

© 2020 Nigel Adam Lucas Pynn-Coates

ON ASYMPTOTIC VALUED DIFFERENTIAL FIELDS  
WITH SMALL DERIVATION

BY

NIGEL ADAM LUCAS PYNN-COATES

DISSERTATION

Submitted in partial fulfillment of the requirements  
for the degree of Doctor of Philosophy in Mathematics  
in the Graduate College of the  
University of Illinois at Urbana-Champaign, 2020

Urbana, Illinois

Doctoral Committee:

Associate Professor Philipp Hieronymi, Chair  
Professor Lou van den Dries, Director of Research  
Assistant Professor Anush Tserunyan  
Assistant Professor James Freitag, University of Illinois at Chicago

## Abstract

This thesis is a contribution to the algebra and model theory of certain valued differential fields and ordered valued differential fields. We focus on those with small derivation, which is a strong form of continuity of the derivation with respect to the valuation topology, and especially on those that are also asymptotic, which is a weak valuation-theoretic analogue of l'Hôpital's Rule.

The first component of this thesis concerns three conjectures for valued differential fields  $K$  with small derivation and linearly surjective differential residue field: the uniqueness of maximal immediate extensions of  $K$ , the equivalence of differential-algebraic maximality and differential-henselianity for asymptotic  $K$ , and the existence and uniqueness of differential-henselizations of asymptotic  $K$ . First, we show that any two maximal immediate extensions of  $K$  are isomorphic over  $K$  whenever the value group of  $K$  has only finitely many convex subgroups. More significantly, we also establish this conjecture when  $K$  is asymptotic. Next, we show that if  $K$  is asymptotic and differential-henselian, then it is differential-algebraically maximal; this is optimal, as Aschenbrenner, van den Dries, and van der Hoeven have shown that the asymptoticity assumption is necessary. They have also shown that if  $K$  is differential-algebraically maximal, then it is differential-henselian, so this establishes the equivalence of differential-algebraic maximality and differential-henselianity for asymptotic  $K$ . Finally, we use this equivalence to show that if  $K$  is asymptotic, then it has a differential-henselization, and that differential-henselizations are unique.

The second component of this thesis builds on the first to study the model theory of pre- $H$ -fields with gap 0, which are certain asymptotic ordered valued differential fields with small derivation that are transexponential in some sense. We show that the theory  $T^*$  of differential-henselian, real closed pre- $H$ -fields that have exponential integration and closed ordered differential residue field (such pre- $H$ -fields necessarily have gap 0) has quantifier elimination in the language  $\{+, -, \cdot, 0, 1, \leq, \preceq, \partial\}$ . From quantifier elimination, we deduce that this theory is complete and is the model completion of the theory of pre- $H$ -fields with gap 0 (equivalently, it axiomatizes the class of existentially closed pre- $H$ -fields with gap 0). Moreover, we show that it is combinatorially tame in the sense that it is distal, and hence has NIP. Finally, we consider a two-sorted structure with one sort for a model of  $T^*$  and one sort for its residue field in a language  $\mathcal{L}_{\text{res}}$  expanding the language  $\{+, -, \cdot, 0, 1, \leq, \partial\}$  of ordered differential rings, and show that the theory of this two-sorted structure is model complete when the theory of the residue field is model complete in  $\mathcal{L}_{\text{res}}$ .

## Acknowledgements

My deepest gratitude goes to my advisor, Lou van den Dries, for his guidance and help over the last several years. Much of the research in this thesis was pursued at his suggestion, and it has benefited from his care and precision through many discussions and drafts. In particular, Chapter 3 contains joint work. Thanks are due to James Freitag, Philipp Hieronymi, and Anush Tserunyan for their service on my thesis committee.

I would also like to thank Anush, Lou, and Philipp, as well as Sławek Solecki, for all that I have learnt from them in courses and seminars, and for fostering the stimulating and collegial environment in the logic group. The other students and postdocs have also done a great deal to contribute to this environment, and I have particularly benefited from informal discussions and reading groups with the model theory students: Neer Bhardwaj, Alexi Block Gorman, Santiago Camacho, Erin Caulfield, Allen Gehret, Tigran Hakobyan, Elliot Kaplan, Travis Nell, and Minh Tran. Allen deserves special thanks for organizing many of these reading groups and for providing guidance to me while I was reading the book. In addition, I have had many helpful discussions with Anton Bernshteyn, and I thank him for his interest in my research.

In practical matters, I am indebted to Marci Blocher for her unparalleled proficiency in administering the logistical aspects of the mathematics graduate program. Financially, my work in graduate school was supported by an NSERC Postgraduate Scholarship, research assistantships from Lou van den Dries, and a University Fellowship from the mathematics department, in addition to the usual teaching assistantships.

Outside the University of Illinois, my graduate school experience has been enriched by my participation in numerous summer schools, workshops, and conferences, as well as several opportunities to present my research at seminars and conferences. I thank the organizers of these events and the speakers, and of course the many sources of travel funding, including the Association for Symbolic Logic, the National Science Foundation, the mathematics department, and the Graduate College. I am also grateful to Deirdre Haskell and Patrick Speissegger, who supported my applications to graduate school and NSERC, and contributed to my mathematical development, including by first exposing me to valued fields and differential fields, respectively.

On a personal note, I would like to thank my mother, Jo Ann Pynn, and my sister, Marieke Pynn-Coates, for their enduring love and encouragement, and for many fine articles of clothing. To the many friends I have made during graduate school, foremost among them Anton Bernshteyn, Ravi Donepudi, and Xiao Li, I cherish our shared experiences and the relationships that we have developed. And finally to Yuki: Thanks for five years of adventures; I would never have gone anywhere without you.

*We must beware of needless innovation, especially when guided by logic.*

—Winston Churchill

# Contents

Chapter 1. Introduction . . . . .	1
1.1. Some history and motivation . . . . .	1
1.2. Summary of results . . . . .	3
Chapter 2. Preliminaries . . . . .	9
2.1. Conventions . . . . .	9
2.2. Valued differential fields . . . . .	9
2.3. Immediate extensions and pseudocauchy sequences . . . . .	12
2.4. Asymptotic valued differential fields . . . . .	15
2.5. Algebraic extensions . . . . .	16
2.6. Standing assumptions . . . . .	17
Chapter 3. Differential-henselianity and maximality, part I . . . . .	18
3.1. Introduction . . . . .	18
3.2. Preliminaries . . . . .	19
3.3. The differential-henselian configuration property . . . . .	23
3.4. Main results . . . . .	26
Chapter 4. Differential-henselianity and maximality, part II . . . . .	29
4.1. Introduction . . . . .	29
4.2. Preliminaries . . . . .	30
4.3. Main results . . . . .	34
4.4. Newton diagrams . . . . .	37
4.5. Asymptotic differential equations . . . . .	40
4.6. Reducing complexity . . . . .	49
4.7. Proof of Proposition 4.15 . . . . .	57
Chapter 5. Newtonian valued differential fields with arbitrary value group . . . . .	59
5.1. Introduction . . . . .	59
5.2. Preliminaries . . . . .	61
5.3. Removing divisibility . . . . .	64
5.4. Uniqueness . . . . .	66
Chapter 6. Pre- $H$ -fields with gap 0 . . . . .	68
6.1. Introduction . . . . .	68
6.2. Preliminaries on pre- $H$ -fields . . . . .	70

6.3. Extensions controlled by the residue field . . . . .	71
6.4. Asymptotic couples with small derivation . . . . .	73
6.5. Extensions controlled by the asymptotic couple . . . . .	78
6.6. Extending the constant field . . . . .	82
6.7. Differential-Hensel-Liouville closures . . . . .	84
6.8. Main results . . . . .	90
Chapter 7. Concluding remarks and questions . . . . .	94
Bibliography . . . . .	96

## CHAPTER 1

# Introduction

### 1.1. Some history and motivation

In the early 20th century, Hardy studied “logarithmico-exponential” functions, which are germs at infinity of certain real-valued functions obtained from exponentials and logarithms [Har12]. This led Bourbaki to introduce the notion of a Hardy field: a field of germs at infinity of real-valued functions that is closed under differentiation. Thus such structures provide a framework for studying rates of growth of solutions to differential equations as time goes to infinity. Hardy’s field of logarithmico-exponential functions seems to cover all the rates of growth occurring naturally in mathematics but lacks certain closure properties, and Hardy therefore sought a “universal domain” that would be as large as possible for studying the asymptotics of such functions.

In the 1980s, Dahn–Göring and Écalé independently introduced transseries for different purposes: Dahn and Göring constructed them as a possible non-standard model of the theory of the real field with exponentiation [DG87], while Écalé used them to solve Dulac’s conjecture about plane analytic vector fields, connected with Hilbert’s 16th Problem [Éca90; Éca92]. In [DMM97], van den Dries, Macintyre, and Marker showed that  $\mathbb{T}$ , the field of logarithmic-exponential transseries, is indeed a model of the theory of the real exponential field (construing  $\mathbb{T}$  as an exponential field). Moreover, in [DMM01], the same authors conjectured that  $\mathbb{T}$  is the kind of universal domain that Hardy sought. This was substantiated by Aschenbrenner, van den Dries, and van der Hoeven over a decade later in their book [ADH17a]. As context for the work in this thesis, especially that of Chapter 6, we briefly discuss their results here; interested readers are encouraged to consult [ADH17a] and its extensive introduction for more details.

Making this conjecture precise involves model theory, a branch of mathematical logic. One task is to identify the right language, or essential functions and relations, in which to study Hardy fields and transseries. Hardy fields are naturally ordered differential fields, but they also can be equipped with a canonical valuation induced by the ordering, which measures the rate of growth at infinity. Although essentially going back to du Bois-Reymond, this canonical valuation is explicated in Rosenlicht’s paper [Ros83]. For a valuation we use the symbol  $\preceq$ , where  $f \preceq g$  is viewed as “ $f$  grows at most as fast as  $g$ .” Below, we use the more suggestive terms “bounded,” “infinite,” and “infinitesimal,” to indicate that  $f \preceq 1$ ,  $f \succ 1$  (that is,  $f \not\preceq 1$ ), and  $f \prec 1$ , respectively. The set of bounded elements forms a subring called the *valuation ring*. Thus the natural language in which to study these structures is that of ordered valued differential fields,  $\{+, -, \cdot, 0, 1, \leq, \preceq, \partial\}$ , where  $\partial$  is interpreted as differentiation,  $\preceq$  is as above, and the other symbols have their usual interpretation in ordered rings.



Aschenbrenner and van den Dries introduced certain ordered valued differential fields,  $H$ -fields and pre- $H$ -fields, as a framework for studying Hardy fields from an algebraic and model-theoretic perspective [AD02]. A pre- $H$ -field is a field equipped with an ordering, valuation, and derivation that interact in four ways, capturing some relations from Hardy fields. First, constants (elements with derivative zero) are bounded; second, infinite positive elements have positive derivative; third, the valuation ring is convex; and fourth, an analogue of l'Hôpital's Rule holds. An  $H$ -field is a pre- $H$ -field such that every bounded element is infinitely close to a constant, and in fact, pre- $H$ -fields are exactly the ordered valued differential subfields of  $H$ -fields. All Hardy fields, construed as ordered valued differential fields, are pre- $H$ -fields, while those that contain  $\mathbb{R}$  are  $H$ -fields. The ordered valued differential field  $\mathbb{T}$  is an  $H$ -field with small derivation, where we say that a valued differential field has *small derivation* if derivatives of infinitesimals are infinitesimal.

To be a universal domain for asymptotic differential algebra,  $\mathbb{T}$  should contain as many solutions to asymptotic differential equations as possible; we make this precise via the model-theoretic term “existentially closed.” Let  $K$  be an  $H$ -field with small derivation. We use the notation  $K\{Y\} := K[Y, Y', Y'', \dots]$  for the ring of differential polynomials over  $K$ , with the obvious evaluation map. Then  $K$  is *existentially closed* if, for any  $H$ -field extension  $L$  of  $K$  with small derivation, whenever  $y = (y_1, \dots, y_n) \in L^n$  satisfies a boolean combination  $R$  of relations of the form

$$P(y) = 0, \quad P(y) > 0, \quad \text{and} \quad P(y) \preceq Q(y),$$

where  $P$  and  $Q$  range over  $K\{Y_1, \dots, Y_n\}$ , there is  $z = (z_1, \dots, z_n) \in K^n$  satisfying  $R$ . In [ADH13], Aschenbrenner, van den Dries, and van der Hoeven conjecture that  $\mathbb{T}$  is existentially closed, and moreover that the models of the theory of  $\mathbb{T}$  (that is, the ordered valued differential fields that have the same first-order properties as  $\mathbb{T}$ ) are exactly the existentially closed  $H$ -fields with small derivation. An equivalent formulation is that the theory of  $\mathbb{T}$  is the *model companion* of the theory of  $H$ -fields with small derivation.

They prove this conjecture in [ADH17a], substantiating the notion that  $\mathbb{T}$  is a universal domain for asymptotic differential algebra. To do this, they introduce a certain theory  $T^{\text{nl}}$  and show that  $T^{\text{nl}}$  is the model companion of the theory of pre- $H$ -fields; the details of this theory are not germane here, although the results of Chapter 6 are modelled on those for  $T^{\text{nl}}$ . What is important is that  $T^{\text{nl}}$  + “small derivation” axiomatizes the theory of  $\mathbb{T}$ . The conjecture from [ADH13] then follows from these two results.

Much of [ADH17a] is developed in greater generality than the setting of pre- $H$ -fields. For example, some of a Hardy field's structure is captured just by the valuation and derivation, for which Rosenlicht [Ros80] developed a theory of differentially valued (or differential-valued) fields. Generalizing this, [ADH17a] introduces asymptotic valued differential fields, in which the asymptotic relation  $f \preceq g$  is preserved under differentiation and integration (when this is possible); all pre- $H$ -fields are asymptotic. Second, [ADH17a] studies valued differential fields with small derivation. A key property of some such fields is differential-henselianity, which generalizes the well-studied notion of henselianity for valued fields. The first part of this thesis, Chapters 3 and 4, deals with three conjectures from [ADH17a; ADH18] in the general theory of valued differential fields with small derivation, especially asymptotic valued differential fields with small derivation.

The results of [ADH17a] describe the model companions of the theories of pre- $H$ -fields and of  $H$ -fields with small derivation, but what about other pre- $H$ -fields with small derivation? There are two subclasses of pre- $H$ -fields with small derivation not covered by the results above, and we focus on the model theory of one of these, the class of pre- $H$ -fields with gap 0, in Chapter 6. In contrast to  $\mathbb{T}$ , pre- $H$ -fields with gap 0 contain transexponential elements, and their valuations only distinguish infinite elements that differ transexponentially. This penultimate chapter isolates a model companion for the theory of pre- $H$ -fields with gap 0, making use of the results of Chapter 4. Thus this thesis is a contribution to the study of valued differential fields with small derivation and of asymptotic valued differential fields, and to the model theory of pre- $H$ -fields with small derivation.

## 1.2. Summary of results

Here we give a relatively self-contained overview of the results in this thesis. For the sake of readability, some definitions are given only informally here, but keen readers can consult the precise definitions in Chapter 2. Some explanation of model-theoretic terms is given, although knowledge of the rudiments of logic would be helpful (such as language, sentence, formula/definable set).

The first component of the thesis, comprising Chapters 3 and 4, is an attempt to generalize fundamental results from the theory of valued fields of equicharacteristic 0 to the setting of valued differential fields with small derivation. We are concerned with the following three results: the uniqueness of maximal immediate extensions, the equivalence of henselianity with algebraic maximality, and the existence of henselizations. We achieve appropriate analogues of these results in Theorems 3.1, 4.18, 4.19, and 4.20, primarily under the assumption that the valued differential fields with small derivation are asymptotic.

The second component of this thesis concerns the model theory of certain transexponential pre- $H$ -fields, called pre- $H$ -fields with gap 0. We obtain a model companion for this theory in Corollary 6.55, using Theorems 4.19 and 4.20 and new results on differential-Hensel-Liouville closures. In fact, this model companion result follows from quantifier elimination, Theorem 6.53, which we also use to study the combinatorial complexity of these structures in Theorem 6.57.

Let  $K$  be a valued field, that is, a field equipped with a subring  $\mathcal{O}$  containing  $f$  or  $f^{-1}$  for all  $f \in K^\times$ , called a *valuation ring*. When  $K$  is additionally a differential field, that is, equipped with an additive map  $\partial: K \rightarrow K$  satisfying the product rule, we always assume that  $\mathcal{O}$  contains (an isomorphic copy of) the field  $\mathbb{Q}$  as a subring. When this holds, we say that  $K$  has *equicharacteristic* 0. Motivated by Hardy fields, we think of  $\mathcal{O}$  as comprising the bounded elements of  $K$ . One goal of valuation theory is to try to understand  $K$  in terms of two simpler associated objects, its value group and residue field. The *value group* of  $K$  is  $K^\times/\mathcal{O}^\times$ , an ordered abelian group; in a Hardy field, this encodes the possible rates of growth. The ring  $\mathcal{O}$  has a unique maximal ideal  $\mathfrak{o} = \mathcal{O} \setminus \mathcal{O}^\times$ , and  $\mathcal{O}/\mathfrak{o}$  is the *residue field* of  $K$ . We refer to elements of  $\mathfrak{o}$  as “infinitesimals” and to elements of  $K \setminus \mathcal{O}$  as “infinite;” in Hardy fields, these are exactly the elements with limit 0 and  $\pm\infty$  respectively.

**1.2.1. Maximal immediate extensions.** We say that a valued field extension of  $K$  is *immediate* if it has the same value group and residue field as  $K$ . By Zorn, any valued field has an immediate valued field extension that is *maximal* in the sense that it has no proper immediate valued field

extension. Kaplansky [Kap42] shows that when  $K$  has equicharacteristic 0, all maximal immediate valued field extensions of  $K$  are isomorphic over  $K$ .

Now let  $K$  be a valued differential field with small derivation. Recall that “small derivation” means that the derivatives of infinitesimals are infinitesimal; if  $K$  has small derivation, then its derivation induces a derivation on the residue field of  $K$ , and the differential field structure of the residue field plays an important role in studying  $K$ . Again by Zorn,  $K$  has an immediate valued differential field extension with small derivation that is *maximal* in the sense that it has no proper valued differential field extension with small derivation. By [ADH18], such extensions are maximal as valued fields, and hence any two such extensions of  $K$  are isomorphic as valued fields over  $K$ . Can this isomorphism be strengthened to an isomorphism of valued differential fields? The answer in general is “no,” as [ADH18, Corollary 8.12] shows, but those authors conjecture that the answer is “yes” in the case that the residue field of  $K$  is *linearly surjective* (that is, all linear differential equations have solutions).

This conjecture is established for monotone fields in [ADH17a, Theorem 7.4.3], with an earlier case due to Scanlon [Sca00]. The next theorem is the first step towards this conjecture outside of the monotone setting, where we instead assume that the value group is small in the following sense. If an ordered abelian group has only finitely many nontrivial convex subgroups, then we call the number of such subgroups its (*archimedean*) *rank*. For brevity, in this theorem and the rest of the introduction, we use “extension” to mean “valued differential field extension with small derivation” and “maximal” to mean having no proper immediate extension of this kind.

**Theorem 3.1.** *If  $K$  has linearly surjective residue field and its value group is the union of its convex subgroups of finite rank, then any two maximal immediate extensions of  $K$  are isomorphic as valued differential fields over  $K$ .*

This and other results from Chapter 3 are joint work with van den Dries and appeared earlier in [DP19]. Our proof involves reducing to the monotone case, using that if  $K$  has small derivation and its value group has rank 1, then  $K$  is monotone [ADH17a, Corollary 6.1.2]. Thus new techniques are needed to remove the rank assumption on the value group. By adapting the differential Newton diagram method of [ADH17a, Chapters 13 and 14] to the setting of asymptotic valued differential fields with small derivation, we prove the following. This and other results from Chapter 4 appear in [Pyn20b], to be published in the Pacific Journal of Mathematics.

**Theorem 4.18.** *If  $K$  is asymptotic and has linearly surjective residue field, then any two maximal immediate extensions of  $K$  are isomorphic as valued differential fields over  $K$ .*

**1.2.2. Differential-henselianity and maximality.** The uniqueness in the above results also holds for differentially algebraic extensions of  $K$  that are *differential-algebraically maximal*, that is, have no proper differentially algebraic immediate extension. If  $K$  is differential-algebraically maximal and has linearly surjective residue field, then it is *differential-henselian* [ADH17a, Theorem 7.0.1]. Henselianity is a central notion in the study of valued fields and their model theory. For instance, it appears in the theorem of Ax–Kochen and Ershov that the theory of a henselian valued field of equicharacteristic 0 is determined by the theories of its ordered value group and its residue field [AK66; Ers65].

Differential-henselianity, which generalizes henselianity to the setting of valued differential fields with small derivation, was introduced by Scanlon in [Sca00; Sca03] for monotone fields and studied more systematically in [ADH17a]. We say that  $K$  is *differential-henselian* if every quasilinear differential polynomial over  $\mathcal{O}$  has a zero in  $\mathcal{O}$ , where a differential polynomial  $P \in \mathcal{O}\{Y\}$  with not all coefficients infinitesimal is *quasilinear* if its image under the natural map to  $(\mathcal{O}/\mathfrak{o})\{Y\}$ , effectively neglecting infinitesimals, has degree 1. Note that if  $K$  is differential-henselian, then its residue field is clearly linearly surjective.

An analogue of [ADH17a, Theorem 7.0.1] holds for valued fields: algebraic maximality implies henselianity, and its converse holds for valued fields of equicharacteristic 0. Does the converse of [ADH17a, Theorem 7.0.1] hold? In [ADH17a, Theorem 7.0.3], a positive answer is obtained for  $K$  that are asymptotic and monotone, and the authors suggest that monotonicity should be unnecessary. They also show that it fails outside of the asymptotic setting (see [ADH17a, example after Corollary 7.4.5]). Using the same techniques as in the proof of Theorem 4.18, we remove the monotonicity assumption altogether. In the next result, and throughout the thesis, we use “d” to abbreviate “differential” or “differentially” as appropriate.

**Theorem 4.19.** *If  $K$  is asymptotic and d-henselian, then it is d-algebraically maximal.*

In joint work with van den Dries, we first established this under the same assumption on the value group as in Theorem 3.1; see Theorem 3.2. Theorem 4.19 is perhaps the central result of this thesis. It underlies the next theorem about differential-henselizations, and these two results are used in essential ways in the model-theoretic study of pre- $H$ -fields with gap 0 carried out in Chapter 6.

**1.2.3. Differential-henselizations.** If the residue field of  $K$  is linearly surjective, then by taking any d-algebraically maximal immediate extension of  $K$ , we obtain an immediate d-henselian extension of  $K$ . When does  $K$  have a smallest such extension? If  $K$  is asymptotic with linearly surjective residue field, then it has an immediate d-henselian extension that is d-algebraic over  $K$ , asymptotic, and minimal in the sense that it has no proper differential subfield containing  $K$  that is d-henselian [ADH17a, Corollary 9.4.11]. The authors conjecture that such minimal d-henselian extensions of  $K$  are unique up to isomorphism over  $K$ . Each valued field has a *henselization*, which is a henselian extension with a universal property. For asymptotic  $K$  with linearly surjective residue field, we defined with van den Dries the notion of a *differential-henselization*: a d-henselian extension of  $K$  that is asymptotic and embeds over  $K$  into every asymptotic d-henselian extension of  $K$ . Thus we establish a strong form of the uniqueness conjecture for minimal d-henselian extensions.

**Theorem 4.20.** *If  $K$  is asymptotic with linearly surjective residue field, then  $K$  has a d-henselization, and any two d-henselizations of  $K$  are isomorphic over  $K$ .*

As before, this was first established in joint work with van den Dries under the same assumption on the value group as in Theorem 3.1; see Theorem 3.4.

**1.2.4. Removing divisibility.** To prove Theorems 4.18, 4.19, and 4.20, we essentially first establish them under the assumption that the value group is divisible and then reduce to that case. In doing this reduction, we noticed that a similar lemma could be applied to remove the assumption that the value group is divisible from three results of [ADH17a, §14.5] paralleling those above. These

results, which first appeared in [Pyn19], make up Chapter 5, and are something of a digression from the main thrust of this thesis.

**1.2.5. Pre- $H$ -fields with gap 0.** Returning to that story, Chapter 6 studies pre- $H$ -fields with gap 0. Introduced in the previous section, recall that a pre- $H$ -field is an ordered valued differential field in which constants (elements with derivative zero) are bounded, infinite positive elements have positive derivative, the valuation ring is convex, and an analogue of l'Hôpital's Rule holds. A pre- $H$ -field  $K$  has gap 0 if it has small derivation and for every infinite  $f \in K$  the logarithmic derivative  $\partial(f)/f$  of  $f$  is infinite. Thus in such structures, infinite elements are transexponential in a certain sense, and the valuation gives us a coarser notion of rate of growth than in Hardy fields.

To obtain an example of a pre- $H$ -field with gap 0, start by taking an  $\aleph_0$ -saturated elementary extension  $\mathbb{T}^*$  of  $\mathbb{T}$ . Here, saturation is a model-theoretic notion of largeness; in particular,  $\mathbb{T}^*$  contains a transexponential element. Then enlarging the valuation ring of  $\mathbb{T}^*$  so that it is the set of elements bounded in absolute value by some finite iterate of the exponential yields a pre- $H$ -field with gap 0 (see [ADH17a, Example 10.1.7]); the transexponential element ensures that the enlarged valuation ring is a proper subring. Another, more concrete, example is given by considering the functional equation  $f(x+1) = e^{f(x)}$ . It has a solution lying in a Hardy field [Bos86], and any solution is clearly transexponential, so performing the same enlargement of the valuation ring of this Hardy field also yields a pre- $H$ -field with gap 0.

We aim to achieve similar model-theoretic results for the theory of pre- $H$ -fields with gap 0 to those obtained for the theory of pre- $H$ -fields or  $H$ -fields with small derivation in [ADH17a]. Namely, does the theory of pre- $H$ -fields with gap 0 have a model companion? Recall that finding a model companion of the theory of pre- $H$ -fields with gap 0 is equivalent to determining which first-order conditions need to be imposed on a pre- $H$ -field with gap 0 to ensure that it is existentially closed. In this way, a model companion for this theory is a universal domain for studying transexponential growth rates in pre- $H$ -fields with small derivation.

In fact, whenever a pre- $H$ -field  $K$  has small derivation and the induced derivation on its residue field is nontrivial,  $K$  must have gap 0. By the convexity of the valuation ring, the ordering of a pre- $H$ -field induces an ordering on its residue field. Thus it is reasonable to expect that the residue fields of existentially closed pre- $H$ -fields with gap 0 are existentially closed as ordered differential fields, and hence as large as possible. This contrasts with the case of  $H$ -fields with small derivation, where the residue field is always canonically isomorphic to the constant field, and thus as small as possible. The theory of ordered differential fields, where no interaction is assumed between the valuation and the derivation, has a model companion. This theory, called the theory of closed ordered differential fields, was introduced by Singer and has quantifier elimination [Sin78]. We also expect that an existentially closed pre- $H$ -field  $K$  with gap 0 should be real closed and have *exponential integration*, that is, for every  $f \in K$  there is  $z \in K^\times$  with  $\partial(z)/z = f$ . Finally, it should also be closed under solutions to some differential equations, such as quasilinear ones, which is ensured by d-henselianity.

It turns out that these properties are enough to axiomatize the model companion of the theory of pre- $H$ -fields with gap 0; in fact, it is the model completion, a stronger property. The language for the next three model-theoretic results is the natural one,  $\{+, -, \cdot, 0, 1, \leq, \preceq, \partial\}$ , where for a

pre- $H$ -field  $K$  the function symbol  $\partial$  is interpreted as the derivation and the binary relation symbol  $\preceq$  is interpreted by  $f \preceq g \iff f \in g\mathcal{O}$  for  $f, g \in K$ . This theorem and the other results of Chapter 6 appear in the preprint [Pyn20a].

**Corollary 6.55.** *The theory of  $d$ -henselian, real closed pre- $H$ -fields that have exponential integration and closed ordered differential residue field is the model completion of the theory of pre- $H$ -fields with gap 0.*

In fact, in the same language we achieve quantifier elimination, which says that every definable set is equivalent to a quantifier-free definable set. To get some sense of what this means, we first give examples from the settings of fields and ordered fields before explaining this more fully in the setting of pre- $H$ -fields. In fields the quantifier-free definable sets are exactly the Zariski-constructible sets, so that the theory of algebraically closed fields has quantifier elimination says that the projections of Zariski-constructible sets are Zariski-constructible. Likewise, in ordered fields the quantifier-free definable sets are exactly the semi-algebraic sets, so that the theory of real closed fields has quantifier elimination says that projections of semi-algebraic sets are semi-algebraic.

In an ordered valued differential field  $K$ , the quantifier-free definable sets are solutions of boolean combinations of relations of the form

$$P(y) = 0, \quad P(y) > 0, \quad \text{and} \quad P(y) \preceq Q(y),$$

where  $P$  and  $Q$  range over  $K\{Y_1, \dots, Y_n\}$  and  $y = (y_1, \dots, y_n) \in K^n$ . As we saw for  $T^{\text{nl}}$  and  $\mathbb{T}$ , sometimes it is necessary to enlarge the class of quantifier-free definable sets by adding certain predicates or function symbols to the language in order to obtain quantifier elimination. This is not necessary for the theory in Corollary 6.55, for which the class of quantifier-free definable sets are closed under projections. To summarize:

**Theorem 6.53.** *The theory of  $d$ -henselian, real closed pre- $H$ -fields that have exponential integration and closed ordered differential residue field has quantifier elimination.*

Quantifier elimination enables us to get some handle on the definable sets in models of this theory. An active direction of model theory is to study the combinatorial complexity of definable families of sets and to classify theories. It turns out that when certain patterns are not present, this implies strong structural properties. One condition that has been well-studied, especially in the last two decades, is “not the independence property,” abbreviated NIP. Many theories have NIP, including algebraically closed (valued) fields, real closed (valued) fields,  $p$ -adic fields, the real exponential field, the theory of closed ordered differential fields, and the theory of  $\mathbb{T}$ . This notion is connected to theoretical computer science, as a theory has NIP if and only if, in every model of the theory, every definable family of sets has finite Vapnik–Chervonenkis dimension. Our model companion from Corollary 6.55 also belongs to this class; in fact, it has a stronger property called distality.

**Theorem 6.57.** *The theory of  $d$ -henselian, real closed pre- $H$ -fields that have exponential integration and closed ordered differential residue field is distal, and hence has NIP.*

**1.2.6. A multi-sorted result.** Let  $K$  be either an  $\aleph_0$ -saturated elementary extension of  $\mathbb{T}$  or a transexponential Hardy field. Then our examples of pre- $H$ -fields with gap 0 were obtained by enlarging the valuation ring  $\mathcal{O}$  of  $K$  to  $\mathcal{O}^*$ . Letting  $\mathfrak{o}^*$  denote the unique maximal ideal of  $\mathcal{O}^*$ , the original valuation given by  $\mathcal{O}$  induces a valuation on the residue field  $\mathcal{O}^*/\mathfrak{o}^*$  of  $(K, \mathcal{O}^*)$ ; equivalently,  $\mathcal{O}/\mathfrak{o}^*$  is a valuation ring of  $\mathcal{O}^*/\mathfrak{o}^*$ . This suggests that we should consider the residue field as a structure in some language expanding that of ordered differential rings (so it remains an ordered differential field), and consider it a part of our structure distinct from  $K$ . The model-theoretic framework for this is that of *multi-sorted structures* (in this case, we only need two-sorted structures).

More precisely: Let  $K$  be a pre- $H$ -field with gap 0,  $\mathbf{k}$  be an expansion of an ordered differential field, and  $\pi: \mathcal{O} \rightarrow \mathbf{k}$  be a map inducing an isomorphism of ordered differential fields between the residue field  $\mathcal{O}/\mathfrak{o}$  of  $K$  and  $\mathbf{k}$  (and extend  $\pi$  to  $K$  by  $\pi(K \setminus \mathcal{O}) = \{0\}$ ). We consider the two-sorted structure  $(K, \mathbf{k}; \pi)$  where the language on the sort of  $K$  is  $\{+, -, \cdot, 0, 1, \leq, \preceq, \partial\}$  and the language  $\mathcal{L}_{\text{res}}$  on the sort of  $\mathbf{k}$  expands  $\{+, -, \cdot, 0, 1, \leq, \partial\}$ .

**Theorem 6.58.** *If  $K$  is a d-henselian, real closed pre- $H$ -field with exponential integration, and the  $\mathcal{L}_{\text{res}}$ -theory of  $\mathbf{k}$  is model complete, then the theory of  $(K, \mathbf{k}; \pi)$  is model complete.*

If the  $\mathcal{L}_{\text{res}}$ -theory of  $\mathbf{k}$  is actually the model companion of an  $\mathcal{L}_{\text{res}}$ -theory of ordered differential fields, then the theory of  $(K, \mathbf{k}; \pi)$  with  $K$  as in Theorem 6.58 is in fact the model companion of the expected two-sorted theory; this is Corollary 6.60.

## CHAPTER 2

### Preliminaries

#### 2.1. Conventions

We let  $d, m, n$ , and  $r$  range over  $\mathbb{N} = \{0, 1, 2, \dots\}$  and  $\rho, \lambda$ , and  $\mu$  be ordinals.

#### 2.2. Valued differential fields

The main objects of this thesis are valued differential fields. For any field  $K$ , we set  $K^\times := K \setminus \{0\}$ .

##### 2.2.1. Valued fields.

**Definition.** A *valued field* is a field  $K$  equipped with a surjective map  $v: K^\times \rightarrow \Gamma$ , where  $\Gamma$  is an ordered abelian group, satisfying for  $f, g \in K^\times$ :

- (V1)  $v(fg) = v(f) + v(g)$ ;
- (V2)  $v(f + g) \geq \min\{v(f), v(g)\}$  whenever  $f + g \neq 0$ .

By an *ordered abelian group* (written additively), we mean that its (total) ordering is preserved by addition. Let  $K$  be a valued field. We add a new symbol  $\infty$  to  $\Gamma$  and extend the addition and ordering to  $\Gamma_\infty := \Gamma \cup \{\infty\}$  by  $\infty + \gamma = \gamma + \infty = \infty + \infty = \infty$  and  $\infty > \gamma$  for all  $\gamma \in \Gamma$ . This allows us to extend  $v$  to  $K$  by setting  $v(0) := \infty$ . We often use the following more intuitive notation:

$$\begin{aligned} f \preceq g &: \Leftrightarrow v(f) \geq v(g), & f \prec g &: \Leftrightarrow v(f) > v(g), \\ f \asymp g &: \Leftrightarrow v(f) = v(g), & f \sim g &: \Leftrightarrow f - g \prec g. \end{aligned}$$

The relation  $\preceq$  is called a *dominance relation*. Both  $\asymp$  and  $\sim$  are equivalence relations on  $K$  and  $K^\times$  respectively, with a consequence of (V2) being that if  $f \sim g$ , then  $f \asymp g$ . We set  $\mathcal{O} := \{f \in K : f \preceq 1\}$  and call it the *valuation ring* of  $K$ . It has a unique maximal ideal  $\mathfrak{o} := \{f \in K : f \prec 1\}$ , and we call  $\text{res}(K) := \mathcal{O}/\mathfrak{o}$  the *residue field* of  $K$ , usually denoting it  $\mathbf{k}$ . We say that  $K$  has *equicharacteristic* 0 if  $\mathbf{k}$  has characteristic 0; equivalently,  $\mathcal{O}$  contains (an isomorphic copy of) the field  $\mathbb{Q}$  as a subring. We let  $\bar{a}$  or  $\text{res}(a)$  denote the image of  $a \in \mathcal{O}$  under the map to  $\mathbf{k}$ . For another valued field  $L$ , we denote these objects by  $\mathcal{O}_L, \Gamma_L, \mathbf{k}_L$ , etc.

Let  $G$  be an ordered abelian group. We set  $G^> := \{g \in G : g > 0\}$  and likewise  $G^<$ , as well as  $G^\neq := G \setminus \{0\}$ . For  $g \in G$  we let  $[g]$  denote its archimedean class. That is,

$$[g] := \{h \in G : |h| \leq n|g| \text{ and } |g| \leq n|h| \text{ for some } n\}.$$

We order the set  $[G] := \{[g] : g \in G\}$  by  $[h] < [g]$  if  $n|h| < |g|$  for all  $n$ . Then the map  $g \mapsto [g]$  is a convex valuation on  $G$  (technically, we must give the set of archimedean classes its reverse order) in the sense that, for  $g, h \in G^\neq$ , we have  $[-g] = [g]$ ,  $[g + h] \leq \max\{[g], [h]\}$ , and  $[g] \leq [h]$  whenever  $0 < g \leq h$  (see [ADH17a, §2.2] for more on this). In particular, if  $[g] < [h]$ , then  $[g + h] = [h]$ . For  $g, h \in G$  we write  $g = o(h)$  if  $[g] < [h]$ .



### 2.2.2. Differential fields.

**Definition.** A *differential field* is a field  $K$  equipped with a *derivation*  $\partial: K \rightarrow K$ , which satisfies for  $f, g \in K$ :

$$(D1) \quad \partial(f + g) = \partial(f) + \partial(g);$$

$$(D2) \quad \partial(fg) = f\partial(g) + g\partial(f).$$

Let  $K$  be a differential field. For  $f \in K$ , we often write  $f'$  for  $\partial(f)$  if the derivation is clear from the context and set  $f^\dagger := f'/f$  if  $f \neq 0$ . We say that  $K$  has *exponential integration* if  $(K^\times)^\dagger = K$ . The *field of constants* of  $K$  is  $C := \{f \in K : f' = 0\}$ . For another differential field  $L$ , we denote this object by  $C_L$ . We let  $K\{Y\} := K[Y, Y', Y'', \dots]$  be the ring of differential polynomials over  $K$  and set  $K\{Y\}^\neq := K\{Y\} \setminus \{0\}$ , extending the derivation of  $K$  to  $K\{Y\}$  in the natural way. Let  $P$  range over  $K\{Y\}^\neq$ . The *order* of  $P$  is the smallest  $r$  such that  $P \in K[Y, Y', \dots, Y^{(r)}]$ , and its *degree* is its total degree. If  $r$  is the order of  $P$ ,  $m$  its degree in  $Y^{(r)}$ , and  $n$  its total degree, then the *complexity* of  $P$  is the triple  $c(P) := (r, m, n)$ ; complexities are ordered lexicographically.

For  $\mathbf{i} = (i_0, \dots, i_r) \in \mathbb{N}^{1+r}$ , we set  $Y^{\mathbf{i}} := Y^{i_0}(Y')^{i_1} \dots (Y^{(r)})^{i_r}$ . If  $P$  has order at most  $r$ , then we decompose  $P$  as  $\sum_{\mathbf{i}} P_{\mathbf{i}} Y^{\mathbf{i}}$ , where  $\mathbf{i}$  ranges over  $\mathbb{N}^{1+r}$ . We also sometimes decompose  $P$  into its homogeneous parts, so let  $P_d$  denote the homogeneous part of  $P$  of degree  $d$  and set  $P_{\leq d} := \sum_{i \leq d} P_i$  and  $P_{> d} := \sum_{i > d} P_i$ . Letting  $|\mathbf{i}| := i_0 + \dots + i_r$ , we note that  $P_d = \sum_{|\mathbf{i}|=d} P_{\mathbf{i}} Y^{\mathbf{i}}$ , where  $P_d$  denotes the homogeneous part of  $P$  of degree  $d$ . The *multiplicity of  $P$  at 0*, denoted by  $\text{mul } P$ , is the least  $d$  with  $P_d \neq 0$ .

We often use, for  $a \in K$ , the additive and multiplicative conjugates of  $P$  by  $a$  defined by  $P_{+a}(Y) := P(a + Y)$  and  $P_{\times a}(Y) := P(aY)$ . For convenience, we also write  $P_{-a}$  for  $P_{+(-a)}$ . Note that  $(P_{+a})_{+b} = (P_{+b})_{+a} = P_{+(a+b)}$  for  $b \in K$ , which we write  $P_{+a+b}$ . We define  $P_{+a-b}$  likewise. The *multiplicity of  $P$  at  $a$*  is  $\text{mul } P_{+a}$ . Note that  $(P_d)_{\times a} = (P_{\times a})_d$ , which we denote by  $P_{d, \times a}$ . For more on such conjugation, see [ADH17a, §4.3].

If  $a$  in some differential field extension of  $K$  is *differentially algebraic* (d-algebraic for short) over  $K$ , by which we mean it satisfies  $P(a) = 0$  for some  $P \in K\{Y\}^\neq$ , then it has a *minimal annihilator*  $P$  over  $K$ . That is,  $P \in K\{Y\}^\neq$  is irreducible of order  $r$  such that  $P(a) = 0$  and  $Q(a) \neq 0$  for all  $Q \in K\{Y\}^\neq$  of order at most  $r$  and  $\deg_{Y^{(r)}} Q < \deg_{Y^{(r)}} P$ . Then  $K\langle a \rangle$ , the differential field generated by  $a$  over  $K$ , is isomorphic to the fraction field of  $K[Y_0, \dots, Y_r]/(p)$ , where  $p \in K[Y_0, \dots, Y_r]$  with  $P = p(Y, Y', \dots, Y^{(r)})$ , equipped with the derivation given by [ADH17a, Lemma 1.9.1]. If  $a$  is *differentially transcendental* (d-transcendental for short) over  $K$ , that is, not d-algebraic over  $K$ , then  $K\langle a \rangle$  is isomorphic to the fraction field  $K\langle Y \rangle$  of  $K\{Y\}$ . See [ADH17a, §4.1] for more details.

**2.2.3. Small derivation and differential-henselianity.** Now suppose that  $K$  is a *valued differential field*, by which we mean that  $K$  is both a valued field of equicharacteristic 0 and a differential field. The central condition relating the valuation and the derivation in this thesis is small derivation, a strong form of continuity.

**Definition.** We say that  $K$  has *small derivation* if  $\partial\mathcal{O} \subseteq \mathcal{O}$ .

As a valued field,  $K$  can be equipped with its *valuation topology*, which has as a basis the open balls  $\{f \in K : v(f - a) > \gamma\}$  for  $a \in K$  and  $\gamma \in \Gamma$ . When is  $\partial$  continuous with respect to this topology? Here is a characterization showing that it holds whenever  $K$  has small derivation.

**Lemma 2.1** ([ADH17a, 4.4.7]). *The derivation  $\partial$  is continuous with respect to the valuation topology on  $K$  if and only if there is  $a \in K^\times$  such that  $\partial\mathcal{O} \subseteq a\mathcal{O}$ .*

Most of the valued differential fields in this thesis have small derivation. We also sometimes discuss, particularly in Chapter 5, asymptotic fields that may not have small derivation (see §2.4 for the definition of asymptotic). Asymptotic fields also have continuous derivation, and hence all the valued differential fields in this thesis have this property. A study of valued differential fields with continuous derivation was initiated in [ADH18].

