \log f_n'(0) > 0$ and $\sum_{n=1}^N p_n \log f_n'(1) > 0$, then there exists a unique invariant measure μ_* for $(\mathcal{F}, p)$ with $\mu_*((0,1)) = 1$.*

The main result [3, Corollary 4.4] of this talk reads as follows.

Main Theorem. *Assume that $(\mathcal{F}, p)$ is such that for any $x \in (0,1)$ there exists $n \in \{1, \ldots, N\}$ with $f_n(x) < x$, or for any $x \in (0,1)$ there exists $m \in \{1, \ldots, N\}$ with $x < f_m(x)$. Then there exists a unique invariant measure μ_* for $(\mathcal{F}, p)$ with $\mu_*((0,1)) = 1$ if and only if there exist non-decreasing and continuous $\rho_1, \rho_2 \colon [0,1] \to [0,1]$ such that $\rho_1 \le \rho_2$, $\rho_2(0) = 0$, $\rho_1(1) = 1$, $\rho_1(x) \le \sum_{n=1}^N p_n \rho_1(f_n^{-1}(x))$ and $\sum_{n=1}^N p_n \rho_2(f_n^{-1}(x)) \le \rho_2(x)$ for every $x \in [0,1]$.*

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An indicator for a new pandemic

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(joint work with **Rogério Colaço**, **Carlos Robalo Cordeiro**, **José Rui Figueira**, **Filipe Froes**, **Miguel Guimarães** and **Ana Paula Serro**)

In this talk we present the methodology of the construction of a multicriteria indicator in epidemiology. This indicator combines incidence, transmissibility and severity. We present the rubustness analysis of this indicator. We make a historical perspective of the results of this indicator when applied to the COVID-19 case in Portugal until now. Finally, we give an overview of discrete models in epidemiology of contagion.

On intersections of orbits of rational functions

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Let A be a rational function of degree at least two on $\mathbb{CP}^1$. For a point $z_1 \in \mathbb{CP}^1$ we denote by $O_A(z_1)$ the forward orbit of A, that is, the set $\{z_1, A(z_1), A^{\circ 2}(z_1), \dots\}$. In the talk, we address the following problem: *given two rational functions A and B of degree at least two, under what conditions do there exist orbits $O_A(z_1)$ and $O_B(z_2)$ having an infinite intersection?* We show that under a mild restriction on A and B this happens if and only if A and B have an iterate in common, that is, if and only if $A^{\circ k} = B^{\circ l}$ for some $k, l \geq 1$. Put another way, unless rational functions A and B have the same global dynamics, an orbit of A may intersect an orbit of B at most at finitely many places.

Border collision bifurcations of chaotic attractors in 1D maps with multiple discontinuities

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(joint work with **Viktor Avrutin** and **Iryna Sushko**)

In the present paper we consider a 1D piecewise linear map with *multiple* discontinuities and describe several bifurcations for chaotic attractors, which can not be associated with a homoclinic bifurcation of any cycle. Recall that border collision bifurcations of fixed points and cycles are widely investigated in contrast to those of chaotic attractors, for which their transformations are usually related to homoclinic bifurcations. For the most extensively studied class of piecewise smooth maps, i.e., 1D piecewise monotone maps with a *single* discontinuity, a chaotic attractor must include the border point, and thus, cannot collide with it.

Besides the "direct" border collision—when a chaotic attractor collides with a discontinuity point, which does not belong to this attractor—called an *exterior border collision bifurcation*, we also analyze more sophisticated cases. Some of them are grouped under the term *interior border collision bifurcations* that are related to disappearance of a preimage of a critical point (located inside a chaotic attractor). In addition, an *expansion border collision bifurcation* and *expansion-like* transitions of other type are reported.

The first return map: revealing bifurcation mechanisms in a 2D nonsmooth map

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(joint work with **Raffaella Coppier**, **Elisabetta Michetti** and **Iryna Sushko**)

The current paper is devoted to describing dynamics of a two-dimensional piecewise smooth map, which models the effects of fraud in a public procurement procedure, in particular, the level of dishonest behavior in society. Our main goal is to show how properly constructed first return map can help to describe formation principles of the bifurcation structure in the parameter space of the map studied. We prove that under certain conditions the asymptotic dynamics is reduced to a one-dimensional set in the phase space, which is composed by segments of critical lines. We report bifurcation structure of the map formed by the periodicity regions associated with attracting cycles of different periods. By means of the first return map, defined on the proper segment of the critical line LC, we show that the boundaries of these regions are related to border collision bifurcations typical for nonsmooth maps, as well as to the standard fold and flip bifurcations. The main economic result is related to the emergence of unexpected changes in qualitative non-compliant behavior over time.

A Multiscale Network with Percolation Model to Describe the Spreading of Forest Fires

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(joint work with **Maria Clara Grácio** and **Nuno de Almeida Ribeiro**)

Forest fires are currently the main menace to the environment in general, especially in the Mediterranean climate regions. To reduce the hazard of fire occurrence, understanding fire behaviour has been a core aspect of fire modelling throughout the years.