Suppose in the rest of this section that  $K$  has small derivation. Then  $\partial\mathcal{O} \subseteq \mathcal{O}$  [ADH17a, Lemma 4.4.2], so  $\partial$  induces a derivation on  $\mathbf{k}$ . We always construe  $\mathbf{k}$  as a differential field with this induced derivation and are typically interested in the case that it is nontrivial, in contrast to the  $H$ -fields with small derivation studied in [ADH17a]. Moreover, we are interested in the case that  $\mathbf{k}$  has solutions to certain differential equations. We say that  $\mathbf{k}$  is  *$r$ -linearly surjective* if, for all  $a_0, \dots, a_r \in \mathbf{k}$  with  $a_n \neq 0$  for some  $n \leq r$ , the equation  $1 + a_0y + a_1y' + \dots + a_ry^{(r)} = 0$  has a solution in  $\mathbf{k}$ . We say that  $\mathbf{k}$  is *linearly surjective* if it is  $r$ -linearly surjective for all  $r$ .

Let  $P \in K\{Y\}$ . We extend  $v$  to  $K\langle Y \rangle$ , the fraction field of  $K\{Y\}$ , by setting  $v(P)$  to be the minimum valuation of the coefficients of  $P$ ; this is called the *Gaussian valuation* on  $K\langle Y \rangle$ . The relations  $\preceq$ ,  $\prec$ ,  $\succ$ , and  $\sim$  are extended to  $K\langle Y \rangle$  in the corresponding way, and the image of  $P \in \mathcal{O}\{Y\}$  under the canonical map to  $\mathbf{k}\{Y\}$  is denoted by  $\overline{P}$ .

**Definition.** We say that  $K$  is  *$r$ -differential-henselian* ( *$r$ -d-henselian* for short) if:

- (rDH1)  $\mathbf{k}$  is  $r$ -linearly surjective;
- (rDH2) whenever  $P \in \mathcal{O}\{Y\}$  of order at most  $r$  satisfies  $P_0 \prec 1$  and  $P_1 \succ 1$ , there is  $y \in \mathcal{O}$  with  $P(y) = 0$ .

We say that  $K$  is *differential-henselian* (*d-henselian* for short) if it is  $r$ -d-henselian for every  $r$ .

Differential-henselianity was introduced by Scanlon in [Sca00; Sca03] and studied more systematically in [ADH17a]. We give a useful equivalent formulation, for which we first describe how  $v(P)$  changes as we additively and multiplicatively conjugate  $P$ .

**Lemma 2.2** ([ADH17a, 4.5.1]). *Let  $P \in K\{Y\}^\neq$  and  $f \in K$ .*

- (i) *If  $f \preceq 1$ , then  $P_{+f} \succcurlyeq P$ ; if  $f \prec 1$ , then  $P_{+f} \sim P$ .*
- (ii) *If  $f \neq 0$ , then  $v(P_{\times f}) \in \Gamma$  depends only on  $vf \in \Gamma$ .*

Item (ii) allows us to define a function  $v_P: \Gamma \rightarrow \Gamma$  by  $vf \mapsto v(P_{\times f})$ . The main properties of this function are recorded in the following lemma.

**Lemma 2.3** ([ADH17a, 6.1.3, 6.1.5]). *Let  $P, Q \in K\{Y\}^\neq$  be homogeneous of degrees  $m$  and  $n$ , respectively. For  $\alpha, \beta \in \Gamma$  with  $\alpha \neq \beta$ , we have*

$$v_P(\alpha) - v_P(\beta) = m(\alpha - \beta) + o(\alpha - \beta).$$

It follows that, for  $\gamma \in \Gamma^\neq$ ,

$$v_P(\gamma) - v_Q(\gamma) = v(P) - v(Q) + (m - n)\gamma + o(\gamma),$$

and if  $m > n$ , then  $v_P - v_Q$  is strictly increasing.

The most consequential result about this function, and a key result of [ADH17a], is the following Equalizer Theorem, which underlies the results of Chapter 4.

**Theorem 2.4** ([ADH17a, 6.0.1]). *Let  $P, Q \in K\{Y\}^\neq$  be homogeneous of degrees  $m$  and  $n$ , respectively, with  $m > n$ . If  $(m - n)\Gamma = \Gamma$ , then there exists a unique  $\alpha \in \Gamma$  such that  $v_P(\alpha) = v_Q(\alpha)$ .*

In particular, it says that if  $P \in K\{Y\}$  is homogeneous of degree 1, then  $v_P$  is bijective. Using this theorem, we obtain an equivalent characterization of  $r$ -d-henselianity.

**Lemma 2.5** ([ADH17a, 7.1.1, 7.2.1]). *We have that  $K$  is  $r$ -d-henselian if and only if, for every  $P \in \mathcal{O}\{Y\}$  of order at most  $r$  satisfying  $P_1 \asymp 1$  and  $P_n \prec 1$  for all  $n \geq 2$ , there is  $y \in \mathcal{O}$  with  $P(y) = 0$ .*

### 2.3. Immediate extensions and pseudocauchy sequences

In fact, d-henselianity is closely connected to the notion of differential-algebraic maximality, for which we need to discuss immediate extensions. Let  $K$  be a valued field. Given an extension  $L$  of  $K$ , we identify  $\Gamma$  with a subgroup of  $\Gamma_L$  and  $\mathbf{k}$  with a subfield of  $\mathbf{k}_L$  in the obvious way. Here and throughout we use the word *extension* as follows: if  $F$  is a valued field, “extension” means “valued field extension;” if  $F$  is a valued differential field, “extension” means “valued differential field extension;” if  $F$  is an ordered valued differential field, “extension” means “ordered valued differential field extension;” etc., unless otherwise specified. We hope this will not cause confusion; where there is particular danger, we are more explicit. The words “embedding” and “isomorphism” are used similarly.

We say that an extension  $L$  of  $K$  is *immediate* if  $\Gamma_L = \Gamma$  and  $\mathbf{k}_L = \mathbf{k}$ . If  $K$  is a valued differential field with small derivation and  $L$  is a valued differential field extension of  $K$  with small derivation, then  $\mathbf{k}$  is naturally a differential subfield of  $\mathbf{k}_L$ . By Zorn (and [ADH17a, Lemma 2.2.1]),  $K$  has an immediate extension that is *maximal* in the sense that it has no proper immediate extension. Kaplansky [Kap42] shows that when  $K$  has equicharacteristic 0 (and in a more general setting not relevant here), any two maximal immediate extensions of  $K$  are isomorphic over  $K$ . Such extensions are complete in a certain way, called *spherically complete*, that we now describe. For a more detailed exposition of this material, including proofs of facts stated here, see [ADH17a, §2.2 and 3.2]. Let  $(a_\rho)$  be a sequence in  $K$  indexed by a limit ordinal.

**Definition.** We say that  $(a_\rho)$  is a *pseudocauchy sequence* (*pc-sequence* for short) if, for sufficiently large  $\rho$  and for all  $\mu > \lambda > \rho$ , we have

$$a_\mu - a_\lambda \prec a_\lambda - a_\rho.$$

We say that  $a \in K$  is a *pseudolimit* of  $(a_\rho)$  if  $v(a - a_\rho)$  is eventually strictly increasing and write  $a_\rho \rightsquigarrow a$ .

If  $a \in K$  is a pseudolimit of  $(a_\rho)$ , then either  $a \prec a_\rho$ , eventually, or  $a \sim a_\rho$ , eventually. If  $(a_\rho)$  has a pseudolimit in some extension of  $K$ , then  $(a_\rho)$  is a pc-sequence. Conversely, if  $(a_\rho)$  is a pc-sequence, then it has a pseudolimit in some extension of  $K$ ; in fact, this extension can be taken to be elementary in any language expanding that of valued fields. For instance, if  $K$  is a valued differential field with small derivation and  $(a_\rho)$  is a pc-sequence, then  $(a_\rho)$  has a pseudolimit in a valued differential field extension of  $K$  with small derivation. A pc-sequence in  $K$  with no pseudolimit in  $K$  is called *divergent in  $K$* .

**Lemma 2.6** ([ADH17a, 2.2.18, 2.2.19]). *Let  $a$  lie in some extension of  $K$ .*

- (i) *The set  $v(a - K)$  has no greatest element if and only if  $a$  is a pseudolimit of some divergent pc-sequence in  $K$ .*
- (ii) *If  $v(a - K)$  has no greatest element, then  $v(a - K) \subseteq \Gamma$ .*
- (iii) *If  $a$  lies in an immediate extension of  $K$ , then  $v(a - K)$  has no greatest element.*
- (iv) *If  $a_\rho \rightsquigarrow a$  with  $(a_\rho)$  divergent in  $K$ , then  $v(a - a_\rho)$  is cofinal in  $v(a - K)$ .*

It follows that if  $L$  is an immediate extension of  $K$ , then every element of  $L \setminus K$  is the pseudolimit of a divergent pc-sequence in  $K$ . Conversely, every divergent pc-sequence in  $K$  has a pseudolimit in an immediate extension of  $K$  (see [ADH17a, Corollary 3.2.8]). We call  $K$  *spherically complete* if every pc-sequence in  $K$  has a pseudolimit in  $K$ . Hence:

**Theorem 2.7.** *The valued field  $K$  is maximal if and only if it is spherically complete.*

Let  $(a_\rho)$  be a pc-sequence in  $K$ . The elements  $\gamma_\rho := v(a_{\rho+1} - a_\rho) \in \Gamma_\infty$ , eventually in  $\Gamma$ , play an important role in working with  $(a_\rho)$ . For sufficiently large  $\rho$  and any  $\rho' > \rho$ , we have  $v(a_{\rho'} - a_\rho) = \gamma_\rho$ . We have that  $(\gamma_\rho)$  is eventually strictly increasing, and  $a_\rho \rightsquigarrow a$  if and only if  $v(a - a_\rho) = \gamma_\rho$ , eventually. The *width* of  $(a_\rho)$  is the set  $\{\gamma \in \Gamma_\infty : \gamma > \gamma_\rho, \text{ eventually}\}$ . If  $a, b \in K$  and  $a_\rho \rightsquigarrow a$ , then  $a_\rho \rightsquigarrow b$  if and only if  $v(a - b)$  is in the width of  $(a_\rho)$ . In the setting of valued differential fields, unlike in valued fields, we frequently need to pass to equivalent pc-sequences:

**Definition.** Let  $(a_\rho)$  and  $(b_\lambda)$  be pc-sequences in  $K$ . We say that  $(b_\lambda)$  is *equivalent* to  $(a_\rho)$  if there are arbitrarily large  $\rho$  and  $\lambda$  such that for all  $\rho' > \rho$  and  $\lambda' > \lambda$ :

$$a_{\rho'} - b_{\lambda'} \prec b_{\lambda'} - b_\lambda \quad \text{and} \quad a_{\rho'} - b_{\lambda'} \prec a_{\rho'} - a_\rho.$$

For example, if  $(a_\rho)$  is a pc-sequence, then every cofinal subsequence of  $(a_\rho)$  is a pc-sequence equivalent to  $(a_\rho)$ . By a *cofinal subsequence* of  $(a_\rho)$ , we mean a subsequence of  $(a_\rho)$  indexed by a cofinal subset of the index set of  $(a_\rho)$ , which we identify with a limit ordinal in the usual way. If the reader prefers, they may always index pc-sequences by regular cardinals.

**Lemma 2.8** ([ADH17a, 2.2.17]). *Let  $(a_\rho)$  and  $(b_\lambda)$  be pc-sequences in  $K$ . Then the following are equivalent.*

- (i)  *$(a_\rho)$  and  $(b_\lambda)$  are equivalent;*
- (ii)  *$(a_\rho)$  and  $(b_\lambda)$  have the same pseudolimits in every extension of  $K$ ;*
- (iii)  *$(a_\rho)$  and  $(b_\lambda)$  have the same width and a common pseudolimit in some extension of  $K$ .*

As before, if  $K$  has extra first-order structure, then we can replace “extension of  $K$ ” with “elementary extension of  $K$ .” For example, if  $K$  is a valued differential field with small derivation and  $(a_\rho)$  and  $(b_\lambda)$  are equivalent pc-sequences in  $K$ , then they have a common pseudolimit in some valued differential field extension of  $K$  with small derivation.

**2.3.1. Valued differential fields with small derivation.** Suppose that  $K$  is a valued differential field with small derivation. Then we call  $K$  *maximal* if it has no proper immediate (valued differential field) extension with small derivation.

**Theorem 2.9** ([ADH18]). *The valued differential field  $K$  with small derivation is maximal if and only if it is spherically complete.*

Hence  $K$  is maximal as a valued differential field with small derivation if and only if it is maximal as a valued field, and thus any two maximal immediate valued differential field extensions of  $K$  with small derivation are isomorphic over  $K$  as valued fields. Whether this can be strengthened to an isomorphism of valued differential fields is the subject of Chapters 3 and 4.

We call  $K$  *differential-algebraically maximal* (*d-algebraically maximal* for short) if it has no proper differentially algebraic (“d-algebraic” for short) immediate extension with small derivation.

**Theorem 2.10** ([ADH17a, 7.0.1]). *If  $K$  is d-algebraically maximal and  $\mathbf{k}$  is linearly surjective, then  $K$  is d-henselian.*

They also establish a partial converse in [ADH17a, Theorem 7.0.3], and generalizing that is the focus of Chapter 4. A special case of Theorem 2.10 is in [Sca00].

Suppose that the derivation induced on  $\mathbf{k}$  is nontrivial. We now recall three lemmas about evaluating differential polynomials along pc-sequences and constructing immediate extensions. A pc-sequence  $(a_\rho)$  in  $K$  is of *differential-algebraic type over  $K$*  (*d-algebraic type over  $K$*  for short) if there is an equivalent pc-sequence  $(b_\lambda)$  in  $K$  and a  $P \in K\{Y\}$  such that  $P(b_\lambda) \rightsquigarrow 0$ . We call such a  $P$  of minimal complexity a *minimal differential polynomial of  $(a_\rho)$  over  $K$* . If no such  $(b_\lambda)$  and  $P$  exist, then we say that  $(a_\rho)$  is of *differential-transcendental type over  $K$*  (*d-transcendental type over  $K$*  for short). Combining [ADH17a, Lemmas 6.8.1 and 6.8.3], and the information from their proofs:

**Lemma 2.11** ([ADH17a, 6.8.1, 6.8.3]). *Let  $(a_\rho)$  be a pc-sequence in  $K$  with pseudolimit  $a \in L$ , where  $L$  is an extension of  $K$  with small derivation, and let  $G \in L\{Y\} \setminus L$ . Then we have an equivalent pc-sequence  $(b_\rho)$  in  $K$  such that  $(G(b_\rho))$  is a pc-sequence with  $G(b_\rho) \rightsquigarrow G(a)$ . Moreover, for this  $(b_\rho)$ , setting  $P(Y) := G(a + Y) - G(a) \in L\{Y\}$  we have:*

- (i)  $v(b_\rho - a) = \gamma_\rho$  and  $v(P(b_\rho - a)) = v_P(\gamma_\rho)$ , eventually;
- (ii) for  $n = 1, \dots, \deg P$ ,  $v(P_n(b_\rho - a)) = v_{P_n}(\gamma_\rho)$ , eventually, whenever  $P_n \neq 0$ ;
- (iii) if  $(G(a_\rho))$  is a pc-sequence with  $G(a_\rho) \rightsquigarrow 0$ , then  $G(b_\rho) \rightsquigarrow 0$ .

**Lemma 2.12** ([ADH17a, 6.9.1]). *Let  $(a_\rho)$  be a pc-sequence in  $K$  of d-transcendental type over  $K$ . Then  $K$  has an immediate extension  $K\langle a \rangle$  with small derivation such that:*

- (i)  $a$  is d-transcendental over  $K$ ;
- (ii)  $a_\rho \rightsquigarrow a$ ;

- (iii) for any extension  $L$  of  $K$  with small derivation and any  $b \in L$  with  $a_\rho \rightsquigarrow b$ , there is a unique embedding  $K\langle a \rangle \rightarrow L$  over  $K$  sending  $a$  to  $b$ .

**Lemma 2.13** ([ADH17a, 6.9.3]). *Let  $(a_\rho)$  be a pc-sequence in  $K$  with minimal differential polynomial  $P$  over  $K$ . Then  $K$  has an immediate extension  $K\langle a \rangle$  with small derivation such that:*

- (i)  $P(a) = 0$ ;
- (ii)  $a_\rho \rightsquigarrow a$ ;
- (iii) for any extension  $L$  of  $K$  with small derivation and any  $b \in L$  with  $a_\rho \rightsquigarrow b$  and  $P(b) = 0$ , there is a unique embedding  $K\langle a \rangle \rightarrow L$  over  $K$  sending  $a$  to  $b$ .

Thus (when the derivation on  $\mathbf{k}$  is nontrivial)  $K$  is d-algebraically maximal if and only if there is no divergent pc-sequence in  $K$  of d-algebraic type over  $K$ .

## 2.4. Asymptotic valued differential fields

The other condition we impose relating valuations and derivations throughout much of the thesis is asymptoticity. We say that  $K$  is *asymptotic* if  $f \prec g \iff f' \prec g'$  for all nonzero  $f, g \in \mathcal{O}$ . The broad class of asymptotic fields was studied in [ADH17a], generalizing the differential-valued fields introduced in [Ros80]. If  $K$  is asymptotic, then  $\partial$  is continuous with respect to the valuation topology on  $K$  [ADH17a, Corollary 9.1.5]. It follows immediately from the definition that when  $K$  is asymptotic, it has *few constants* in the sense that  $C \subseteq \mathcal{O}$ . Conversely, if  $K$  is 1-d-henselian and has few constants, then it is asymptotic by [ADH17a, Lemmas 9.1.1 and 7.1.8].

Suppose in the rest of this section that  $K$  is asymptotic. Then for  $g \in K^\times$  with  $g \not\prec 1$ ,  $v(g^\dagger)$  and  $v(g')$  depend only on  $vg$  and not on  $g$ , so for  $\gamma := vg$  we set  $\gamma^\dagger := v(g^\dagger)$  and  $\gamma' := v(g')$ ; note that  $\gamma^\dagger = \gamma' - \gamma$ . Thus logarithmic differentiation induces a map

$$\begin{aligned} \psi: \Gamma^\neq &\rightarrow \Gamma \\ \gamma &\mapsto \gamma^\dagger, \end{aligned}$$

which we extend to  $\psi: \Gamma_\infty \rightarrow \Gamma_\infty$  by setting  $\psi(0) = \psi(\infty) := \infty$ . The map  $\psi$  is a valuation on  $\Gamma$  in the sense of [ADH17a, §2.2], and we set  $\Psi := \psi(\Gamma^\neq)$ . Then we have  $\Psi < (\Gamma^\neq)'$ .

We call  $(\Gamma, \psi)$  the *asymptotic couple* of  $K$ . Conversely, if  $L$  is a valued differential field and logarithmic differentiation induces such a map on its value group satisfying certain axioms specified in §6.4, then  $L$  is asymptotic [ADH17a, Proposition 9.1.3]. Hence  $(\Gamma, \psi)$  encodes various properties of  $K$ . Asymptotic couples were first identified as playing an important role in the study of *differential-valued* fields, which are the asymptotic fields  $K$  with  $\mathcal{O} = C + \mathcal{O}$ , in [Ros80]; for a more thorough exposition of asymptotic couples (as structures in their own right) than given here, consult [ADH17a, §6.5 and 9.2]. We say that  $K$  is *H-asymptotic* or *of H-type* if for  $f, g \in K^\times$ , if  $f \preccurlyeq g \prec 1$  then  $f^\dagger \succcurlyeq g^\dagger$ ; equivalently,  $\psi$  is *convex* with respect to the ordering of  $\Gamma$  in the sense that  $\psi(\gamma) \geq \psi(\delta)$  whenever  $0 < \gamma < \delta$ . Other properties of  $K$  are also determined by its asymptotic couple. For example,  $K$  has small derivation if and only if  $(\Gamma^\neq)' \subseteq \Gamma^\neq$ .

In this thesis, we are interested in the case that  $\sup \Psi = 0$ , because of the following lemma.

**Lemma 2.14** ([ADH17a, 9.4.2]). *If  $K$  has small derivation and the derivation induced on  $\mathbf{k}$  is nontrivial, then  $\sup \Psi = 0$ .*

Conversely, if  $\sup \Psi = 0$ , then  $K$  has small derivation, since  $(\Gamma^>)' \subseteq \Gamma^>$  if and only if there is no  $\gamma < 0$  with  $\Psi \leq \gamma$  [ADH17a, Corollary 9.2.9]. In fact, more is true. By [ADH17a, Theorem 9.2.1],  $\Gamma \setminus (\Gamma^\neq)'$  has at most one element, and if  $\Psi$  has a maximum, then  $\Gamma \setminus (\Gamma^\neq)' = \{\max \Psi\}$ . If  $\sup \Psi = 0$ , then either  $0 \in \Psi$ , so  $\max \Psi = 0$ , or  $0 \notin \Psi$ . It follows that  $\sup \Psi = 0 \iff (\Gamma^>)' = \Gamma^>$ . By [ADH17a, Corollary 9.2.9] again,  $\sup \Psi = 0 \notin \Psi \iff \Psi < 0 < (\Gamma^>)',$  in which case we say that  $K$  has *gap* 0. Equivalently,  $K$  has gap 0 if it has small derivation and  $f^\dagger \succ 1$  for all  $f \in K$  with  $f \succ 1$ . We are particularly interested in the case that  $K$  has gap 0 in Chapter 6, as this condition is satisfied by all pre- $H$ -fields  $K$  with small derivation such that the derivation induced on  $\mathbf{k}$  is nontrivial.

**Lemma 2.15** ([ADH17a, 9.4.5]). *If  $\sup \Psi = 0$  and  $L$  is an immediate extension of  $K$  with small derivation, then  $L$  is asymptotic.*

## 2.5. Algebraic extensions

In this section, we suspend our convention on the word “extension,” and now use it to mean “field extension.” We discuss how to extend valuations and derivations to algebraic extensions of  $K$ , and which properties are preserved in the process; we discuss the case that  $K$  is an ordered valued differential field only when it becomes relevant in Chapter 6. First, the derivation of  $K$  extends uniquely to any algebraic extension of  $K$  by [ADH17a, Lemma 1.9.2], and we thus construe any algebraic extension of  $K$  as a differential field extension of  $K$  with this derivation.

Here is a typical example of how we use this without comment. Let  $F$  be a valued field. We call  $F$  *henselian* if for every  $P \in \mathcal{O}_F[X]$  with  $P_0 \prec 1$  and  $P_1 \succ 1$ , there is  $a \in \mathcal{O}_F$  with  $P(a) = 0$ . Hence if  $K$  has small derivation, then it is 0-d-henselian if and only if it is henselian as a valued field. Then  $F$  has a *henselization*, which is a henselian valued field extension  $F^h$  of  $F$  such that any valued field embedding of  $F$  into a henselian valued field  $L$  extends uniquely to a valued field embedding of  $F^h$  into  $L$  (see [ADH17a, Proposition 3.3.25]). Henselizations are algebraic extensions, so we always construe  $K^h$  as a valued *differential* field extension of  $K$  and any valued field embedding of  $K^h$  into a valued *differential* field extension of  $K$  as a valued *differential* field embedding.

We equip the algebraic closure  $K^{\text{ac}}$  of  $K$  with any valuation extending that of  $K$ , which determines  $K^{\text{ac}}$  as a valued differential field extension of  $K$  up to isomorphism over  $K$ , with value group the divisible hull  $\mathbb{Q}\Gamma$  of  $\Gamma$  and residue field the algebraic closure  $\mathbf{k}^{\text{ac}}$  of  $\mathbf{k}$  (see [ADH17a, Proposition 3.1.21 and Corollary 3.1.18]). If  $K$  has small derivation, then so does  $K^{\text{ac}}$ .

**Proposition 2.16** ([ADH17a, 6.2.1]). *If  $K$  has small derivation and  $L$  is an algebraic valued differential field extension of  $K$ , then  $L$  has small derivation.*

If  $K$  is asymptotic, then so is  $K^{\text{ac}}$ :

**Proposition 2.17** ([ADH17a, 9.5.3]). *If  $K$  is asymptotic and  $L$  is an algebraic valued differential field extension of  $K$ , then  $L$  is asymptotic.*

Let  $K$  be asymptotic. We denote the extension of  $\psi$  to  $\mathbb{Q}\Gamma$  also by  $\psi$ , which satisfies  $\psi(q\gamma) = \psi(\gamma)$  for  $\gamma \in \Gamma^\neq$  and  $q \in \mathbb{Q}^\times$ , so  $\psi(\mathbb{Q}\Gamma^\neq) = \psi(\Gamma^\neq)$ . Hence if  $K$  has gap 0, then so does  $K^{\text{ac}}$ .

## 2.6. Standing assumptions

Throughout this thesis, we let  $K$  be a valued differential field (by definition of equicharacteristic 0) with nontrivial valuation  $v: K^\times \rightarrow \Gamma$  and nontrivial derivation  $\partial: K \rightarrow K$ , with the additional notation specified in this chapter. That  $v$  is nontrivial means that  $\Gamma \neq \{0\}$ , or, equivalently, that  $\mathcal{O} \neq K$ . That  $\partial$  is nontrivial means that  $\partial(K) \neq \{0\}$ .



## CHAPTER 3

# Differential-henselianity and maximality, part I

### 3.1. Introduction

This chapter and the next, like [ADH17a, Chapters 6 and 7] and [ADH18], are meant to contribute to an emerging theory of valued *differential* fields, in analogy with the theory of valued fields. Namely, we make progress towards three conjectures discussed in the introduction: the uniqueness of maximal immediate extensions, the equivalence of d-henselianity and d-algebraic maximality, and the uniqueness of minimal d-henselian extensions. We concentrate here on the case that the value group is small in the sense that it has *finite (archimedean) rank*, which means that it has only finitely many convex subgroups. One contrast between valued fields and valued differential fields to keep in mind is that valued differential fields typically have rather large (infinite rank) value groups. In the next chapter we handle the case of arbitrary rank value groups, but only in the asymptotic setting. Although the proof of the main technical lemma in this chapter uses the assumption on the value group in an essential way, in deducing the main results we show how they follow from this lemma without using this assumption. The work in this chapter is joint with van den Dries and appeared in [DP19].

In this chapter, suppose that our valued differential field  $K$  has small derivation, that is,  $\partial\mathcal{O} \subseteq \mathcal{O}$ . The residue field  $\mathbf{k}$  of  $K$  is construed throughout as a differential field with the induced derivation. We work in the category of valued differential fields with small derivation, so we assume in this chapter that all (valued differential field) extensions of  $K$  have small derivation. If  $L$  is an extension of  $K$ , then we consider  $\mathbf{k}$  as a differential subfield of  $\mathbf{k}_L$  and  $\Gamma$  as a subgroup of  $\Gamma_L$  in the usual way.

Any two maximal immediate extensions of  $K$  are maximal *as valued fields* by Theorem 2.9. Hence they are isomorphic as valued fields over  $K$ , and the authors of [ADH18] conjecture that if  $\mathbf{k}$  is linearly surjective, then moreover they are isomorphic as valued *differential* fields over  $K$ . Similarly, we expect that if  $\mathbf{k}$  is linearly surjective, then all d-algebraically maximal d-algebraic immediate extensions of  $K$  are isomorphic over  $K$ . Both conjectures hold when  $K$  is *monotone*, i.e.,  $a' \preceq a$  for all  $a \in \mathcal{O}$  [ADH17a, Theorem 7.4.3]; for monotone  $K$  with *many constants*, that is,  $v(C^\times) = \Gamma$ , this is due to Scanlon [Sca00]. We prove these conjectures when  $\Gamma$  has finite rank. More generally:

**Theorem 3.1.** *If  $\mathbf{k}$  is linearly surjective and  $\Gamma$  is the union of its finite rank convex subgroups, then any two maximal immediate extensions of  $K$  are isomorphic over  $K$ , and any two d-algebraically maximal d-algebraic immediate extensions of  $K$  are isomorphic over  $K$ .*

To prove this, we isolate a property, the *differential-henselian configuration property*, used implicitly in [ADH17a], and do the proof in two steps. The first step shows in §3.3 that  $\Gamma$  as in the theorem has the differential-henselian configuration property. The second shows in the same way as

in the monotone case that the conclusion of the theorem holds whenever  $\mathbf{k}$  is linearly surjective and  $\Gamma$  has this property; this is done in §3.4.

Theorem 2.10 says that if  $K$  is d-algebraically maximal and  $\mathbf{k}$  is linearly surjective, then  $K$  is d-henselian. The same authors also proved a partial converse [ADH17a, Theorem 7.0.3]: if  $K$  is monotone, d-henselian, and has few constants ( $C \subseteq \mathcal{O}$ ), then  $K$  is d-algebraically maximal. They asked whether the monotonicity assumption can be dropped. The following, from [DP19], was the first result in that direction.

**Theorem 3.2.** *If  $K$  is d-henselian, has few constants, and  $\Gamma$  is the union of its finite rank convex subgroups, then  $K$  is d-algebraically maximal.*

Recall that if  $K$  is d-henselian, then it has few constants if and only if it is asymptotic. Hence the result above is really about asymptotic fields. If  $\mathbf{k}$  is linearly surjective, then  $K$  has an immediate d-henselian extension: just take any immediate extension of  $K$  that is maximal (or d-algebraically maximal). Conversely, if  $K$  has an immediate d-henselian extension, then  $\mathbf{k}$  must be linearly surjective. Moreover, if  $K$  is asymptotic and  $\mathbf{k}$  is linearly surjective, then  $K$  has a d-henselian extension that is minimal in the following sense.

**Lemma 3.3** ([ADH17a, 9.4.11]). *Suppose that  $K$  is asymptotic and  $\mathbf{k}$  is linearly surjective. Then  $K$  has an immediate asymptotic extension  $L$  such that:*

- (i)  $L$  is d-algebraic over  $K$ ;
- (ii)  $L$  is d-henselian;
- (iii)  $L$  has no proper differential subfield containing  $K$  that is d-henselian.

We use the differential-henselian configuration property to show that this extension is unique in a strong way. For this we introduce differential-henselizations:

**Definition.** If  $K$  is asymptotic, then a *differential-henselization* (d-henselization for short) of  $K$  is an immediate asymptotic d-henselian extension of  $K$  that embeds over  $K$  into every asymptotic d-henselian extension of  $K$ .

**Theorem 3.4.** *If  $K$  is asymptotic,  $\mathbf{k}$  is linearly surjective, and  $\Gamma$  is the union of its finite rank convex subgroups, then  $K$  has, up to isomorphism over  $K$ , a unique d-henselization.*

### 3.2. Preliminaries

In this section,  $P \in K\{Y\}^\neq$ . Recall from [ADH17a, §6.6] the notion of the *dominant part* of  $P$ : To any  $P$  we associate  $\mathfrak{d}_P \in K^\times$  with  $\mathfrak{d}_P \asymp P$  such that  $\mathfrak{d}_P = \mathfrak{d}_Q$  for all  $Q \in K\{Y\}^\neq$  with  $Q \sim P$ . Then  $\mathfrak{d}_P^{-1}P \asymp 1$ , so we can define the *dominant part*  $D_P \in \mathbf{k}\{Y\}^\neq$  of  $P$  to be the image of  $\mathfrak{d}_P^{-1}P$  in  $\mathbf{k}\{Y\}$ . For  $Q = 0 \in K\{Y\}$ , we set  $\mathfrak{d}_Q := 0$  and  $D_Q := 0 \in \mathbf{k}\{Y\}$ . Then we define the *dominant degree* of  $P$  to be  $\text{ddeg } P := \deg D_P$  and the *dominant multiplicity of  $P$  at 0* to be  $\text{dmul } P := \text{mul } D_P$ .

For later use we show that the condition  $\text{ddeg } P \geq 1$  is necessary for the existence of a zero  $f \prec 1$  of  $P$  in an extension of  $K$ :

**Lemma 3.5.** *Let  $g \in K^\times$ , and suppose  $P(f) = 0$  for some  $f \preccurlyeq g$  in some extension of  $K$ . Then  $\text{ddeg } P_{\times g} \geq 1$ .*

PROOF. Let  $L$  be an extension of  $K$  and suppose  $f \in L$ ,  $f \preccurlyeq g$ , and  $P(f) = 0$ . Then  $f = ag$  for some  $a \in L$  with  $a \preccurlyeq 1$ . Setting  $Q := P_{\times g}$ , we have  $Q(a) = 0$ , so  $D_Q(\bar{a}) = 0$ . Hence,  $\text{ddeg } P_{\times g} \geq 1$ .  $\square$

Now we recall below properties of these notions needed in this chapter and the next. First, by [ADH17a, Lemma 6.6.5(ii)]  $\text{ddeg } P_{\times g}$  depends only on  $vg$  and not on  $g$ , for  $g \in K^\times$ .

**Lemma 3.6** ([ADH17a, 6.6.6]). *If  $f, g \in K$  and  $h \in K^\times$  satisfy  $f - g \preccurlyeq h$ , then*

$$\text{ddeg } P_{+f, \times h} = \text{ddeg } P_{+g, \times h}.$$

**Lemma 3.7** ([ADH17a, 6.6.7]). *Let  $f, g \in K^\times$ . Then  $\text{mul } P = \text{mul}(P_{\times f}) \leq \text{ddeg } P_{\times f}$  and*

$$f \prec g \implies \text{dmul } P_{\times f} \leq \text{ddeg } P_{\times f} \leq \text{dmul } P_{\times g} \leq \text{ddeg } P_{\times g}.$$

Below, we let  $\mathcal{E} \subseteq K^\times$  be nonempty and such that for  $f, g \in K^\times$ ,  $f \preccurlyeq g \in \mathcal{E}$  implies  $f \in \mathcal{E}$ . In this case, we say that  $\mathcal{E}$  is  $\preccurlyeq$ -closed, and we consider the *dominant degree of  $P$  on  $\mathcal{E}$*  defined by:

$$\text{ddeg}_{\mathcal{E}} P := \max\{\text{ddeg } P_{\times f} : f \in \mathcal{E}\}.$$

Note that  $\preccurlyeq$ -closed sets correspond to nonempty upward-closed sets in  $\Gamma$ . If  $\mathcal{E} = \{f \in K^\times : vf \geq \gamma\}$  for  $\gamma \in \Gamma$ , then  $\text{ddeg}_{\geq \gamma} P := \text{ddeg}_{\mathcal{E}} P$ . For any  $g \in K^\times$  with  $vg = \gamma$  we set  $\text{ddeg}_{\preccurlyeq g} P := \text{ddeg}_{\geq \gamma} P$ , and by the previous result we have  $\text{ddeg}_{\preccurlyeq g} P = \text{ddeg } P_{\times g}$ . We define  $\text{ddeg}_{> \gamma} P$  and  $\text{ddeg}_{< g} P$  analogously.

**Lemma 3.8** ([ADH17a, 6.6.9]). *If  $v(\mathcal{E})$  has no smallest element, then*

$$\text{ddeg}_{\mathcal{E}} P = \max\{\text{dmul}(P_{\times f}) : f \in \mathcal{E}\}.$$

**Lemma 3.9** ([ADH17a, 6.6.10]). *If  $f \in \mathcal{E}$ , then  $\text{ddeg}_{\mathcal{E}} P_{+f} = \text{ddeg}_{\mathcal{E}} P$ .*

**Corollary 3.10** ([ADH17a, 6.6.11]). *Suppose that  $\text{ddeg}_{\mathcal{E}} P = 1$  and  $y \in \mathcal{E}$  satisfies  $P(y) = 0$ , and let  $f \in \mathcal{E}$ . Then*

$$\text{mul } P_{+y, \times f} = \text{dmul } P_{+y, \times f} = \text{ddeg } P_{+y, \times f} = 1.$$

**Corollary 3.11** ([ADH17a, 6.6.12]). *If  $a, b \in K$  and  $\alpha, \beta \in \Gamma$  satisfy  $v(b - a) \geq \alpha$  and  $\beta \geq \alpha$ , then*

$$\text{ddeg}_{\geq \beta} P_{+b} \leq \text{ddeg}_{\geq \alpha} P_{+a}.$$

**3.2.1. Dominant degree in a cut.** Let  $(a_\rho)$  be a pc-sequence in  $K$  with  $\gamma_\rho := v(a_{\rho+1} - a_\rho)$ . Here we define the crucial notion of “dominant degree in a cut” and record some basic properties. This is a variant of the “newton degree in a cut” from [ADH18].

**Lemma 3.12.** *There is an index  $\rho_0$  and a number  $d(P, (a_\rho)) \in \mathbb{N}$  such that for all  $\rho > \rho_0$ ,*

$$\text{ddeg}_{\geq \gamma_\rho} P_{+a_\rho} = d(P, (a_\rho)).$$

*Whenever  $(b_\lambda)$  is a pc-sequence in  $K$  equivalent to  $(a_\rho)$ , we have  $d(P, (a_\rho)) = d(P, (b_\lambda))$ .*

PROOF. Take  $\rho_0$  such that for all  $\rho' > \rho \geq \rho_0$ , we have  $\gamma_{\rho'} > \gamma_\rho$  and  $\gamma_\rho \in \Gamma$ . Then

$$\text{ddeg}_{\geq \gamma_{\rho'}} P_{+a_{\rho'}} \leq \text{ddeg}_{\geq \gamma_\rho} P_{+a_\rho} \text{ for all } \rho' > \rho \geq \rho_0$$

by Corollary 3.11. This gives the existence of  $d(P, (a_\rho))$ . Set  $d := d(P, (a_\rho))$ . We can assume  $\rho_0$  to be so large that  $\text{ddeg}_{\geq \gamma_\rho} P_{+a_\rho} = d$  for all  $\rho \geq \rho_0$ . Let  $(b_\lambda)$  be a pc-sequence in  $K$  equivalent to  $(a_\rho)$ , and set  $\beta_\lambda := v(b_{\lambda+1} - b_\lambda)$ . Take an index  $\lambda_0$  and  $e \in \mathbb{N}$  so that  $\beta_\lambda \in \Gamma$  and  $\text{ddeg}_{\geq \beta_\lambda} P_{+b_\lambda} = e$  for all  $\lambda \geq \lambda_0$ . Since  $(a_\rho)$  and  $(b_\lambda)$  are equivalent, we can further arrange that  $b_\lambda - a_\rho \prec a_\rho - a_{\rho_0}$  and  $\beta_\lambda \geq \gamma_{\rho_0}$  for all  $\rho > \rho_0$  and  $\lambda > \lambda_0$ . Then for  $\lambda > \lambda_0$  we have  $v(b_\lambda - a_{\rho_0}) = \gamma_{\rho_0}$ , and so

$$e = \text{ddeg}_{\geq \beta_\lambda} P_{+b_\lambda} \leq \text{ddeg}_{\geq \gamma_{\rho_0}} P_{+a_{\rho_0}} = d,$$

by Corollary 3.11. By symmetry, we also have  $d \leq e$ , so  $d = e$ .  $\square$

As in [ADH18, §2] and [ADH17a, §11.2], we associate to each pc-sequence  $(a_\rho)$  in  $K$  its *cut* in  $K$ , denoted by  $c_K(a_\rho)$ , such that if  $(b_\lambda)$  is a pc-sequence in  $K$ , then

$$c_K(a_\rho) = c_K(b_\lambda) \iff (b_\lambda) \text{ is equivalent to } (a_\rho).$$

Below,  $\mathbf{a} := c_K(a_\rho)$ . If we want to emphasize the dependence on  $K$  we write  $\mathbf{a}_K$ . Note that  $c_K(a_\rho + y)$  for  $y \in K$  depends only on  $\mathbf{a}$  and  $y$ , so we let  $\mathbf{a} + y$  denote  $c_K(a_\rho + y)$ . Similarly,  $c_K(a_\rho y)$  for  $y \in K^\times$  depends only on  $\mathbf{a}$  and  $y$ , so we let  $\mathbf{a} \cdot y$  denote  $c_K(a_\rho y)$ .

**Definition.** The *dominant degree of  $P$  in the cut of  $(a_\rho)$* , denoted by  $\text{ddeg}_{\mathbf{a}} P$ , is the natural number  $d(P, (a_\rho))$  from the previous lemma.

**Lemma 3.13.** *Dominant degree in a cut has the following properties:*

- (i)  $\text{ddeg}_{\mathbf{a}} P \leq \deg P$ ;
- (ii)  $\text{ddeg}_{\mathbf{a}} P_{+y} = \text{ddeg}_{\mathbf{a}+y} P$  for  $y \in K$ ;
- (iii) if  $y \in K$  and  $vy$  is in the width of  $(a_\rho)$ , then  $\text{ddeg}_{\mathbf{a}} P_{+y} = \text{ddeg}_{\mathbf{a}} P$ ;
- (iv)  $\text{ddeg}_{\mathbf{a}} P_{\times y} = \text{ddeg}_{\mathbf{a} \cdot y} P$  for  $y \in K^\times$ ;
- (v) if  $Q \in K\{Y\}^\neq$ , then  $\text{ddeg}_{\mathbf{a}} PQ = \text{ddeg}_{\mathbf{a}} P + \text{ddeg}_{\mathbf{a}} Q$ ;
- (vi) if  $P(\ell) = 0$  for some pseudolimit  $\ell$  of  $(a_\rho)$  in an extension of  $K$ , then  $\text{ddeg}_{\mathbf{a}} P \geq 1$ ;
- (vii) if  $L$  is an extension of  $K$ , then  $\text{ddeg}_{\mathbf{a}} P = \text{ddeg}_{\mathbf{a}_L} P$ , where  $\mathbf{a}_L = c_L(a_\rho)$ .

PROOF. Items (i), (ii), (iv), and (v) follow routinely from basic facts about dominant degree, (iii) follows from (ii), and (vii) is obvious.

For (vi), let  $\ell$  be a pseudolimit of  $(a_\rho)$  in an extension of  $K$  with  $P(\ell) = 0$ . Take  $\rho_0$  such that, for all  $\rho \geq \rho_0$ ,  $v(\ell - a_\rho) = \gamma_\rho \in \Gamma$  and  $d(P, (a_\rho)) = \text{ddeg}_{\geq \gamma_\rho} P_{+a_\rho}$ . Let  $\rho \geq \rho_0$  and set  $Q = P_{+a_\rho}$ . Then  $Q(\ell - a_\rho) = 0$ , so  $\text{ddeg}_{\mathbf{a}} Q_{\times g} \geq 1$  for any  $g \in K$  with  $vg = \gamma_\rho$ , by Lemma 3.5.  $\square$

**3.2.2. Coarsening and specialization.** Coarsening and specializing are central to the arguments in this chapter, so we review the definitions here. Details and proofs can be found in [ADH17a, §3.4].

Let  $\Delta \neq \{0\}$  be a proper convex subgroup of  $\Gamma$ . Then we have another valuation on  $K$ ,

$$\begin{aligned} v_\Delta: K^\times &\rightarrow \Gamma/\Delta \\ a &\mapsto v(a) + \Delta. \end{aligned}$$

We denote  $K$  with this valuation by  $K_\Delta$  and call it the *coarsening of  $K$  by  $\Delta$* . Set  $\dot{v} := v_\Delta$ ,  $\dot{\Gamma} := \Gamma/\Delta$ , and  $\dot{\gamma} := \gamma + \Delta$ , so if  $v(a) = \gamma$ , then  $\dot{v}(a) = \dot{\gamma}$ . The valuation ring of  $K_\Delta$  is

$$\dot{\mathcal{O}} = \{a \in K : \dot{v}(a) \geq 0\} = \{a \in K : v(a) \geq \delta \text{ for some } \delta \in \Delta\} \supseteq \mathcal{O}$$

with maximal ideal

$$\dot{\mathcal{o}} = \{a \in K : \dot{v}(a) > 0\} = \{a \in K : v(a) > \Delta\} \subseteq \mathcal{o}.$$

With the same derivation,  $\partial$ ,  $K_\Delta$  is a valued differential field. By [ADH17a, Corollary 4.4.4],  $\partial\dot{\mathcal{O}} \subseteq \dot{\mathcal{O}}$ , so  $K_\Delta$  has small derivation. By the choice of  $\Delta$ ,  $\dot{\Gamma} \neq \{0\}$ , so  $K_\Delta$  satisfies our standing assumptions from §2.6.

Its residue field,  $\dot{K} := \dot{\mathcal{O}}/\dot{\mathcal{o}}$ , is also a valued field, called the *specialization of  $K$  to  $\Delta$* . For  $a \in \dot{\mathcal{O}}$ , let  $\dot{a} := a + \dot{\mathcal{o}}$ . Then the valuation  $v: \dot{K}^\times \rightarrow \Delta$  is defined by  $v(\dot{a}) = v(a)$  for  $a \in \dot{\mathcal{O}} \setminus \dot{\mathcal{o}}$ . Note that  $\dot{K}$  is also a differential field because  $K_\Delta$  has small derivation, and  $\dot{K}$  has small derivation because  $K$  does. As  $\Delta \neq \{0\}$ ,  $\dot{K}$  also satisfies our standing assumptions.

To distinguish between pseudoconvergence in  $K$  and in a coarsening of  $K$  with valuation  $\dot{v}$ , we write  $\rightsquigarrow_v$  for the former and  $\rightsquigarrow_{\dot{v}}$  for the latter. We use  $\rightsquigarrow$  for pseudoconvergence in both  $K$  and specializations of  $K$ .

**Lemma 3.14** ([ADH17a, 2.2.21]). *If  $(a_\rho)$  is a pc-sequence with respect to  $\dot{v}$ , then  $(a_\rho)$  is a pc-sequence with respect to  $v$ , and for any  $a \in K$ :*

$$a_\rho \rightsquigarrow_{\dot{v}} a \iff a_\rho \rightsquigarrow_v a.$$

**Lemma 3.15** ([ADH17a, 3.4.1]). *If  $(a_\rho)$  is a sequence in  $\dot{\mathcal{O}}$  indexed by a limit ordinal such that  $(\dot{a}_\rho)$  is a pc-sequence in  $\dot{K}$ , then  $(a_\rho)$  is a pc-sequence in  $K$ , and for any  $a \in \dot{\mathcal{O}}$ :*

$$a_\rho \rightsquigarrow a \iff \dot{a}_\rho \rightsquigarrow \dot{a}.$$

Conversely, some pc-sequences in  $K$  remain pc-sequences after coarsening or specializing. To discuss this, let  $(a_\rho)$  be a pc-sequence in  $K$ . We say that  $(a_\rho)$  is  $\Delta$ -*fluent* if for some index  $\rho_0$  we have  $\gamma_{\rho'} - \gamma_\rho > \Delta$ , for all  $\rho' > \rho > \rho_0$ ; in that case  $(a_\rho)$  is still a pc-sequence in  $K_\Delta$ . We say that  $(a_\rho)$  is  $\Delta$ -*jammed* if for some index  $\rho_0$  we have  $\gamma_{\rho'} - \gamma_\rho \in \Delta$ , for all  $\rho' > \rho > \rho_0$ . If  $a_\rho \in \dot{\mathcal{O}}$  and  $\gamma_\rho \in \Delta$ , eventually, then  $(a_\rho)$  is  $\Delta$ -jammed and  $(\dot{a}_\rho)$  is a pc-sequence in  $\dot{K}$ , where by convention we drop the indices  $\rho$  for which  $a_\rho \notin \dot{\mathcal{O}}$ . If  $(a_\rho)$  is not  $\Delta$ -jammed, then it has a  $\Delta$ -fluent cofinal subsequence.

Let  $\text{ddeg}^\Delta P$  be the dominant degree of  $P$  in  $K_\Delta$ , that is, with respect to the valuation  $\dot{v} = v_\Delta$  on  $K$ . Here is how dominant degree in a cut behaves under coarsening:

**Lemma 3.16.** *Suppose that  $(a_\rho)$  is  $\Delta$ -fluent. Let  $\mathbf{a} := c_K(a_\rho)$  and  $\mathbf{a}_\Delta := c_{K_\Delta}(a_\rho)$ . Then*

$$\text{ddeg}_{\mathbf{a}} P \leq \text{ddeg}_{\mathbf{a}_\Delta}^\Delta P.$$

PROOF. For  $g \in K^\times$  with  $vg = \gamma_\rho$  and  $Q := P_{+a_\rho}$  we have

$$\text{ddeg}_{\geq \dot{\gamma}_\rho}^\Delta Q = \text{ddeg}^\Delta Q_{\times g} \geq \text{ddeg} Q_{\times g} = \text{ddeg}_{\geq \gamma_\rho} Q.$$

It follows that  $\text{ddeg}_{\mathbf{a}} P \leq \text{ddeg}_{\mathbf{a}_\Delta}^\Delta P$ . □

**Lemma 3.17.** *Suppose that  $P \in \dot{\mathcal{O}}\{Y\} \setminus \dot{\mathcal{O}}\{Y\}$ ,  $b \in \dot{\mathcal{O}}$ , and  $h \in \dot{\mathcal{O}} \setminus \dot{\mathcal{O}}$ . Then*

$$\text{ddeg } \dot{P}_{+b} = \text{ddeg } P_{+b} \quad \text{and} \quad \text{ddeg } \dot{P}_{\times h} = \text{ddeg } P_{\times h}.$$

PROOF. By [ADH17a, Lemma 4.3.1], we have  $\dot{P}_{+b} = (\dot{P}_{+b})$  and  $\dot{P}_{\times h} = (\dot{P}_{\times h})$ . It remains to note that  $\text{ddeg } Q = \text{ddeg } \dot{Q}$ , for all  $Q \in \dot{\mathcal{O}}\{Y\} \setminus \dot{\mathcal{O}}\{Y\}$ .  $\square$

**Corollary 3.18.** *Suppose that  $P \in \dot{\mathcal{O}}\{Y\} \setminus \dot{\mathcal{O}}\{Y\}$  and that  $a_\rho \in \dot{\mathcal{O}}$ ,  $\gamma_\rho \in \Delta$ , eventually. Let  $\mathbf{a} := c_K(a_\rho)$  and  $\dot{\mathbf{a}} := c_{\dot{K}}(\dot{a}_\rho)$ . Then  $\text{ddeg}_{\mathbf{a}} P = \text{ddeg}_{\dot{\mathbf{a}}} \dot{P}$ .*

PROOF. For  $g \in K^\times$  with  $vg = \gamma_\rho \in \Delta$  we have

$$\text{ddeg}_{\geq \gamma_\rho} \dot{P}_{+\dot{a}_\rho} = \text{ddeg } \dot{P}_{+\dot{a}_\rho, \times g} \quad \text{and} \quad \text{ddeg}_{\geq \gamma_\rho} P_{+a_\rho} = \text{ddeg } P_{+a_\rho, \times g},$$

so the desired result follows from Lemma 3.17.  $\square$

### 3.3. The differential-henselian configuration property

**Assumption.** In this section, we assume that the induced derivation on  $\mathbf{k}$  is nontrivial.

This assumption makes available tools from §2.3 on constructing immediate extensions, which are fundamental to our results. We introduce here the differential-henselian configuration property. In [ADH17a, Proposition 7.4.1], the authors proved that monotone valued differential fields have this property, from which they deduced the uniqueness of maximal immediate extensions of monotone fields, and of d-algebraically maximal d-algebraic immediate extensions of monotone fields. In §3.4, we show that those results depend only on the differential-henselian configuration property, not on monotonicity.

The goal of this section then is to prove that  $K$  has the differential-henselian configuration property when  $\Gamma$  has finite rank, in Corollary 3.20. Corollary 3.23 extends this to  $K$  such that  $\Gamma$  is the union of its finite rank convex subgroups. The theorems in the introduction then follow from the results of §3.4 combined with Corollary 3.23.

**Definition.** We say that  $K$  has the *differential-henselian configuration property* (*dh-configuration property* for short) if, for every divergent pc-sequence  $(a_\rho)$  in  $K$  with minimal differential polynomial  $G(Y)$  over  $K$ , we have  $\text{ddeg}_{\mathbf{a}} G = 1$ .

This property is so named because of its connection to the notion of  $G \in K\{Y\}$  being in dh-configuration at  $a \in K$  from [ADH17a, §7.3], exploited in §3.4.

Rephrasing [ADH17a, Proposition 7.4.1] gives that monotone  $K$  have the dh-configuration property. There, the assumption on  $G$  was weaker, but this form was all that was necessary for the consequences mentioned above. If  $\Gamma$  has finite rank, then we call the number of nontrivial convex subgroups of  $\Gamma$  its (*archimedean*) *rank*. If  $\Gamma$  has rank 1, then  $K$  is monotone since it has small derivation [ADH17a, Corollary 6.1.2]. Thus  $K$  has the dh-configuration property whenever  $\Gamma$  has rank 1, which will be the base case for an inductive proof of Corollary 3.20. To prove the inductive step we examine how the dh-configuration property relates to coarsening and specialization.

**Proposition 3.19.** *Let  $\Delta \neq \{0\}$  be a proper convex subgroup of  $\Gamma$ . Let  $K_\Delta$  be the coarsening of  $K$  by  $\Delta$  and  $\dot{K}$  be the specialization of  $K$  to  $\Delta$ . Suppose that  $K_\Delta$  and  $\dot{K}$  have the dh-configuration property. Then so does  $K$ .*

PROOF. Let  $(a_\rho)$  be a pc-sequence in  $K$  with minimal differential polynomial  $G(Y)$  over  $K$ . We need to show that  $\text{ddeg}_{\mathbf{a}} G = 1$ . By the definition of minimal differential polynomial, we can replace  $(a_\rho)$  by an equivalent pc-sequence to arrange that  $G(a_\rho) \rightsquigarrow 0$ .

At this point we distinguish between Case 1 and Case 2 below, but first we indicate a construction that is needed in both cases and which depends only on the assumptions and arrangements that are now in place. Using Lemma 2.13 we obtain a pseudolimit  $a$  of  $(a_\rho)$  in an immediate extension  $K\langle a \rangle$  of  $K$  with  $G(a) = 0$ . Hence  $\text{ddeg}_{\mathbf{a}} G \geq 1$  by Lemma 3.13(vi), so it is enough to show that  $\text{ddeg}_{\mathbf{a}} G \leq 1$ . Set  $P := G_{+a}$ , so  $\deg P = \deg G$  by [ADH17a, Corollary 4.3.2], and  $\gamma_\rho := v(a_{\rho+1} - a_\rho)$ .

By Lemma 2.11, we have a pc-sequence  $(b_\rho)$  in  $K$  equivalent to  $(a_\rho)$  such that:

- (i)  $G(b_\rho) \rightsquigarrow 0$ ;
- (ii)  $v(b_{\rho+1} - b_\rho) = v(b_\rho - a) = \gamma_\rho$ , eventually;
- (iii)  $v(G(b_\rho)) = v(P(b_\rho - a)) = v_P(\gamma_\rho)$ , eventually.

Then  $\mathbf{a} = c_K(b_\rho)$ , as  $(b_\rho)$  is equivalent to  $(a_\rho)$ . From [ADH17a, Corollary 6.1.10] and  $P(0) = 0$  we obtain an  $e \in \mathbb{N}$  with  $1 \leq e \leq \deg P$  such that  $P_e \neq 0$  and

$$v(G(b_\rho)) = v_P(\gamma_\rho) = \min_d v_{P_d}(\gamma_\rho) = v_{P_e}(\gamma_\rho), \text{ eventually.}$$

Then Lemma 2.3 gives, for all sufficiently large  $\rho$  and all  $\rho' > \rho$ ,

$$(*) \quad v(G(b_{\rho'})) - v(G(b_\rho)) = v_{P_e}(\gamma_{\rho'}) - v_{P_e}(\gamma_\rho) = e(\gamma_{\rho'} - \gamma_\rho) + o(\gamma_{\rho'} - \gamma_\rho).$$

*Case 1:  $(a_\rho)$  is not  $\Delta$ -jammed.* Then  $(a_\rho)$  has a  $\Delta$ -fluent cofinal subsequence. Replacing  $(a_\rho)$  by such a subsequence we arrange that  $(a_\rho)$  is  $\Delta$ -fluent, preserving  $G(a_\rho) \rightsquigarrow 0$ . Next we do the construction above of  $a$ ,  $(b_\rho)$ ,  $P$ , and  $e$ . Note that then  $(b_\rho)$  is also  $\Delta$ -fluent by (ii).

We claim that  $(G(b_\rho))$  is  $\Delta$ -fluent, that is, for all sufficiently large  $\rho$  and all  $\rho' > \rho$ ,

$$v(G(b_{\rho'+1}) - G(b_{\rho'})) - v(G(b_{\rho+1}) - G(b_\rho)) > \Delta.$$

Since  $G(b_\rho) \rightsquigarrow 0$ ,  $v(G(b_\rho))$  is eventually strictly increasing, and thus  $v(G(b_{\rho+1}) - G(b_\rho)) = v(G(b_\rho))$ , eventually. Hence it is enough to show that, for all sufficiently large  $\rho$  and all  $\rho' > \rho$ ,

$$v(G(b_{\rho'})) - v(G(b_\rho)) > \Delta.$$

This inequality holds by (\*), since  $\gamma_{\rho'} - \gamma_\rho > \Delta$ , eventually. So  $(G(b_\rho))$  is  $\Delta$ -fluent and  $G(b_\rho) \rightsquigarrow_{\dot{v}} 0$ .

Next we show that  $G$  remains a minimal differential polynomial of  $(b_\rho)$  over  $K_\Delta$ . So let  $(e_\lambda)$  be a pc-sequence in  $K_\Delta$  that is equivalent to  $(a_\rho)$  (with respect to  $\dot{v}$ ) and suppose  $H \in K\{Y\}$  is such that  $H(e_\lambda) \rightsquigarrow_{\dot{v}} 0$ . Since  $\dot{v}$  is a coarsening of  $v$ , Lemma 3.14 gives that  $(e_\lambda)$  is a pc-sequence in  $K$  that is equivalent to  $(b_\rho)$  (with respect to  $v$ ). From  $H(e_\lambda) \rightsquigarrow_{\dot{v}} 0$  we get  $H(e_\lambda) \rightsquigarrow_v 0$ , and hence  $c(H) \geq c(G)$ . As  $K_\Delta$  has the dh-configuration property, we obtain  $\text{ddeg}_{\mathbf{a}_\Delta}^\Delta G = 1$ , so  $\text{ddeg}_{\mathbf{a}} G \leq 1$  by Lemma 3.16, as desired.

*Case 2:  $(a_\rho)$  is  $\Delta$ -jammed.* Take an index  $\rho_0$  so large that, for all  $\rho' > \rho \geq \rho_0$ ,

$$\gamma_\rho \in \Gamma, \quad v(a_{\rho'} - a_\rho) = \gamma_\rho, \quad \gamma_{\rho'} > \gamma_\rho, \quad \text{and} \quad \gamma_\rho - \gamma_{\rho_0} \in \Delta.$$

Take  $g \in K$  with  $vg = \gamma_{\rho_0}$ . Then, replacing  $(a_\rho)$  by  $((a_\rho - a_{\rho_0})/g)$  and  $G$  by  $G_{+a_{\rho_0} \times g}$ , we arrange that  $v(a_\rho) = 0$  and  $\gamma_\rho \in \Delta^>$ , eventually, preserving  $G(a_\rho) \rightsquigarrow 0$ . This is possible by Lemma 3.13(ii)

and (iv). By scaling  $G$  we also arrange that  $v(G) = 0 \in \Delta$ . At this point we do the construction above of  $a$ ,  $(b_\rho)$ ,  $P$ , and  $e$ . From  $v(a) = 0$  and (ii) we get  $v(b_\rho) = 0$ , eventually.

We claim that  $\dot{G}(\dot{b}_\rho)$  is a pc-sequence in  $\dot{K}$  with  $\dot{G}(\dot{b}_\rho) \rightsquigarrow 0$ . First, by Lemma 2.2,  $v(P) = 0$ . This yields  $v(P_e) \in \Delta$ : otherwise,  $v(P_e) > \Delta$  and so by taking  $d \neq e$  with  $v(P_d) = 0$  we obtain  $v_{P_e}(\gamma_\rho) > v_{P_d}(\gamma_\rho)$ , eventually, by Lemma 2.3 and because  $\gamma_\rho \in \Delta$ , eventually; but this contradicts the choice of  $e$ . Next, from Lemma 2.3 and  $v(P_e) \in \Delta$  we get

$$v(G(b_\rho)) = v_{P_e}(\gamma_\rho) = v(P_e) + e\gamma_\rho + o(\gamma_\rho) \in \Delta, \text{ eventually,}$$

so  $v(\dot{G}(\dot{b}_\rho)) = v(P_e) + e\gamma_\rho + o(\gamma_\rho)$ , eventually, and thus  $\dot{G}(\dot{b}_\rho)$  is a pc-sequence in  $\dot{K}$  with  $\dot{G}(\dot{b}_\rho) \rightsquigarrow 0$ .

Finally, we show that  $\dot{G}$  is a minimal differential polynomial of  $(\dot{a}_\rho)$  over  $\dot{K}$ . So let  $(e_\lambda)$  be a sequence in  $\dot{\mathcal{O}}$  indexed by a limit ordinal such that  $(\dot{e}_\lambda)$  is a pc-sequence in  $\dot{K}$  equivalent to  $(\dot{a}_\rho)$ , and thus to  $(\dot{b}_\rho)$ , and let  $H \in \dot{\mathcal{O}}\{Y\}$  be such that  $\dot{H} \in \dot{K}\{Y\}^\neq$ ,  $\dot{H}(\dot{e}_\lambda) \rightsquigarrow 0$  and  $c(H) = c(\dot{H})$ . Then Lemma 3.15 gives  $H(e_\lambda) \rightsquigarrow 0$ . Moreover,  $(e_\lambda)$  is equivalent to  $(b_\rho)$ , so  $c(H) \geq c(G)$ , and thus  $c(\dot{H}) \geq c(\dot{G})$ . The hypothesis on  $\dot{K}$  therefore yields  $\text{ddeg}_{\dot{a}} \dot{G} = 1$ , and so  $\text{ddeg}_{\mathbf{a}} G = 1$  by Corollary 3.18.  $\square$

Note that the part of the proof preceding Case 1 does not use the assumptions on  $K_\Delta$  and  $\dot{K}$ , that Case 1 only uses the assumption on  $K_\Delta$ , and Case 2 only the assumption on  $\dot{K}$ .

**Corollary 3.20.** *Suppose that  $\Gamma$  has finite rank. Then  $K$  has the dh-configuration property.*

PROOF. By induction on the rank of  $\Gamma$ . If  $\Gamma$  has rank 1, then  $K$  is monotone, so has the dh-configuration property. If  $\Gamma$  has rank  $n > 1$ , then it has a proper convex subgroup  $\Delta \neq \{0\}$ . Both  $\Gamma/\Delta$  and  $\Delta$  have rank  $< n$ , so the coarsening of  $K$  by  $\Delta$  and the specialization of  $K$  to  $\Delta$  have the dh-configuration property by the induction hypothesis. Then Proposition 3.19 gives the result.  $\square$

**Definition.** An ordered abelian group  $G$  has the *differential-henselian configuration property* (dh-configuration property for short) if every valued differential field with small derivation, nontrivial induced derivation on its residue field, and value group  $G$  has the dh-configuration property.

Thus any  $G$  of finite rank has the dh-configuration property by Corollary 3.20. This property is inherited by “convex” unions, but for that we need the following.

**Lemma 3.21.** *Let  $(a_\rho)$  be a pc-sequence in  $K$  of width  $\{\infty\}$  and  $G(Y)$  a minimal differential polynomial of  $(a_\rho)$  over  $K$ . Then  $\text{ddeg}_{\mathbf{a}} G = 1$ .*

PROOF. As in the proof of Proposition 3.19 we arrange  $G(a_\rho) \rightsquigarrow 0$ , take a pseudolimit  $a$  of  $(a_\rho)$  in an immediate extension of  $K$  with  $G(a) = 0$ , and set  $P := G_{+a}$  (with  $\deg P = \deg G$ ) and  $\gamma_\rho := v(a_{\rho+1} - a_\rho)$ . We also arrange that  $(\gamma_\rho)$  is strictly increasing and  $v(a - a_\rho) = \gamma_\rho$  for all  $\rho$ . We claim that  $P_1 \neq 0$ . To prove this claim, let  $r$  be the order of  $G$ . Then  $\partial G / \partial Y^{(r)} \neq 0$ . Hence  $(\partial G / \partial Y^{(r)})(a) \neq 0$ , since otherwise Lemma 2.11 would give a pc-sequence  $(b_\rho)$  in  $K$  equivalent to  $(a_\rho)$  such that  $\frac{\partial G}{\partial Y^{(r)}}(b_\rho) \rightsquigarrow 0$ , contradicting the minimality of  $G$ . In view of  $P_1 = \sum_{i=0}^r \frac{\partial G}{\partial Y^{(i)}}(a) \cdot Y^{(i)}$ , this gives  $P_1 \neq 0$ .

For  $\text{ddeg}_{\mathbf{a}} G = 1$ , it is enough by Lemma 3.6 that  $\text{ddeg}_{+a, \times g} G = 1$  for some  $\rho$  and  $g \in K^\times$  with  $v(g) = \gamma_\rho$ . As  $(P_d)_{\times g} = (G_{+a, \times g})_d$  for  $g \in K^\times$ , it suffices to show that for all  $d > 1$  with  $P_d \neq 0$  we



have  $v_{P_d}(\gamma_\rho) > v_{P_1}(\gamma_\rho)$ , eventually. Now  $P_1 \neq 0$ , so if  $d > 1$  and  $P_d \neq 0$ , then

$$v_{P_d}(\gamma_\rho) - v_{P_1}(\gamma_\rho) = v(P_d) - v(P_1) + (d-1)\gamma_\rho + o(\gamma_\rho),$$

by Lemma 2.3, and thus  $v_{P_d}(\gamma_\rho) > v_{P_1}(\gamma_\rho)$ , eventually, since  $(\gamma_\rho)$  is cofinal in  $\Gamma$ .  $\square$

**Lemma 3.22.** *Suppose that  $\Gamma$  is a union of convex subgroups with the dh-configuration property. Then  $\Gamma$  has the dh-configuration property.*

PROOF. Let  $\Gamma = \bigcup_{i \in I} \Delta_i$  for some index set  $I$ , where each  $\Delta_i$  is a nontrivial convex subgroup of  $\Gamma$  with the dh-configuration property. The case that  $\Gamma = \Delta_i$  for some  $i \in I$  is trivial, so suppose that  $\Gamma \neq \Delta_i$  for each  $i \in I$ . Let  $(a_\rho)$  be a pc-sequence in  $K$  with a minimal differential polynomial  $G(Y)$  over  $K$ . Set  $\mathbf{a} := c_K(a_\rho)$  and  $\gamma_\rho := v(a_{\rho+1} - a_\rho)$ ; we can assume that  $(\gamma_\rho)$  is strictly increasing.

If  $(a_\rho)$  is  $\Delta_i$ -jammed for some  $i$ , then  $\text{ddeg}_{\mathbf{a}} G = 1$  by the argument from Case 2 in the proof of Proposition 3.19. If  $(a_\rho)$  is not  $\Delta_i$ -jammed for any  $i$ , then  $(\gamma_\rho)$  is cofinal in  $\Gamma$ , so  $\text{ddeg}_{\mathbf{a}} G = 1$  by Lemma 3.21.  $\square$

**Corollary 3.23.** *If  $\Gamma$  is the union of its finite rank convex subgroups, then  $\Gamma$  has the dh-configuration property.*

### 3.4. Main results

We now show how the desired results follow from the dh-configuration property, without other assumptions on  $\Gamma$ . The results claimed in the introduction then follow in view of Corollary 3.23. We first use the dh-configuration property to find pseudolimits of pc-sequences that are also zeroes of their minimal differential polynomials. The argument is the same as in [ADH17a, Lemma 7.4.2].

**Lemma 3.24.** *Suppose  $K$  has the dh-configuration property. Let  $(a_\rho)$  be a divergent pc-sequence in  $K$  with minimal differential polynomial  $G(Y)$  over  $K$ . Let  $L$  be a d-algebraically maximal extension of  $K$  such that  $\mathbf{k}_L$  is linearly surjective. Then  $a_\rho \rightsquigarrow b$  and  $G(b) = 0$  for some  $b \in L$ .*

PROOF. Note that  $L$  is d-henselian by Theorem 2.10. Since  $L$  is d-algebraically maximal and the derivation of  $\mathbf{k}_L$  is nontrivial, every pc-sequence in  $L$  of d-algebraic type over  $L$  has a pseudolimit in  $L$  by Lemma 2.13. Thus we get  $a \in L \setminus K$  with  $a_\rho \rightsquigarrow a$ . Passing to an equivalent pc-sequence we arrange that  $G(a_\rho) \rightsquigarrow 0$ . With  $\gamma_\rho := v(a_{\rho+1} - a_\rho) = v(a - a_\rho)$ , eventually, the dh-configuration property gives  $g_\rho \in K$  with  $v(g_\rho) = \gamma_\rho$  and  $\text{ddeg } G_{+a_\rho, \times g_\rho} = 1$ , eventually. By Lemma 3.6,  $\text{ddeg } G_{+a, \times g_\rho} = 1$ , eventually. We have  $G(a + Y) = G(a) + A(Y) + R(Y)$  where  $A$  is linear and homogeneous and all monomials in  $R$  have degree  $\geq 2$ , and so

$$G_{+a, \times g_\rho}(Y) = G(a) + A_{\times g_\rho}(Y) + R_{\times g_\rho}(Y).$$

Now  $\text{ddeg } G_{+a, \times g_\rho} = 1$  eventually, so  $v(G(a)) \geq v_A(\gamma_\rho) < v_R(\gamma_\rho)$  eventually. With  $v(G, a)$  as in [ADH17a, §7.3] we get  $v(G, a) > v(a - a_\rho)$ , eventually. Then [ADH17a, Lemma 7.3.1] gives  $b \in L$  with  $v_L(a - b) = v(G, a)$  and  $G(b) = 0$ , so  $v_L(a - b) > v(a - a_\rho)$  eventually. Thus  $a_\rho \rightsquigarrow b$ .  $\square$

The assumption in the next theorem and Theorem 3.29, that every immediate extension of  $K$  has the dh-configuration property, is satisfied for example if  $\Gamma$  has the dh-configuration property.

**Theorem 3.25.** *Suppose that  $\mathbf{k}$  is linearly surjective and every immediate extension of  $K$  has the dh-configuration property. Then any two maximal immediate extensions of  $K$  are isomorphic over  $K$ . Also, any two d-algebraically maximal d-algebraic immediate extensions of  $K$  are isomorphic over  $K$ .*

PROOF. Let  $L_0$  and  $L_1$  be maximal immediate extensions of  $K$ . By Zorn we have a maximal isomorphism  $\varphi: F_0 \cong_K F_1$  between valued differential subfields  $F_i \supseteq K$  of  $L_i$  for  $i = 0, 1$ , where “maximal” means that  $\varphi$  does not extend to an isomorphism between strictly larger such valued differential subfields. Suppose towards a contradiction that  $F_0 \neq L_0$  (equivalently,  $F_1 \neq L_1$ ). Then  $F_0$  is not spherically complete, so we have a divergent pc-sequence  $(a_\rho)$  in  $F_0$ .

Suppose  $(a_\rho)$  is of d-transcendental type over  $F_0$ . The spherical completeness of  $L_0$  and  $L_1$  then yields  $f_0 \in L_0$  and  $f_1 \in L_1$  such that  $a_\rho \rightsquigarrow f_0$  and  $\varphi(a_\rho) \rightsquigarrow f_1$ . Hence by Lemma 2.12 we obtain an isomorphism  $F_0\langle f_0 \rangle \cong F_1\langle f_1 \rangle$  extending  $\varphi$ , contradicting the maximality of  $\varphi$ .

Suppose  $(a_\rho)$  is of d-algebraic type over  $F_0$ , with minimal differential polynomial  $G$  over  $F_0$ . Then Lemma 3.24 gives  $f_0 \in L_0$  with  $a_\rho \rightsquigarrow f_0$  and  $G(f_0) = 0$ , and  $f_1 \in L_1$  with  $\varphi(a_\rho) \rightsquigarrow f_1$  and  $G^\varphi(f_1) = 0$ . Now Lemma 2.13 gives an isomorphism  $F_0\langle f_0 \rangle \cong F_1\langle f_1 \rangle$  extending  $\varphi$ , and we have again a contradiction. Thus  $F_0 = L_0$  and hence  $F_1 = L_1$  as well.

The proof of the second statement is the same, without needing Lemma 2.12.  $\square$

In the case of few constants, we have the following additional results. Underlying them is the next lemma, a consequential property of  $r$ -d-henselian fields with few constants.

**Lemma 3.26** ([ADH17a, 7.5.5]). *Let  $r \geq 1$  and suppose that  $K$  is  $r$ -d-henselian with  $C \subseteq \mathcal{O}$ . Let  $G \in K\{Y\} \setminus K$  of order at most  $r$ . Then there do not exist  $y_0, \dots, y_{r+1} \in K$  such that:*

- (i)  $y_{i-1} - y_i \succ y_i - y_{i+1}$  for all  $i \in \{1, \dots, r\}$ , and  $y_r \neq y_{r+1}$ ;
- (ii)  $G(y_i) = 0$  for all  $i \in \{0, \dots, r+1\}$ ;
- (iii)  $\text{ddeg } G_{+y_{r+1}, \times g} = 1$  and  $y_0 - y_{r+1} \preccurlyeq g$  for some  $g \in K^\times$ .

The next theorem has the same proof as [ADH17a, Theorem 7.0.3].

**Theorem 3.27.** *Suppose that  $K$  has the dh-configuration property. Let  $L$  be a d-henselian extension of  $K$  with few constants. Let  $(a_\rho)$  be a pc-sequence in  $K$  with minimal differential polynomial  $G(Y)$  over  $K$ . Then  $(a_\rho)$  has a pseudolimit in  $L$ . In particular, if  $K$  itself is d-henselian and has few constants, then it is d-algebraically maximal.*

PROOF. Towards a contradiction, assume that  $(a_\rho)$  is divergent in  $L$ . Then Lemma 2.13 shows that  $G$  has order at least 1 (since  $L$  is henselian) and provides a proper immediate extension  $L\langle a \rangle$  of  $L$  with  $a_\rho \rightsquigarrow a$ . Replacing  $(a_\rho)$  by an equivalent pc-sequence in  $K$ , we arrange that  $G(a_\rho) \rightsquigarrow 0$ .

By the dh-configuration property,  $\text{ddeg}_a G = 1$ . Taking  $g_\rho \in K$  with  $v(g_\rho) = \gamma_\rho$  we have  $\text{ddeg } G_{+a_\rho, \times g_\rho} = 1$ , eventually. By removing some initial terms of the sequence, we arrange that this holds for all  $\rho$  and that  $v(a - a_\rho) = \gamma_\rho$  for all  $\rho$ . By d-henselianity and Lemma 2.5, we have  $z_\rho \in L$  with  $G(z_\rho) = 0$  and  $a_\rho - z_\rho \preccurlyeq g_\rho$ . From  $a - a_\rho \asymp g_\rho$  we get  $a - z_\rho \preccurlyeq g_\rho$ .

Let  $r \geq 1$  be the order of  $G$ . By Lemma 2.6,  $(\gamma_\rho)$  is cofinal in  $v(a - K)$ , so there are indices  $\rho_0 < \rho_1 < \dots < \rho_{r+1}$  such that  $a - z_{\rho_j} \prec a - z_{\rho_i}$ , for  $0 \leq i < j \leq r+1$ . Then

$$z_{\rho_i} - z_{\rho_{i-1}} \asymp a - z_{\rho_{i-1}} \succ a - z_{\rho_i} \asymp z_{\rho_{i+1}} - z_{\rho_i}, \text{ for } 1 \leq i \leq r.$$

We have  $a - z_{\rho_{r+1}} \prec a - z_{\rho_0} \preccurlyeq g_{\rho_0}$ , so  $z_{\rho_0} - z_{\rho_{r+1}} \preccurlyeq g_{\rho_0}$ , and thus also  $a_{\rho_0} - z_{\rho_{r+1}} \preccurlyeq g_{\rho_0}$ . Hence

$$\text{ddeg } G_{+z_{\rho_{r+1}}, \times g_{\rho_0}} = \text{ddeg } G_{+z_{\rho_0}, \times g_{\rho_0}} = \text{ddeg } G_{+a_{\rho_0}, \times g_{\rho_0}} = 1$$

by Lemma 3.6. Thus, with  $z_{\rho_i}$  in the role of  $y_i$ , for  $0 \leq i \leq r+1$ , and  $g_{\rho_0}$  in the role of  $g$ , we have reached a contradiction with Lemma 3.26.  $\square$

Next a result playing the same role in Theorem 3.29 as Lemma 3.24 played in Theorem 3.25.

**Corollary 3.28.** *If  $K$  has the dh-configuration property,  $L$  is a d-henselian extension of  $K$  with few constants, and  $(a_\rho)$  is a divergent pc-sequence in  $K$  with minimal differential polynomial  $G(Y)$  over  $K$ , then  $a_\rho \rightsquigarrow b$  and  $G(b) = 0$  for some  $b \in L$ .*

PROOF. Follow the proof of Lemma 3.24, invoking Theorem 3.27 instead of d-algebraic maximality.  $\square$

The following, from [DP19], was the first result about differential-henselizations.

**Theorem 3.29.** *Suppose that  $K$  is asymptotic,  $\mathbf{k}$  is linearly surjective, and every immediate extension of  $K$  has the dh-configuration property. Then  $K$  has a d-henselization, and any two d-henselizations of  $K$  are isomorphic over  $K$ .*

PROOF. By Lemma 3.3 we have an immediate asymptotic d-henselian extension  $K^{\text{dh}}$  of  $K$  that is d-algebraic over  $K$  and has no proper d-henselian subfields containing  $K$ . By Theorems 3.27 and 3.25, there is up to isomorphism over  $K$  just one such extension.

Let  $L$  be an asymptotic d-henselian extension of  $K$ . To see that  $K^{\text{dh}}$  embeds over  $K$  into  $L$ , use an argument similar to that in the proof of Theorem 3.25, using Corollary 3.28 in place of Lemma 3.24. Thus  $K^{\text{dh}}$  is a d-henselization of  $K$  and any d-henselization of  $K$  is isomorphic over  $K$  to  $K^{\text{dh}}$ .  $\square$

**Corollary 3.30.** *Suppose that  $K$  is asymptotic,  $\mathbf{k}$  is linearly surjective, and every immediate extension of  $K$  has the dh-configuration property. Then any immediate d-henselian extension of  $K$  that is d-algebraic over  $K$  is a d-henselization of  $K$ .*

PROOF. Let  $K^{\text{dh}}$  be the d-henselization of  $K$  from the proof of Theorem 3.29 and let  $L$  be an immediate d-henselian extension of  $K$  that is d-algebraic over  $K$ . Then  $L$  is asymptotic by Lemmas 2.14 and 2.15. Hence we have an embedding  $K^{\text{dh}} \rightarrow L$  over  $K$ , which is surjective since  $K^{\text{dh}}$  is d-algebraically maximal by Theorem 3.27.  $\square$

## CHAPTER 4

# Differential-henselianity and maximality, part II

### 4.1. Introduction

In the previous chapter, we established three conjectures for  $K$  with finite rank value group: the uniqueness of maximal immediate extensions, the equivalence of d-henselianity and d-algebraic maximality, and the uniqueness of minimal d-henselian extensions. We now turn to the setting that the value group can have arbitrary rank, and establish these conjectures in the setting of asymptotic fields. (This is only a restriction for the first conjecture, as it is an assumption in the other two.) The results of this chapter appear in [Pyn20b], to be published in the Pacific Journal of Mathematics.

Suppose that  $K$  has small derivation. We continue to work in the category of valued differential fields with small derivation, so (valued differential field) extensions of  $K$  are assumed to have small derivation. In addition to the convention that  $d$ ,  $m$ ,  $n$ , and  $r$  are in  $\mathbb{N}$ , in this chapter we also let  $i$  range over  $\mathbb{N}$ .

**Theorem 4.18.** *If  $K$  is asymptotic and  $\mathbf{k}$  is linearly surjective, then any two maximal immediate extensions of  $K$  are isomorphic over  $K$ .*

**Theorem 4.19.** *If  $K$  is d-henselian and has few constants, then it is d-algebraically maximal.*

Recall that any  $K$  as in the theorem above is necessarily asymptotic.

**Theorem 4.20.** *If  $K$  is asymptotic and  $\mathbf{k}$  is linearly surjective, then  $K$  has a d-henselization, and any two d-henselizations of  $K$  are isomorphic over  $K$ .*

To prove these results, we show in Proposition 4.17 that henselian asymptotic  $K$  with linearly surjective differential residue field have the dh-configuration property, and then deduce the results above from this in the same way as in the previous chapter, modulo some arguments involving henselizations. We first prove that asymptotic  $K$  with linearly surjective differential residue field and divisible value group have the dh-configuration property in Proposition 4.15; deducing Proposition 4.17 uses the assumption of henselianity for technical reasons. We also use Proposition 4.15 to obtain analogues of Theorems 4.18, 4.19, and 4.20 relativized to differential polynomials of a fixed order in §4.3.3 (under the assumption of divisible value group). The rest of the chapter after §4.3 is devoted to the proof of Proposition 4.15. Our strategy closely follows the approach taken to prove [ADH17a, Proposition 14.5.1], which is an analogue of Proposition 4.15 in the setting of  $\omega$ -free  $H$ -asymptotic differential-valued fields (see Chapter 5 and Proposition 5.17).

First, we adapt the differential newton diagram method of [ADH17a, §13.5] to the setting of valued differential fields with small derivation and divisible value group in §4.4, which relies in an

essential way on the Equalizer Theorem [ADH17a, Theorem 6.0.1]; this is where divisibility is used. The main results are Proposition 4.28 and Corollary 4.29, which are then connected to pc-sequences in §4.4.1.

From there, we proceed to study asymptotic differential equations in §4.5, with the main technical notion being that of an unraveller, adapted from [ADH17a, §13.8]. There are three key steps in this section. First, the existence of an unraveller that is a pseudolimit of a pc-sequence in Lemma 4.39, via Proposition 4.36. Second, reducing the degree of an asymptotic differential equation in Lemma 4.41. Third, finding a solution of an asymptotic differential equation in a d-henselian field that approximates an element in an extension of that field in §4.5.3.

The penultimate section, §4.6, based on [ADH17a, §14.4], is quite technical. It combines many results from the previous sections and culminates in Proposition 4.47 and its Corollary 4.60, which is essential to the proof of Proposition 4.15. One of the main steps here is Lemma 4.57, which allows us to use Lemma 4.41 to reduce the degree of an asymptotic differential equation.

There are four salient differences from the approach in [ADH17a]. The first is that the “dominant part” and “dominant degree” of differential polynomials replace their more technical cousins “newton polynomial” and “newton degree,” leading to the simplification of some proofs. The second is that since  $K$  is not assumed to be  $H$ -asymptotic, we replace the convex valuation  $\psi$  on  $\Gamma^\neq$  with that given by considering archimedean classes. Third, under the assumption of  $\omega$ -freeness, newton polynomials are in  $C[Y](Y')^n$  for some  $n$ , but dominant parts need not have this special form. This leads to changes in §4.6, such as the need to take partial derivatives with respect to higher order derivatives of  $Y$  than just  $Y'$ .

## 4.2. Preliminaries

Throughout this section,  $P \in K\{Y\}^\neq$ .

**4.2.1. Dominant parts of differential polynomials.** In §3.2 we defined the dominant part of a differential polynomial over  $K$ . We present in this subsection slightly improved versions of lemmas from [ADH17a, §6.6] under the assumption that  $K$  has a *monomial group*  $\mathfrak{M}$ , i.e., a subgroup of  $K^\times$  that is mapped bijectively onto  $\Gamma$  by  $v$ . The proofs are essentially the same, so are omitted.

**Assumption.** In this subsection,  $K$  has a monomial group  $\mathfrak{M}$ .

We let  $\mathfrak{d}_P \in \mathfrak{M}$  be the unique monomial such that  $\mathfrak{d}_P \asymp P$ . For  $Q = 0 \in K\{Y\}$ , we set  $\mathfrak{d}_Q := 0$ .

**Definition.** Since  $\mathfrak{d}_P^{-1}P \in \mathcal{O}\{Y\}$ , we define the *dominant part* of  $P$  to be the differential polynomial

$$D_P := \overline{\mathfrak{d}_P^{-1}P} = \sum_i (\overline{P_i/\mathfrak{d}_P}) Y^i \in \mathbf{k}\{Y\}^\neq,$$

where  $i$  ranges over  $\mathbb{N}^{1+r}$  and  $r$  is the order of  $P$ . For  $Q = 0 \in K\{Y\}$ , we set  $D_Q := 0 \in \mathbf{k}\{Y\}$ .

Then  $\deg D_P \leq \deg P$  and  $\text{ord } D_P \leq \text{ord } P$ . We call  $\text{ddeg } P := \deg D_P$  the *dominant degree* of  $P$  and  $\text{dmul } P := \text{mul } D_P$  the *dominant multiplicity* of  $P$  at 0.

Note that if  $P$  is homogeneous of degree  $d$ , then so is  $D_P$ .

**Lemma 4.1.** *Let  $Q \in K\{Y\}$ . Then:*

- (i) if  $P \succ Q$ , then  $D_{P+Q} = D_P$ ;
- (ii) if  $P \asymp Q$  and  $P + Q \asymp P$ , then  $D_{P+Q} = D_P + D_Q$ ;
- (iii)  $D_{PQ} = D_P D_Q$ .

PROOF. Part (ii) is not in [ADH17a], so we give a proof. Suppose  $P \asymp Q$  and  $P + Q \asymp P$ , so  $\mathfrak{d}_{P+Q} = \mathfrak{d}_P = \mathfrak{d}_Q$ . Then, letting  $\mathbf{i} \in \mathbb{N}^{1+r}$  with  $r$  at least the orders of  $P$  and  $Q$ ,

$$D_{P+Q} = \sum_{\mathbf{i}} \overline{(P+Q)_{\mathbf{i}} / \mathfrak{d}_{P+Q}} Y^{\mathbf{i}} = \sum_{\mathbf{i}} \left( \overline{P_{\mathbf{i}} / \mathfrak{d}_P} + \overline{Q_{\mathbf{i}} / \mathfrak{d}_Q} \right) Y^{\mathbf{i}} = D_P + D_Q. \quad \square$$

**Lemma 4.2.** *Let  $a \in K$  with  $a \preccurlyeq 1$ . Then:*

- (i)  $D_{P+a} = (D_P)_{+a}$ , and thus  $\text{ddeg } P_{+a} = \text{ddeg } P$ ;
- (ii) if  $a \asymp 1$ , then  $D_{P \times a} = (D_P)_{\times a}$ ,  $\text{dmul } P_{\times a} = \text{dmul } P$ , and  $\text{ddeg } P_{\times a} = \text{ddeg } P$ .