Fire behaves according to three interacting physical factors: fuel availability (morphological and physiological characteristics of vegetation), weather (wind speed and direction, temperature, and relative humidity) and terrain (slope and aspect) [11, 14] – FWT conditions. Fire models such as Rothermel's [15] predict fire's local behaviour using fuel model parameters as input. Fuel models, such as [1, 18] are sets of parameters that describe the characteristics that classify certain fuel types.

The field of percolation has played an important role in developing models and strategies to model fire spreading. Works [6, 8, 9] are examples of percolation frameworks to predict fire spreading, based on FWT conditions, using cellular automata (CA).

One of the existing strategies and very important structure in mitigating fire spreading is the implementation of firebreaks [16]. These are gaps in the available combustible material that prevent the fire from advancing. In [16, 17], the authors use CA modelling to identify efficient fuel break partitions for fire containment and study the efficiency of various centrality statistics, considering GIS meteorological and landscape information data.

Laboratory experiments can be designed to simulate a small-scale fire, but with limitations. An interesting study [2] compares theory results with laboratory simulations, using matchsticks. The theory predicts that at critical percolation a fire front decelerates, whereas experiments indicate acceleration. This discrepancy shows that percolation theory models of forest-fire propagation using simple site percolation are unlikely to be accurate. On the opposite side, large-scale experiments have their limitations in terms of reproducibility as well. Still, percolation modelling of fire-spread is of considerable importance because it describes the transition regime between extinction (spanning fires) and uncontrolled spread (penetrating fires) [2].

To overcome scale limitations, some examples using Multilayer Networks [5, 3, 13] and cellular automata with different approaches have come up with important findings. Work [4], applies multiplex networks to model fire propagation, simulating a 3-layer of possible fire development: ground, surface, and crown, where each node of the multilayer represents a group of trees. At a larger (landscape) scale, work [12] presents a network-of-networks structure, where the nodes are local land patches, with their own spreading dynamics each and, as such, presenting different spreading times.

Our work follows the line of research of the fields of percolation and complex networks. First, we define the local scale as the range in which is possible to delimit a land patch with a measurable set of characteristics, and landscape scale as the scale at which each patch of land is the element of study.

Following the works [12, 13], we present a 2-scale network structure applied to the region of Serra de Ossa, in Portugal. The nodes of the landscape network correspond to territory division in irregular-shaped polygons, based on land characteristics. Within each polygon, SIR simulations occur on a CA, whose cells constitute the local network of the corresponding polygon. Using a classical percolation algorithm, our results for the percolation threshold, p_c 0.407, are consistent with the literature [7, 10]. We then introduce a neighbourhood of warm trees, which change this value to p_c 0.725. The landscape network is then parametrized with p_c values.

The main goal is to find an efficient fire-break structure that mitigates the spreading, to complement the efforts of civil protection forces. Given the application geography of this model, the complexity of the problem is limited by restricting FWT conditions to those specific to that area. Still, the spatial extent to which our model can be applied strongly depends on previous land monitoring, which implies a demand for the inclusion of other areas of expertise.

As future work, after calibrating the spreading network dynamics, we intend to introduce classifications of autochthonous biomes. Afterwards, vectorial influence inherent to meteorological and orographic factors are to be considered in the simulations in addition to the existing model.

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Ambiguity aversion as a route to randomness in a duopoly game

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The global dynamics is investigated for a duopoly game where the perfect-foresight hypothesis is relaxed and firms are worst-case maximizers, see [1]. Ignoring that products are perfect substitute and ignoring the sensitivity of price to quantity, the unique and globally stable Cournot-Nash equilibrium of the complete-information duopoly game losses stability when firms are not aware if they are playing a duopoly game, as it is, or an oligopoly game with more than two competitors. This finding resembles Theocharis's condition for the stability of the Cournot-Nash equilibrium in oligopolies without uncertainty. Differently from complete-information oligopoly games, coexisting attractors, disconnected basins of attractions and chaotic dynamics emerge when the Cournot-Nash equilibrium losses stability. This difference in the global dynamics is due to the nonlinearities introduced by the worst-case approach to uncertainty, which mirror in bimodal best-reply functions and piecewise-smooth map, see, e.g., [2]. The investigation reveals that chaotic dynamics are caused, and at the same time source, of uncertainty.

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Transitivity of expanding Lorenz maps on the interval

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Let $f : [0, 1] \to [0, 2]$ be a continuous strictly increasing function which is differentiable on $(0, 1) \setminus F$ where F is a finite set. Moreover, assume that $\beta := \inf f' := \inf_{x \in (0,1) \setminus F} f'(x) > 1$. Then there exists a unique $c \in (0, 1)$ with $f(c) = 1$. Set $T_f x := f(x) - \lfloor f(x) \rfloor$, where $\lfloor y \rfloor$ is the largest integer smaller or equal to y. Such a map T_f is called an expanding Lorenz map. Observe that T_f has a discontinuity at c.