We also use some of Lemmas 3.6–3.11 from the previous chapter.

#### 4.2.2. Constructing immediate extensions and vanishing.

**Assumption.** In this subsection, the derivation induced on  $\mathbf{k}$  is nontrivial and  $\Gamma$  has no least positive element.

The notion of minimal differential polynomial is not first-order, so we include here a first-order variant of Lemma 2.13 that is a special case of [ADH18, Lemma 5.3]. We then connect it to dominant degree in a cut. Under the assumptions above, all extensions of  $K$  are “strict” [ADH18, Lemma 1.3], and  $K$  is “flexible” [ADH18, Lemma 1.15 and Corollary 3.4]. These notions are defined in [ADH18] but are incidental here, and mentioned only since they occur in the corresponding lemmas of [ADH18].

Let  $\ell \notin K$  be an element in an extension of  $K$  such that  $v(\ell - K) := \{v(\ell - a) : a \in K\}$  has no largest element (equivalently,  $\ell$  is the pseudolimit of some divergent pc-sequence in  $K$ ). We say that  $P$  *vanishes at*  $(K, \ell)$  if for all  $a \in K$  and  $\mathfrak{v} \in K^\times$  with  $a - \ell \prec \mathfrak{v}$ ,  $\text{ddeg}_{\prec \mathfrak{v}} P_{+a} \geq 1$ . We let  $Z(K, \ell)$  be the set of nonzero differential polynomials over  $K$  vanishing at  $(K, \ell)$ .

**Lemma 4.3** ([ADH18, 4.6]). *Suppose that  $(a_\rho)$  is a divergent pc-sequence in  $K$  with  $a_\rho \rightsquigarrow \ell$ . If  $P(a_\rho) \rightsquigarrow 0$ , then  $P \in Z(K, \ell)$ .*

**Lemma 4.4** ([ADH18, 4.7]). *Suppose  $(a_\rho)$  is a divergent pc-sequence in  $K$  with  $a_\rho \rightsquigarrow \ell$ . Then*

$$\text{ddeg}_{\mathbf{a}} P = \min\{\text{ddeg}_{\prec \mathfrak{v}} P_{+a} : a - \ell \prec \mathfrak{v}\}.$$

*In particular,  $\text{ddeg}_{\mathbf{a}} P \geq 1 \iff P \in Z(K, \ell)$ .*

**Corollary 4.5.** *Suppose that  $(a_\rho)$  is a divergent pc-sequence in  $K$  with  $a_\rho \rightsquigarrow \ell$ . The following conditions on  $P$  are equivalent:*

- (i)  $P \in Z(K, \ell)$  and has minimal complexity in  $Z(K, \ell)$ ;
- (ii)  $P$  is a minimal differential polynomial of  $(a_\rho)$  over  $K$ .

PROOF. The proof is the same as that of [ADH17a, Corollary 11.4.13], using Lemma 4.6 in place of [ADH17a, Lemma 11.4.8], Lemma 2.11 in place of [ADH17a, Lemma 11.3.8], and Lemma 4.3 in place of [ADH17a, Lemma 11.4.11].  $\square$

In particular,  $Z(K, \ell) = \emptyset$  if and only if  $(a_\rho)$  is of d-transcendental type over  $K$ , and  $Z(K, \ell) \neq \emptyset$  if and only if  $(a_\rho)$  is of d-algebraic type over  $K$ .

**Lemma 4.6** ([ADH18, 5.3]). *Suppose that  $Z(K, \ell) \neq \emptyset$ , and  $P \in Z(K, \ell)$  has minimal complexity. Then  $K$  has an immediate extension  $K\langle f \rangle$  such that  $P(f) = 0$  and  $v(a - f) = v(a - \ell)$  for all  $a \in K$ . Moreover, if  $M$  is an extension of  $K$  and  $s \in M$  satisfies  $P(s) = 0$  and  $v(a - s) = v(a - \ell)$  for all  $a \in K$ , then there is a unique embedding  $K\langle f \rangle \rightarrow M$  over  $K$  sending  $f$  to  $s$ .*

**4.2.3. Archimedean classes and coarsening.** Recall from §2.2.1 the set of archimedean classes  $[\Gamma] := \{[\gamma] : \gamma \in \Gamma\}$ , ordered in the natural way. Giving  $[\Gamma]$  its reverse order, the map  $\gamma \mapsto [\gamma]$  is a convex valuation on  $\Gamma$ , and we often use the implication  $[\delta] < [\gamma] \implies [\delta + \gamma] = [\gamma]$ . Then for  $\phi \in K^\times$  with  $\phi \neq 1$ , the set  $\Gamma_\phi := \{\gamma : [\gamma] < [v\phi]\}$  is a convex subgroup of  $\Gamma$ . We will use  $v_\phi$ , the coarsening of  $v$  by  $\Gamma_\phi$ , and its corresponding dominance relation,  $\preceq_\phi$ , defined by

$$\begin{aligned} v_\phi : K^\times &\rightarrow \Gamma/\Gamma_\phi \\ a &\mapsto va + \Gamma_\phi, \end{aligned}$$

and  $a \preceq_\phi b$  if  $v_\phi(a) \geq v_\phi(b)$ . Note that the symbols  $v_\phi$  and  $\preceq_\phi$  also appeared in [ADH17a, §9.4], where they indicated a different coarsening of  $v$ .

The lemmas from the rest of this subsection will play an important role in §4.6. Lemmas 4.7–4.12 are variants of lemmas from the end of [ADH17a, §9.4]. The first two are facts about valued fields, not involving the derivation.

**Lemma 4.7.** *Let  $f, g \in K^\times$  with  $f, g \neq 1$ . Then  $f \prec_g g \implies f \prec_f g$ .*

PROOF. From  $f/g \prec_g 1$ , we obtain  $[vf - vg] \geq [vg]$ . If  $[vf - vg] < [vf]$ , then

$$[vf] = [vf - (vf - vg)] = [vg],$$

contradicting  $[vf - vg] \geq [vg]$ . Thus  $[vf - vg] \geq [vf]$ . But since  $vf - vg > 0$ , we have  $f/g \prec_f 1$ , that is,  $f \prec_f g$ .  $\square$

**Lemma 4.8.** *Let  $\phi_1, \phi_2 \in K^\times$  with  $\phi_1, \phi_2 \neq 1$  and  $[v\phi_1] \leq [v\phi_2]$ . Then for all  $f, g \in K$*

$$f \preceq_{\phi_1} g \implies f \preceq_{\phi_2} g \quad \text{and} \quad f \prec_{\phi_2} g \implies f \prec_{\phi_1} g.$$

*In particular,  $f \succ_{\phi_1} g \implies f \succ_{\phi_2} g$ .*

PROOF. Note that for  $\phi \in K^\times$  with  $\phi \neq 1$ ,  $f \preceq_\phi g$  if and only if  $vf - vg \in \Gamma_\phi$  or  $vf > vg$ , and  $f \prec_\phi g$  if and only if  $vf - vg > \Gamma_\phi$ . Both implications then follow from  $\Gamma_{\phi_1} \subseteq \Gamma_{\phi_2}$ .  $\square$

**Lemma 4.9.** *Suppose that  $P$  is homogeneous of degree  $d$ , and let  $g \in K^\times$  with  $g \neq 1$ . Then*

$$P_{\times g} \asymp_g g^d P.$$

PROOF. By Lemma 2.3,  $v(P_{\times g}) = vP + dvg + o(vg)$ , so  $v_g(P_{\times g}) = v_g(g^d P)$ .  $\square$

**Lemma 4.10.** *Suppose that  $g \in K^\times$  with  $g \prec 1$  and  $d := \text{dmul } P = \text{mul } P$ . Then  $P_{\times g} \asymp_g g^d P$ .*

PROOF. Since  $d = \text{dmul } P$ , we have  $P_i \preceq P_d$  for  $i \geq d$ . Since  $g \prec 1$ , we also have  $g \prec_g 1$ . Hence  $g^i P_i \prec_g g^d P_d$  for  $i > d$ , so

$$P_{\times g} \asymp_g P_{d, \times g} \asymp_g g^d P_d$$

by Lemma 4.9. In view of  $P_d \asymp P$ , this yields  $P_{\times g} \asymp_g g^d P$ .  $\square$

**Lemma 4.11.** *Suppose that  $g \in K^\times$  with  $g \succ 1$  and  $d := \text{ddeg } P = \text{ddeg } P_{\times g}$ . Then  $gP_{>d} \preccurlyeq_g P$ .*

PROOF. If  $P_{>d} = 0$ , then the result holds trivially, so assume that  $P_{>d} \neq 0$ . Take  $i > d$  such that  $P_i \asymp P_{>d}$ . Then Lemma 4.9 and the fact that  $g \succ 1$  give

$$(P_{\times g})_i = (P_i)_{\times g} \asymp_g g^i P_i \asymp g^{d+1} P_i \asymp g^{d+1} P_{>d},$$

so  $(P_{\times g})_{>d} \asymp_g g^{d+1} P_{>d}$ . Since  $\text{ddeg } P_{\times g} = d$ , we also have  $(P_{\times g})_d \succ (P_{\times g})_{>d}$ . As  $\text{ddeg } P = d$ ,  $P_d \asymp P$ , and so

$$g^d P \asymp g^d P_d \asymp_g (P_{\times g})_d \succ (P_{\times g})_{>d} \asymp_g g^{d+1} P_{>d},$$

using Lemma 4.9 again. Hence  $P \asymp_g gP_{>d}$ .  $\square$

**Lemma 4.12.** *Let  $f, g \in K^\times$  with  $f, g \not\asymp 1$  and  $[vf] < [vg]$ . Then  $P_{\times f} \asymp_g P$ .*

PROOF. Take  $d$  with  $P_{\times f} \asymp (P_{\times f})_d$ , so  $P_{\times f} \asymp_f f^d P_d$  by Lemma 4.9. Then Lemma 4.8 gives  $P_{\times f} \asymp_g f^d P_d$ . As  $[vf] < [vg]$ , we get  $f \asymp_g 1$ , and thus  $P_{\times f} \asymp_g P_d \preccurlyeq P$ , so  $P_{\times f} \preccurlyeq_g P$ . Now, apply the same argument to  $P_{\times f}$  and  $f^{-1}$  in place of  $P$  and  $f$ , using that  $[v(f^{-1})] = [-vf] = [vf]$ , to get  $P = (P_{\times f})_{\times f^{-1}} \preccurlyeq_g P_{\times f}$ , and hence  $P \asymp_g P_{\times f}$ .  $\square$

**Assumption.** In the next two results,  $K$  has a monomial group  $\mathfrak{M}$ .

Let  $\mathfrak{m}$  and  $\mathfrak{n}$  range over  $\mathfrak{M}$ . These results are based on [ADH17a, Lemma 13.2.3 and Corollary 13.2.4].

**Lemma 4.13.** *Let  $\mathfrak{n} \neq 1$  and  $[v\mathfrak{n}] < [v\mathfrak{n}]$ . Suppose that  $P = Q + R$  with  $R \prec_{\mathfrak{n}} P$ . Then*

$$D_{P_{\times \mathfrak{m}}} = D_{Q_{\times \mathfrak{m}}}.$$

PROOF. From  $R \prec_{\mathfrak{n}} Q$ , we get  $R \prec Q$ , so if  $\mathfrak{m} = 1$ , then  $D_P = D_Q$ . Now assume that  $\mathfrak{m} \neq 1$ . Then Lemma 4.12 gives

$$R_{\times \mathfrak{m}} \asymp_{\mathfrak{n}} R \prec_{\mathfrak{n}} Q \asymp_{\mathfrak{n}} Q_{\times \mathfrak{m}},$$

so  $R_{\times \mathfrak{m}} \prec Q_{\times \mathfrak{m}}$ , and hence  $D_{P_{\times \mathfrak{m}}} = D_{Q_{\times \mathfrak{m}}}$ .  $\square$

**Corollary 4.14.** *Suppose that  $\mathfrak{n} \succ 1$  and  $d := \text{ddeg } P = \text{ddeg } P_{\times \mathfrak{n}}$ . Let  $Q := P_{\leq d}$ . Then for all  $\mathfrak{m}$  with  $[v\mathfrak{m}] < [v\mathfrak{n}]$  and all  $g \preccurlyeq 1$  in  $K$ , we have*

$$D_{P_{+g, \times \mathfrak{m}}} = D_{Q_{+g, \times \mathfrak{m}}}.$$

PROOF. Let  $R := P - Q = P_{>d}$ . Then Lemma 4.11 gives

$$R \preccurlyeq_{\mathfrak{n}} \mathfrak{n}^{-1} P \prec_{\mathfrak{n}} P.$$

Let  $g \preccurlyeq 1$ . Then  $R_{+g} \asymp R$  and  $P_{+g} \asymp P$  by Lemma 2.2(i). Thus we have  $R_{+g} \prec_{\mathfrak{n}} P_{+g}$ , so it remains to apply the previous lemma.  $\square$



### 4.3. Main results

Assuming Proposition 4.15, we prove here the main results of this chapter concerning the uniqueness of maximal immediate extensions, the equivalence of d-algebraic maximality and d-henselianity, and the existence and uniqueness of d-henselizations. The proof of Proposition 4.15 is given in §4.7.

**Proposition 4.15.** *Suppose that  $K$  is asymptotic,  $\Gamma$  is divisible, and  $\mathbf{k}$  is  $r$ -linearly surjective. Let  $(a_\rho)$  be a pc-sequence in  $K$  with minimal differential polynomial  $G$  over  $K$  of order at most  $r$ . Then  $\text{ddeg}_a G = 1$ .*

**4.3.1. Removing divisibility.** In the next lemmas, we construe the algebraic closure  $K^{\text{ac}}$  of  $K$  as a valued differential field extension of  $K$  with small derivation as in §2.5.

**Lemma 4.16.** *Suppose that  $K$  is henselian and the derivation induced on  $\mathbf{k}$  is nontrivial. Let  $(a_\rho)$  be a pc-sequence in  $K$  with minimal differential polynomial  $P$  over  $K$ . Then  $P$  remains a minimal differential polynomial of  $(a_\rho)$  over  $K^{\text{ac}}$ .*

PROOF. We may suppose that  $(a_\rho)$  is divergent in  $K$ , the other case being trivial. Then  $(a_\rho)$  is still divergent in  $K^{\text{ac}}$ : If it had a pseudolimit  $a \in K^{\text{ac}}$ , then we would have  $Q(a_\rho) \rightsquigarrow 0$ , where  $Q \in K[Y]$  is the minimum polynomial of  $a$  over  $K$  (see [ADH17a, Proposition 3.2.1]). But since  $K$  is henselian, it is algebraically maximal (see [ADH17a, Corollary 3.3.21]), and then  $(a_\rho)$  would have a pseudolimit in  $K$ .

Now suppose to the contrary that  $Q$  is a minimal differential polynomial of  $(a_\rho)$  over  $K^{\text{ac}}$  with  $c(Q) < c(P)$ . Take an extension  $L \subseteq K^{\text{ac}}$  of  $K$  with  $[L : K] = n$  and  $Q \in L\{Y\}$ . Then as  $K$  is henselian,

$$[L : K] = [\Gamma_L : \Gamma] \cdot [\mathbf{k}_L : \mathbf{k}]$$

(see [ADH17a, Corollary 3.3.49]), so we have a valuation basis  $\mathcal{B} = \{e_1, \dots, e_n\}$  of  $L$  over  $K$  (see [ADH17a, Proposition 3.1.7]). That is,  $\mathcal{B}$  is a basis of  $L$  over  $K$ , and for all  $a_1, \dots, a_n \in K$ ,

$$v\left(\sum_{i=1}^n a_i e_i\right) = \min_{1 \leq i \leq n} v(a_i e_i).$$

Then by expressing the coefficients of  $Q$  in terms of the valuation basis,

$$Q(Y) = \sum_{i=1}^n R_i(Y) \cdot e_i,$$

where  $R_i \in K\{Y\}$  for  $1 \leq i \leq n$ .

Since  $Q$  is a minimal differential polynomial of  $(a_\rho)$  over  $K^{\text{ac}}$ , by Lemma 2.13 we have an immediate extension  $K^{\text{ac}}\langle a \rangle$  of  $K^{\text{ac}}$  with  $a_\rho \rightsquigarrow a$  and  $Q(a) = 0$ . Then by Lemma 2.11, there is a pc-sequence  $(b_\rho)$  in  $K$  equivalent to  $(a_\rho)$  such that  $Q(b_\rho) \rightsquigarrow Q(a) = 0$ . Finally, by passing to a cofinal subsequence, we can assume that we have  $i$  with  $Q(b_\rho) \asymp R_i(b_\rho) \cdot e_i$  for all  $\rho$ . Then  $R_i(b_\rho) \rightsquigarrow 0$  and  $c(R_i) < c(P)$ , contradicting the minimality of  $P$ .  $\square$

Since minimal differential polynomials are irreducible, a corollary of this lemma is that minimal differential polynomials over henselian  $K$  (with nontrivial induced derivation on  $\mathbf{k}$ ) are absolutely

irreducible. We can now replace the divisibility assumption in the main proposition with that of henselianity.

**Proposition 4.17.** *Suppose that  $K$  is asymptotic and henselian, and that  $\mathbf{k}$  is linearly surjective. Let  $(a_\rho)$  be a pc-sequence in  $K$  with minimal differential polynomial  $G$  over  $K$ . Then  $\text{ddeg}_{\mathbf{a}} G = 1$ .*

PROOF. By the previous lemma,  $G$  remains a minimal differential polynomial of  $(a_\rho)$  over  $K^{\text{ac}}$ . Note that the value group of  $K^{\text{ac}}$  is divisible, and its differential residue field is linearly surjective, as an algebraic extension of  $\mathbf{k}$  [ADH17a, Corollary 5.4.3]. Then  $\text{ddeg}_{\mathbf{a}_{K^{\text{ac}}}} G = 1$  by Proposition 4.15, and hence  $\text{ddeg}_{\mathbf{a}} G = 1$  by Lemma 3.13(vii).  $\square$

**4.3.2. Main results.** For the next result, we copy the proof of Theorem 3.25, except for an argument involving henselizations.

**Theorem 4.18.** *Suppose that  $K$  is asymptotic and  $\mathbf{k}$  is linearly surjective. Then any two maximal immediate extensions of  $K$  are isomorphic over  $K$ . Also, any two d-algebraically maximal d-algebraic immediate extensions of  $K$  are isomorphic over  $K$ .*

PROOF. Let  $L_0$  and  $L_1$  be maximal immediate extensions of  $K$ . By Zorn's lemma we have a maximal isomorphism  $\varphi: F_0 \cong_K F_1$  between valued differential subfields  $F_i \supseteq K$  of  $L_i$  for  $i = 0, 1$ , where “maximal” means that  $\varphi$  does not extend to an isomorphism between strictly larger such valued differential subfields. First,  $F_i$  is asymptotic by Lemmas 2.14 and 2.15, and  $\mathbf{k}_{F_i}$  is linearly surjective, as  $F_i$  is an immediate extension of  $K$  for  $i = 0, 1$ . Next, they must be henselian, because the henselization of  $F_i$  in  $L_i$  is an algebraic extension of  $F_i$ , and thus a valued differential subfield of  $L_i$  for  $i = 0, 1$ . Now suppose towards a contradiction that  $F_0 \neq L_0$  (equivalently,  $F_1 \neq L_1$ ). Then the proof continues as in the proof of Theorem 3.25; Lemma 3.24 is available by Proposition 4.17.  $\square$

By Proposition 4.17 and Theorem 3.27, we can now remove the monotonicity assumption from [ADH17a, Theorem 7.0.3], removing also the assumption on the value group from our earlier Theorem 3.2.

**Theorem 4.19.** *If  $K$  is asymptotic and d-henselian, then it is d-algebraically maximal.*

The following generalizes Theorem 3.4, removing the assumption on the value group. Its proof is essentially the same, except for the use of the henselization.

**Theorem 4.20.** *Suppose  $K$  is asymptotic and  $\mathbf{k}$  is linearly surjective. Then  $K$  has a d-henselization, and any two d-henselizations of  $K$  are isomorphic over  $K$ .*

PROOF. We can assume that  $K$  is henselian, as  $K$  has a henselization that embeds (uniquely) over  $K$  into any d-henselian extension of  $K$ . By Lemma 3.3 we have an immediate asymptotic d-henselian extension  $K^{\text{dh}}$  of  $K$  that is d-algebraic over  $K$  and has no proper d-henselian subfields containing  $K$ . By Theorems 4.19 and 4.18, there is up to isomorphism over  $K$  just one such extension.

Let  $L$  be an asymptotic d-henselian extension of  $K$ ; by Theorem 4.19,  $L$  is d-algebraically maximal. To see that  $K^{\text{dh}}$  embeds over  $K$  into  $L$ , use an argument similar to that in the proof of Theorem 4.18, using henselizations as in that proof. Thus  $K^{\text{dh}}$  is a d-henselization of  $K$  and any d-henselization of  $K$  is isomorphic over  $K$  to  $K^{\text{dh}}$ .  $\square$

This corollary has the same proof as Corollary 3.30.

**Corollary 4.21.** *If  $K$  is asymptotic and  $\mathbf{k}$  is linearly surjective, then any immediate d-henselian extension of  $K$  that is d-algebraic over  $K$  is a d-henselization of  $K$ .*

**4.3.3. Additional results.** We also record versions of the above results relativized to differential polynomials of a given order. In these results, we assume that  $\Gamma$  is divisible but not that  $K$  is henselian. The proofs are essentially the same as in §3.4, except for using Proposition 4.15 instead of the dh-configuration property.

To state these results, we make some definitions. If  $E$  and  $F$  are differential fields, then we say that  $F$  is  *$r$ -differentially algebraic* ( *$r$ -d-algebraic* for short) over  $E$  if for each  $a \in F$  there are  $a_1, \dots, a_n \in E$  such that  $a \in E\langle a_1, \dots, a_n \rangle$  and, for  $i = 0, \dots, n-1$ ,  $a_{i+1}$  is d-algebraic over  $E\langle a_1, \dots, a_i \rangle$  with minimal annihilator of order at most  $r$ . It is routine to prove that if  $L$  is  $r$ -d-algebraic over  $F$  and  $F$  is  $r$ -d-algebraic over  $E$ , then  $L$  is  $r$ -d-algebraic over  $E$ .

**Definition.** We call  $K$   *$r$ -differential-algebraically maximal* ( *$r$ -d-algebraically maximal* for short) if it has no proper immediate  $r$ -d-algebraic extension.

By Zorn,  $K$  has an immediate  $r$ -d-algebraic extension that is  $r$ -d-algebraically maximal. Note that  $K$  is d-algebraically maximal if and only if it is  $r$ -d-algebraically maximal for all  $r$ . In addition, if the derivation induced on  $\mathbf{k}$  is nontrivial, then by Lemmas 2.11 and 2.13  $K$  is  $r$ -d-algebraically maximal if and only if every pc-sequence in  $K$  with minimal differential polynomial over  $K$  of order at most  $r$  has a pseudolimit in  $K$ .

Note that  $K$  being 0-d-algebraically maximal means that it has no proper immediate valued differential field extension with small derivation that is algebraic over  $K$ . Since each algebraic field extension of  $K$ , given any valuation extending that of  $K$ , has small derivation by Proposition 2.16,  $K$  is 0-d-algebraically maximal if and only if it is algebraically maximal *as a valued field*. Thus the results below for  $r = 0$  follow from the corresponding results for valued fields and hence we may assume that  $r \geq 1$ , so the derivation induced on  $\mathbf{k}$  is nontrivial, as was used in §3.4.

**Theorem 4.22.** *If  $K$  is asymptotic,  $\Gamma$  is divisible, and  $\mathbf{k}$  is  $r$ -linearly surjective, then any two  $r$ -d-algebraically maximal  $r$ -d-algebraic immediate extensions of  $K$  are isomorphic over  $K$ .*

The proof of [ADH17a, Theorem 7.0.1] shows that if  $K$  is  $r$ -d-algebraically maximal and  $\mathbf{k}$  is  $r$ -linearly surjective, then  $K$  is  $r$ -d-henselian. Conversely:

**Theorem 4.23.** *If  $K$  is asymptotic and  $r$ -d-henselian, and  $\Gamma$  is divisible, then  $K$  is  $r$ -d-algebraically maximal.*

We say that an extension  $L$  of  $K$  is an  *$r$ -differential-henselization* ( *$r$ -d-henselization* for short) of  $K$  if it is an immediate asymptotic  $r$ -d-henselian extension of  $K$  that embeds over  $K$  into any asymptotic  $r$ -d-henselian extension of  $K$ . For the following, we need that Lemma 3.3 goes through with “ $r$ -linearly surjective,” “ $r$ -d-henselian,” and “ $r$ -d-algebraic” replacing “linearly surjective,” “d-henselian,” and “d-algebraic,” respectively, when  $\Gamma$  is divisible.

**Theorem 4.24.** *If  $K$  is asymptotic,  $\Gamma$  is divisible, and  $\mathbf{k}$  is  $r$ -linearly surjective, then  $K$  has an  $r$ -d-henselization, and any two  $r$ -d-henselizations of  $K$  are isomorphic over  $K$ .*

**Corollary 4.25.** *If  $K$  is asymptotic,  $\Gamma$  is divisible, and  $\mathbf{k}$  is  $r$ -linearly surjective, then any immediate  $r$ -d-henselian extension of  $K$  that is  $r$ -d-algebraic over  $K$  is an  $r$ -d-henselization of  $K$ .*

#### 4.4. Newton diagrams

We develop a differential newton diagram method for valued differential fields with small derivation. This approach is closely modelled on the differential newton diagram method for a certain class of asymptotic fields developed in [ADH17a, §13.5]. In §4.4.1, we connect this to dominant degree in a cut, adapting two lemmas from [ADH17a, §13.6]. The assumption of divisible value group allows us to use the Equalizer Theorem (Theorem 2.4), which underlies this method.

**Assumption.** In this section,  $K$  has a monomial group  $\mathfrak{M}$ .

Let  $P$  range over  $K\{Y\}^\neq$ ,  $f$  and  $g$  over  $K$ , and  $\mathfrak{m}$  and  $\mathfrak{n}$  over  $\mathfrak{M}$ . For  $f \in K^\times$ , let  $\mathfrak{d}_f$  be the unique monomial with  $\mathfrak{d}_f \asymp f$  and  $u_f := f/\mathfrak{d}_f \asymp 1$ .

**Lemma 4.26.** *Suppose that  $\Gamma^>$  has no least element and  $f \preccurlyeq \mathfrak{m}$ . If  $f \prec \mathfrak{m}$ , let  $u := 0$ ; if  $f \asymp \mathfrak{m}$ , let  $u := u_f$ . Then*

$$\text{ddeg}_{\prec \mathfrak{m}} P_{+f} = \text{mul}(D_{P_{\times \mathfrak{m}}})_{+\bar{u}}.$$

In particular,  $\text{ddeg}_{\prec \mathfrak{m}} P = \text{dmul } P_{\times \mathfrak{m}}$ .

PROOF. For  $\mathfrak{n} \prec \mathfrak{m}$ , let  $\mathfrak{e} := \mathfrak{n}\mathfrak{m}^{-1} \in \mathfrak{M}$ . Then

$$P_{+f, \times \mathfrak{n}} = P_{\times \mathfrak{m}, +\mathfrak{m}^{-1}f, \times \mathfrak{e}},$$

so by replacing  $P$  with  $P_{\times \mathfrak{m}}$  and  $f$  with  $\mathfrak{m}^{-1}f$ , we may assume that  $\mathfrak{m} = 1$ . Set  $Q := P_{+f}$ , so by Lemma 4.2(i),  $D_Q = (D_P)_{+\bar{f}} = (D_P)_{+\bar{u}}$ . Thus  $\text{mul}(D_P)_{+\bar{u}} = \text{dmul } Q$ , so it remains to show that

$$\text{ddeg}_{\prec 1} Q = \text{dmul } Q.$$

First,  $\text{ddeg}_{\prec 1} Q \leq \text{dmul } Q$  by Lemma 3.7. For the other direction, let  $d := \text{dmul } Q$ . We have  $v(Q_d) < v(Q_i)$  for all  $i < d$ , so take  $g \prec 1$  with  $vg$  small enough that

$$v(Q_d) + (d+1)vg < v(Q_i) \quad \text{for all } i < d.$$

It follows that

$$v(Q_d) + dvg + o(vg) < v(Q_i) + ivg + o(vg) \quad \text{for all } i < d,$$

so  $v(Q_{d, \times g}) < v(Q_{i, \times g})$  for all  $i < d$  by Lemma 2.3. Hence  $\text{dmul } Q_{\times g} \geq d$ . But

$$\text{ddeg}_{\prec 1} Q = \max\{\text{dmul } Q_{\times g} : g \prec 1\}$$

by Lemma 3.8, so  $\text{ddeg}_{\prec 1} Q \geq d$ , as desired.  $\square$

We call  $y \in K^\times$  an *approximate zero* of  $P$  if, for  $\mathfrak{m} := \mathfrak{d}_y$  and  $u := u_y$ ,  $D_{P_{\times \mathfrak{m}}}(\bar{u}) = 0$ . If  $y$  is an approximate zero of  $P$ , we define its *multiplicity* to be  $\text{mul}(D_{P_{\times \mathfrak{m}}})_{+\bar{u}}$ . We call  $\mathfrak{m}$  an *algebraic starting monomial* for  $P$  if  $D_{P_{\times \mathfrak{m}}}$  is not homogeneous. In particular, if  $\mathfrak{m}$  is an algebraic starting monomial for  $P$ , then  $\text{ddeg } P_{\times \mathfrak{m}} \geq 1$ . Note that  $\mathfrak{m}$  is an algebraic starting monomial for  $P$  if and only if  $\mathfrak{m}/\mathfrak{n}$  is an algebraic starting monomial for  $P_{\times \mathfrak{n}}$ . By Lemma 3.7,  $P$  has at most  $\text{deg } P - \text{mul } P$  algebraic starting monomials.

**Assumption.** In the rest of this section,  $\Gamma$  is divisible.

The existence of algebraic starting monomials is an easy corollary of the Equalizer Theorem, and is crucial to what follows.

**Lemma 4.27.** *Let  $P, Q \in K\{Y\}^\neq$  be homogeneous of different degrees. Then there exists a unique  $\mathfrak{m}$  such that  $D_{(P+Q) \times \mathfrak{m}}$  is not homogeneous.*

PROOF. By Theorem 2.4, there is a unique  $\mathfrak{m}$  such that  $P_{\times \mathfrak{m}} \asymp Q_{\times \mathfrak{m}}$ . Then

$$D_{(P+Q) \times \mathfrak{m}} = D_{P_{\times \mathfrak{m}} + Q_{\times \mathfrak{m}}} = D_{P_{\times \mathfrak{m}}} + D_{Q_{\times \mathfrak{m}}}$$

by Lemma 4.1(ii), so  $D_{(P+Q) \times \mathfrak{m}}$  is not homogeneous. For  $\mathfrak{n} \neq \mathfrak{m}$ , we have  $D_{(P+Q) \times \mathfrak{n}} = D_{P_{\times \mathfrak{n}}}$  or  $D_{(P+Q) \times \mathfrak{n}} = D_{Q_{\times \mathfrak{n}}}$  by Lemma 4.1(i), since  $P_{\times \mathfrak{n}} \succ Q_{\times \mathfrak{n}}$  or  $P_{\times \mathfrak{n}} \prec Q_{\times \mathfrak{n}}$ .  $\square$

For  $P$  and  $Q$  as in Lemma 4.27, we let  $\mathfrak{e}(P, Q)$  denote the unique monomial which that lemma yields and call it the *equalizer* for  $P, Q$ . We are interested in the case that these two differential polynomials are homogeneous parts of the same differential polynomial. Let  $J := \{j \in \mathbb{N} : P_j \neq 0\}$  and note that  $\text{ddeg } P_{\times \mathfrak{m}} \in J$  for all  $\mathfrak{m}$ . For distinct  $i, j \in J$ , let  $\mathfrak{e}(P, i, j) := \mathfrak{e}(P_i, P_j)$ , and so any algebraic starting monomial for  $P$  is of the form  $\mathfrak{e}(P, i, j)$  for some distinct  $i, j \in J$ .

In the next two results, let  $\mathcal{E} \subseteq K^\times$  be  $\preccurlyeq$ -closed. Recall this means that  $\mathcal{E} \neq \emptyset$  and  $f \in \mathcal{E}$  whenever  $0 \neq f \preccurlyeq g$  with  $g \in \mathcal{E}$ .

**Proposition 4.28.** *There exist  $i_0, \dots, i_n \in J$  and equalizers*

$$\mathfrak{e}(P, i_0, i_1) \prec \mathfrak{e}(P, i_1, i_2) \prec \dots \prec \mathfrak{e}(P, i_{n-1}, i_n)$$

*with  $\text{mul } P = i_0 < \dots < i_n = \text{ddeg}_{\mathcal{E}} P$  such that*

- (i) *the algebraic starting monomials for  $P$  in  $\mathcal{E}$  are the  $\mathfrak{e}(P, i_m, i_{m+1})$  for  $m < n$ ;*
- (ii) *for  $m < n$  and  $\mathfrak{m} = \mathfrak{e}(P, i_m, i_{m+1})$ , we have  $\text{dmul } P_{\times \mathfrak{m}} = i_m$  and  $\text{ddeg } P_{\times \mathfrak{m}} = i_{m+1}$ .*

PROOF. Let  $i$  and  $j$  range over  $J$  and  $d := \text{ddeg}_{\mathcal{E}} P$ . Then  $\text{mul } P \leq d \leq \deg P$ , and we proceed by induction on  $d - \text{mul } P$ . If  $d = \text{mul } P$ , then for all  $\mathfrak{m} \in \mathcal{E}$ ,  $D_{P_{\times \mathfrak{m}}}$  is homogeneous of degree  $d$ , so there is no algebraic starting monomial for  $P$  in  $\mathcal{E}$ .

Now assume that  $d > \text{mul } P$  and take  $i < d$  such that  $\mathfrak{e} := \mathfrak{e}(P, i, d) \succcurlyeq \mathfrak{e}(P, j, d)$  for all  $j < d$ . First, we show that  $\mathfrak{e} \in \mathcal{E}$ . We have  $P_{d, \times \mathfrak{e}} \asymp P_{i, \times \mathfrak{e}}$  by the previous lemma, so  $v_{P_d}(v\mathfrak{e}) = v_{P_i}(v\mathfrak{e})$ . By Lemma 2.3, the function  $v_{P_d} - v_{P_i}$  is strictly increasing, so for any  $g \prec \mathfrak{e}$ ,  $v_{P_d}(vg) > v_{P_i}(vg)$ , that is,  $P_{d, \times g} \prec P_{i, \times g}$ . Hence  $\text{ddeg } P_{\times g} < d$ . To obtain  $\mathfrak{e} \in \mathcal{E}$ , take  $g \in \mathcal{E}$  with  $\text{ddeg } P_{\times g} = d$ , so  $\mathfrak{e} \preccurlyeq g$ .

Next, we show that  $\text{ddeg } P_{\times \mathfrak{e}} = d$ . If  $\text{ddeg } P_{\times \mathfrak{e}} = j < d$ , then  $P_{d, \times \mathfrak{e}} \prec P_{j, \times \mathfrak{e}}$ . By Lemma 2.3 again, the function  $v_{P_d} - v_{P_j}$  is strictly increasing, so it follows that  $\mathfrak{e} \prec \mathfrak{e}(P, j, d)$ , contradicting the maximality of  $\mathfrak{e}$ .

From this and  $P_{i, \times \mathfrak{e}} \asymp P_{d, \times \mathfrak{e}}$ , we get  $(D_{P_{\times \mathfrak{e}}})_i = D_{P_{i, \times \mathfrak{e}}} \neq 0$  and  $(D_{P_{\times \mathfrak{e}}})_d = D_{P_{d, \times \mathfrak{e}}} \neq 0$ , and hence  $\mathfrak{e}$  is an algebraic starting monomial for  $P$ . In fact,  $\mathfrak{e}$  is the largest algebraic starting monomial for  $P$  in  $\mathcal{E}$ . Suppose to the contrary that  $\mathfrak{n} \in \mathcal{E}$  is an algebraic starting monomial for  $P$  with  $\mathfrak{n} \succ \mathfrak{e}$ . Then  $d = \text{ddeg } P_{\times \mathfrak{e}} \leq \text{ddeg } P_{\times \mathfrak{n}}$  by Lemma 3.7, so  $\text{ddeg } P_{\times \mathfrak{n}} = d$ . It follows that  $\mathfrak{n} = \mathfrak{e}(P, j, d)$  for some  $j < d$ , contradicting the maximality of  $\mathfrak{e}$ .

If  $i > \text{dmul } P_{\times \mathfrak{e}}$ , then for  $j := \text{dmul } P_{\times \mathfrak{e}}$ , the uniqueness in Lemma 4.27 yields  $\mathfrak{e}(P, j, d) = \mathfrak{e}$ . By replacing  $i$  with  $j$ , we assume that  $i = \text{dmul } P_{\times \mathfrak{e}}$ . Then by Lemma 4.26, we also have  $\text{ddeg}_{\prec \mathfrak{e}} P = i$ . To complete the proof, we apply the inductive assumption with  $\{g \in K^\times : g \prec \mathfrak{e}\}$  replacing  $\mathcal{E}$ .  $\square$

The tuple  $(i_0, \dots, i_n)$  from Proposition 4.28 is uniquely determined by  $K$ ,  $P$ , and  $\mathcal{E}$ . Note that if  $\text{mul } P = \text{ddeg}_{\mathcal{E}} P$ , then  $n = 0$  and the tuple is  $(\text{mul } P)$ . For  $1 \leq m \leq n$ , set  $\mathfrak{e}_m := \mathfrak{e}(P, i_{m-1}, i_m)$ . We now show how  $\text{dmul } P_{\times g}$  and  $\text{ddeg } P_{\times g}$  behave as  $g$  ranges over  $\mathcal{E}$ .

**Corollary 4.29.** *Suppose that  $\text{mul } P \neq \text{ddeg}_{\mathcal{E}} P$ , so  $n \geq 1$ . Let  $g$  range over  $\mathcal{E}$ . Then  $\text{dmul } P_{\times g}$  and  $\text{ddeg } P_{\times g}$  are in  $\{i_0, \dots, i_n\}$  and we have:*

$$\begin{aligned} \text{dmul } P_{\times g} = i_0 &\iff g \preceq \mathfrak{e}_1; \\ \text{ddeg } P_{\times g} = i_0 &\iff g \prec \mathfrak{e}_1; \\ \text{dmul } P_{\times g} = i_m &\iff \mathfrak{e}_m \prec g \preceq \mathfrak{e}_{m+1}, \quad (1 \leq m < n); \\ \text{ddeg } P_{\times g} = i_m &\iff \mathfrak{e}_m \preceq g \prec \mathfrak{e}_{m+1}, \quad (1 \leq m < n); \\ \text{dmul } P_{\times g} = i_n &\iff \mathfrak{e}_n \prec g; \\ \text{ddeg } P_{\times g} = i_n &\iff \mathfrak{e}_n \preceq g. \end{aligned}$$

PROOF. We first prove the third equivalence, so let  $1 \leq m < n$ . Then for  $\mathfrak{e}_m \prec g \prec \mathfrak{e}_{m+1}$ , Proposition 4.28 and Lemma 3.7 give

$$i_m = \text{ddeg } P_{\times \mathfrak{e}_m} \leq \text{dmul } P_{\times g} \leq \text{ddeg } P_{\times g} \leq \text{dmul } P_{\times \mathfrak{e}_{m+1}} = i_m,$$

which yields the right-to-left direction since if  $g \succ \mathfrak{e}_{m+1}$ , then  $\text{dmul } P_{\times g} = \text{dmul } P_{\times \mathfrak{e}_{m+1}} = i_m$ . For the converse, note that similarly, if  $g \preceq \mathfrak{e}_m$ , then  $\text{dmul } P_{\times g} \leq \text{dmul } P_{\times \mathfrak{e}_m} = i_{m-1}$ , and if  $g \succ \mathfrak{e}_{m+1}$ , then  $\text{dmul } P_{\times g} \geq \text{ddeg } P_{\times \mathfrak{e}_{m+1}} = i_{m+1}$ . The fourth equivalence is proved in the same way.

For the first equivalence, if  $g \prec \mathfrak{e}_1$ , then

$$i_0 = \text{mul } P \leq \text{dmul } P_{\times g} \leq \text{ddeg } P_{\times g} \leq \text{dmul } P_{\times \mathfrak{e}_1} = i_0,$$

and if  $g \succ \mathfrak{e}_1$ , then  $\text{dmul } P_{\times g} = \text{dmul } P_{\times \mathfrak{e}_1} = i_0$ . The converse follows as in the third equivalence. The remaining equivalences are proved similarly.  $\square$

#### 4.4.1. Application to dominant degree in a cut.

**Lemma 4.30.** *Suppose that  $(a_\rho)$  is a pc-sequence in  $K$  with  $a_\rho \rightsquigarrow 0$ , and let*

$$\mathcal{E}_{\mathbf{a}} := \{g \in K^\times : g \prec a_\rho, \text{ eventually}\}.$$

- (i) *If  $\mathcal{E}_{\mathbf{a}} \neq \emptyset$ , then  $\text{ddeg}_{\mathbf{a}} P = \text{ddeg}_{\mathcal{E}_{\mathbf{a}}} P$ .*
- (ii) *If  $\mathcal{E}_{\mathbf{a}} = \emptyset$ , then  $\text{ddeg}_{\mathbf{a}} P = \text{mul } P$ .*

PROOF. Set  $\gamma_\rho := v(a_{\rho+1} - a_\rho)$ . By removing some initial terms, we may assume that  $\gamma_\rho$  is strictly increasing and that  $v(a_\rho) = \gamma_\rho \in \Gamma$  for all  $\rho$ . Then by Lemma 3.9,

$$\text{ddeg}_{\geq \gamma_\rho} P_{+a_\rho} = \text{ddeg}_{\geq \gamma_\rho} P = \text{ddeg } P_{\times a_\rho},$$

so  $\text{ddeg}_{\mathbf{a}} P$  is the eventual value of  $\text{ddeg } P_{\times a_\rho}$ . If  $P$  is homogeneous, then  $\text{ddeg } P_{\times g} = \deg P = \text{mul } P$  for all  $g \in K^\times$ , so the statements about  $\text{ddeg}_{\mathbf{a}} P$  are immediate.

Suppose now that  $P$  is not homogeneous, so  $\text{mul } P < \deg P$ . If  $\mathcal{E}_a \neq \emptyset$ , we use Corollary 4.29 with  $K^\times$  in the role of  $\mathcal{E}$ , and we have the tuple  $(i_0, \dots, i_n)$  with  $i_n = \deg P$ . By removing further initial terms, we may assume that  $\text{ddeg } P_{\times a_\rho}$  is constant. If  $\text{ddeg } P_{\times a_\rho} = i_0$ , then  $a_\rho \prec \mathfrak{e}_1$ . Thus for any  $g \in \mathcal{E}_a$ , we have  $g \prec \mathfrak{e}_1$ , and hence  $\text{ddeg}_{\mathcal{E}_a} P = i_0$ . If  $\text{ddeg } P_{\times a_\rho} = i_m$  for any  $1 \leq m \leq n$ , then  $\mathfrak{e}_m \preccurlyeq a_\rho$ . As  $\gamma_\rho$  is strictly increasing,  $\mathfrak{e}_m \prec a_\rho$  for all  $\rho$ , so  $\mathfrak{e}_m \in \mathcal{E}_a$ . Hence  $\text{ddeg}_{\mathcal{E}_a} P \geq \text{ddeg } P_{\times \mathfrak{e}_m} = i_m$ . But by Lemma 3.7,  $\text{ddeg } P_{\times a_\rho} \geq \text{ddeg}_{\mathcal{E}_a} P$ , so  $\text{ddeg}_{\mathcal{E}_a} P = i_m$ .

If  $\mathcal{E}_a = \emptyset$ , let  $i_0 := \text{mul } P$ . Then for all  $i > i_0$ , by Lemma 2.3,

$$v_{P_i}(\gamma_\rho) - v_{P_{i_0}}(\gamma_\rho) = v(P_i) - v(P_{i_0}) + (i - i_0)\gamma_\rho + o(\gamma_\rho).$$

As  $\gamma_\rho$  is cofinal in  $\Gamma$ , we thus have  $v_P(\gamma_\rho) = v_{P_{i_0}}(\gamma_\rho) < v_{P_i}(\gamma_\rho)$ , eventually, for all  $i > i_0$ , and so  $\text{ddeg } P_{\times a_\rho} = i_0$ , eventually.  $\square$

With  $(a_\rho)$  and  $\mathcal{E}_a$  as in the above lemma, if  $\mathcal{E}_a = \emptyset$ , then  $(a_\rho)$  is in fact a Cauchy sequence in  $K$  (see [ADH17a, §2.2]), since  $\gamma_\rho$  is cofinal in  $\Gamma$ ; this is not used later.

**Corollary 4.31.** *Suppose  $(b_\rho)$  is a pc-sequence in  $K$  with pseudolimit  $b \in K$ . Let  $\mathbf{b} := c_K(b_\rho)$  and*

$$\mathcal{E}_{\mathbf{b}} := \{g \in K^\times : g \prec b_\rho - b, \text{ eventually}\}.$$

- (i) *If  $\mathcal{E}_{\mathbf{b}} \neq \emptyset$ , then  $\text{ddeg}_{\mathbf{b}} P = \text{ddeg}_{\mathcal{E}_{\mathbf{b}}} P_{+b}$ .*
- (ii) *If  $\mathcal{E}_{\mathbf{b}} = \emptyset$ , then  $\text{ddeg}_{\mathbf{b}} P = \text{mul } P_{+b}$ .*

PROOF. Set  $a_\rho := b_\rho - b$ . By Lemma 3.13(ii), we have

$$\text{ddeg}_{\mathbf{b}} P = \text{ddeg}_{a+b} P = \text{ddeg}_a P_{+b}.$$

It remains to apply the previous lemma with  $P_{+b}$  in place of  $P$ .  $\square$

## 4.5. Asymptotic differential equations

**Assumption.** In this section,  $K$  has a monomial group  $\mathfrak{M}$  and  $\Gamma^>$  has no least element.

Let  $\mathfrak{m}$  range over  $\mathfrak{M}$  and  $P \in K\{Y\}^\neq$  have order at most  $r$ . An *asymptotic differential equation* over  $K$  is something of the form

$$(E) \quad P(Y) = 0, \quad Y \in \mathcal{E},$$

where  $\mathcal{E} \subseteq K^\times$  is  $\preccurlyeq$ -closed. That is, it consists of an algebraic differential equation with an asymptotic condition on solutions. If  $\mathcal{E} = \{g \in K^\times : g \prec f\}$  for some  $f \in K^\times$ , then we write  $Y \prec f$  for the asymptotic condition instead of  $Y \in \mathcal{E}$ , and similarly with “ $\preccurlyeq$ .”

For the rest of this section, we fix this asymptotic differential equation (E). Then the *dominant degree* of (E) is defined to be  $\text{ddeg}_{\mathcal{E}} P$ . A *solution* of (E) is a  $y \in \mathcal{E}$  such that  $P(y) = 0$ . An *approximate solution* of (E) is an approximate zero of  $P$  that lies in  $\mathcal{E}$ , and the *multiplicity* of an approximate solution of (E) is its multiplicity as an approximate zero of  $P$ . The following is used frequently and follows from Lemma 4.26.

**Corollary 4.32.** *Let  $y \in \mathcal{E}$ . Then*

- (i)  *$y$  is an approximate solution of (E)  $\iff \text{ddeg}_{\prec y} P_{+y} \geq 1$ ;*

(ii) if  $y$  is an approximate solution of (E), then its multiplicity is  $\text{ddeg}_{\prec_y} P_{+y}$ .

A *starting monomial* for (E) is a starting monomial for  $P$  that lies in  $\mathcal{E}$ . An *algebraic starting monomial* for (E) is an algebraic starting monomial for  $P$  that lies in  $\mathcal{E}$ . So if  $\text{ddeg}_{\mathcal{E}} P = 0$ , then (E) has no algebraic starting monomials. By Proposition 4.28, if  $\Gamma$  is divisible and  $\text{mul } P < \text{ddeg}_{\mathcal{E}} P$ , then there is an algebraic starting monomial for (E) and  $\text{ddeg}_{\mathcal{E}} P = \text{ddeg } P_{\times \epsilon}$ , where  $\epsilon$  is the largest algebraic starting monomial for (E).

It will be important to alter  $P$  and  $\mathcal{E}$  in certain ways. Namely, let  $\mathcal{E}' \subseteq \mathcal{E}$  be  $\prec$ -closed and let  $f \in \mathcal{E} \cup \{0\}$ . We call the asymptotic differential equation

$$(E') \quad P_{+f}(Y) = 0, \quad Y \in \mathcal{E}'$$

a *refinement* of (E). Below,  $(E')$  refers to a refinement of this form. By Lemma 3.9,

$$\text{ddeg}_{\mathcal{E}} P = \text{ddeg}_{\mathcal{E}} P_{+f} \geq \text{ddeg}_{\mathcal{E}'} P_{+f},$$

so the dominant degree of  $(E')$  is at most the dominant degree of (E). Note also that if  $y$  is a solution of  $(E')$  and  $f + y \neq 0$ , then  $f + y$  is a solution of (E). The same is true with “approximate solution” replacing “solution,” provided that  $y \not\prec -f$ .

Here is a sufficient condition for being an approximate solution.

**Lemma 4.33.** *Let  $f \neq 0$  with  $f \succ g$  for all  $g \in \mathcal{E}'$ , and suppose that*

$$\text{ddeg}_{\mathcal{E}'} P_{+f} = \text{ddeg}_{\mathcal{E}} P \geq 1.$$

*Then  $f$  is an approximate solution of (E).*

PROOF. We have, using Lemma 3.9 for the equality,

$$\text{ddeg}_{\mathcal{E}'} P_{+f} \leq \text{ddeg}_{\prec_f} P_{+f} \leq \text{ddeg}_{\preccurlyeq f} P_{+f} = \text{ddeg}_{\preccurlyeq f} P \leq \text{ddeg}_{\mathcal{E}} P.$$

Hence  $\text{ddeg}_{\prec_f} P_{+f} = \text{ddeg}_{\mathcal{E}} P \geq 1$ , so  $f$  is an approximate solution of (E).  $\square$

Note that by the previous proof,  $\text{ddeg}_{\prec_f} P_{+f} \leq \text{ddeg}_{\mathcal{E}} P$  for all  $f \in \mathcal{E}$ .

**Lemma 4.34.** *Suppose that  $d := \text{ddeg}_{\mathcal{E}} P \geq 1$ . Then the following are equivalent:*

- (i)  $\text{ddeg}_{\prec_f} P_{+f} < d$  for all  $f \in \mathcal{E}$ ;
- (ii)  $\text{ddeg}_{\prec_f} P_{+f} < d$  for all  $f \in \mathcal{E}$  with  $\text{ddeg } P_{\times f} = d$ ;
- (iii) there is no approximate solution of (E) of multiplicity  $d$ .

PROOF. The equivalence of (i) and (iii) is given by Corollary 4.32. Now, let  $f \in \mathcal{E}$  and suppose that  $\text{ddeg } P_{\times f} < d$ . Then, using Lemma 3.9 for the first equality,

$$\text{ddeg}_{\prec_f} P_{+f} \leq \text{ddeg}_{\preccurlyeq f} P_{+f} = \text{ddeg}_{\preccurlyeq f} P = \text{ddeg } P_{\times f} < d.$$

This gives (ii)  $\implies$  (i), and the converse is trivial.  $\square$

We say that (E) is *unravelled* if  $d := \text{ddeg}_{\mathcal{E}} P \geq 1$  and the conditions in Lemma 4.34 hold. In particular, if  $d \geq 1$  and (E) does not have an approximate solution, then (E) is unravelled. And if (E) is unravelled and has an approximate solution, then  $d \geq 2$  by Lemma 4.34(iii).



We now introduce unravellers and partial unravellers, which correspond to special refinements of (E). In the proof of Proposition 4.36, we construct a sequence of partial unravellers ending in an unravelled asymptotic differential equation. Suppose that  $d \geq 1$ , and let  $f \in \mathcal{E} \cup \{0\}$  and  $\mathcal{E}' \subseteq \mathcal{E}$  be  $\preceq$ -closed. We say that  $(f, \mathcal{E}')$  is a *partial unraveller* for (E) if  $\text{ddeg}_{\mathcal{E}'} P_{+f} = d$ . By Lemma 3.9,  $(f, \mathcal{E})$  is a partial unraveller for (E). Note that if  $(f, \mathcal{E}')$  is a partial unraveller for (E) and  $(f_1, \mathcal{E}_1)$  is a partial unraveller for (E'), then  $(f + f_1, \mathcal{E}_1)$  is a partial unraveller for (E). An *unraveller* for (E) is a partial unraveller  $(f, \mathcal{E}')$  for (E) such that (E') is unravelled.

The following is routine but is used later.

**Lemma 4.35.** *Suppose that  $\text{ddeg}_{\mathcal{E}} P \geq 1$ . Let  $a \in K^\times$  and set  $a\mathcal{E} := \{ay \in K^\times : y \in \mathcal{E}\}$ . Consider the asymptotic differential equation*

$$(aE) \quad P_{\times a^{-1}}(Y) = 0, \quad Y \in a\mathcal{E}.$$

- (i) *The dominant degree of (aE) equals the dominant degree of (E).*
- (ii) *If  $(f, \mathcal{E}')$  is a partial unraveller for (E), then  $(af, a\mathcal{E}')$  is a partial unraveller for (aE).*
- (iii) *If  $(f, \mathcal{E}')$  is an unraveller for (E), then  $(af, a\mathcal{E}')$  is an unraveller for (aE).*
- (iv) *If  $a \in \mathfrak{M}$ , then the algebraic starting monomials for (aE) are exactly the elements  $a\mathfrak{e}$ , where  $\mathfrak{e}$  ranges over the algebraic starting monomials for (E).*

The next proposition is about the existence of unravellers, and is a key ingredient in the proof of Proposition 4.15. Recall the notion of  $r$ -d-algebraic maximality from §4.3.3, and in particular that if the derivation induced on  $\mathbf{k}$  is nontrivial, then  $K$  is  $r$ -d-algebraically maximal if and only if every pc-sequence with a minimal differential polynomial over  $K$  of order at most  $r$  has a pseudolimit in  $K$ .

**Proposition 4.36.** *Suppose that  $K$  is  $r$ -d-algebraically maximal,  $\Gamma$  is divisible, and the derivation induced on  $\mathbf{k}$  is nontrivial. Suppose that  $d := \text{ddeg}_{\mathcal{E}} P \geq 1$  and that there is no  $f \in \mathcal{E} \cup \{0\}$  with  $\text{mul } P_{+f} = d$ . Then there exists an unraveller for (E).*

PROOF. In this proof, we let  $\nu$  be an ordinal, in addition to  $\rho$ ,  $\lambda$ , and  $\mu$ . We construct a sequence  $((f_\lambda, \mathcal{E}_\lambda))_{\lambda < \rho}$  of partial unravellers for (E) indexed by  $\rho > 0$  such that:

- (i)  $\mathcal{E}_\lambda \supseteq \mathcal{E}_\mu$  for all  $\lambda < \mu < \rho$ ;
- (ii)  $f_\mu - f_\lambda \succ f_\nu - f_\mu$  for all  $\lambda < \mu < \nu < \rho$ ;
- (iii)  $f_{\lambda+1} - f_\lambda \in \mathcal{E}_\lambda \setminus \mathcal{E}_{\lambda+1}$  for all  $\lambda$  with  $\lambda + 1 < \rho$ .

For  $\rho = 1$ , we set  $(f_0, \mathcal{E}_0) := (0, \mathcal{E})$  and these conditions are vacuous. Below, we frequently use that by (ii) we have  $f_\mu - f_\lambda \asymp f_{\lambda+1} - f_\lambda$  for all  $\lambda < \mu < \rho$ .

First, suppose that  $\rho$  is a successor ordinal, so  $\rho = \nu + 1$ , and consider the refinement

$$(E_\nu) \quad P_{+f_\nu}(Y) = 0, \quad Y \in \mathcal{E}_\nu$$

of (E). If  $(E_\nu)$  is unravelled, then  $(f_\nu, \mathcal{E}_\nu)$  is an unraveller for (E) and we are done, so suppose that  $(E_\nu)$  is not unravelled. Take  $f \in \mathcal{E}_\nu$  such that  $\text{ddeg}_{\prec f}(P_{+f_\nu})_{+f} = d$ . Then

$$\mathcal{E}_\rho := \{y \in K^\times : y \prec f\} \subset \mathcal{E}_\nu$$

is  $\preccurlyeq$ -closed with

$$\text{ddeg}_{\mathcal{E}_\rho}(P_{+f_\nu})_{+f} = d,$$

so  $(f, \mathcal{E}_\rho)$  is a partial unraveller for  $(E_\nu)$ . Thus, setting  $f_\rho := f_\nu + f$ , we have that  $(f_\rho, \mathcal{E}_\rho)$  is a partial unraveller for  $(E)$ . Conditions (i) and (iii) on  $((f_\lambda, \mathcal{E}_\lambda))_{\lambda < \rho+1}$  with  $\rho + 1$  in place of  $\rho$  are obviously satisfied. For (ii), it is sufficient to check that  $f_{\lambda+1} - f_\lambda \succ f_\rho - f_\nu = f$  for  $\lambda < \nu$ , which follows from  $f_{\lambda+1} - f_\lambda \notin \mathcal{E}_\nu$ .

Now suppose that  $\rho$  is a limit ordinal. By (ii),  $(f_\lambda)_{\lambda < \rho}$  is a pc-sequence in  $K$ , so we let  $\mathbf{f} := c_K(f_\lambda)$  and claim that  $\text{ddeg}_{\mathbf{f}} P = d$ . To see this, set  $g_\lambda := f_{\lambda+1} - f_\lambda$  for  $\lambda$  with  $\lambda + 1 < \rho$ . By (iii), we have, using Lemma 3.9 in the third line,

$$\begin{aligned} d &= \text{ddeg}_{\mathcal{E}_{\lambda+1}} P_{+f_{\lambda+1}} \leq \text{ddeg}_{\preccurlyeq g_\lambda} P_{+f_{\lambda+1}} \\ &= \text{ddeg}_{\preccurlyeq g_\lambda} (P_{+f_\lambda})_{+(f_{\lambda+1}-f_\lambda)} \\ &= \text{ddeg}_{\preccurlyeq g_\lambda} P_{+f_\lambda} \\ &\leq \text{ddeg}_{\mathcal{E}_\lambda} P_{+f_\lambda} = d. \end{aligned}$$

Thus  $\text{ddeg}_{\preccurlyeq g_\lambda} P_{+f_\lambda} = d$  for all  $\lambda < \rho$ , so  $\text{ddeg}_{\mathbf{f}} P = d$ . By Lemma 4.4 and Corollary 4.5,  $(f_\lambda)_{\lambda < \rho}$  has a minimal differential polynomial over  $K$  of order at most the order of  $P$  (which is at most  $r$ ), so since  $K$  is  $r$ -d-algebraically maximal, we may take  $f_\rho \in K$  with  $f_\lambda \rightsquigarrow f_\rho$ . Now set

$$\mathcal{E}_\rho := \bigcap_{\lambda < \rho} \mathcal{E}_\lambda = \{y \in K^\times : y \prec g_\lambda \text{ for all } \lambda < \rho\},$$

where the equality follows from (iii). If  $\mathcal{E}_\rho = \emptyset$ , then by Corollary 4.31,

$$d = \text{ddeg}_{\mathbf{f}} P = \text{mul } P_{+f_\rho},$$

contradicting the hypothesis. So  $\mathcal{E}_\rho \neq \emptyset$ , and thus Corollary 4.31 yields

$$d = \text{ddeg}_{\mathbf{f}} P = \text{ddeg}_{\mathcal{E}_\rho} P_{+f_\rho},$$

so  $(f_\rho, \mathcal{E}_\rho)$  is a partial unraveller for  $(E)$ . For  $((f_\lambda, \mathcal{E}_\lambda))_{\lambda < \rho+1}$ , conditions (i) and (iii) with  $\rho + 1$  in place of  $\rho$  are obviously satisfied. For (ii), it is enough to check that  $f_{\lambda+1} - f_\lambda \succ f_\rho - f_\mu$  for  $\lambda < \mu < \rho$ , which follows from  $f_\rho - f_\mu \prec f_{\mu+1} - f_\mu$ .

This inductive construction must end, and therefore there exists an unraveller for  $(E)$ .  $\square$

**4.5.1. Behaviour of unravellers under immediate extensions.** In this subsection, we fix an immediate extension  $L$  of  $K$ , and we use the monomial group of  $K$  as a monomial group for  $L$ . We consider how unravellers change under passing from  $K$  to  $L$  and connect this to pseudolimits of pc-sequences. Lemma 4.39 is a key step in the proof of Proposition 4.15.

Given  $\mathcal{E}$ , the set  $\mathcal{E}_L := \{y \in L^\times : vy \in v\mathcal{E}\}$  is also  $\preccurlyeq$ -closed with  $\mathcal{E}_L \cap K = \mathcal{E}$ . Consider the asymptotic differential equation

$$(E_L) \quad P(Y) = 0, \quad Y \in \mathcal{E}_L$$

over  $L$ , which has the same dominant degree as  $(E)$ , i.e.,  $\text{ddeg}_{\mathcal{E}_L} P = \text{ddeg}_{\mathcal{E}} P$ . Note that  $y \in K^\times$  is an approximate solution of  $(E)$  if and only if it is an approximate solution of  $(E_L)$ . If so, its multiplicities in both settings agree. Thus if  $(E_L)$  is unravelled, then  $(E)$  is unravelled. For the

other direction, if  $y \in L^\times$  is an approximate solution of  $(E_L)$  of multiplicity  $\text{ddeg}_{\mathcal{E}_L} P$ , then any  $z \in K^\times$  with  $z \sim y$  is an approximate solution of  $(E)$  of multiplicity  $\text{ddeg}_{\mathcal{E}} P = \text{ddeg}_{\mathcal{E}_L} P$ . The next lemma follows from this.

**Lemma 4.37.** *Suppose that  $\text{ddeg}_{\mathcal{E}} P \geq 1$ , and let  $f \in \mathcal{E} \cup \{0\}$  and  $\mathcal{E}' \subseteq \mathcal{E}$  be  $\preceq$ -closed. Then:*

- (i)  *$(f, \mathcal{E}')$  is a partial unraveller for  $(E)$  if and only if  $(f, \mathcal{E}'_L)$  is a partial unraveller for  $(E_L)$ ;*
- (ii)  *$(f, \mathcal{E}')$  is an unraveller for  $(E)$  if and only if  $(f, \mathcal{E}'_L)$  is an unraveller for  $(E_L)$ .*

This next lemma does not use the assumptions of this section and could have been included earlier, but is only used in the proofs of the next lemma and Proposition 4.15.

**Lemma 4.38.** *Suppose that the derivation induced on  $\mathbf{k}$  is nontrivial. Let  $(a_\rho)$  be a divergent pc-sequence in  $K$  with minimal differential polynomial  $G$  over  $K$ , and let  $a_\rho \rightsquigarrow \ell \in L$ . Then  $\text{mul}(G_{+\ell}) \leq 1$ .*

PROOF. Let  $H \in K\{Y\}^\neq$  be of lower complexity than  $G$ . If  $H(\ell) = 0$ , then by Lemma 2.11 there would be an equivalent pc-sequence  $(b_\rho)$  in  $K$  with  $H(b_\rho) \rightsquigarrow 0$ , contradicting the minimality of  $G$ .

In particular,  $S_G(\ell) \neq 0$ , where  $S_G := \partial G / \partial Y^{(n)}$  is the separant of  $G$  and  $n$  is the order of  $G$ . To see that  $\text{mul}(G_{+\ell}) \leq 1$ , decompose

$$G_{+\ell} = \sum_{i=0}^m F_i \cdot (Y^{(n)})^i \quad \text{and} \quad S_{G_{+\ell}} = \sum_{i=1}^m i F_i \cdot (Y^{(n)})^{i-1},$$

with  $F_i \in K[Y, \dots, Y^{(n-1)}]$ ,  $i = 0, \dots, m$ . From  $S_{G_{+\ell}} = (S_G)_{+\ell}$  and  $S_G(\ell) \neq 0$  we get  $F_1(0) \neq 0$ , so  $\text{mul } F_1 = 0$ , and thus  $\text{mul } G_{+\ell} \leq 1$ .  $\square$

**Lemma 4.39.** *Suppose that  $\Gamma$  is divisible and the derivation induced on  $\mathbf{k}$  is nontrivial. Let  $(a_\rho)$  be a divergent pc-sequence in  $K$  that has  $P$  as a minimal differential polynomial over  $K$ , and let  $a_\rho \rightsquigarrow \ell \in L$ . Suppose that  $L$  is  $r$ -d-algebraically maximal and  $\text{ddeg}_{\mathbf{a}} P \geq 2$ . Let  $a \in K$  and  $\mathbf{v} \in K^\times$  be such that  $a - \ell \prec \mathbf{v}$  and  $\text{ddeg}_{\prec \mathbf{v}} P_{+a} = \text{ddeg}_{\mathbf{a}} P$ . (Such  $a$  and  $\mathbf{v}$  exist by Lemma 4.4.) Consider the asymptotic differential equation*

$$(4.5.1) \quad P_{+a}(Y) = 0, \quad Y \prec \mathbf{v}.$$

*Then there exists an unraveller  $(f, \mathcal{E})$  for (4.5.1) over  $L$  such that:*

- (i)  $f \neq 0$ ;
- (ii)  $\text{ddeg}_{\prec f} P_{+a+f} = \text{ddeg}_{\mathbf{a}} P$ ;
- (iii)  $a_\rho \rightsquigarrow a + f + z$  for all  $z \in \mathcal{E} \cup \{0\}$ .

PROOF. We first show how to arrange that  $a = 0$  and (ii) holds. Take  $g \in K^\times$  with  $a - \ell \sim -g$ , so  $g \prec \mathbf{v}$ . Then, using Lemma 3.9, we have

$$\text{ddeg}_{\prec g} P_{+a+g} \leq \text{ddeg}_{\prec \mathbf{v}} P_{+a+g} = \text{ddeg}_{\prec \mathbf{v}} P_{+a} = \text{ddeg}_{\mathbf{a}} P.$$

Conversely, as  $(a + g) - \ell \prec g$ , Lemma 4.4 gives  $\text{ddeg}_{\mathbf{a}} P \leq \text{ddeg}_{\prec g} P_{+a+g}$ , so

$$\text{ddeg}_{\mathbf{a}} P = \text{ddeg}_{\prec \mathbf{v}} P_{+a} = \text{ddeg}_{\prec g} P_{+a+g}.$$

Also,  $P_{+a+g}$  is a minimal differential polynomial of  $(a_\rho - (a + g))$  over  $K$  and, by Lemma 3.13(ii),

$$\text{ddeg}_{\mathbf{a}-(a+g)} P_{+a+g} = \text{ddeg}_{\mathbf{a}} P.$$

We can now replace  $P$ ,  $(a_\rho)$ ,  $\ell$ , and  $\mathbf{v}$  with  $P_{+a+g}$ ,  $(a_\rho - (a + g))$ ,  $\ell - (a + g)$ , and  $g$ , respectively, to arrange that  $a = 0$  and (ii) holds. To see that this works, suppose that  $\mathcal{E} \subseteq L^\times$  is  $\preceq$ -closed in  $L$  with  $\mathcal{E} \prec g$ , and  $(h, \mathcal{E})$  is an unraveller for the asymptotic differential equation

$$P_{+a+g}(Y) = 0, \quad Y \prec g$$

over  $L$  with  $a_\rho - (a + g) \rightsquigarrow h + z$  for all  $z \in \mathcal{E} \cup \{0\}$ . In particular,  $h \prec g$ , so  $g + h \neq 0$ , and it is clear from  $\text{ddeg}_{\prec \mathbf{v}} P_{+a} = \text{ddeg}_{\prec g} P_{+a+g}$  that  $(g + h, \mathcal{E})$  is an unraveller for (4.5.1). Condition (iii) is also obviously satisfied. For condition (ii), note that as  $h \prec g$ , using Lemma 3.9 in the middle equality,

$$\text{ddeg}_{\prec g+h} P_{+a+g+h} = \text{ddeg}_{\prec g} P_{+a+g+h} = \text{ddeg}_{\prec g} P_{+a+g} = \text{ddeg}_{\mathbf{a}} P.$$

Thus it remains to show that there is an unraveller  $(f, \mathcal{E})$  for (4.5.1) in  $L$  (with  $a = 0$ ) such that  $a_\rho \rightsquigarrow f + z$  for all  $z \in \mathcal{E} \cup \{0\}$ . Consider the set

$$\mathcal{Z} := \{z \in L^\times : z \prec a_\rho - \ell, \text{ eventually}\}.$$

For any  $z \in \mathcal{Z} \cup \{0\}$ , we have  $a_\rho \rightsquigarrow z + \ell$ , so by Lemma 4.38,

$$\text{mul}(P_{+\ell+z}) \leq 1 < 2 \leq \text{ddeg}_{\mathbf{a}} P.$$

By Corollary 4.31,  $\mathcal{Z} \neq \emptyset$ , so  $\mathcal{Z}$  is  $\preceq$ -closed and  $\text{ddeg}_{\mathcal{Z}} P_{+\ell} = \text{ddeg}_{\mathbf{a}} P$ . Then Proposition 4.36 yields an unraveller  $(s, \mathcal{E})$  for the asymptotic differential equation

$$P_{+\ell}(Y) = 0, \quad Y \in \mathcal{Z}$$

over  $L$ . Setting  $f := \ell + s$ ,  $(f, \mathcal{E})$  is an unraveller for (4.5.1) with  $a_\rho \rightsquigarrow f + z$  for all  $z \in \mathcal{E} \cup \{0\}$ .  $\square$

**4.5.2. Reducing degree.** In this subsection, we consider a refinement of (E) and then truncate it by removing monomials of degree higher than the dominant degree of (E). Given an unraveller for (E), we show how to find an unraveller for this truncated refinement in Lemma 4.41, an essential component in the proof of Proposition 4.47.

**Assumption.** In this subsection,  $\Gamma$  is divisible.

Suppose that  $d := \text{ddeg}_{\mathcal{E}} P \geq 1$  and we have an unraveller  $(f, \mathcal{E}')$  for (E). That is, the refinement

$$(E') \quad P_{+f}(Y) = 0, \quad Y \in \mathcal{E}'$$

of (E) is unravelled with dominant degree  $d$ . Now suppose that  $d > \text{mul}(P_{+f})$ , so  $(E')$  has an algebraic starting monomial, and let  $\mathfrak{e}$  be its largest algebraic starting monomial. Suppose that  $g \in K^\times$  satisfies  $\mathfrak{e} \prec g \prec f$ , and consider another refinement of (E):

$$(E_g) \quad P_{+f-g}(Y) = 0, \quad Y \preceq g.$$

Set  $\mathcal{E}'_g := \{y \in \mathcal{E}' : y \prec g\}$ , so  $\mathfrak{e} \in \mathcal{E}'_g$ .

**Lemma 4.40.** *The asymptotic differential equation  $(E_g)$  has dominant degree  $d$  and  $(g, \mathcal{E}'_g)$  is an unraveller for  $(E_g)$ .*

PROOF. First, since  $\mathfrak{e}$  is the largest algebraic starting monomial for  $(E')$ , Proposition 4.28 gives

$$d = \text{ddeg } P_{+f, \times \mathfrak{e}} = \text{ddeg}_{\preccurlyeq \mathfrak{e}} P_{+f}.$$

Note that  $f - g \sim f \in \mathcal{E}$ . Now, by Lemma 3.9 we obtain

$$d = \text{ddeg}_{\preccurlyeq \mathfrak{e}} P_{+f} \leq \text{ddeg}_{\preccurlyeq g} P_{+f} = \text{ddeg}_{\preccurlyeq g} P_{+f-g} \leq \text{ddeg}_{\mathcal{E}} P_{+f-g} = \text{ddeg}_{\mathcal{E}} P = d,$$

which gives that  $(E_g)$  has dominant degree  $d$ . Similarly,

$$d = \text{ddeg}_{\preccurlyeq \mathfrak{e}} P_{+f} \leq \text{ddeg}_{\mathcal{E}'_g} P_{+f} \leq \text{ddeg}_{\mathcal{E}} P_{+f} = \text{ddeg}_{\mathcal{E}} P = d,$$

and thus the asymptotic differential equation

$$P_{+f}(Y) = 0, \quad Y \in \mathcal{E}'_g,$$

which is a refinement of both  $(E_g)$  and  $(E')$ , has dominant degree  $d$ . Finally, since  $(E')$  is unravelled, the pair  $(g, \mathcal{E}'_g)$  is an unraveller for  $(E_g)$ .  $\square$

We now turn to ignoring terms of degree higher than the dominant degree of  $(E)$ . First, some notation. Recall that for  $F \in K\{Y\}$ , we set  $F_{\leq n} := F_0 + F_1 + \cdots + F_n$ . Note that if  $n \geq \text{ddeg } F$ , then  $D_F = D_{F_{\leq n}}$ . Now set  $F := P_{+f-g}$ , so  $d \geq \text{ddeg } F_{\times \mathfrak{m}}$  for all  $\mathfrak{m} \preccurlyeq g$ . Consider the “truncation”

$$(E_{g, \leq d}) \quad F_{\leq d}(Y) = 0, \quad Y \preccurlyeq g$$

of  $(E_g)$  as an asymptotic differential equation over  $K$ . We have, for all  $\mathfrak{m} \preccurlyeq g$ ,

$$D_{F_{\times \mathfrak{m}}} = D_{(F_{\times \mathfrak{m}})_{\leq d}} = D_{(F_{\leq d})_{\times \mathfrak{m}}},$$

so  $(E_{g, \leq d})$  has the same algebraic starting monomials and dominant degree as  $(E_g)$ . Next, we show that under suitable conditions the unraveller  $(g, \mathcal{E}'_g)$  for  $(E_g)$  from the previous lemma remains an unraveller for  $(E_{g, \leq d})$ . Recall that  $[\gamma]$  denotes the archimedean class of  $\gamma \in \Gamma$  and that such classes are ordered in the natural way; see §2.2.1.

**Lemma 4.41.** *Suppose that  $[v(\mathfrak{e}/g)] < [v(g/f)]$ . Then  $(g, \mathcal{E}'_g)$  is an unraveller for  $(E_{g, \leq d})$ , and  $\mathfrak{e}$  is the largest algebraic starting monomial for the unravelled asymptotic differential equation*

$$(E'_{g, \leq d}) \quad (F_{\leq d})_{+g}(Y) = 0, \quad Y \in \mathcal{E}'_g.$$

PROOF. First, we reduce to the case  $g \asymp 1$ : set  $\mathfrak{g} := \mathfrak{d}_g$  and replace  $P$ ,  $f$ ,  $g$ ,  $\mathcal{E}$ , and  $\mathcal{E}'$  by  $P_{\times \mathfrak{g}}$ ,  $f/\mathfrak{g}$ ,  $g/\mathfrak{g}$ ,  $\mathfrak{g}^{-1}\mathcal{E}$ , and  $\mathfrak{g}^{-1}\mathcal{E}'$ , respectively, and use Lemma 4.35. Note that now  $\mathfrak{e} \prec 1 \prec f$  and  $[v\mathfrak{e}] < [vf]$ .

Since  $F = P_{+f-g}$  and  $g \asymp 1$ , we have  $\text{ddeg } F = \text{ddeg}_{\preccurlyeq 1} F = d$  by Lemma 4.40, so

$$d \leq \text{ddeg } F_{\times f} \leq \text{ddeg}_{\mathcal{E}} F = d,$$

using Lemma 3.7 and Lemma 3.9. This yields  $d = \text{ddeg } F = \text{ddeg } F_{\times f}$ . For  $\mathfrak{m}$  with  $[v\mathfrak{m}] < [vf]$  we may thus apply Corollary 4.14 with  $F$  and  $\mathfrak{d}_f$  in place of  $P$  and  $\mathfrak{n}$  to get

$$(4.5.2) \quad D_{P_{+f, \times \mathfrak{m}}} = D_{F_{+g, \times \mathfrak{m}}} = D_{(F_{\leq d})_{+g, \times \mathfrak{m}}}.$$

In particular, this holds if  $\mathfrak{e} \preccurlyeq \mathfrak{m} \prec 1$ , as then  $[v\mathfrak{m}] \leq [v\mathfrak{e}] < [vf]$ . Thus  $\mathfrak{e}$  is the largest algebraic starting monomial for  $(E'_{g, \leq d})$ , since it is the largest algebraic starting monomial for  $(E')$ .

For  $(g, \mathcal{E}'_g)$  to be an unraveller for  $(E_{g, \leq d})$ , we now show:

- (i)  $\text{ddeg}_{\mathcal{E}'_g}(F_{\leq d})_{+g} = d$ ;
- (ii)  $\text{ddeg}_{\prec h}(F_{\leq d})_{+g+h} < d$  for all  $h \in \mathcal{E}'_g$ .

For (i), if  $\mathfrak{e} \preceq \mathfrak{m} \in \mathcal{E}'_g$ , then by Corollary 4.29 and (4.5.2) we have

$$d = \text{ddeg } P_{+f, \times \mathfrak{m}} = \text{ddeg}(F_{\leq d})_{+g, \times \mathfrak{m}}.$$

For (ii), let  $h \in \mathcal{E}'_g$ , so  $h \in \mathcal{E}'$  and  $h \prec 1$ . Set  $\mathfrak{h} := \mathfrak{d}_h$  and  $u := h/\mathfrak{h}$ . Applying Lemma 4.26, we have

$$(4.5.3) \quad \text{ddeg}_{\prec h}(F_{\leq d})_{+g+h} = \text{mul}(D_{(F_{\leq d})_{+g, \times \mathfrak{h}}})_{+\bar{u}};$$

$$(4.5.4) \quad \text{ddeg}_{\prec h} P_{+f+h} = \text{mul}(D_{P_{+f, \times \mathfrak{h}}})_{+\bar{u}}.$$

First suppose that  $\mathfrak{e} \preceq h$ , so then combining (4.5.2), for  $\mathfrak{m} = \mathfrak{h}$ , with (4.5.3) and (4.5.4) we have

$$\text{ddeg}_{\prec h}(F_{\leq d})_{+g+h} = \text{ddeg}_{\prec h} P_{+f+h} < d,$$

since (E') is unravelled. Now suppose that  $h \prec \mathfrak{e}$ . If  $\mathfrak{e}^2 \preceq h \prec \mathfrak{e}$ , then  $[vh] = [v\mathfrak{e}] < [vf]$ , and thus by (4.5.2) and Corollary 4.29,

$$\text{ddeg}(F_{\leq d})_{+g, \times \mathfrak{h}} = \text{ddeg } P_{+f, \times \mathfrak{h}} < \text{ddeg } P_{+f, \times \mathfrak{e}} = d.$$

By Lemma 3.7,  $\text{ddeg}(F_{\leq d})_{+g, \times \mathfrak{h}} < d$  remains true for any  $h \prec \mathfrak{e}$ . Hence, by (4.5.3),

$$\text{ddeg}_{\prec h}(F_{\leq d})_{+g+h} = \text{mul}(D_{(F_{\leq d})_{+g, \times \mathfrak{h}}})_{+\bar{u}} \leq \text{ddeg}(F_{\leq d})_{+g, \times \mathfrak{h}} < d,$$

which completes the proof of (ii).  $\square$

**4.5.3. Finding solutions in differential-henselian fields.** We now use d-henselianity to find solutions of asymptotic differential equations. Given an element of an extension of  $K$ , when  $K$  has few constants we find a solution closest to that element. The only result in this subsection that uses the assumption that  $\Gamma^>$  has no least element is Lemma 4.46.

We say that (E) is *quasilinear* if  $\text{ddeg}_{\mathcal{E}} P = 1$ . Note that if  $K$  is  $r$ -d-henselian and (E) is quasilinear, then  $P$  has a zero in  $\mathcal{E} \cup \{0\}$  by Lemma 2.5. Note that even in this case, (E) may not have a solution, since those are required to be nonzero.

**Lemma 4.42.** *Suppose that  $K$  is  $r$ -d-henselian. Let  $g \in K^\times$  be an approximate zero of  $P$  such that  $\text{ddeg } P_{\times g} = 1$ . Then there exists  $y \sim g$  in  $K$  such that  $P(y) = 0$ .*

PROOF. Let  $\mathfrak{m} := \mathfrak{d}_g$  and  $u := u_g = g/\mathfrak{m}$ , so  $D_{P_{\times \mathfrak{m}}}(\bar{u}) = 0$  and thus

$$\text{dmul } P_{\times \mathfrak{m}, +u} = \text{mul}(D_{P_{\times \mathfrak{m}}})_{+\bar{u}} \geq 1.$$

By Lemma 4.2(i), we also have

$$\text{dmul } P_{\times \mathfrak{m}, +u} \leq \text{ddeg } P_{\times \mathfrak{m}, +u} = \text{ddeg } P_{\times \mathfrak{m}} = 1.$$

Thus  $\text{dmul } P_{\times \mathfrak{m}, +u} = 1$ , so as  $K$  is  $r$ -d-henselian, there is  $z \prec 1$  with  $P_{\times \mathfrak{m}, +u}(z) = 0$ . Setting  $y := (u + z)\mathfrak{m}$ , we have  $P(y) = 0$  and  $y \sim u\mathfrak{m} = g$ .  $\square$

Now let  $f$  be an element of an extension of  $K$ . We say that a solution  $y$  of (E) *best approximates*  $f$  (among solutions of (E)) if  $y - f \preceq z - f$  for each solution  $z$  of (E). Note that if  $f \in K^\times$  is a solution of (E), then  $f$  is the unique solution of (E) that best approximates  $f$ . Also, if  $f \succ \mathcal{E}$ , then

$y - f \asymp f$  for all  $y \in \mathcal{E}$ , and so each solution of (E) best approximates  $f$ . First, we see that this is preserved under multiplicative conjugation.

**Lemma 4.43.** *Let  $f$  be an element of an extension of  $K$  and let  $g \in \mathcal{E}$  with  $f \preccurlyeq g$ . Suppose that  $y$  is a solution of the asymptotic differential equation*

$$P_{\times g}(Y) = 0, \quad Y \preccurlyeq 1$$

*that best approximates  $g^{-1}f$ . Then the solution  $gy$  of (E) best approximates  $f$ .*

PROOF. Let  $z$  be a solution of (E). If  $z \succ g \succcurlyeq f$ , then  $z - f \sim z$ . As  $y \preccurlyeq 1$  and  $f \preccurlyeq g$ , we have  $gy - f \preccurlyeq g$ . Combining these two yields  $gy - f \preccurlyeq z - f$ . If  $z \preccurlyeq g$ , then  $g^{-1}z \preccurlyeq 1$  is a solution of the above asymptotic differential equation and so by the assumption on  $y$ , we have  $y - g^{-1}f \preccurlyeq g^{-1}z - g^{-1}f$ , and hence  $gy - f \preccurlyeq z - f$ .  $\square$

The following lemma is the only place in this section that we assume few constants.

**Lemma 4.44.** *Suppose that  $r \geq 1$  and  $K$  is  $r$ -d-henselian with  $C \subseteq \mathcal{O}$ . Suppose that (E) is quasilinear and has a solution. Let  $f$  be an element of an extension of  $K$ . Then  $f$  is best approximated by some solution of (E).*

PROOF. By the comment above Lemma 4.43, we may assume that  $f \not\asymp \mathcal{E}$ . Thus we may take  $g \in \mathcal{E}$  with  $f \preccurlyeq g$  such that (E) has a solution  $y \preccurlyeq g$  and

$$\text{ddeg } P_{\times g} = \text{ddeg}_{\mathcal{E}} P = 1.$$

By Lemma 4.43, we may replace  $P$  by  $P_{\times g}$  and  $\mathcal{E}$  by  $\mathcal{O}^\neq$  in order to assume that  $\mathcal{E} = \mathcal{O}^\neq$ . Suppose that  $f$  is not best approximated by any solution of (E). Then we get  $y_0, \dots, y_{r+1} \in K^\times$  such that:

- (i)  $y_i$  is a solution of (E), i.e.,  $P(y_i) = 0$  and  $y_i \preccurlyeq 1$ , for all  $i \in \{0, \dots, r+1\}$ ;
- (ii)  $y_i - f \succ y_{i+1} - f$  for all  $i \in \{0, \dots, r\}$ ;
- (iii)  $\text{ddeg } P_{+y_i} = \text{ddeg } P = 1$  for all  $i \in \{0, \dots, r+1\}$  (by Lemma 4.2(i)).

Item (ii) implies  $y_{i+1} - y_i \asymp y_i - f$  for all  $i \in \{0, \dots, r\}$ , so we have reached a contradiction with Lemma 3.26.  $\square$

**Lemma 4.45.** *Suppose that  $K$  is  $r$ -d-henselian, (E) is quasilinear, and  $f \in \mathcal{E}$  is an approximate solution of (E). Then (E) has a solution  $y_0 \sim f$ , and every solution  $y$  of (E) that best approximates  $f$  satisfies  $y \sim f$ .*

PROOF. Let  $\mathfrak{m} := \mathfrak{d}_f$  and  $u := f/\mathfrak{m}$ . Then by Lemma 4.2 we have

$$\text{ddeg } P_{\times f} = \text{ddeg } P_{\times \mathfrak{m}, +u} \geq \text{dmul } P_{\times \mathfrak{m}, +u} \geq 1,$$

and so since (E) is quasilinear,  $\text{ddeg } P_{\times f} = 1$ . Thus Lemma 4.42 yields a solution  $y_0 \sim f$  of (E). If  $y$  is a solution of (E) that best approximates  $f$ , then  $y \sim f$ , as

$$y - f \preccurlyeq y_0 - f \prec f. \quad \square$$

For the next lemma, recall from §4.5.1 that given an immediate extension  $L$  of  $K$ , we extend the asymptotic differential equation (E) over  $K$  to  $(E_L)$  over  $L$ . Note that if (E) is quasilinear, then so is  $(E_L)$ .