For the case $\beta \geq \sqrt[3]{2}$ topological transitivity and topological mixing of T_f is investigated. In the case $\beta \geq \sqrt[3]{2}$ and $f(0) \geq \frac{1}{\beta+1}$ the map T_f is topologically transitive. Furthermore it is also topologically mixing except in the case $f(x) = \sqrt[3]{2}x + \frac{2+\sqrt[3]{4}-2\sqrt[3]{2}}{2}$ for all $x \in [0, 1]$.

Better results are obtained in the special case $f(x) = \beta x + \alpha$. One can completely describe the set of all (β, α) with $\sqrt[3]{2} \leq \beta \leq 2$ and $0 \leq \alpha \leq 2 - \beta$ such that T_f is topologically transitive in this case. With three exceptions all of these topologically transitive maps are also topologically mixing.

According to Glendinning the map T_f is called locally eventually onto if every nonempty open $U \subseteq [0, 1]$ contains open intervals

$U_1, U_2 \subseteq U$ and there are $n_1, n_2 \in \mathbb{N}$ such that ${T_f}^{n_1}$ maps U_1 homeomorphically to $(0, c)$ and ${T_f}^{n_2}$ maps U_2 homeomorphically to $(c, 1)$. The map T_f is called renormalizable if there are $0 \leq u_1 < c < u_2 \leq 1$ and $l, r \in \mathbb{N}$ with $l + r \geq 3$ such that ${T_f}^l$ is continuous on (u_1, c), ${T_f}^r$ is continuous on (c, u_2), $\lim_{x \to c^-} {T_f}^l x = u_2$ and $\lim_{x \to c^+} {T_f}^r x = u_1$. An example of a renormalizable and locally eventually onto expanding Lorenz map is given. Using a condition closely related to "locally eventually onto" it is shown that this condition is equivalent to T_f is not renormalizable.

The life and mathematics of Jaroslav Smítal

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Jaroslav Smítal (1942–2022) was an important Czech and Slovak mathematician. In this talk a short outline of his life will be given.

Bifurcations of closed invariant curves in a 2D piecewise smooth map

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(joint work with **Viktor Avrutin**, **Laura Gardini** and **Zhanybai T. Zhusubaliyev**)

We consider a 2D piecewise smooth map originating from an application (acting as a discrete-time model of a DC/DC converter with pulse-width modulated multilevel control). We focus on several nontrivial transformations occurring in the phase space of this map under parameter variation. In particular, we describe the effect of a fold border collision bifurcation leading to the appearance of a pair of cycles, an attracting and a saddle one, a sequence of transformations of the basins of coexisting attractors, as well as heteroclinic bifurcations which result first in the destruction of an attracting closed resonant curve and then in the creation of another one such curve.

We also discuss a border-collision bifurcation of a repelling resonant closed invariant curve (a repelling saddle-node connection) colliding with the border by a point of the repelling cycle. As a result, this cycle becomes attracting and the curve is destroyed, while a new repelling closed invariant curve appears (not in a neighborhood of the previously existing invariant curve), being associated with quasiperiodic dynamics. This leads to a global restructuring of the phase portrait since both curves mentioned above belong to basin boundaries of coexisting attractors.

Classification of Floyd-Auslander systems with fixed pattern

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Floyd-Auslander systems form an essential class of minimal nonhomogeneous systems, moreover, their dynamics is realised by homeomorphisms. We give a full classification up to topological conjugacy of Floyd-Auslander systems with fixed subdivision pattern. Besides the trivial identity conjugacy, any Floyd-Auslander system is conjugate only to the Floyd-Auslander system defined by the mirror subdivision pattern.

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Some properties in discrete dynamical system of Lorenz map

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(joint work with **Laura Gardini**)

In presentation, we study some properties of a two-dimensional noninvertible map $T : (x', y') = (x(1 + a\tau - y), (1 - \tau)y + \tau x^2)$ proposed by Lorenz in 1989, depending on the two parameters a and τ. We show the two different bifurcation scenarios occurring for $a > 0$ and $a < 0$. Two particular degenerate cases are investigated, at $\tau = 1$ and $\tau = 2$, describing the related bifurcations as a function of the parameter $a > 0$. For $\tau = 2$ a straight line filled with 2-cycles and the occurrence of a resonant case with rotation number $\pi/2$ in the Neimark-Sacker bifurcation of two fixed points determine particular bifurcations, associated with the existence of four invariant sets of the map. For $\tau = 1$ the critical curve degenerates into one point, the origin, which is also fixed for map T and focal point for the inverse map, leading to chaotic attracting sets with infinitely many lobes issuing from the origin. The transition to chaos from attracting closed curves is also analyzed, evidencing the occurrence of homoclinic tangles.

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