**Lemma 4.46.** *Suppose that  $K$  is  $r$ -d-henselian and let  $L$  be an immediate extension of  $K$ . Suppose that (E) is quasilinear,  $\mathcal{E}' \subseteq \mathcal{E}$  is  $\preccurlyeq$ -closed, and  $f \in \mathcal{E}_L$  is such that the refinement*

$$(E'_L) \quad P_{+f}(Y) = 0, \quad Y \in \mathcal{E}'_L$$

*of (E<sub>L</sub>) is also quasilinear. Let  $y \preccurlyeq f$  be a solution of (E) that best approximates  $f$ . Then  $f - y \in \mathcal{E}'_L \cup \{0\}$ .*

PROOF. The case  $f = y$  being trivial, suppose that  $f \neq y$  and set  $\mathfrak{m} := \mathfrak{d}_{f-y}$ . As  $f - y \in \mathcal{E}_L$ , we have  $\mathfrak{m} \in \mathcal{E}$ . Now suppose towards a contradiction that  $f - y \notin \mathcal{E}'_L$ . Then  $\mathcal{E}'_L \prec \mathfrak{m} \in \mathcal{E}$ , so by quasilinearity and Lemma 3.9,

$$1 = \text{ddeg}_{\mathcal{E}'_L} P_{+f} \leq \text{ddeg}_{\preccurlyeq \mathfrak{m}} P_{+f} = \text{ddeg}_{\preccurlyeq \mathfrak{m}} P_{+y} \leq \text{ddeg}_{\mathcal{E}} P_{+y} = \text{ddeg}_{\mathcal{E}} P = 1.$$

Hence the asymptotic differential equation

$$(4.5.5) \quad P_{+y}(Y) = 0, \quad Y \preccurlyeq \mathfrak{m}$$

over  $K$  is also quasilinear. Also, by the quasilinearity of  $(E'_L)$ , we have

$$\text{ddeg}_{\prec \mathfrak{m}} (P_{+y})_{+(f-y)} = \text{ddeg}_{\prec \mathfrak{m}} P_{+f} \geq \text{ddeg}_{\mathcal{E}'_L} P_{+f} = 1,$$

so  $f - y$  is an approximate solution of (4.5.5) over  $L$ , by Corollary 4.32. Take  $g \in K^\times$  with  $g \sim f - y$ , so  $g$  is an approximate solution of (4.5.5) over  $K$ , and, by the quasilinearity of (4.5.5),

$$\text{ddeg} P_{+y, \times g} = \text{ddeg}_{\preccurlyeq \mathfrak{m}} P_{+y} = 1.$$

Then by Lemma 4.42 we have  $z \sim g \sim f - y$  in  $K$  such that  $P(y + z) = 0$ . We must have  $y + z \neq 0$ , as otherwise  $f \prec y - f$ , contradicting  $y \preccurlyeq f$ . From  $y \preccurlyeq f$ , we also obtain  $y + z \preccurlyeq f$ , so  $y + z \in \mathcal{E}$ . Since  $y + z - f \prec y - f$ , this contradicts that  $y$  best approximates  $f$ .  $\square$

## 4.6. Reducing complexity

This is a technical section whose main goal is Proposition 4.47. This proposition, or rather its consequence Corollary 4.60, is the linchpin of Proposition 4.15, and its proof uses all of the previous sections and some additional results from [ADH17a].

**Assumption.** In this section,  $K$  is asymptotic and has a monomial group  $\mathfrak{M}$ ,  $\Gamma$  is divisible, and  $\mathbf{k}$  is  $r$ -linearly surjective with  $r \geq 1$ .

Let  $\mathfrak{m}$  and  $\mathfrak{n}$  range over  $\mathfrak{M}$ . We let  $P \in K\{Y\}^\neq$  with order at most  $r$ . As in the previous section, let  $\mathcal{E} \subseteq K^\times$  be  $\preccurlyeq$ -closed, so we have an asymptotic differential equation

$$(E) \quad P(Y) = 0, \quad Y \in \mathcal{E}$$

over  $K$ . Set  $d := \text{ddeg}_{\mathcal{E}} P$  and suppose that  $d \geq 2$ . We fix an immediate asymptotic  $r$ -d-henselian extension  $\hat{K}$  of  $K$  and use  $\mathfrak{M}$  as a monomial group of  $\hat{K}$ .

Let  $\hat{\mathcal{E}} := \mathcal{E}_{\hat{K}} = \{y \in \hat{K}^\times : vy \in v\mathcal{E}\}$ , so we have the asymptotic differential equation

$$(\hat{E}) \quad P(Y) = 0, \quad Y \in \hat{\mathcal{E}}$$



over  $\hat{K}$  with dominant degree  $d$ . Suppose that  $(\hat{E})$  is not unravelled, and that this is witnessed by  $\hat{f} \in \hat{\mathcal{E}}$  such that  $(\hat{f}, \hat{\mathcal{E}}')$  is an unraveller for  $(\hat{E})$ . That is,  $\text{ddeg}_{\prec \hat{f}} P_{+\hat{f}} = d$ , and the refinement

$$(\hat{E}') \quad P_{+\hat{f}}(Y) = 0, \quad Y \in \hat{\mathcal{E}}'$$

of  $(\hat{E})$  is unravelled with dominant degree  $d$ . By Corollary 4.32,  $\hat{f}$  is an approximate solution of  $(\hat{E})$  of multiplicity  $d$ . Note also that  $\hat{\mathcal{E}}' = \mathcal{E}'_{\hat{K}}$  for the  $\preceq$ -closed set  $\mathcal{E}' := \hat{\mathcal{E}}' \cap K \subseteq \mathcal{E}$ . Since  $(\hat{E})$  is not unravelled, neither is  $(E)$  by the discussion preceding Lemma 4.37. Suppose also that  $\text{mul } P_{+\hat{f}} < d$ , so by Proposition 4.28,  $(\hat{E}')$  has an algebraic starting monomial; let  $\mathfrak{e}$  be the largest algebraic starting monomial of  $(\hat{E}')$ .

**Proposition 4.47.** *There exists  $f \in \hat{K}$  such that one of the following holds:*

- (i)  $\hat{f} - f \preceq \mathfrak{e}$  and  $A(f) = 0$  for some  $A \in K\{Y\}$  with  $c(A) < c(P)$  and  $\deg A = 1$ ;
- (ii)  $\hat{f} \sim f$ ,  $\hat{f} - a \preceq f - a$  for all  $a \in K$ , and  $A(f) = 0$  for some  $A \in K\{Y\}$  with  $c(A) < c(P)$  and  $\text{ddeg } A_{\times f} = 1$ .

**4.6.1. Special case.** We first prove Proposition 4.47 in the special case that  $\text{ddeg}_{\mathcal{E}} P = \deg P$  and later reduce to this case using Lemma 4.59. Below, we consider the differential polynomial

$$P_{+\hat{f}, \times \mathfrak{e}} \in \hat{K}\{Y\};$$

note that  $\text{ddeg } P_{+\hat{f}, \times \mathfrak{e}} = d$  by the choice of  $\mathfrak{e}$ . Let  $s \leq r$  be the order of  $P$ . For  $\mathbf{i} \in \mathbb{N}^{1+s}$ , we let

$$\partial^{\mathbf{i}} := \frac{\partial^{|\mathbf{i}|}}{\partial Y^{i_0} \dots \partial (Y^{(s)})^{i_s}}$$

denote the partial differential operator on  $\hat{K}\{Y\}$  that differentiates  $i_n$  times with respect to  $Y^{(n)}$  for  $n = 0, \dots, s$ . (We also use additive and multiplicative conjugates of partial differential operators; see [ADH17a, §12.8].) For any partial differential operator (in the sense of [ADH17a, §12.7])  $\Delta$  on  $\hat{K}\{Y\}$ , any  $Q \in \hat{K}\{Y\}$ , and any  $a \in \hat{K}$ ,

$$\Delta(Q_{+a}) = (\Delta Q)_{+a}$$

by [ADH17a, Lemma 12.8.7], so we write  $\Delta Q_{+a}$  and do not distinguish between these. If  $a \in \hat{K}^\times$ , note that, by [ADH17a, Lemma 12.8.8],

$$\Delta Q_{\times a} := \Delta(Q_{\times a}) = (\Delta_{\times a} Q)_{\times a},$$

Note that, when no parentheses are used, we intend additive and multiplicative conjugation of  $Q$  to take place before  $\Delta$  is applied, in order to simplify notation.

Now, choose  $\mathbf{i} \in \mathbb{N}^{1+s}$  such that  $\deg(\partial^{\mathbf{i}} Y^{\mathbf{j}}) = 1$  for some  $\mathbf{j} \in \mathbb{N}^{1+s}$  with  $|\mathbf{j}| = d$  and

$$(P_{+\hat{f}, \times \mathfrak{e}})_{\mathbf{j}} \asymp P_{+\hat{f}, \times \mathfrak{e}}.$$

In particular,  $|\mathbf{i}| = d - 1$  and

$$\text{ddeg } \partial^{\mathbf{i}} P_{+\hat{f}, \times \mathfrak{e}} = \deg D_{\partial^{\mathbf{i}} P_{+\hat{f}, \times \mathfrak{e}}} = \deg \partial^{\mathbf{i}} D_{P_{+\hat{f}, \times \mathfrak{e}}} = 1.$$

We consider the partial differential operator  $\Delta := (\partial^i)_{\times \mathfrak{e}}$  on  $\hat{K}\{Y\}$ . We have

$$(\Delta P)_{+\hat{f}, \times \mathfrak{e}} = \partial^i P_{+\hat{f}, \times \mathfrak{e}}$$

by [ADH17a, Lemmas 12.8.7 and 12.8.8]. Hence the asymptotic differential equation

$$\Delta P_{+\hat{f}}(Y) = 0, \quad Y \preccurlyeq \mathfrak{e}$$

is quasilinear. In [ADH17a, §14.4], partial differentiation is also used to obtain a quasilinear asymptotic differential equation. Under the powerful assumption of  $\mathfrak{w}$ -freeness made in that setting, newton polynomials have a very special form, and so a specific choice of  $\Delta$  was needed. Here, and in the next subsection, a similar technique works despite the lack of restrictions on dominant parts.

**Lemma 4.48.** *Suppose that  $\mathfrak{e} \prec \hat{f}$  and the asymptotic differential equation*

$$(4.6.1) \quad \Delta P(Y) = 0, \quad Y \in \hat{\mathcal{E}}$$

*over  $\hat{K}$  is quasilinear. Then (4.6.1) has a solution  $y \sim \hat{f}$ , and if  $f$  is any solution of (4.6.1) that best approximates  $\hat{f}$ , then  $f - \hat{f} \preccurlyeq \mathfrak{e}$ .*

PROOF. By Lemma 3.9, we have

$$\text{ddeg}_{\prec \hat{f}} \Delta P_{+\hat{f}} \leq \text{ddeg}_{\hat{\mathcal{E}}} \Delta P_{+\hat{f}} = \text{ddeg}_{\hat{\mathcal{E}}} \Delta P = 1.$$

But from  $\mathfrak{e} \prec \hat{f}$ , we also have

$$1 = \text{ddeg}_{\prec \mathfrak{e}} \Delta P_{+\hat{f}} \leq \text{ddeg}_{\prec \hat{f}} \Delta P_{+\hat{f}},$$

so  $\text{ddeg}_{\prec \hat{f}} \Delta P_{+\hat{f}} = 1$ . Then since  $\hat{K}$  is  $r$ -d-henselian, we get  $y \sim \hat{f}$  with  $\Delta P(y) = 0$ .

For the second statement, the refinement

$$(4.6.2) \quad \Delta P_{+\hat{f}}(Y) = 0, \quad Y \preccurlyeq \mathfrak{e}$$

of (4.6.1) is quasilinear, so we can apply Lemma 4.46 with  $\hat{K}$  in the roles of both  $L$  and  $K$ , and  $\Delta P$ ,  $\hat{f}$ ,  $f$ , (4.6.1), and (4.6.2) in the roles of  $P$ ,  $f$ ,  $y$ , (E), and  $(E'_L)$ , respectively.  $\square$

We now conclude the proof of Proposition 4.47 in the case that  $\deg P = d$ . Recall that  $d \geq 2$ .

**Lemma 4.49.** *Suppose that  $\deg P = d$ . Then there exist  $f \in \hat{K}$  and  $A \in K\{Y\}$  such that  $\hat{f} - f \preccurlyeq \mathfrak{e}$ ,  $A(f) = 0$ ,  $c(A) < c(P)$ , and  $\deg A = 1$ .*

PROOF. Since  $\deg P = d$ , we also have  $\deg P_{+\hat{f}, \times \mathfrak{e}} = d$ , and hence

$$\deg \Delta P = \deg (\Delta P)_{+\hat{f}, \times \mathfrak{e}} = \deg \partial^i P_{+\hat{f}, \times \mathfrak{e}} = 1,$$

by the choice of  $i$ . Hence (4.6.1) is quasilinear.

If  $\hat{f} \preccurlyeq \mathfrak{e}$ , then  $f := 0$  and  $A := Y$  work, so assume that  $\mathfrak{e} \prec \hat{f}$ . First, Lemma 4.48 yields a solution  $y \sim \hat{f}$  of (4.6.1). As  $\hat{K}$  has few constants, Lemma 4.44 gives that  $\hat{f}$  is best approximated by some solution  $f$  of (4.6.1). So applying Lemma 4.48 again, we have  $f - \hat{f} \preccurlyeq \mathfrak{e}$ . Then  $\Delta P(f) = 0$ ,  $c(\Delta P) < c(P)$ , and  $\deg \Delta P = 1$ , so we may take  $A := \Delta P$ .  $\square$

**4.6.2. Tschirnhaus refinements.** Set  $\mathfrak{f} := \mathfrak{d}_{\hat{f}}$ , and we now consider the differential polynomial  $P_{\times \mathfrak{f}} \in K\{Y\}^{\neq}$ . If  $\mathfrak{e} \succ \mathfrak{f}$ , then the first case of Proposition 4.47 holds for  $f := 0$  and  $A := Y$ , so in

the rest of this subsection and in §4.6.3 and §4.6.4 we suppose that  $\mathfrak{e} \prec \mathfrak{f}$ . Then we have, by the choice of  $\mathfrak{e}$  and Lemma 3.9,

$$d = \text{ddeg}_{\mathfrak{g} \prec \mathfrak{e}} P_{+\hat{f}} \leq \text{ddeg}_{\mathfrak{g} \prec \mathfrak{f}} P_{+\hat{f}} = \text{ddeg}_{\mathfrak{g} \prec \mathfrak{f}} P \leq \text{ddeg}_{\hat{\mathfrak{e}}} P = d,$$

and thus  $\text{ddeg} P_{\times \mathfrak{f}} = d$ .

Now, let  $s \leq r$  be the order of  $P$  and choose  $\mathbf{i} \in \mathbb{N}^{1+s}$  so that  $\deg(\partial^{\mathbf{i}} Y^{\mathbf{j}}) = 1$  for some  $\mathbf{j} \in \mathbb{N}^{1+s}$  with  $|\mathbf{j}| = d$  and  $(P_{\times \mathfrak{f}})_{\mathbf{j}} \asymp P_{\times \mathfrak{f}}$ . Thus we have  $|\mathbf{i}| = d - 1$  and

$$D_{\partial^{\mathbf{i}} P_{\times \mathfrak{f}}} = \partial^{\mathbf{i}} D_{P_{\times \mathfrak{f}}},$$

and so  $\text{ddeg} \partial^{\mathbf{i}} P_{\times \mathfrak{f}} = 1$ . We consider the partial differential operator  $\Delta := (\partial^{\mathbf{i}})_{\times \mathfrak{f}}$  on  $\hat{K}\{Y\}$ . By [ADH17a, Lemma 12.8.8],

$$(\Delta P)_{\times \mathfrak{f}} = \partial^{\mathbf{i}} P_{\times \mathfrak{f}},$$

and thus the asymptotic differential equation

$$(4.6.3) \quad \Delta P(Y) = 0, \quad Y \preccurlyeq \mathfrak{f}$$

over  $\hat{K}$  is quasilinear. The next lemma follows immediately from this by Corollary 3.10.

**Lemma 4.50.** *Suppose  $f \in \hat{K}^{\times}$  is a solution of (4.6.3). Then for all  $g \in \hat{K}^{\times}$  with  $g \preccurlyeq \mathfrak{f}$  we have*

$$\text{mul}(\Delta P)_{+f, \times g} = \text{ddeg}(\Delta P)_{+f, \times g} = 1,$$

and hence  $(\Delta P)_{+f}$  has no algebraic starting monomial  $\mathfrak{g} \in \mathfrak{M}$  with  $\mathfrak{g} \preccurlyeq \mathfrak{f}$ .

**Lemma 4.51.** *The element  $\hat{f} \in \hat{K}^{\times}$  is an approximate solution of (4.6.3).*

PROOF. Set  $u := \hat{f}/\mathfrak{f}$ . Since  $\hat{f}$  is an approximate zero of  $P$  of multiplicity  $d = \text{ddeg} P_{\times \mathfrak{f}} = \deg D_{P_{\times \mathfrak{f}}}$ ,

$$(D_{P_{\times \mathfrak{f}}})_{+\bar{u}} = \sum_{|\mathbf{j}|=d} (D_{P_{\times \mathfrak{f}}})_{\mathbf{j}} Y^{\mathbf{j}},$$

by [ADH17a, Lemma 4.3.1], where  $\mathbf{j}$  ranges over  $\mathbb{N}^{1+s}$ . Then

$$(\partial^{\mathbf{i}} D_{P_{\times \mathfrak{f}}})_{+\bar{u}} = \partial^{\mathbf{i}} (D_{P_{\times \mathfrak{f}}})_{+\bar{u}} = \sum_{|\mathbf{j}|=d} (D_{P_{\times \mathfrak{f}}})_{\mathbf{j}} \partial^{\mathbf{i}} Y^{\mathbf{j}},$$

so the multiplicity of  $\partial^{\mathbf{i}} D_{P_{\times \mathfrak{f}}}$  at  $\bar{u}$  is 1 by the choice of  $\mathbf{i}$ . In view of

$$D_{(\Delta P)_{\times \mathfrak{f}}} = D_{\partial^{\mathbf{i}} P_{\times \mathfrak{f}}} = \partial^{\mathbf{i}} D_{P_{\times \mathfrak{f}}},$$

$\hat{f}$  is an approximate solution of (4.6.3). □

Let  $f \in \hat{K}^{\times}$  with  $f \sim \hat{f}$ , so  $\text{ddeg}_{\prec \mathfrak{f}} P_{+f} = \text{ddeg}_{\prec \mathfrak{f}} P_{+\hat{f}} = d$  by Lemma 3.9. That is, the refinement

$$(T) \quad P_{+f}(Y) = 0, \quad Y \prec \mathfrak{f}$$

of  $(\hat{E})$  still has dominant degree  $d$ . As  $\hat{f}$  is an approximate solution of (4.6.3), Lemmas 4.45 and 4.44 give a solution  $f_0 \in \hat{K}^{\times}$  of (4.6.3) that best approximates  $\hat{f}$  with  $f_0 \sim \hat{f} \sim f$ . Thus

$$\text{ddeg}_{\prec \mathfrak{f}} \Delta P_{+f} = \text{ddeg}_{\prec \mathfrak{f}} \Delta P_{+f_0} = 1$$

by Lemmas 3.9 and 4.50. Hence the refinement

$$(\Delta T) \quad \Delta P_{+f}(Y) = 0, \quad Y \prec \mathfrak{f}$$

of (4.6.3) is also quasilinear.

**Definition.** A *Tschirnhaus refinement* of  $(\hat{E})$  is an asymptotic differential equation  $(T)$  over  $\hat{K}$  as above with  $\hat{f} \sim f \in \hat{K}^\times$  such that some solution  $f_0 \in \hat{K}^\times$  of (4.6.3) over  $\hat{K}$  best approximates  $\hat{f}$  and satisfies  $f_0 - \hat{f} \sim f - \hat{f}$ .

**Definition.** Let  $f, \hat{g} \in \hat{K}^\times$  and  $\mathfrak{m}$  satisfy

$$\mathfrak{m} \prec f - \hat{f} \preceq \hat{g} \prec \mathfrak{f},$$

so in particular  $f \sim \hat{f}$ . With  $(T)$  as above, but not necessarily a Tschirnhaus refinement of  $(\hat{E})$ , we say that the refinement

$$(TC) \quad P_{+f+\hat{g}}(Y) = 0, \quad Y \preceq \mathfrak{m}$$

of  $(T)$  is *compatible with*  $(T)$  if it has dominant degree  $d$  and  $\hat{g}$  is not an approximate solution of  $(\Delta T)$ .

**Lemma 4.52.** *Let  $f, f_0, \hat{g} \in \hat{K}^\times$  and  $\mathfrak{m}$  be such that*

$$\mathfrak{m} \prec f_0 - \hat{f} \sim f - \hat{f} \preceq \hat{g} \prec \mathfrak{f},$$

*and  $(TC)$  has dominant degree  $d$ . Then  $\hat{g}$  is an approximate solution of  $(T)$  and of*

$$(T_0) \quad P_{+f_0}(Y) = 0, \quad Y \prec \mathfrak{f}.$$

PROOF. First,  $\hat{g}$  is an approximate solution of  $(T)$  by Lemma 4.33, since  $\mathfrak{m} \prec \hat{g}$  and

$$\text{ddeg}_{\preceq \mathfrak{m}} P_{+f+\hat{g}} = d = \text{ddeg}_{\prec \mathfrak{f}} P_{+f}.$$

From  $f_0 - f \prec f - \hat{f} \preceq \hat{g}$  and  $\mathfrak{m} \prec \hat{g}$ , we obtain, using Lemma 3.9 in the first equality,

$$\text{ddeg}_{\prec \hat{g}} P_{+f_0+\hat{g}} = \text{ddeg}_{\prec \hat{g}} P_{+f+\hat{g}} \geq \text{ddeg}_{\preceq \mathfrak{m}} P_{+f+\hat{g}} = d \geq 1,$$

so  $\hat{g}$  is an approximate solution of  $(T_0)$  by Corollary 4.32. □

**Lemma 4.53.** *Let  $f, f_0, \hat{g} \in \hat{K}^\times$  with*

$$f_0 - \hat{f} \sim f - \hat{f} \preceq \hat{g} \prec \mathfrak{f}.$$

*Then  $\hat{g}$  is an approximate solution of  $(\Delta T)$  if and only if  $\hat{g}$  is an approximate solution of*

$$(\Delta T_0) \quad \Delta P_{+f_0}(Y) = 0, \quad Y \prec \mathfrak{f}.$$

PROOF. Again, since  $f_0 - f \prec f - \hat{f} \preceq \hat{g}$ , by Lemma 3.9 we have

$$\text{ddeg}_{\prec \hat{g}} \Delta P_{+f_0+\hat{g}} = \text{ddeg}_{\prec \hat{g}} \Delta P_{+f+\hat{g}}.$$

The result then follows from Corollary 4.32, since  $\hat{g} \prec \mathfrak{f}$ . □

Note that, for any  $f_0 \sim f$ , the equation  $(\Delta T_0)$  in the previous lemma is quasilinear by Lemma 3.9, since  $(\Delta T)$  is. We now exhibit compatible refinements of  $(T)$  when  $\mathfrak{e} \prec f - \hat{f}$ .

**Lemma 4.54.** *Suppose that (T) is a Tschirnhaus refinement of  $(\hat{E})$  and  $\mathfrak{e} \prec f - \hat{f}$ . Then, with  $\hat{g} := \hat{f} - f$  and  $\mathfrak{m} := \mathfrak{e}$ , the refinement (TC) of (T) is compatible with (T).*

PROOF. Since  $\mathfrak{e}$  is the largest algebraic starting monomial for  $(\hat{E}')$ ,

$$\text{ddeg}_{\preceq \mathfrak{e}} P_{+f+\hat{g}} = \text{ddeg}_{\preceq \mathfrak{e}} P_{+\hat{f}} = \text{ddeg}_{+f, \times \mathfrak{e}} P_{+\hat{f}} = d,$$

and so (TC) has dominant degree  $d$ .

As (T) is a Tschirnhaus refinement of  $(\hat{E})$ , let  $f_0 \in \hat{K}^\times$  be a solution of (4.6.3) that best approximates  $\hat{f}$  and satisfies  $f - \hat{f} \sim f_0 - \hat{f}$ . Suppose towards a contradiction that  $\hat{g}$  is an approximate solution of  $(\Delta T)$ , so by Lemma 4.53,  $\hat{g}$  is also an approximate solution of  $(\Delta T_0)$ . Then by Lemma 4.45,  $(\Delta T_0)$  has a solution  $y \sim \hat{g} \sim \hat{f} - f_0$ . Thus  $\Delta P(f_0 + y) = 0$ , so  $f_0 + y$  is a solution of (4.6.3), since  $f_0 + y \preceq \mathfrak{f}$ . But also

$$f_0 + y - \hat{f} = y - (\hat{f} - f_0) \prec \hat{f} - f_0,$$

contradicting that  $f_0$  best approximates  $\hat{f}$ . Hence  $\hat{g}$  is not an approximate solution of  $(\Delta T)$ , and so (TC) is compatible with (T).  $\square$

In fact, the proof above shows that  $(\Delta T_0)$  has no approximate solution  $h$  with  $h \sim \hat{f} - f_0$ . We now consider the effect of multiplicative conjugation by  $\mathfrak{f}$  on the asymptotic differential equations considered so far.

**Lemma 4.55.** *Consider the asymptotic differential equation*

$$(\mathfrak{f}^{-1}E) \quad P_{\times \mathfrak{f}}(Y) = 0, \quad Y \in \mathfrak{f}^{-1}\mathcal{E}$$

*over  $K$ . Then  $(\mathfrak{f}^{-1}\hat{f}, \mathfrak{f}^{-1}\hat{\mathcal{E}}')$  is an unraveller for*

$$(\mathfrak{f}^{-1}\hat{E}) \quad P_{\times \mathfrak{f}}(Y) = 0, \quad Y \in \mathfrak{f}^{-1}\hat{\mathcal{E}}$$

*over  $\hat{K}$ , and  $\text{ddeg}_{\prec 1}(P_{\times \mathfrak{f}})_{+\mathfrak{f}^{-1}\hat{f}} = d = \text{ddeg}_{\mathfrak{f}^{-1}\hat{\mathcal{E}}} P_{\times \mathfrak{f}}$ . Moreover, if (T) is a Tschirnhaus refinement of  $(\hat{E})$ , then*

$$(\mathfrak{f}^{-1}T) \quad (P_{\times \mathfrak{f}})_{+\mathfrak{f}^{-1}f}(Y) = 0, \quad Y \prec 1$$

*is a Tschirnhaus refinement of  $(\mathfrak{f}^{-1}\hat{E})$ . If (TC) is a compatible refinement of (T), then*

$$(\mathfrak{f}^{-1}TC) \quad (P_{\times \mathfrak{f}})_{+\mathfrak{f}^{-1}(f+\hat{g})}(Y) = 0, \quad Y \preceq \mathfrak{f}^{-1}\mathfrak{m}$$

*is a compatible refinement of  $(\mathfrak{f}^{-1}T)$ .*

PROOF. The claims in the second sentence follow directly from Lemma 4.35. The other claims are direct but tedious calculations; however, it is important to recall that  $\Delta = (\partial^i)_{\times \mathfrak{f}}$ , so depends on  $\mathfrak{f}$ , and, by [ADH17a, Lemma 12.8.8],

$$((\partial^i)_{\times \mathfrak{f}} P)_{\times \mathfrak{f}} = \partial^i P_{\times \mathfrak{f}}. \quad \square$$

**4.6.3. The Slowdown Lemma.** In this subsection, we assume that (T) is a Tschirnhaus refinement of  $(\hat{E})$  and (TC) is a compatible refinement of (T). Set  $\mathfrak{g} := \mathfrak{d}_{\hat{g}}$ , with  $\hat{g}$  as in (TC). The main result of this subsection is Lemma 4.57, called the Slowdown Lemma. A consequence of this, Lemma 4.59,

gives the reduction to the special case of Proposition 4.47 considered in §4.6.1. We first prove the following preliminary lemma. Recall from §4.2.3 the coarsening  $\preceq_\phi$  of  $\preceq$  for  $\phi \in K^\times$  with  $\phi \neq 1$ .

**Lemma 4.56.** *Suppose that  $\mathfrak{f} = 1$ . Then*

$$\Delta P_{+f}(\hat{g}) \preceq_{\mathfrak{g}} \mathfrak{g} \Delta P_{+f}.$$

PROOF. Let  $f_0 \in \hat{K}^\times$  be a solution of (4.6.3) that best approximates  $\hat{f}$  and satisfies  $f - \hat{f} \sim f_0 - \hat{f}$ ; in particular,  $f_0 \sim f \sim \hat{f} \preceq 1$ . For this proof, set  $Q := \Delta P$ .

Since (TC) is compatible with (T),  $\hat{g}$  is not an approximate solution of  $(\Delta T)$ , and thus, with  $u := \hat{g}/\mathfrak{g}$ ,

$$D_{Q_{+f}, \times \mathfrak{g}}(\bar{u}) \neq 0.$$

This yields

$$Q_{+f}(\hat{g}) = Q_{+f, \times \mathfrak{g}}(u) \preceq Q_{+f, \times \mathfrak{g}}.$$

Now, since  $f - f_0 \prec \mathfrak{g}$ , Lemma 2.2(i) gives

$$Q_{+f, \times \mathfrak{g}} = Q_{\times \mathfrak{g}, +f/\mathfrak{g}} \sim Q_{\times \mathfrak{g}, +f_0/\mathfrak{g}} = Q_{+f_0, \times \mathfrak{g}}.$$

As  $f_0$  is a solution of (4.6.3), we have

$$\text{mul } Q_{+f_0} = \text{ddeg } Q_{+f_0} = 1$$

by Lemma 4.50. Using Lemma 4.10 and Lemma 2.2(i) again, we get

$$Q_{+f_0, \times \mathfrak{g}} \preceq_{\mathfrak{g}} \mathfrak{g} Q_{+f_0} \sim \mathfrak{g} Q_{+f}.$$

Finally, we obtain the desired result by combining these steps:

$$Q_{+f}(\hat{g}) \preceq_{\mathfrak{g}} \mathfrak{g} Q_{+f}. \quad \square$$

Using this result, we now turn to the proof of the Slowdown Lemma. The idea, as Aschenbrenner, van den Dries, and van der Hoeven note, is that “the step from (E) to (T) is much larger than the step from (T) to (TC)” [ADH17a, p. 661 or arXiv p. 565].

**Lemma 4.57** (Slowdown Lemma). *With  $\mathfrak{m}$  the monomial appearing in (TC), we have*

$$\left[ v \left( \frac{\mathfrak{m}}{\mathfrak{g}} \right) \right] < \left[ v \left( \frac{\mathfrak{g}}{\mathfrak{f}} \right) \right].$$

PROOF. By Lemma 4.55, we may assume that  $\mathfrak{f} = 1$ , so  $\mathfrak{m} \prec f - \hat{f} \preceq \mathfrak{g} \prec 1$  and  $\Delta = \partial^{\mathfrak{i}}$ . Set  $F := P_{+f}$  and note that  $\text{ddeg } F_{+\hat{g}} = \text{ddeg } F = d$  by Lemma 4.2(i).

**Claim 4.57.1.**  $\mathfrak{g} (F_{+\hat{g}})_d \preceq_{\mathfrak{g}} (F_{+\hat{g}})_{d-1}$ .

PROOF OF CLAIM 4.57.1. By Lemma 4.56, we have  $\mathfrak{g} \partial^{\mathfrak{i}} F \preceq_{\mathfrak{g}} \partial^{\mathfrak{i}} F(\hat{g})$ , and hence it suffices to show that  $(F_{+\hat{g}})_d \preceq \partial^{\mathfrak{i}} F$  and  $\partial^{\mathfrak{i}} F(\hat{g}) \preceq (F_{+\hat{g}})_{d-1}$ .

By the choice of  $\mathfrak{i}$ , we have  $\partial^{\mathfrak{i}} P \prec P$ , so  $\partial^{\mathfrak{i}} F \prec F_{+\hat{g}}$  by Lemma 2.2(i). As  $\text{ddeg } F_{+\hat{g}} = d$ , we have  $F_{+\hat{g}} \prec (F_{+\hat{g}})_d$ , and thus  $(F_{+\hat{g}})_d \prec \partial^{\mathfrak{i}} F$ . By Taylor expansion,  $\partial^{\mathfrak{i}} F(\hat{g})$  is, up to a factor from  $\mathbb{Q}^\times$ , the coefficient of  $Y^{\mathfrak{i}}$  in  $F_{+\hat{g}}$ . Since  $|\mathfrak{i}| = d - 1$ , this yields  $\partial^{\mathfrak{i}} F(\hat{g}) \preceq (F_{+\hat{g}})_{d-1}$ . ■

**Claim 4.57.2.**  $\mathfrak{n} \prec_{\mathfrak{n}} \mathfrak{g} \implies \text{ddeg } F_{+\hat{g}, \times \mathfrak{n}} \leq d - 1$ .

PROOF OF CLAIM 4.57.2. Suppose that  $\mathfrak{n} \prec_{\mathfrak{n}} \mathfrak{g}$ . Then  $\mathfrak{n} \prec 1$ , so by Lemma 3.7,

$$\text{ddeg } F_{+\hat{g}, \times \mathfrak{n}} \leq \text{ddeg } F_{+\hat{g}} = d,$$

and hence it suffices to show that  $(F_{+\hat{g}, \times \mathfrak{n}})_d \prec_{\mathfrak{n}} (F_{+\hat{g}})_{d-1}$ . By Lemma 4.9, for all  $i$ ,

$$(F_{+\hat{g}, \times \mathfrak{n}})_i = ((F_{+\hat{g}})_i)_{\times \mathfrak{n}} \asymp_{\mathfrak{n}} \mathfrak{n}^i (F_{+\hat{g}})_i,$$

so we show that  $\mathfrak{n} (F_{+\hat{g}})_d \prec_{\mathfrak{n}} (F_{+\hat{g}})_{d-1}$ . First, as  $(F_{+\hat{g}})_d \neq 0$ , we have  $\mathfrak{n} (F_{+\hat{g}})_d \prec_{\mathfrak{n}} \mathfrak{g} (F_{+\hat{g}})_d$ . Second,  $\mathfrak{n} \prec \mathfrak{g} \prec 1$  implies  $[v\mathfrak{n}] \leq [v\mathfrak{g}]$ , so the first claim and Lemma 4.8 yield  $\mathfrak{g} (F_{+\hat{g}})_d \prec_{\mathfrak{n}} (F_{+\hat{g}})_{d-1}$ . Combining these two relations, we obtain  $\mathfrak{n} (F_{+\hat{g}})_d \prec_{\mathfrak{n}} (F_{+\hat{g}})_{d-1}$ , as desired.  $\blacksquare$

To finish the proof of the lemma, note that  $\text{ddeg } F_{+\hat{g}, \times \mathfrak{m}} = d$ , because (TC) is compatible. Then the second claim gives  $\mathfrak{g} \prec_{\mathfrak{m}} \mathfrak{m}$ , and so  $\mathfrak{g} \prec_{\mathfrak{g}} \mathfrak{m}$  by Lemma 4.7. But since  $\mathfrak{m} \prec \mathfrak{g}$ , we must have  $\mathfrak{m} \asymp_{\mathfrak{g}} \mathfrak{g}$ , which means that  $[v\mathfrak{m} - v\mathfrak{g}] < [v\mathfrak{g}]$ , as desired.  $\square$

#### 4.6.4. Consequences of the Slowdown Lemma.

**Corollary 4.58.** *If (T) is a Tschirnhaus refinement of  $(\hat{\mathbf{E}})$ , then*

$$\mathfrak{e} \prec \hat{f} - f \implies \left[ v \left( \frac{\mathfrak{e}}{\hat{f} - f} \right) \right] < \left[ v \left( \frac{\hat{f} - f}{\hat{f}} \right) \right].$$

PROOF. This follows immediately from Lemmas 4.54 and 4.57.  $\square$

**Lemma 4.59.** *Suppose that (T) is a Tschirnhaus refinement of  $(\hat{\mathbf{E}})$  and  $\mathfrak{e} \prec \hat{f} - f$ . Let  $F := P_{+f}$ ,  $\hat{g} := \hat{f} - f$ , and  $\mathfrak{g} := \mathfrak{d}_{\hat{g}}$ . Then the asymptotic differential equation*

$$(\hat{\mathbf{E}}_{\mathfrak{g}, \leq d}) \quad F_{\leq d}(Y) = 0, \quad Y \asymp \mathfrak{g}$$

*has dominant degree  $d$ . Moreover, with  $\hat{\mathcal{E}}'_g := \{y \in \hat{\mathcal{E}}' : y \prec \mathfrak{g}\}$ ,  $(\hat{g}, \hat{\mathcal{E}}'_g)$  is an unraveller for  $(\hat{\mathbf{E}}_{\mathfrak{g}, \leq d})$  and  $\mathfrak{e}$  is the largest algebraic starting monomial for the unravelled asymptotic differential equation*

$$(\hat{\mathbf{E}}'_{\mathfrak{g}, \leq d}) \quad (F_{\leq d})_{+\hat{g}}(Y) = 0, \quad Y \in \hat{\mathcal{E}}'_g$$

*over  $\hat{K}$ .*

PROOF. This follows from Corollary 4.58 by applying Lemma 4.41 with  $\hat{K}$ ,  $\hat{f}$ ,  $f$ ,  $\hat{g}$ ,  $\hat{\mathcal{E}}$ ,  $\hat{\mathcal{E}}'$ , and  $\hat{\mathcal{E}}'_g$  in the roles of  $K$ ,  $f$ ,  $f - g$ ,  $g$ ,  $\mathcal{E}$ ,  $\mathcal{E}'$ , and  $\mathcal{E}'_g$ , respectively.  $\square$

**4.6.5. Proposition 4.47 and its consequence.** Finally, we return to the proof of the main proposition of this section. Recall the statement:

**Proposition 4.47.** *There exists  $f \in \hat{K}$  such that one of the following holds:*

- (i)  $\hat{f} - f \asymp \mathfrak{e}$  and  $A(f) = 0$  for some  $A \in K\{Y\}$  with  $c(A) < c(P)$  and  $\deg A = 1$ ;
- (ii)  $\hat{f} \sim f$ ,  $\hat{f} - a \asymp f - a$  for all  $a \in K$ , and  $A(f) = 0$  for some  $A \in K\{Y\}$  with  $c(A) < c(P)$  and  $\text{ddeg } A_{\times f} = 1$ .

PROOF. As noted already, if  $\mathfrak{e} \succ \mathfrak{f}$ , then case (i) holds with  $f := 0$  and  $A := Y$ , so suppose that  $\mathfrak{e} \prec \mathfrak{f}$ . By Lemma 4.51,  $\hat{f}$  is an approximate solution of (4.6.3), so by Lemmas 4.45 and 4.44, we have a solution  $f_0 \sim \hat{f}$  in  $\hat{K}^\times$  of (4.6.3) that best approximates  $\hat{f}$ . If  $\hat{f} - a \asymp f_0 - a$  for all  $a \in K$ , then case (ii) holds with  $f := f_0$  and  $A := \Delta P$ . Now suppose to the contrary that we have  $f \in K^\times$

with  $\hat{f} - f \succ f_0 - f$ . That is,  $f_0 - \hat{f} \sim f - \hat{f}$ , so in view of  $f_0 \sim \hat{f}$ , we have  $f \sim \hat{f}$ . Hence (T) is a Tschirnhaus refinement of  $(\hat{E})$ . We are going to show that then case (i) holds.

If  $\hat{f} - f \preceq \mathfrak{e}$ , then case (i) holds with  $A := Y - f$ , so for the rest of the proof, assume that  $\mathfrak{e} \prec \hat{f} - f$ , and set  $F := P_{+f}$ ,  $\hat{g} := \hat{f} - f$ , and  $\mathfrak{g} := \mathfrak{d}_{\hat{g}}$ . This puts us in the situation of the previous lemma, so  $(\hat{E}_{\mathfrak{g}, \leq d})$  has dominant degree  $d$  and  $(\hat{g}, \hat{\mathcal{E}}'_{\mathfrak{g}})$  is an unraveller for  $(\hat{E}_{\mathfrak{g}, \leq d})$ . In particular,  $\text{ddeg}_{\prec \hat{g}}(F_{\leq d})_{+\hat{g}} = d$ , since

$$\text{ddeg}_{\prec \hat{g}}(F_{\leq d})_{+\hat{g}} \geq \text{ddeg}_{\hat{\mathcal{E}}'_{\mathfrak{g}}}(F_{\leq d})_{+\hat{g}} = d.$$

Also,  $\mathfrak{e}$  is the largest algebraic starting monomial for  $(\hat{E}'_{\mathfrak{g}, \leq d})$ . Now since  $f \in K$ , we can view  $(\hat{E}_{\mathfrak{g}, \leq d})$  as an asymptotic differential equation over  $K$ . We also have  $\deg F_{\leq d} = d$  and  $\text{mul}(F_{\leq d})_{+\hat{g}} < d$ , since otherwise  $(F_{\leq d})_{+\hat{g}}$  would be homogeneous and so not have any algebraic starting monomials. Thus with  $(\hat{E}_{\mathfrak{g}, \leq d})$  in place of (E) and  $(\hat{g}, \hat{\mathcal{E}}'_{\mathfrak{g}})$  in place of  $(\hat{f}, \hat{\mathcal{E}}')$ , Lemma 4.49 applies. Hence we have  $g \in \hat{K}$  and  $B \in K\{Y\}$  such that  $\hat{g} - g \preceq \mathfrak{e}$ ,  $B(g) = 0$ ,  $c(B) < c(F_{\leq d})$ , and  $\deg B = 1$ . Finally, case (i) holds with  $f + g$  in place of  $f$  and with  $A := B_{-f}$ , completing the proof.  $\square$

In fact, if  $K$  is  $r$ -d-henselian, then the  $f \in \hat{K}$  in Proposition 4.47 actually lies in  $K$  by [ADH17a, Proposition 7.5.6]. We do not use Proposition 4.47 directly in the proof of Proposition 4.15, but rather this corollary concerning pc-sequences.

**Corollary 4.60.** *Suppose that  $(a_\rho)$  is a divergent pc-sequence in  $K$  with pseudolimit  $\hat{f} \in \hat{K}$  and minimal differential polynomial  $P$  over  $K$ . Then there exist  $f \in \hat{K}$  and  $A \in K\{Y\}$  such that  $\hat{f} - f \preceq \mathfrak{e}$ ,  $A(f) = 0$ ,  $c(A) < c(P)$ , and  $\deg A = 1$ .*

PROOF. Suppose towards a contradiction that there are no such  $f$  and  $A$ . Then Proposition 4.47 gives instead  $f \in \hat{K}$  and  $A \in K\{Y\}^\neq$  such that  $\hat{f} - a \preceq f - a$  for all  $a \in K$ ,  $A(f) = 0$ , and  $c(A) < c(P)$ . Since  $(a_\rho)$  has no pseudolimit in  $K$ ,  $\hat{f} \notin K$ , and so  $f \notin K$ . Hence we may take a divergent pc-sequence  $(b_\lambda)$  in  $K$  such that  $b_\lambda \rightsquigarrow f$ . Since  $\hat{f} - b_\lambda \preceq f - b_\lambda$  for all  $\lambda$ , we have  $b_\lambda \rightsquigarrow \hat{f}$ . By Lemma 2.6, the pc-sequences  $(a_\rho)$  and  $(b_\lambda)$  must also have the same width, since they have no pseudolimit in  $K$  but a common pseudolimit in  $\hat{K}$ , and so  $(a_\rho)$  and  $(b_\lambda)$  are equivalent pc-sequences in  $K$  by Lemma 2.8. Thus  $a_\rho \rightsquigarrow f$ , so applying Lemma 2.11 to  $A$  and  $f$  contradicts that  $P$  is a minimal differential polynomial of  $(a_\rho)$  over  $K$ .  $\square$

#### 4.7. Proof of Proposition 4.15

In this section, we prove the main proposition, derived from the work of the previous sections, thus completing the proof of the main results.

**Proposition 4.15.** *Suppose that  $K$  is asymptotic,  $\Gamma$  is divisible, and  $\mathbf{k}$  is  $r$ -linearly surjective. Let  $(a_\rho)$  be a pc-sequence in  $K$  with minimal differential polynomial  $G$  over  $K$  of order at most  $r$ . Then  $\text{ddeg}_a G = 1$ .*

PROOF. Let  $d := \text{ddeg}_a G$ . We may assume that  $(a_\rho)$  has no pseudolimit in  $K$ , as otherwise, up to scaling,  $G$  is of the form  $Y - a$  for some pseudolimit  $a$  of  $(a_\rho)$ , and hence  $d = 1$ . We may also assume that  $r \geq 1$ , since the case  $r = 0$  is handled by the analogous fact for valued fields of equicharacteristic 0 (see [ADH17a, Proposition 3.3.19]). By Zorn's lemma, we may take a  $d$ -algebraically maximal



immediate extension  $\hat{K}$  of  $K$ . By the proof of [ADH17a, Theorem 7.0.1],  $\hat{K}$  is  $r$ -d-henselian. Note that as an immediate extension of  $K$ ,  $\hat{K}$  is also asymptotic by Lemmas 2.14 and 2.15.

Now, take  $\ell \in \hat{K}$  such that  $a_\rho \rightsquigarrow \ell$ , so  $G$  is an element of  $Z(K, \ell)$  of minimal complexity by Corollary 4.5. Lemma 4.4 gives  $d \geq 1$ , as well as  $a \in K$  and  $\mathfrak{v} \in K^\times$  such that  $a - \ell \prec \mathfrak{v}$  and  $\text{ddeg}_{\prec \mathfrak{v}} G_{+a} = d$ . Towards a contradiction, suppose that  $d \geq 2$ . Lemma 4.39 then yields an unraveller  $(\hat{f}, \hat{\mathcal{E}})$  for the asymptotic differential equation

$$(4.7.1) \quad G_{+a}(Y) = 0, \quad Y \prec \mathfrak{v}$$

over  $\hat{K}$  such that:

- (i)  $\hat{f} \neq 0$ ;
- (ii)  $\text{ddeg}_{\prec \hat{f}} G_{+a+\hat{f}} = d$ ;
- (iii)  $a_\rho \rightsquigarrow a + \hat{f} + g$  for all  $g \in \hat{\mathcal{E}} \cup \{0\}$ ;
- (iv)  $\text{mul } G_{+a+\hat{f}} < d$ ,

where (iv) follows from (iii) by Lemma 4.38.

Suppose first that  $K$  has a monomial group. Consider the pc-sequence  $(a_\rho - a)$  with minimal differential polynomial  $P := G_{+a}$  over  $K$ . Since  $(\hat{f}, \hat{\mathcal{E}})$  is an unraveller for (4.7.1), we have  $\text{ddeg}_{\hat{\mathcal{E}}} P_{+\hat{f}} = d > \text{mul } P_{+\hat{f}}$  by (iv), so let  $\mathfrak{e}$  be the largest algebraic starting monomial for the asymptotic differential equation

$$(4.7.2) \quad P_{+\hat{f}}(Y) = 0, \quad Y \in \hat{\mathcal{E}}$$

over  $\hat{K}$  by Proposition 4.28. Hence all the assumptions of the previous section are satisfied (with (4.7.1) and (4.7.2) in the roles of  $(\hat{\mathcal{E}})$  and  $(\hat{\mathcal{E}}')$ , respectively), so applying Corollary 4.60 to  $(a_\rho - a)$  and  $P$  yields  $f \in \hat{K}$  and  $A \in K\{Y\}^\neq$  such that  $\hat{f} - f \prec \mathfrak{e}$ ,  $A(f) = 0$ , and  $c(A) < c(P)$ . Since  $\mathfrak{e}$  is an algebraic starting monomial for (4.7.2), we have  $\mathfrak{e} \in \hat{\mathcal{E}}$ , and so  $f - \hat{f} \in \hat{\mathcal{E}} \cup \{0\}$ . But then  $a_\rho - a \rightsquigarrow f$  by (iii), so applying Lemma 2.11 to  $A$  and  $f$  contradicts the minimality of  $P$ .

Finally, we reduce to the case that  $K$  has a monomial group. Consider  $\hat{K}$  as a valued differential field with a predicate for  $K$  and pass to an  $\aleph_1$ -saturated elementary extension of this structure. In particular, the new  $K$  has a monomial group [ADH17a, Lemma 3.3.39]. In doing this, we preserve all the relevant first order properties: small derivation,  $r$ -linearly surjective differential residue field, divisible value group, asymptoticity,  $r$ -d-henselianity of  $\hat{K}$ , and that  $G \in Z(K, \ell)$  but  $H \notin Z(K, \ell)$  for all  $H \in K\{Y\}$  with  $c(H) < c(G)$ .

However, it is possible that  $\hat{K}$  is no longer d-algebraically maximal, in which case we pass to a d-algebraically maximal immediate extension of  $\hat{K}$  (and hence of  $K$ ). It is also possible that  $(a_\rho)$  is no longer divergent in  $K$ , in which case we replace  $(a_\rho)$  with a divergent pc-sequence  $(b_\lambda)$  in  $K$  with  $b_\lambda \rightsquigarrow \ell$ . By Corollary 4.5,  $G$  is a minimal differential polynomial of  $(b_\lambda)$  over  $K$ , and by Lemma 4.4,  $\text{ddeg}_{\mathbf{b}} G = d$ , where  $\mathbf{b} := c_K(b_\lambda)$ . By the argument above used in this new structure,  $d = 1$ , as desired.  $\square$

## CHAPTER 5

# Newtonian valued differential fields with arbitrary value group

### 5.1. Introduction

A consequential first-order property of  $\mathbb{T}$  is that it is newtonian. Indeed, newtonianity is an axiom scheme in the theory  $T^{\text{nl}}$  and hence in the axiomatization of  $\mathbb{T}$ . Newtonianity is a generalization of henselianity to the class of ungrounded  $H$ -asymptotic fields, where we say that an asymptotic field  $K$  is *ungrounded* if  $\Psi := \psi(\Gamma^\neq)$  has no maximum element (recall the asymptotic couple  $(\Gamma, \psi)$  of an asymptotic field from §2.4). What is the relationship between newtonianity and d-henselianity? Valued differential fields like  $\mathbb{T}$  cannot be d-henselian because even when they have small derivation, the derivation induced on the residue field is trivial, whereas the residue field of a d-henselian field has nontrivial derivation and contains solutions to all linear differential equations. In some sense, newtonianity is an “eventual” version of d-henselianity; this is made precise in [ADH17a, §14.1].

The book [ADH17a] develops a theory of newtonian fields, including results analogous to Theorems 4.19 and 4.20 for certain ungrounded  $H$ -asymptotic fields. Namely, for these fields newtonianity is equivalent to asymptotic differential-algebraic maximality [ADH17a, Theorems 14.0.1 and 14.0.2] and they have newtonizations [ADH17a, Corollary 14.5.4]. These results played an important role in the proof of model completeness for  $\mathbb{T}$ . This chapter makes minor improvements to these two results. In Chapter 4, we first established the main lemma, Proposition 4.15, under the assumption that the value group was divisible and then reduced the main results to that case via Lemma 4.16. In doing that reduction, we noticed that a similar lemma could be proved to remove the divisibility assumption in [ADH17a, Theorem 14.0.2 and Corollary 14.5.4]. The results of this chapter appear in [Pyn19].

This chapter is about certain kinds of asymptotic fields not considered earlier in the thesis, so before stating the main theorems, we recall some necessary definitions. We are interested in ungrounded asymptotic fields, and it turns out that these satisfy an additional condition relating the valuation and derivation, namely they are pre-d-valued [ADH17a, Corollary 10.1.3]. We are also interested in pre-d-valued fields in Chapter 6, since all pre- $H$ -fields are pre-d-valued.

**Definition.** We call  $K$  *pre-differential-valued* (*pre-d-valued* for short) if for all  $f, g \in K^\times$  with  $f \preccurlyeq 1$  and  $g \prec 1$ , we have  $f' \prec g^\dagger$ .

By [ADH17a, Lemma 10.1.1], pre-d-valued fields are asymptotic. Here is an equivalent characterization showing that pre-d-valued fields are exactly those with few constants that satisfy an analogue of l’Hôpital’s Rule.

**Lemma 5.1** ([ADH17a, 10.1.4]). *The valued differential field  $K$  is pre-d-valued if and only if the following two conditions are satisfied.*

- (i)  $C \subseteq \mathcal{O}$ ;
- (ii) for all  $f, g \in K^\times$ , if  $f \preccurlyeq g \prec 1$ , then  $\frac{f}{g} - \frac{f'}{g'} \prec 1$ .

For the rest of the introduction,  $K$  will be an asymptotic field; we impose further standing assumptions at the end of this section. We say that  $K$  is *differential-valued* (d-valued for short) if  $\mathcal{O} = C + \mathfrak{o}$ . By [AD02, Theorem 4.4], pre-d-valued fields are exactly the valued differential subfields of d-valued fields. The same is true with “pre-d-valued fields of  $H$ -type” and “d-valued fields of  $H$ -type,” where we say that an asymptotic field is *of  $H$ -type* if it is  $H$ -asymptotic. We are concerned here mainly with ungrounded d-valued fields of  $H$ -type.

Apart from being an ungrounded d-valued field of  $H$ -type, two of the main properties of  $\mathbb{T}$  are newtonianity, already mentioned, and  $\mathfrak{w}$ -freeness. The former has a somewhat technical definition that we delay until §5.2. But granted that, we can define newtonizations; in this definition, we require  $K$  to be ungrounded and  $H$ -asymptotic. We say that an ungrounded  $H$ -asymptotic extension  $L$  of  $K$  is a *newtonization* of  $K$  if it is newtonian and embeds over  $K$  into every ungrounded  $H$ -asymptotic extension of  $K$  that is newtonian.

**Definition.** We say that an ungrounded  $H$ -asymptotic field  $K$  is  $\mathfrak{w}$ -free if for all  $f \in K$ , there is  $g \in K$  such that  $g \succ 1$  and

$$f + (2(-g^{\dagger\dagger})' + (g^{\dagger\dagger})^2) \succ (g^\dagger)^2.$$

Thus  $\mathfrak{w}$ -freeness can be expressed as a universal-existential sentence in the language of valued differential fields, but it is also equivalent to the absence of a pseudolimit of a certain pc-sequence  $(\mathfrak{w}_p)$  related to iterated logarithms, hence the “free” in the name. In  $\mathbb{T}$ , the sequence  $(\mathfrak{w}_n)$  is indexed by  $\mathbb{N}$  and defined by:

$$\mathfrak{w}_n = \frac{1}{(\ell_0)^2} + \frac{1}{(\ell_0 \ell_1)^2} + \cdots + \frac{1}{(\ell_0 \ell_1 \dots \ell_n)^2},$$

where  $\ell_0 := x$  and  $\ell_{n+1} := \log \ell_n$ . See [ADH17a, Corollary 11.7.8] and the surrounding pages for more on these equivalent definitions. The definition of  $\mathfrak{w}$ -freeness is not directly used in this chapter.

We can now state the main theorems. The first connects newtonianity to asymptotic d-algebraic maximality:  $K$  is said to be *asymptotically differential-algebraically maximal* (slightly shorter: *asymptotically d-algebraically maximal*) if it has no proper immediate d-algebraic extension that is asymptotic. Note that any  $K$  satisfying the assumptions of the first theorem is in fact d-valued by [ADH17a, Lemma 14.2.5] and Lemma 5.3.

**Theorem 5.20.** *If  $K$  is an ungrounded  $H$ -asymptotic field that is  $\mathfrak{w}$ -free and newtonian, then it is asymptotically d-algebraically maximal.*

**Theorem 5.21.** *If  $K$  is an ungrounded  $H$ -asymptotic field that is  $\mathfrak{w}$ -free and d-valued, then it has a newtonization.*

The case that the value group is divisible is covered by [ADH17a, Theorem 14.0.2 and Corollary 14.5.4]. The converse of Theorem 5.20 holds for ungrounded  $H$ -asymptotic  $K$  that are  $\lambda$ -free, a weaker notion than  $\mathfrak{w}$ -freeness to be defined in §5.2; see Theorem 5.4 ([ADH17a, Theorem 14.0.1]).

Finally, the next theorem is a corollary of the previous two results, but we provide an alternative proof in §5.4. The case that the value group is divisible is not stated in [ADH17a], but follows from the corresponding results.

**Theorem 5.25.** *If  $K$  is an ungrounded  $H$ -asymptotic field that is  $\omega$ -free and  $d$ -valued, then any two immediate  $d$ -algebraic extensions of  $K$  that are asymptotically  $d$ -algebraically maximal are isomorphic over  $K$ .*

**5.1.1. Assumptions.** As we are primarily concerned with ungrounded  $H$ -asymptotic fields in this chapter, to avoid repetition we assume throughout that  $K$  is an ungrounded  $H$ -asymptotic field.

## 5.2. Preliminaries

We review here some definitions and results from [ADH17a] that are used later. Most of these are from Chapters 9, 11, 13, and 14, and although proofs are omitted, references are provided.

**5.2.1. More on asymptotic fields.** The first lemma does not require the  $H$ -asymptotic or ungrounded assumptions on  $K$  and is valid for any asymptotic  $K$ .

**Lemma 5.2** ([ADH17a, 9.1.2]). *If  $K$  is  $d$ -valued and  $L$  is an asymptotic extension of  $K$  with  $k_L = k$ , then  $L$  is  $d$ -valued and  $C_L = C$ .*

In the main results we will assume that  $K$  is  $\omega$ -free, but at various places we weaken that assumption for generality. For instance,  $K$  has *asymptotic integration* if  $\Gamma = (\Gamma^\neq)'$ . We also consider this property for the algebraic closure  $K^{\text{ac}}$  of  $K$ . Recall how we construe  $K^{\text{ac}}$  as a valued differential field extension of  $K$ , so  $K^{\text{ac}}$  is  $H$ -asymptotic and ungrounded. If  $K$  is  $d$ -valued, then so is  $K^{\text{ac}}$  [ADH17a, Corollary 10.1.23]. We say that  $K$  has *rational asymptotic integration* if  $K^{\text{ac}}$  has asymptotic integration, that is,  $\mathbb{Q}\Gamma = (\mathbb{Q}\Gamma^\neq)'$ .

**Definition.** We call  $K$   $\lambda$ -free if for all  $f \in K$ , there is  $g \in K$  such that  $g \succ 1$  and  $f - g^{\dagger\dagger} \succcurlyeq g^\dagger$ .

Thus  $\lambda$ -freeness can be expressed as a universal-existential sentence in the language of valued differential fields, but it is also equivalent to the absence of a pseudolimit of a certain pseudocauchy sequence  $(\lambda_\rho)$  related to iterated logarithms, hence the “free” in the name. In  $\mathbb{T}$ , the sequence  $(\lambda_n)$  is indexed by  $\mathbb{N}$  and defined by:

$$\lambda_n = -\left(\ell_n^{\dagger\dagger}\right) = \frac{1}{\ell_0} + \frac{1}{\ell_0\ell_1} + \cdots + \frac{1}{\ell_0\ell_1\cdots\ell_n}.$$

See [ADH17a, Corollary 11.6.1] and the surrounding pages for more on these equivalent definitions. This definition is also not directly used. Here is the relationship among these notions, collecting [ADH17a, Corollaries 11.6.8 and 11.7.3].

**Lemma 5.3.** *We have*

$$\omega\text{-free} \implies \lambda\text{-free} \implies \text{rational asymptotic integration} \implies \text{asymptotic integration}.$$

It is also worth noting that all  $H$ -asymptotic fields with asymptotic integration are ungrounded by [ADH17a, Corollary 9.2.16].

**Theorem 5.4** ([ADH17a, 14.0.1]). *If  $K$  is  $\lambda$ -free and asymptotically  $d$ -algebraically maximal, then it is  $\omega$ -free and newtonian.*

Both  $\lambda$ -freeness and  $\omega$ -freeness are preserved under algebraic extensions, and even more is true for  $\omega$ -freeness. In particular, these properties are preserved when passing to the henselization, a fact that is used in the main results.

**Lemma 5.5** ([ADH17a, 11.6.8]). *The algebraic closure  $K^{\text{ac}}$  of  $K$  is  $\lambda$ -free if and only if  $K$  is  $\lambda$ -free.*

**Lemma 5.6** ([ADH17a, 11.7.23]). *The algebraic closure  $K^{\text{ac}}$  of  $K$  is  $\omega$ -free if and only if  $K$  is  $\omega$ -free.*

**Theorem 5.7** ([ADH17a, 13.6.1]). *If  $K$  is  $\omega$ -free and  $L$  is a pre-d-valued extension of  $K$  of  $H$ -type that is d-algebraic over  $K$ , then  $L$  is  $\omega$ -free.*

**5.2.2. Compositional conjugation and newton degree.** We recall the notion of newton degree from [ADH17a, §11.1 and §11.2], a more subtle version of dominant degree for asymptotic fields that may not have small derivation.<sup>1</sup> We say that  $\phi \in K^\times$  is *active (in  $K$ )* if  $v\phi \in \Psi^\downarrow$ , where  $\Psi^\downarrow$  denotes the downward closure of  $\Psi$  in  $\Gamma$ . Below, we let  $\phi$  range over the active elements of  $K^\times$ . To  $K$  we associate the valued differential field  $K^\phi$ , which is simply the field  $K$  with the derivation  $\phi^{-1}\partial$  and unchanged valuation; it is still  $H$ -asymptotic and ungrounded, and moreover it has small derivation by [ADH17a, Lemma 9.2.9]. We call  $K^\phi$  the *compositional conjugate of  $K$  by  $\phi$* .

This leads to the ring  $K^\phi\{Y\}$  of differential polynomials over  $K^\phi$ , which is viewed as a differential ring with derivation extending  $\phi^{-1}\partial$ . We then have a ring isomorphism  $K\{Y\} \rightarrow K^\phi\{Y\}$  given by associating to  $P \in K\{Y\}$  an appropriate element  $P^\phi \in K^\phi\{Y\}$ , called the *compositional conjugate of  $P$  by  $\phi$* , with the property that  $P^\phi(y) = P(y)$  for all  $y \in K$ . The details of this map are not used here and can be found in [ADH17a, §5.7], but it is the identity on the common subring  $K[Y] = K^\phi[Y]$  of  $K\{Y\}$  and  $K^\phi\{Y\}$ . What is important here is that  $\text{ddeg } P^\phi$  eventually stabilizes, that is, there is an active  $\phi_0 \in K^\times$  such that for all  $\phi \preceq \phi_0$ ,  $\text{ddeg } P^\phi = \text{ddeg } P^{\phi_0}$ . We call this eventual value of  $\text{ddeg } P^\phi$  the *newton degree of  $P$*  and denote it by  $\text{ndeg } P$ . With this, we can finally define newtonianity.

**Definition.** We call  $K$  *newtonian* if each  $P \in K\{Y\}$  with  $\text{ndeg } P = 1$  has a zero in  $\mathcal{O}$ .

We know that if  $K$  has an immediate newtonian extension, then it must have a minimal one:

**Lemma 5.8** ([ADH17a, 14.1.9]). *If  $K$  has an immediate newtonian extension, then  $K$  has a d-algebraic such extension that has no proper newtonian differential subfield containing  $K$ .*

Newton degree is connected to pc-sequences in an important way. Recall from §3.2.1 how we associate to each pc-sequence  $(a_\rho)$  in  $K$  its cut in  $K$ , denoted by  $c_K(a_\rho)$ , such that if  $(b_\lambda)$  is a pc-sequence in  $K$ , then

$$c_K(a_\rho) = c_K(b_\lambda) \iff (b_\lambda) \text{ is equivalent to } (a_\rho).$$

In the rest of the chapter, let  $(a_\rho)$  be a pc-sequence in  $K$  with  $\mathbf{a} = c_K(a_\rho)$ . If  $L$  is an extension of  $K$ , then we let  $\mathbf{a}_L$  denote  $c_L(a_\rho)$ . We now define newton degree in a cut in the same way that we defined dominant degree in a cut in §3.2.1. To do so, we set, for any  $\gamma \in \Gamma$  and  $P \in K\{Y\}^\neq$ ,

$$\text{ndeg}_{\geq \gamma} P := \max\{\text{ndeg } P_{\times g} : g \in K^\times, v g \geq \gamma\}.$$

<sup>1</sup>Newton degree has since been extended to valued differential fields with continuous derivation in [ADH18].

Note that  $\text{ndeg}_{\geq \gamma} P = \text{ndeg } P_{\times g}$  for any  $g \in K^\times$  with  $vg = \gamma$ . With  $\gamma_\rho := v(a_{\rho+1} - a_\rho)$ , there is  $\rho_0$  such that

$$\text{ndeg}_{\geq \gamma_\rho} P_{+a_\rho} = \text{ndeg}_{\geq \gamma_{\rho_0}} P_{+a_{\rho_0}}$$

for all  $\rho > \rho_0$ , and this value depends only on  $c_K(a_\rho)$ , not the choice of pc-sequence (see [ADH17a, Lemma 11.2.11]). In the next definition and two lemmas,  $P \in K\{Y\}^\neq$ .

**Definition.** The *newton degree of  $P$  in the cut of  $(a_\rho)$*  is the eventual value of  $\text{ndeg}_{\geq \gamma_\rho} P_{+a_\rho}$ , denoted by  $\text{ndeg}_{\mathbf{a}} P$ .

**Lemma 5.9** ([ADH17a, 11.2.12]). *Newton degree in a cut has the following properties:*

- (i)  $\text{ndeg}_{\mathbf{a}} P \leq \deg P$ ;
- (ii)  $\text{ndeg}_{\mathbf{a}} P^\phi = \text{ndeg}_{\mathbf{a}} P$ ;
- (iii)  $\text{ndeg}_{\mathbf{a}} P_{+y} = \text{ndeg}_{\mathbf{a}+y} P$  for  $y \in K$ ;
- (iv) if  $y \in K$  and  $vy$  is in the width of  $(a_\rho)$ , then  $\text{ndeg}_{\mathbf{a}} P_{+y} = \text{ndeg}_{\mathbf{a}} P$ ;
- (v)  $\text{ndeg}_{\mathbf{a}} P_{\times y} = \text{ndeg}_{\mathbf{a} \cdot y} P$  for  $y \in K^\times$ ;
- (vi) if  $Q \in K\{Y\}^\neq$ , then  $\text{ndeg}_{\mathbf{a}} PQ = \text{ndeg}_{\mathbf{a}} P + \text{ndeg}_{\mathbf{a}} Q$ ;
- (vii) if there is a pseudolimit  $\ell$  of  $(a_\rho)$  in an asymptotic extension of  $K$  with  $P(\ell) = 0$ , then  $\text{ndeg}_{\mathbf{a}} P \geq 1$ ;
- (viii) if  $L$  is an asymptotic extension of  $K$  with  $\Psi$  cofinal in  $\Psi_L$ , then  $\text{ndeg}_{\mathbf{a}} P = \text{ndeg}_{\mathbf{a}_L} P$ .

Recall from §2.3.1 that  $P \in K\{Y\}^\neq$  is a *minimal differential polynomial of  $(a_\rho)$  over  $K$*  if  $P(b_\lambda) \rightsquigarrow 0$  for some pc-sequence  $(b_\lambda)$  in  $K$  equivalent to  $(a_\rho)$  and  $c(P)$  is minimal among differential polynomials with this property. Note that then  $P \notin K$ .

We say  $K$  is *strongly newtonian* if it is newtonian and, for every  $P \in K\{Y\}$  and every pc-sequence  $(a_\rho)$  in  $K$  with minimal differential polynomial  $P$  over  $K$ ,  $\text{ndeg}_{\mathbf{a}} P = 1$ . Here is the usefulness of this notion.

**Lemma 5.10** ([ADH17a, 14.1.10]). *If  $K$  is newtonian and  $\text{ndeg}_{\mathbf{a}} P = 1$ , then  $P(a) = 0$  for some  $a \in K$  with  $a_\rho \rightsquigarrow a$ .*

### 5.2.3. Constructing immediate asymptotic extensions.

**Assumption.** Suppose for the rest of this section that  $K$  has rational asymptotic integration.

We recall some lemmas from [ADH17a, §11.4] on how to construct immediate asymptotic extensions of such fields with appropriate embedding properties, similar to those from §4.2.2. The first lemma is about evaluating differential polynomials at pc-sequences. Although the statement of [ADH17a, Lemma 11.3.8] is slightly less general than that given here, the same proof gives the following.

**Lemma 5.11** ([ADH17a, 11.3.8]). *Let  $E$  be an  $H$ -asymptotic extension of  $K$  with rational asymptotic integration such that  $\Gamma^<$  is cofinal in  $\Gamma_E^<$ . Suppose that  $(a_\rho)$  has a pseudolimit  $\ell$  in  $E$  and let  $G \in E\{Y\} \setminus E$ . Then there a pc-sequence  $(b_\lambda)$  in  $K$  equivalent to  $(a_\rho)$  such that  $(G(b_\lambda))$  is a pc-sequence in  $E$  with  $G(b_\lambda) \rightsquigarrow G(\ell)$ .*

Let  $\ell \notin K$  be an element in an asymptotic extension of  $K$  such that  $v(\ell - K) := \{v(\ell - a) : a \in K\}$  has no largest element (equivalently,  $\ell$  is the pseudolimit of some divergent pc-sequence in  $K$ ). Let  $P \in K\{Y\}^\neq$ . We say that  $P$  *vanishes at*  $(K, \ell)$  if for all  $a \in K$  and  $\mathfrak{v} \in K^\times$  with  $a - \ell \prec \mathfrak{v}$ ,  $\text{ndeg}_{\prec \mathfrak{v}} P_{+a} \geq 1$ , where  $\text{ndeg}_{\prec \mathfrak{v}} P_{+a} := \max\{\text{ndeg } P_{+a, \times g} : g \in K^\times, g \prec \mathfrak{v}\}$ . Then  $Z(K, \ell)$  denotes the set of nonzero differential polynomials over  $K$  vanishing at  $(K, \ell)$ .

**Lemma 5.12** ([ADH17a, 11.4.7]). *Suppose that  $Z(K, \ell) = \emptyset$ . Then  $P(\ell) \neq 0$  for all  $P \in K\{Y\}^\neq$  and  $K\langle \ell \rangle$  is an immediate asymptotic extension of  $K$ . If  $g$  in an asymptotic extension  $L$  of  $K$  satisfies  $v(a - g) = v(a - \ell)$  for all  $a \in K$ , then there is a unique embedding  $K\langle \ell \rangle \rightarrow L$  over  $K$  sending  $\ell$  to  $g$ .*

**Lemma 5.13** ([ADH17a, 11.4.8]). *Suppose that  $Z(K, \ell) \neq \emptyset$  and  $P \in Z(K, \ell)$  has minimal complexity. Then  $K$  has an immediate asymptotic extension  $K\langle f \rangle$  with  $P(f) = 0$  and  $v(a - f) = v(a - \ell)$  for all  $a \in K$ . Moreover, for any asymptotic extension  $L$  of  $K$  with  $g \in L$  satisfying  $P(g) = 0$  and  $v(a - g) = v(a - \ell)$  for all  $a \in K$ , there is a unique embedding  $K\langle f \rangle \rightarrow L$  over  $K$  sending  $f$  to  $g$ .*

**Lemma 5.14** ([ADH17a, 11.4.11]). *Suppose that  $(a_\rho)$  is divergent in  $K$  with  $a_\rho \rightsquigarrow \ell$ . If  $(P(a_\rho))$  is a pc-sequence such that  $P(a_\rho) \rightsquigarrow 0$ , then  $P \in Z(K, \ell)$ .*

The notion of vanishing is connected to newton degree in a cut. Although the first lemma is not directly used, it is worth noting.

**Lemma 5.15** ([ADH17a, 11.4.12]). *Suppose that  $(a_\rho)$  is divergent in  $K$  with  $a_\rho \rightsquigarrow \ell$ . Then*

$$\text{ndeg}_{\mathbf{a}} P = \min\{\text{ndeg}_{\prec \mathfrak{v}} P_{+a} : a - \ell \prec \mathfrak{v}\}.$$

*In particular,  $\text{ndeg}_{\mathbf{a}} P \geq 1 \iff P \in Z(K, \ell)$ .*

**Corollary 5.16** ([ADH17a, 11.4.13]). *Suppose that  $(a_\rho)$  is divergent in  $K$ . Then the following are equivalent:*

- (i)  $P \in Z(K, \ell)$  and has minimal complexity in  $Z(K, \ell)$ ;
- (ii)  $P$  is a minimal differential polynomial of  $(a_\rho)$  over  $K$ .

### 5.3. Removing divisibility

The following proposition is the main tool in proving the desired results in the case that  $\Gamma$  is divisible. Its proof uses the differential newton diagram method of [ADH17a] and is spread over several chapters of that book. The proof of Proposition 4.15 from the previous chapter was based on the proof of Proposition 5.17.

**Proposition 5.17** ([ADH17a, 14.5.1]). *Suppose that  $K$  is  $\omega$ -free and  $\mathbf{d}$ -valued with divisible  $\Gamma$ . Let  $P$  be a minimal differential polynomial of  $(a_\rho)$  over  $K$ . Then  $\text{ndeg}_{\mathbf{a}} P = 1$ .*

Based on Lemma 4.16, the next lemma allows us to replace the divisibility of  $\Gamma$  in the previous result with the assumption that  $K$  is henselian. The main theorems are then proven using the new proposition and arguments with henselizations as in §4.3.

**Lemma 5.18.** *Suppose that  $K$  is henselian and has rational asymptotic integration. Let  $P$  be a minimal differential polynomial of  $(a_\rho)$  over  $K$ . Then  $P$  remains a minimal differential polynomial of  $(a_\rho)$  over the algebraic closure  $K^{\text{ac}}$  of  $K$ .*

PROOF. Recall that  $K^{\text{ac}}$  is an  $H$ -asymptotic extension of  $K$  with  $\Psi_{K^{\text{ac}}} = \Psi$ . Since  $K$  has rational asymptotic integration, so does  $K^{\text{ac}}$ . Note also that  $\Gamma^\neq$  has no smallest archimedean class because  $\psi$  is constant on each archimedean class and  $K$  is ungrounded. It follows that  $\Gamma^<$  is cofinal in  $\Gamma_{K^{\text{ac}}}^< = (\mathbb{Q}\Gamma)^<$ .

We may suppose that  $(a_\rho)$  is divergent in  $K$ , the other case being trivial. Then  $(a_\rho)$  must still be divergent in  $K^{\text{ac}}$ : If it had a pseudolimit  $a \in K^{\text{ac}}$ , then we would have  $Q(a_\rho) \rightsquigarrow 0$ , where  $Q \in K[Y]$  is the minimum polynomial of  $a$  over  $K$  (see [ADH17a, Proposition 3.2.1]). But since  $K$  is henselian, it is algebraically maximal (see [ADH17a, Corollary 3.3.21]), and then  $(a_\rho)$  would have a pseudolimit in  $K$ .

Now, suppose to the contrary that  $Q$  is a minimal differential polynomial of  $(a_\rho)$  over  $K^{\text{ac}}$  with  $c(Q) < c(P)$ . Take an extension  $L \subseteq K^{\text{ac}}$  of  $K$  with  $Q \in L\{Y\}$  and  $[L : K] = n < \infty$ . Since  $K$  is henselian,  $[L : K] = [\Gamma_L : \Gamma] \cdot [\mathbf{k}_L : \mathbf{k}]$  (see [ADH17a, Corollary 3.3.49]), and thus we have a valuation basis  $\mathcal{B} = \{e_1, \dots, e_n\}$  of  $L$  over  $K$  (see [ADH17a, Proposition 3.1.7]). That is,  $\mathcal{B}$  is a basis of  $L$  over  $K$ , and for all  $a_1, \dots, a_n \in K$ ,

$$v\left(\sum_{i=1}^n a_i e_i\right) = \min_{1 \leq i \leq n} v(a_i e_i).$$

Then by expressing the coefficients of  $Q$  in terms of the valuation basis,

$$Q(Y) = \sum_{i=1}^n R_i(Y) \cdot e_i,$$

where  $R_i(Y) \in K\{Y\}$  for  $1 \leq i \leq n$ .

Since  $Q$  is a minimal differential polynomial of  $(a_\rho)$  over  $K^{\text{ac}}$ , by Corollary 5.16 and Lemma 5.13 we have an immediate asymptotic extension  $K^{\text{ac}}\langle a \rangle$  of  $K^{\text{ac}}$  with  $a_\rho \rightsquigarrow a$  and  $Q(a) = 0$ . As an immediate asymptotic extension of  $K^{\text{ac}}$ ,  $K^{\text{ac}}\langle a \rangle$  is of  $H$ -type with rational asymptotic integration and  $\Gamma^<$  remains cofinal in  $\Gamma_{K^{\text{ac}}\langle a \rangle}^< = (\mathbb{Q}\Gamma)^<$ . Then by Lemma 5.11, there is a pc-sequence  $(b_\lambda)$  in  $K$  equivalent to  $(a_\rho)$  such that  $Q(b_\lambda) \rightsquigarrow Q(a) = 0$ . Finally, after passing to a cofinal subsequence, we have  $i$  with  $Q(b_\lambda) \asymp R_i(b_\lambda) \cdot e_i$  for all  $\lambda$ . Then  $R_i(b_\lambda) \rightsquigarrow 0$  and  $c(R_i) < c(P)$ , contradicting the minimality of  $P$ .  $\square$

Note that in the next results, we assume that  $K$  is  $\omega$ -free, and so has rational asymptotic integration by Lemma 5.5. This makes the previous lemma available.

**Proposition 5.19.** *Suppose that  $K$  is  $\omega$ -free, d-valued, and henselian. Let  $P$  be a minimal differential polynomial of  $(a_\rho)$  over  $K$ . Then  $\text{ndeg}_a P = 1$ .*

PROOF. By the previous lemma,  $P$  remains a minimal differential polynomial of  $(a_\rho)$  over  $K^{\text{ac}}$ . Also,  $K^{\text{ac}}$  is  $\omega$ -free by Lemma 5.6 and d-valued by [ADH17a, Corollary 10.1.23]. But then  $\text{ndeg}_{a_{K^{\text{ac}}}} P = 1$  by Proposition 5.17, and since  $\Psi_{K^{\text{ac}}} = \Psi$ , Lemma 5.9(viii) gives  $\text{ndeg}_a P = 1$ .  $\square$



Since being newtonian implies being henselian, we then can improve [ADH17a, Corollary 14.5.2] using the same proof; this result contains the version presented in the introduction.

**Theorem 5.20.** *If  $K$  is  $\omega$ -free, then the following are equivalent:*

- (i)  $K$  is newtonian;
- (ii)  $K$  is strongly newtonian;
- (iii)  $K$  is asymptotically  $d$ -algebraically maximal.

The next result has the same proof as that of [ADH17a, Corollary 14.5.4] after reducing to the henselian case, but is given for completeness. Note that when an extension of  $K$  newtonian, by definition it is ungrounded and  $H$ -asymptotic.

**Theorem 5.21.** *Suppose that  $K$  is  $\omega$ -free and  $d$ -valued, and let  $L$  be an immediate  $d$ -algebraic extension of  $K$  that is newtonian. Then  $L$  is a newtonization of  $K$ , and any other newtonization of  $K$  is isomorphic to  $L$  over  $K$ .*

PROOF. Let  $E$  be a newtonian extension of  $K$ . Note that the henselization  $K^h$  of  $K$  is an immediate algebraic extension of  $K$ , so is asymptotic by Proposition 2.17 and thus  $d$ -valued by Lemma 5.2. By Lemma 5.6,  $K^h$  is  $\omega$ -free. Since  $E$  is newtonian, it is henselian, so by embedding  $K^h$  in  $E$ , we may assume that  $K$  is henselian.

The case  $K = L$  being trivial, we may suppose that  $K \neq L$ . It is sufficient to find  $a \in L \setminus K$  such that  $K\langle a \rangle$  embeds into  $E$  over  $K$ , since any extension  $F \subseteq L$  of  $K$  is still  $\omega$ -free and  $d$ -valued by Theorem 5.7 and Lemma 5.2, respectively. Let  $\ell \in L \setminus K$ , so there is a divergent pc-sequence  $(a_\rho)$  in  $K$  with  $a_\rho \rightsquigarrow \ell$ . Since  $L$  is a  $d$ -algebraic extension of  $K$ ,  $Z(K, \ell) \neq \emptyset$  by Lemma 5.12, and so there is a minimal differential polynomial  $P$  of  $(a_\rho)$  over  $K$  by Corollary 5.16. Hence by Proposition 5.19,  $\text{ndeg}_a P = 1$ . By the  $\omega$ -freeness of  $K$  and [ADH17a, Corollary 13.6.13], newton degree remains the same in  $L$  and  $E$ , so  $\text{ndeg}_{a_L} P = \text{ndeg}_{a_E} P = 1$ . Then Lemma 5.10 gives  $a \in L \setminus K$  with  $a_\rho \rightsquigarrow a$  and  $P(a) = 0$ , and  $b \in E \setminus K$  with  $a_\rho \rightsquigarrow b$  and  $P(b) = 0$ . Then Lemma 5.13 gives an embedding of  $K\langle a \rangle$  into  $E$  over  $K$ .

The uniqueness of  $L$  follows from its embedding property and Lemma 5.8.  $\square$

Here is one last corollary, an improvement of [ADH17a, Corollary 14.5.6] with the same proof.

**Corollary 5.22.** *Suppose that  $K$  is  $\omega$ -free and  $d$ -valued. If  $L = K(C_L)$  is an algebraic extension of  $K$  that is newtonian, then so is  $K$ .*

## 5.4. Uniqueness

For  $\omega$ -free  $d$ -valued  $K$ , it follows from Theorems 5.20 and 5.21 that any two immediate  $d$ -algebraic extensions of  $K$  that are asymptotically  $d$ -algebraically maximal are isomorphic over  $K$ . In this section, we provide an alternative argument, making explicit some ideas only tacitly present in [ADH17a].

**Lemma 5.23.** *Suppose that  $K$  has rational asymptotic integration. Then every pc-sequence in  $K$  of  $d$ -algebraic type over  $K$  has a pseudolimit in  $K$  if and only if  $K$  is asymptotically  $d$ -algebraically maximal.*

PROOF. Suppose first that there is  $\ell \notin K$  in some immediate asymptotic extension of  $K$  that is d-algebraic over  $K$ . Let  $P$  be a minimal annihilator of  $\ell$  over  $K$ . There is a divergent pc-sequence  $(a_\rho)$  in  $K$  with  $a_\rho \rightsquigarrow \ell$ , so by Lemma 5.11, there is an equivalent pc-sequence  $(b_\lambda)$  in  $K$  with  $P(b_\lambda) \rightsquigarrow P(\ell) = 0$ , so  $(a_\rho)$  is of d-algebraic type over  $K$ .

For the other direction, let  $(a_\rho)$  be a divergent pc-sequence in  $K$  with minimal differential polynomial  $P$  over  $K$ . Take a pseudolimit  $\ell$  of  $(a_\rho)$  in an asymptotic extension of  $K$ . By Corollary 5.16,  $Z(K, \ell) \neq \emptyset$ , so Lemma 5.13 gives a proper immediate asymptotic extension of  $K$  that is d-algebraic over  $K$ .  $\square$

**Lemma 5.24.** *Suppose that  $K$  is  $\omega$ -free, d-valued, and henselian. Let  $P$  be a minimal differential polynomial of  $(a_\rho)$  over  $K$ . Let  $L$  be a d-valued extension of  $K$  that is  $\lambda$ -free and asymptotically d-algebraically maximal. Then  $a_\rho \rightsquigarrow b$  and  $P(b) = 0$  for some  $b \in L$ .*

PROOF. By Proposition 5.19,  $\text{ndeg}_a P = 1$ . By the  $\omega$ -freeness of  $K$  and [ADH17a, Corollary 13.6.13],  $\text{ndeg}_{a_L} P = 1$ . By Theorem 5.4,  $L$  is newtonian, so we get  $a_\rho \rightsquigarrow b$  and  $P(b) = 0$  for some  $b \in L$  from Lemma 5.10.  $\square$

**Theorem 5.25.** *Suppose that  $K$  is  $\omega$ -free and d-valued. Then any two immediate d-algebraic extensions of  $K$  that are asymptotically d-algebraically maximal are isomorphic over  $K$ .*

PROOF. Let  $L_0$  and  $L_1$  be immediate d-algebraic extensions of  $K$  that are asymptotically d-algebraically maximal. Note that they are both  $\omega$ -free and d-valued by Theorem 5.7 and Lemma 5.2, respectively. By Zorn's lemma we have a maximal isomorphism  $\varphi: F_0 \cong_K F_1$  between valued differential subfields  $F_i \supseteq K$  of  $L_i$  for  $i = 0, 1$ , where “maximal” means that  $\varphi$  does not extend to an isomorphism between strictly larger such subfields. As before,  $F_i$  is  $\omega$ -free and d-valued for  $i = 0, 1$ . Next, they must be henselian, because the henselization of  $F_i$  in  $L_i$  is an algebraic field extension of  $F_i$ , and thus a valued differential subfield of  $L_i$  that is  $\omega$ -free and d-valued for  $i = 0, 1$ .

Now suppose towards a contradiction that  $F_0 \neq L_0$  (equivalently,  $F_1 \neq L_1$ ). Then  $F_0$  is not asymptotically d-algebraically maximal, so we have a divergent pc-sequence  $(a_\rho)$  in  $F_0$  with a minimal differential polynomial  $P$  over  $F_0$ . Then Lemma 5.24 gives  $f_0 \in L_0$  with  $a_\rho \rightsquigarrow f_0$  and  $P(f_0) = 0$ , and  $f_1 \in L_1$  with  $\varphi(a_\rho) \rightsquigarrow f_1$  and  $P^\varphi(f_1) = 0$ . Now Lemma 5.13 gives an isomorphism  $F_0 \langle f_0 \rangle \cong F_1 \langle f_1 \rangle$  extending  $\varphi$ , and we have a contradiction. Thus  $F_0 = L_0$  and hence  $F_1 = L_1$ .  $\square$

## CHAPTER 6

### Pre- $H$ -fields with gap 0

#### 6.1. Introduction

This chapter is primarily concerned with certain ordered pre-d-valued fields called pre- $H$ -fields. We continue to assume that  $K$  is a valued differential field, but in contrast to Chapters 3 and 4, we do not assume throughout that  $K$  has small derivation; instead, we place such assumptions on  $K$  at the beginning of sections or where needed. Here,  $K$  is an *ordered valued differential field* if, in addition to its valuation and derivation, it is also equipped with a (total) ordering  $\leq$  making it an ordered field (in the sense that the ordering is preserved by addition and by multiplication by positive elements). If  $\mathcal{O}$  is convex with respect to the ordering, then  $\leq$  induces an ordering on  $\mathbf{k}$  making it an ordered field. Relating the ordering, valuation, and derivation (recall the notion of a pre-d-valued field from the previous chapter):

**Definition.** We call an ordered valued differential field  $K$  a *pre- $H$ -field* if:

- (PH1)  $K$  is pre-d-valued;
- (PH2)  $\mathcal{O}$  is convex (with respect to  $\leq$ );
- (PH3) for all  $f \in K$ , if  $f > \mathcal{O}$ , then  $f' > 0$ .

Pre- $H$ -fields were introduced by Aschenbrenner and van den Dries in [AD02]; examples include all Hardy fields. Together with van der Hoeven, they showed in [ADH17a] that the theory  $T^{\text{nl}}$  of  $\omega$ -free, newtonian, Liouville closed  $H$ -fields is the model companion of the theory of pre- $H$ -fields in the language  $\{+, -, \cdot, 0, 1, \partial, \preceq, \leq\}$ , where  $H$ -fields are exactly the pre- $H$ -fields  $K$  with  $\mathcal{O} = C + \mathcal{o}$ . Moreover,  $T^{\text{nl}}$  admits quantifier elimination with the addition of a function symbol for field inversion and two unary predicates identifying the parameters for which two second-order differential equations have solutions. This theory has a natural model  $\mathbb{T}$ , the ordered valued differential field of logarithmic-exponential transseries, and in fact  $T^{\text{nl}}_{\text{small}} = T^{\text{nl}} + \text{“small derivation”}$  axiomatizes the theory of  $\mathbb{T}$  and is one of two completions of  $T^{\text{nl}}$ . They also deduce that  $T^{\text{nl}}_{\text{small}}$  is the model companion of the theory of  $H$ -fields with small derivation.

This raises the question of the model theory of other kinds of pre- $H$ -fields with small derivation. In this chapter we concentrate on pre- $H$ -fields with gap 0. Recall from §2.4 that an asymptotic field (such as a pre- $H$ -field)  $K$  has gap 0 if  $\sup \Psi = 0 \notin \Psi$ , where  $\Psi := \psi(\Gamma^{\neq})$ . Equivalently, an asymptotic field  $K$  has gap 0 if it has small derivation and for all  $f \succ 1$  in  $K$ ,  $f^{\dagger} \succ 1$ . It follows that in such structures, infinite elements are transexponential in a certain sense and the valuation is coarser than the usual valuation of a Hardy field. An example of a pre- $H$ -field with gap 0 is obtained by taking an  $\aleph_0$ -saturated elementary extension of  $\mathbb{T}$  and enlarging the valuation ring so that it is the set of elements bounded in absolute value by some finite iterate of the exponential (see [ADH17a, Example 10.1.7]). Alternatively, consider the functional equation  $f(x+1) = e^{f(x)}$ . This

has a solution lying in a Hardy field [Bos86], and performing the same enlargement of its valuation ring yields another pre- $H$ -field with gap 0.

The goal of this chapter is to find a model companion for the theory of pre- $H$ -fields with gap 0, or, equivalently, to axiomatize the class of existentially closed pre- $H$ -fields with gap 0. One major distinction between the pre- $H$ -fields considered in this chapter and  $T^{\text{nl}}$  is that here the derivation induced on the residue field can be nontrivial, and always is in existentially closed pre- $H$ -fields with gap 0. Conversely, let  $K$  be a pre-d-valued field (such as a pre- $H$ -field) with small derivation and nontrivial induced derivation on  $\mathbf{k}$ ; then  $K$  has gap 0. To see this, take  $u \in K$  with  $u' \asymp u \asymp 1$ . If  $f \in K^\times$  with  $f \prec 1$ , then since  $K$  is pre-d-valued,  $1 \asymp u' \prec f^\dagger$ , and so  $K$  has gap 0.

In pre- $H$ -fields with gap 0, the residue field is construed as an ordered differential field, and hence it is reasonable to expect that an existentially closed pre- $H$ -field with gap 0 has an existentially closed ordered differential residue field. The class of existentially closed ordered differential fields, where no interaction is assumed between the ordering and the derivation, is axiomatized by the theory of closed ordered differential fields introduced by Singer [Sin78]; in fact, this theory is the model completion of the theory of ordered differential fields, and has quantifier elimination and NIP. Our main result is the following, in the language  $\{+, -, \cdot, 0, 1, \partial, \preceq, \leq\}$ . First, we say that a differential field  $F$  has *exponential integration* if for each  $f \in F$  there is  $z \in F^\times$  such that  $z^\dagger = f$  (this  $z$  behaves like  $e^{\int f}$ , whence the name).

**Theorem 6.53.** *The theory of d-henselian, real closed pre- $H$ -fields with exponential integration and closed ordered differential residue field has quantifier elimination.*

By Lemma 6.54, every pre- $H$ -field with gap 0 extends to a model of the theory in Theorem 6.53, so we obtain the desired model companion result characterizing the existentially closed pre- $H$ -fields with gap 0.

**Corollary 6.55.** *The theory of d-henselian, real closed pre- $H$ -fields with exponential integration and closed ordered differential residue field is the model completion of the theory of pre- $H$ -fields with gap 0.*

It also follows from quantifier elimination in the usual way that this theory is complete (Corollary 6.56). Finally, we study the combinatorial complexity of the theory.

**Theorem 6.57.** *The theory of d-henselian, real closed pre- $H$ -fields with exponential integration and closed ordered differential residue field has NIP.*

One of the examples of a pre- $H$ -field with gap 0 given above, call it  $F$ , was obtained by enlarging the valuation ring of an elementary extension  $\mathbb{T}^*$  of  $\mathbb{T}$ . Then the valuation of  $\mathbb{T}^*$  induces a valuation on the residue field of  $F$ , which suggests that we should consider theories where the residue field has structure in addition to its ordered differential field structure. Let  $K$  be a pre- $H$ -field with gap 0,  $\mathbf{k}$  be an expansion of an ordered differential field, and  $\pi: \mathcal{O} \rightarrow \mathbf{k}$  be a surjective differential ring homomorphism with kernel  $\mathfrak{o}$ , which induces an isomorphism of ordered differential fields between the residue field of  $K$  and  $\mathbf{k}$ , extended to  $K$  by  $\pi(K \setminus \mathfrak{o}) = \{0\}$ . We consider the two-sorted structure  $(K, \mathbf{k}; \pi)$  where the language on the sort of  $K$  is  $\{+, -, \cdot, 0, 1, \partial, \preceq, \leq\}$  and the language on the sort of  $\mathbf{k}$  is  $\mathcal{L}_{\text{res}} \supseteq \{+, -, \cdot, 0, 1, \partial, \leq\}$ .

**Theorem 6.58.** *If  $K$  is a d-henselian, real closed pre- $H$ -field with exponential integration, and the  $\mathcal{L}_{\text{res}}$ -theory of  $\mathbf{k}$  is model complete, then the theory of  $(K, \mathbf{k}; \pi)$  is model complete.*

If the  $\mathcal{L}_{\text{res}}$ -theory of  $\mathbf{k}$  is actually the model companion of an  $\mathcal{L}_{\text{res}}$ -theory of ordered differential fields, then the theory of  $(K, \mathbf{k}; \pi)$  with  $K$  as in Theorem 6.58 is in fact the model companion of the expected two-sorted theory; this is Corollary 6.60.

**6.1.1. Outline.** After some preliminary facts about pre- $H$ -fields, we show how to extend embeddings of ordered valued differential fields by first extending the residue field in §6.3. We study the theory of  $H$ -asymptotic couples with gap 0 as structures in their own right in §6.4, isolating the model completion of this theory and proving that this model completion has quantifier elimination in Theorem 6.7. Since models include the asymptotic couples of real closed pre- $H$ -fields with gap 0 and exponential integration, Theorem 6.7 gets used in §6.5, which studies extensions of pre- $H$ -fields with gap 0 controlled by the asymptotic couple, via Corollary 6.8. The main result in that section is Theorem 6.22, a maximality statement that strengthens the conclusion of Theorem 4.19 under additional hypotheses. Section 6.6 deals with extending the constant field for use in §6.7. That section builds towards Theorem 6.49, which shows the existence of differential-Hensel-Liouville closures; these are extensions that are d-henselian, real closed, and have exponential integration, and that satisfy a semi-universal property. Their construction makes use of Theorem 4.20 about the existence of d-henselizations. We use Theorem 6.22 to prove that differential-Hensel-Liouville closures are unique in Corollary 6.51. Finally, §6.8 contains the quantifier elimination, model completion, and distality results advertised above.

## 6.2. Preliminaries on pre- $H$ -fields

Here are two basic but useful facts about pre- $H$ -fields.

**Lemma 6.1** ([ADH17a, 10.5.2]). *If  $K$  is a pre- $H$ -field and  $f, g \in K^\times$ , then:*

- (i)  $f \prec g \implies f^\dagger < g^\dagger$ ;
- (ii)  $f \preceq g \prec 1 \implies f^\dagger \succcurlyeq g^\dagger$ .

Part (ii) says that pre- $H$ -fields are  $H$ -asymptotic.

In the rest of this paragraph,  $K$  is an ordered valued differential field with convex valuation ring. We equip the real closure  $K^{\text{rc}}$  of  $K$  with the unique valuation extending that of  $K$  whose valuation ring is convex (see [ADH17a, Corollary 3.5.18]) and the unique derivation extending that of  $K$  (see [ADH17a, Lemma 1.9.2]), and always construe  $K^{\text{rc}}$  as an ordered valued differential field extension of  $K$  in this way. Then  $\mathcal{O}_{K^{\text{rc}}}$  is the convex hull of  $\mathcal{O}$  in  $K^{\text{rc}}$ ,  $\Gamma_{K^{\text{rc}}}$  is the divisible hull  $\mathbb{Q}\Gamma$  of  $\Gamma$ ,  $\mathbf{k}_{K^{\text{rc}}}$  is the real closure of  $\mathbf{k}$ , and  $C_{K^{\text{rc}}}$  is the real closure of  $C$ . If  $K$  is a pre- $H$ -field, then so is  $K^{\text{rc}}$  [ADH17a, Proposition 10.5.4]. If  $K$  is asymptotic with gap 0, then so is  $K^{\text{rc}}$  (see §2.5).

For immediate asymptotic valued differential field extensions of pre- $H$ -fields, we use:

**Lemma 6.2** ([ADH17a, 10.5.8]). *Suppose that  $K$  is a pre- $H$ -field and  $L$  is an immediate valued differential field extension of  $K$  that is asymptotic. Then there is a unique ordering on  $L$  making it an ordered field extension of  $K$  with respect to which  $\mathcal{O}_L$  is convex. This ordering makes  $L$  a pre- $H$ -field and  $\mathcal{O}_L$  the convex hull of  $\mathcal{O}$  in  $L$ .*

### 6.3. Extensions controlled by the residue field

Here are ordered variants of [ADH17a, Theorem 6.3.2] and [ADH17a, Lemma 6.3.1]. Let  $F \in K\{Y\}^\neq$  have order  $r$  and set  $d := \deg_{Y^{(r)}} F$ . Decomposing  $F(Y) = \sum_{n=0}^d G_n(Y) \cdot (Y^{(r)})^n$  with  $G_n \in K[Y, \dots, Y^{(r-1)}]$  for  $n \in \{0, \dots, d\}$ , the *initial* of  $F$  is  $G_d$ . Below we let  $I$  be the initial of  $F$ . First we recall [ADH17a, Theorem 6.3.2]:

**Theorem 6.3** ([ADH17a, 6.3.2]). *Suppose that  $K$  has small derivation. Let  $K\langle a \rangle$  be a differential field extension of  $K$  such that  $a$  has minimal annihilator  $F$  as above satisfying  $F \asymp 1$  and  $I \asymp 1$ , and such that  $\bar{F}$  is irreducible in  $\mathbf{k}\{Y\}$ . Then there is a unique valuation  $v: K\langle a \rangle^\times \rightarrow \Gamma$  extending that of  $K$  such that:*

- (i)  $K\langle a \rangle$  has small derivation;
- (ii)  $a \preccurlyeq 1$ ;
- (iii)  $\bar{a}$  has minimal annihilator  $\bar{F}$  over  $\mathbf{k}$ .

*It is given by  $P(a)/Q(a) \mapsto vP - vQ \in \Gamma$ , for  $P \in K[Y, \dots, Y^{(r)}]^\neq$  with  $\deg_{Y^{(r)}} P < d$  and  $Q \in K[Y, \dots, Y^{(r-1)}]^\neq$ . The residue field of  $K\langle a \rangle$  is  $\mathbf{k}\langle \bar{a} \rangle$ .*

Below, we equip  $K\langle a \rangle$  with this valuation.

**Lemma 6.4.** *Suppose that  $K$  is an ordered valued differential field with small derivation and convex valuation ring. Let  $K\langle a \rangle$  be as above. Suppose that  $\mathbf{k}\langle \bar{a} \rangle$  is an ordered differential field extension of  $\mathbf{k}$ . Then there exists a unique ordering on  $K\langle a \rangle$  making it an ordered field extension of  $K$  with convex valuation ring such that the induced ordering on  $\mathbf{k}\langle \bar{a} \rangle$  agrees with the given one. If  $K$  is a pre- $H$ -field with gap 0, then so is  $K\langle a \rangle$ .*

PROOF. Suppose that  $K\langle a \rangle$  is equipped with an ordering making it an ordered field extension of  $K$  with convex valuation ring. Let  $P \in K[Y, \dots, Y^{(r)}]^\neq$  and  $\deg_{Y^{(r)}} P < d$ . By scaling  $P$  by an element of  $K^>$ , we may assume that  $v(P) = 0$ , and thus  $\bar{P}(\bar{a}) \neq 0$ . We have  $P(a) > 0 \iff \bar{P}(\bar{a}) > 0$ , which shows that there is at most one ordering on  $K\langle a \rangle$  making it an ordered field extension of  $K$  with convex valuation ring such that the induced ordering on  $\mathbf{k}\langle \bar{a} \rangle$  agrees with the given one.

To construct such an ordering, let  $b \in K\langle a \rangle^\times$ , so  $b = P(a)/Q(a)$  for  $P \in K[Y, \dots, Y^{(r)}]^\neq$  with  $\deg_{Y^{(r)}} P < d$  and  $Q \in K[Y, \dots, Y^{(r-1)}]^\neq$ . By scaling  $b$  by an element of  $K^>$ , it suffices to define  $b > 0$  when  $b \asymp 1$ . Similarly, we may assume that  $P \asymp Q \asymp 1$ . Then the condition  $\bar{P}(\bar{a})/\bar{Q}(\bar{a}) > 0$  in  $\mathbf{k}\langle \bar{a} \rangle$  depends only on  $b$  and not on the choice of  $P$  and  $Q$ , so we can define  $b > 0 :\Leftrightarrow \bar{P}(\bar{a})/\bar{Q}(\bar{a}) > 0$ . Then  $b > 0$  or  $-b > 0$ .

Next, assume that  $b, c \in K\langle a \rangle^\times$  and  $b, c > 0$ ; we show that  $b + c > 0$  and  $bc > 0$ . We have  $b = sP(a)/Q(a)$  and  $c = tG(a)/H(a)$  with  $s, t \in K^>$ ,  $P$  and  $Q$  as above, and  $G \in K[Y, \dots, Y^{(r)}]^\neq$  and  $H \in K[Y, \dots, Y^{(r-1)}]^\neq$  such that  $\deg_{Y^{(r)}} G < d$  and  $G \asymp H \asymp 1$ . First we show that  $b + c > 0$ . Without loss of generality,  $s \preccurlyeq t$ , so we have  $b + c = t(st^{-1}HP + QG)(a)/QH(a)$  with  $\deg_{Y^{(r)}}(st^{-1}HP + QG) < d$ . Then

$$\frac{\overline{(st^{-1}HP + QG)}(\bar{a})}{\overline{QH}(\bar{a})} = \frac{\overline{st^{-1}HP}(\bar{a})}{\overline{QH}(\bar{a})} + \frac{\overline{QG}(\bar{a})}{\overline{QH}(\bar{a})} > 0,$$

so  $b + c > 0$ . Now we show that  $bc > 0$ ; we may assume that  $s = t = 1$ . If  $G \in K[Y, \dots, Y^{(r-1)}]$ , then  $bc = PG(a)/QH(a)$  with  $\deg_{Y^{(r)}} PG < d$  and

$$\frac{\overline{PG}(\bar{a})}{\overline{QH}(\bar{a})} = \frac{\overline{P}(\bar{a})}{\overline{Q}(\bar{a})} \cdot \frac{\overline{G}(\bar{a})}{\overline{H}(\bar{a})} > 0.$$

It therefore suffices to consider the case that  $b = P(a)$  and  $c = G(a)$ . By division with remainder in  $\mathcal{O}[Y, \dots, Y^{(r)}]$  we have  $I^m PG = BF + R$  with  $B, R \in \mathcal{O}[Y, \dots, Y^{(r)}]$  and  $\deg_{Y^{(r)}} R < d$ , and thus  $bc = R(a)/I^m(a)$ . But  $\overline{R}(\bar{a})/\overline{I^m}(\bar{a}) = \overline{P}(\bar{a}) \cdot \overline{G}(\bar{a}) > 0$ , so  $bc > 0$ . Thus we have defined an ordering on  $K\langle a \rangle$  making it an ordered field extension of  $K$ . An easy calculation shows that if  $b \prec 1$ , then  $-1 < b < 1$ , so the valuation ring of  $K\langle a \rangle$  is convex with respect to this ordering (see [ADH17a, Lemma 3.5.11]), and by construction it induces the given ordering on  $\mathbf{k}\langle \bar{a} \rangle$ .

Finally, suppose that  $K$  is a pre- $H$ -field with gap 0. As a valued differential field extension of  $K$  with small derivation and the same value group,  $K\langle a \rangle$  is pre-d-valued [ADH17a, Lemma 10.1.9], so it has the same asymptotic couple as  $K$  and thus has gap 0. By [ADH17a, Lemma 10.5.5] (with  $T = K^\times$ ),  $K\langle a \rangle$  is in fact a pre- $H$ -field.  $\square$

Recall that the gaussian valuation on  $K\langle Y \rangle$  is defined by setting  $v(P)$ , for  $P \in K\{Y\}^\neq$ , to be the minimum valuation of the coefficients of  $P$ ; for more details, see [ADH17a, §4.5 and §6.3].

**Lemma 6.5.** *Suppose that  $K$  is an ordered valued differential field with small derivation and convex valuation ring. Consider  $K\langle Y \rangle$  with the gaussian valuation. Suppose that  $\mathbf{k}\langle \bar{Y} \rangle$  is an ordered differential field extension of  $\mathbf{k}$ . Then there exists a unique ordering on  $K\langle Y \rangle$  making it an ordered field extension of  $K$  with convex valuation ring such that the induced ordering on  $\mathbf{k}\langle \bar{Y} \rangle$  agrees with the given one. If  $K$  is a pre- $H$ -field with gap 0, then so is  $K\langle Y \rangle$ .*

PROOF. The proof is very similar to that of the previous lemma, but easier.  $\square$

**Corollary 6.6.** *Suppose that  $K$  is an ordered valued differential field with small derivation and convex valuation ring. Let  $\mathbf{k}_L$  be an ordered differential field extension of  $\mathbf{k}$ . Then  $K$  has an ordered valued differential field extension  $L$  with the following properties:*

- (i)  $\Gamma_L = \Gamma$ ;
- (ii)  $L$  has small derivation;
- (iii)  $\mathcal{O}_L$  is convex;
- (iv)  $\text{res}(L) \cong \mathbf{k}_L$  over  $\mathbf{k}$  (as ordered differential fields);
- (v) for any ordered valued differential field extension  $M$  of  $K$  with convex valuation ring that is d-henselian, any embedding  $\text{res}(L) \rightarrow \mathbf{k}_M$  over  $\mathbf{k}$  is induced by an embedding  $L \rightarrow M$  over  $K$ .

Moreover, if  $K$  is a pre- $H$ -field with gap 0, then so is  $L$ .

PROOF. First, note that we can reduce to the case that  $\mathbf{k}_L = \mathbf{k}\langle y \rangle$ .

Suppose that  $y$  is d-transcendental over  $\mathbf{k}$ . Set  $L := K\langle Y \rangle$ , equipped with the gaussian valuation and the ordering from Lemma 6.5 so that  $\text{res}(L) = \mathbf{k}\langle \bar{Y} \rangle \cong \mathbf{k}\langle y \rangle$  over  $\mathbf{k}$ . Let  $M$  be an ordered valued differential field extension of  $K$  with convex valuation ring, and suppose that  $M$  is d-henselian and  $i: \mathbf{k}\langle \bar{Y} \rangle \rightarrow \mathbf{k}_M$  is an embedding over  $\mathbf{k}$ . Take  $b \in M$  with  $b \asymp 1$  and  $\bar{b} = i(\bar{Y})$ . Then [ADH17a,

Lemma 6.3.1] provides a valued differential field embedding  $L \rightarrow M$  over  $K$  sending  $Y$  to  $b$ ; this is an ordered field embedding by the uniqueness in Lemma 6.5.

Now suppose that  $y$  is d-algebraic over  $\mathbf{k}$ . Let  $\bar{F} \in \mathbf{k}\{Y\}$  be the minimal annihilator of  $y$  over  $\mathbf{k}$  and take a lift  $F \in \mathcal{O}\{Y\}$  of  $\bar{F}$  with the same complexity. Note that  $F \asymp I \asymp S \asymp 1$ , where  $I$  is the initial of  $F$  and  $S := \partial F / \partial Y^{(r)}$  is the separant of  $F$ . Take a differential field extension  $L := K\langle a \rangle$  of  $K$  such that  $a$  has minimal annihilator  $F$  over  $K$ . We equip  $L$  with the valuation extending that of  $K$  from Theorem 6.3, so  $L$  has small derivation and  $a \preccurlyeq 1$ , and the ordering from Lemma 6.4, making it an ordered field extension of  $K$  with convex valuation ring and  $\text{res}(L) = \mathbf{k}\langle \bar{a} \rangle \cong \mathbf{k}\langle y \rangle$  over  $\mathbf{k}$ . Let  $M$  be an ordered valued differential field extension of  $K$  with convex valuation ring, and suppose that  $M$  is d-henselian and  $i: \mathbf{k}\langle \bar{a} \rangle \rightarrow \mathbf{k}_M$  is an embedding over  $\mathbf{k}$ . Let  $z \in M$  with  $z \preccurlyeq 1$  and  $\bar{z} = i(\bar{a})$ . By the minimality of  $\bar{F}$ , we have  $\bar{S}(i(\bar{a})) \neq 0$ , so  $S(z) \asymp 1$ . In particular,  $(F_{+z})_1 \asymp 1$ , so by the d-henselianity of  $M$ , there is  $b \in M$  with  $F(b) = 0$ ,  $b \preccurlyeq 1$ , and  $\bar{b} = i(\bar{a})$ . Note that then  $F$  is a minimal annihilator of  $b$  over  $K$  by the minimality of  $\bar{F}$ . Hence by Theorem 6.3 and Lemma 6.4 we may embed  $L$  into  $M$  over  $K$  sending  $a$  to  $b$ .  $\square$

#### 6.4. Asymptotic couples with small derivation

Towards our quantifier elimination and model completion results for pre- $H$ -fields with gap 0, we first study their associated asymptotic couples, and prove quantifier elimination and model completion results for the theory of such structures. We suspend in this section the convention that  $\Gamma$  is the value group of  $K$ . Instead, throughout the section  $(\Gamma, \psi)$  is an  $H$ -asymptotic couple, which means that  $\Gamma$  is an ordered abelian group and  $\psi: \Gamma^\neq \rightarrow \Gamma$  is a map satisfying for all  $\gamma, \delta \in \Gamma^\neq$ :

- (AC1) if  $\gamma + \delta \neq 0$ , then  $\psi(\gamma + \delta) \geq \min\{\psi(\gamma), \psi(\delta)\}$ ;
- (AC2)  $\psi(k\gamma) = \psi(\gamma)$  for all  $k \in \mathbb{Z}^\neq$ ;
- (AC3) if  $\gamma > 0$ , then  $\gamma + \psi(\gamma) > \psi(\delta)$ ;
- (HC) if  $0 < \gamma \leq \delta$ , then  $\psi(\gamma) \geq \psi(\delta)$ .

It follows from (AC2) and (HC) that  $\psi$  is constant on archimedean classes of  $\Gamma$ . For  $\gamma \in \Gamma$ , recall from §2.2.1 that  $[\gamma] = \{\delta \in \Gamma : |\delta| \leq n|\gamma| \text{ and } |\gamma| \leq n|\delta| \text{ for some } n\}$  denotes its archimedean class; we set  $[\Gamma] := \{[\gamma] : \gamma \in \Gamma\}$ , ordering it in the natural way. The map  $\psi$  extends uniquely to the divisible hull  $\mathbb{Q}\Gamma$  of  $\Gamma$ , defined by  $\psi(q\gamma) = \psi(\gamma)$  for  $\gamma \in \Gamma^\neq$  and  $q \in \mathbb{Q}^\times$  (for uniqueness see [ADH17a, Lemma 6.5.3]), and in this way we always construe  $\mathbb{Q}\Gamma$  as an  $H$ -asymptotic couple  $(\mathbb{Q}\Gamma, \psi)$  extending  $(\Gamma, \psi)$ ; it satisfies  $\psi(\mathbb{Q}\Gamma^\neq) = \psi(\Gamma^\neq)$ .

Keeping in mind that in later sections  $(\Gamma, \psi)$  will be the asymptotic couple of an  $H$ -asymptotic field (such as a pre- $H$ -field), we let  $\gamma^\dagger := \psi(\gamma)$  and  $\gamma' := \gamma^\dagger + \gamma$  for  $\gamma \in \Gamma^\neq$ . We let  $\Psi := \psi(\Gamma^\neq)$  and let  $\Psi^\downarrow$  be the downward closure of  $\Psi$  in  $\Gamma$ . Thus (AC3) says that  $\Psi < (\Gamma^>)'$ . For  $\beta \in \Gamma$ , we say that  $(\Gamma, \psi)$  has *gap*  $\beta$  if  $\Psi < \beta < (\Gamma^>)'$  and *max*  $\beta$  if  $\max \Psi = \beta$ . Using the same reasoning as in §2.4, having either  $\max 0$  or  $\text{gap } 0$  is equivalent to  $\sup \Psi = 0$ , and having  $\text{gap } 0$  is equivalent to  $\sup \Psi = 0 \notin \Psi$ ; we use these formulations throughout the rest of the section.

We are concerned primarily with  $H$ -asymptotic couples having  $\text{gap } 0$ , but using similar techniques we prove analogous results for asymptotic couples with  $\max 0$ , although we do not use them later in the chapter. Before stating the quantifier elimination and model completion results, we specify the language  $\mathcal{L}_{\text{ac}} = \{+, -, \leq, 0, \infty, \psi\}$  of asymptotic couples. The underlying set of an  $H$ -asymptotic



couple  $(\Gamma, \psi)$  in this language is  $\Gamma_\infty := \Gamma \cup \{\infty\}$ , and we interpret  $\infty$  in the following way: for all  $\gamma \in \Gamma$ ,  $\infty + \gamma = \gamma + \infty := \infty$  and  $\gamma < \infty$ ;  $\infty + \infty := \infty$ ;  $-\infty := \infty$ ;  $\psi(0) = \psi(\infty) := \infty$ . The other symbols have the expected interpretation.

**Theorem 6.7.** *The theory of nontrivial divisible  $H$ -asymptotic couples  $(\Gamma, \psi)$  with  $\Psi = \Gamma^<$  has quantifier elimination, and is the model completion of the theory of  $H$ -asymptotic couples with gap 0.*

In this chapter, we use this theorem via the following corollary. For  $n \geq 1$ ,  $\alpha_1, \dots, \alpha_n \in \Gamma$ , and  $\gamma \in \Gamma$ , we define the function  $\psi_{\alpha_1, \dots, \alpha_n} : \Gamma_\infty \rightarrow \Gamma_\infty$  recursively by

$$\psi_{\alpha_1}(\gamma) := \psi(\gamma - \alpha_1) \quad \text{and} \quad \psi_{\alpha_1, \dots, \alpha_n}(\gamma) := \psi(\psi_{\alpha_1, \dots, \alpha_{n-1}}(\gamma) - \alpha_n) \text{ for } n \geq 2.$$

**Corollary 6.8.** *Let  $(\Gamma, \psi)$  be a nontrivial divisible  $H$ -asymptotic couple with  $\Psi = \Gamma^<$  and let  $(\Gamma^*, \psi^*)$  be an  $H$ -asymptotic couple extending  $(\Gamma, \psi)$  with gap 0. Suppose that  $n \geq 1$ ,  $\alpha_1, \dots, \alpha_n \in \Gamma$ ,  $q_1, \dots, q_n \in \mathbb{Q}$ , and  $\gamma^* \in \Gamma^*$  are such that:*

- (i)  $\psi_{\alpha_1, \dots, \alpha_n}^*(\gamma^*) \neq \infty$  (so  $\psi_{\alpha_1, \dots, \alpha_i}^*(\gamma^*) \neq \infty$  for  $i = 1, \dots, n$ );
- (ii)  $\gamma^* + q_1 \psi_{\alpha_1}^*(\gamma^*) + \dots + q_n \psi_{\alpha_1, \dots, \alpha_n}^*(\gamma^*) \in \Gamma$  (in  $\mathbb{Q}\Gamma^*$ ).

Then  $\gamma^* \in \Gamma$ .

PROOF. By Theorem 6.7,  $(\Gamma, \psi)$  is an existentially closed  $H$ -asymptotic couple with gap 0 (see [ADH17a, Lemma B.10.10]), so we have  $\gamma \in \Gamma$  with

$$\gamma + q_1 \psi_{\alpha_1}(\gamma) + \dots + q_n \psi_{\alpha_1, \dots, \alpha_n}(\gamma) = \gamma^* + q_1 \psi_{\alpha_1}^*(\gamma^*) + \dots + q_n \psi_{\alpha_1, \dots, \alpha_n}^*(\gamma^*).$$

It remains to use [ADH17a, Lemma 9.9.3] to obtain  $\gamma^* = \gamma \in \Gamma$ .  $\square$

The rest of the section is devoted to proving Theorem 6.7, as well as an analogue for  $H$ -asymptotic couples with max 0 that is not used later. The material in this section is based on [ADH17c], which improves [AD00]; in those two papers similar quantifier elimination and model completion results are obtained for a different theory of asymptotic couples. Here we do not need to expand the language by a predicate for the  $\Psi$ -set or by functions for divisibility by nonzero natural numbers. Additionally, those authors work over an arbitrary ordered scalar field  $\mathbf{k}$ , but here we work over  $\mathbb{Q}$  for concreteness (the results of this section hold in that setting in the language  $\mathcal{L}_{ac}$  expanded by function symbols for scalar multiplication). Since the paper [ADH17c] is in preparation, we quote the results that we use and give their proofs (also from [ADH17c]). Moreover, many of the proofs of results specific to the setting of gap 0 or max 0 are very similar to proofs of analogous results from [ADH17c]; we are indebted to those authors for providing their manuscript.

#### 6.4.1. Preliminaries.

**Lemma 6.9** ([ADH17c, 2.7]). *Suppose that  $\Psi$  is downward closed. Let  $(\Gamma_1, \psi_1)$  and  $(\Gamma_*, \psi_*)$  be  $H$ -asymptotic couples extending  $(\Gamma, \psi)$  such that  $\Gamma^<$  is cofinal in  $\Gamma_1^<$ . Suppose that  $\gamma_1 \in \Gamma_1 \setminus \Gamma$  and  $\gamma_* \in \Gamma_* \setminus \Gamma$  realize the same cut in  $\Gamma$  with  $\gamma_1^\dagger \notin \Gamma$ . Then  $\gamma_*^\dagger \notin \Gamma$ , and  $\gamma_1^\dagger$  and  $\gamma_*^\dagger$  realize the same cut in  $\Gamma$ .*

PROOF. Let  $\alpha \in \Gamma^\neq$ , and we show:

$$\gamma_1^\dagger < \alpha^\dagger \implies \gamma_*^\dagger < \alpha^\dagger \quad \text{and} \quad \gamma_1^\dagger > \alpha^\dagger \implies \gamma_*^\dagger > \alpha^\dagger.$$

First, suppose that  $\gamma_1^\dagger < \alpha^\dagger$ . Then  $|\gamma_1| > |\alpha|$ , so  $|\gamma_*| > |\alpha|$ , and thus  $\gamma_*^\dagger \leq \alpha^\dagger$ . Since  $\Gamma^<$  is cofinal in  $\Gamma_1^<$ , there is  $\delta \in \Gamma$  with  $\gamma_1^\dagger < \delta < \alpha^\dagger$ . By taking  $\beta \in \Gamma^\neq$  with  $\beta^\dagger = \delta$ , since  $\Psi = \Psi^\downarrow$ , we can replace  $\alpha$  by  $\beta$  in the argument to get  $\gamma_*^\dagger \leq \beta^\dagger < \alpha^\dagger$ .

Now suppose that  $\gamma_1^\dagger > \alpha^\dagger$ , so we obtain  $\gamma_*^\dagger \geq \alpha^\dagger$  in the same way. By the cofinality assumption, there is  $\delta \in \Gamma$  with  $\gamma_1^\dagger > \delta > \alpha^\dagger$ . Note that  $\delta \in \Psi^\downarrow$  since there is  $\beta \in \Gamma^\neq$  with  $|\gamma_1| \geq |\beta|$ , which gives  $\gamma_1^\dagger < \beta^\dagger \in \Psi = \Psi^\downarrow$ . Hence similar reasoning as in the first case works. This also shows how the lemma follows from the claim.  $\square$

**Lemma 6.10** ([ADH17c, 2.8]). *Suppose that  $(\Gamma_1, \psi_1)$  is an  $H$ -asymptotic couple extending  $(\Gamma, \psi)$ . Let  $\gamma_1 \in \Gamma_1 \setminus \Gamma$  and  $\alpha_1, \alpha_2 \in \Gamma$  with  $\beta_1 := \gamma_1 - \alpha_1$  and  $\beta_2 := \beta_1^\dagger - \alpha_2$ . If  $|\beta_1| \geq |\gamma|$  for some  $\gamma \in \Gamma^\neq$ ,  $\beta_1^\dagger \notin \Gamma$ , and  $\beta_2^\dagger \notin \Psi$ , then  $\beta_1^\dagger < \beta_2^\dagger$ .*

PROOF. Take  $\gamma \in \Gamma^\neq$  with  $|\beta_1| \geq |\gamma|$ , so  $\beta_1^\dagger \leq \gamma^\dagger$ . From  $\beta_2^\dagger \notin \Psi$  we obtain  $[\beta_1^\dagger - \alpha_2] \notin [\Gamma]$ . Thus  $[\beta_1^\dagger - \gamma^\dagger] \geq [\beta_1^\dagger - \alpha_2]$ , as otherwise  $[\beta_1^\dagger - \alpha_2] = [\gamma^\dagger - \alpha_2] \in [\Gamma]$ . Putting this together and using [ADH17a, Lemma 6.5.4(i)] for the first inequality, we get

$$\beta_1^\dagger = \min\{\beta_1^\dagger, \gamma^\dagger\} < (\beta_1^\dagger - \gamma^\dagger)^\dagger \leq (\beta_1^\dagger - \alpha_2)^\dagger = \beta_2^\dagger. \quad \square$$

Next is the variant of [ADH17c, Lemma 3.1] needed here.

**Lemma 6.11** ([ADH17c, 3.1]). *Let  $\beta \in \Psi^\downarrow \setminus \Psi$  or  $\beta$  be a gap in  $(\Gamma, \psi)$ . Then there is an  $H$ -asymptotic couple  $(\Gamma \oplus \mathbb{Z}\alpha, \psi^\alpha)$  extending  $(\Gamma, \psi)$  such that:*

- (i)  $\alpha > 0$  and  $\psi^\alpha(\alpha) = \beta$ ;
- (ii) *given any embedding  $i: (\Gamma, \psi) \rightarrow (\Gamma^*, \psi^*)$  and  $\alpha^* \in \Gamma^*$  with  $\alpha^* > 0$  and  $\psi^*(\alpha^*) = i(\beta)$ , there is a unique embedding  $j: (\Gamma \oplus \mathbb{Z}\alpha, \psi^\alpha) \rightarrow (\Gamma^*, \psi^*)$  extending  $i$  with  $j(\alpha) = \alpha^*$ .*

PROOF. Apply [ADH17a, Lemma 9.8.7] with  $C = \{[\gamma] : \gamma \in \Gamma^\neq, \psi(\gamma) > \beta\}$ .  $\square$

We call  $(\Gamma, \psi)$  *gap-closed* if  $\Gamma$  is nontrivial and divisible, and  $\Psi = \Gamma^<$ . Similarly, we call  $(\Gamma, \psi)$  *max-closed* if  $\Gamma$  is divisible and  $\Psi = \Gamma^{\leq}$  ( $0 \in \Psi$  implies that  $\Gamma$  is nontrivial). Then we call an  $H$ -asymptotic couple  $(\Gamma_1, \psi_1)$  extending  $(\Gamma, \psi)$  a *gap-closure* of  $(\Gamma, \psi)$  if it is gap-closed and it embeds over  $(\Gamma, \psi)$  into every gap-closed  $H$ -asymptotic couple extending  $(\Gamma, \psi)$ . Similarly, we call an  $H$ -asymptotic couple  $(\Gamma_1, \psi_1)$  extending  $(\Gamma, \psi)$  a *max-closure* of  $(\Gamma, \psi)$  if it is max-closed and it embeds over  $(\Gamma, \psi)$  into every max-closed  $H$ -asymptotic couple extending  $(\Gamma, \psi)$ .

**Corollary 6.12.** *Every  $H$ -asymptotic couple  $(\Gamma, \psi)$  with  $\sup \Psi = 0 \notin \Psi$  has a gap-closure. Every  $H$ -asymptotic couple  $(\Gamma, \psi)$  with  $\sup \Psi = 0$  has a max-closure.*

PROOF. This follows by alternating applications of Lemma 6.11 and taking the divisible hull.  $\square$

**6.4.2. Quantifier elimination with gap 0.** We now turn to the proof of quantifier elimination for gap-closed  $H$ -asymptotic couples. To that end, suppose that  $(\Gamma, \psi)$  is a divisible  $H$ -asymptotic couple with gap 0, and let  $(\Gamma_1, \psi_1)$  and  $(\Gamma_*, \psi_*)$  be gap-closed  $H$ -asymptotic couples extending  $(\Gamma, \psi)$  such that  $(\Gamma_*, \psi_*)$  is  $|\Gamma|^{+}$ -saturated. Let  $\gamma_1 \in \Gamma_1 \setminus \Gamma$  and  $(\Gamma\langle\gamma_1\rangle, \psi_1)$  be the divisible  $H$ -asymptotic couple generated by  $\Gamma \cup \{\gamma_1\}$  in  $(\Gamma_1, \psi_1)$ . In light of standard quantifier elimination tests, our goal is to embed  $(\Gamma\langle\gamma_1\rangle, \psi_1)$  into  $(\Gamma_*, \psi_*)$  over  $\Gamma$ .

For convenience, we set  $0^\dagger := \psi(0) = \infty$ , so  $\Gamma^\dagger = \Psi \cup \{\infty\}$ .

The next lemma is adapted from [ADH17c, Lemma 3.4].

**Lemma 6.13.** *Suppose  $(\Gamma + \mathbb{Q}\gamma_1)^\dagger = \Gamma^\dagger$ . Then  $(\Gamma\langle\gamma_1\rangle, \psi_1)$  can be embedded into  $(\Gamma_*, \psi_*)$  over  $\Gamma$ .*

PROOF. From  $(\Gamma + \mathbb{Q}\gamma_1)^\dagger = \Gamma^\dagger$ , we get  $\Gamma\langle\gamma_1\rangle = \Gamma + \mathbb{Q}\gamma_1$ . Note that there is no  $\beta_1 \in \Gamma + \mathbb{Q}\gamma_1$  with  $0 < \beta_1 < \Gamma^>$ : otherwise,  $\psi_1(\beta_1) > \Psi$ , since  $\Psi$  has no greatest element, contradicting that  $(\Gamma + \mathbb{Q}\gamma_1)^\dagger = \Gamma^\dagger$ .

*Case 1:*  $[\Gamma + \mathbb{Q}\gamma_1] = [\Gamma]$ . By saturation, we take  $\gamma_* \in \Gamma_*$  realizing the same cut in  $\Gamma$  as  $\gamma_1$ . Then we have an embedding  $i: \Gamma + \mathbb{Q}\gamma_1 \rightarrow \Gamma_*$  of ordered vector spaces over  $\mathbb{Q}$  that is the identity on  $\Gamma$  and satisfies  $i(\gamma_1) = \gamma_*$  by [ADH17a, Lemma 2.4.16]. Now for  $\gamma \in \Gamma + \mathbb{Q}\gamma_1$  we have  $[i(\gamma)] = [\gamma] \in [\Gamma]$ , so  $i(\gamma)^\dagger = \gamma^\dagger \in \Psi \cup \{\infty\}$ . Hence  $i$  is an embedding of  $(\Gamma\langle\gamma_1\rangle, \psi_1)$  into  $(\Gamma_*, \psi_*)$  over  $\Gamma$ .

*Case 2:*  $[\Gamma + \mathbb{Q}\gamma_1] \neq [\Gamma]$ . Take  $\beta_1 \in \Gamma_1 \setminus \Gamma$  with  $\beta_1 > 0$  and  $[\beta_1] \notin [\Gamma]$ , so  $[\Gamma\langle\gamma_1\rangle] = [\Gamma] \cup \{[\beta_1]\}$ . Let  $D$  be the cut in  $\Gamma$  realized by  $\beta_1$  and  $E := \Gamma \setminus D$ , so  $D < \beta_1 < E$ . First, we claim that  $D$  has no greatest element. If it did have a greatest element  $\delta$ , then  $0 < \beta_1 - \delta < \Gamma^>$ , contradicting the comment at the beginning of the proof. Similarly,  $E$  has no least element. Thus by saturation we have  $\beta_* \in \Gamma_*$  realizing the same cut in  $\Gamma$  as  $\beta_1$  with  $\beta_*^\dagger = \beta_1^\dagger$ . Then [ADH17a, Lemma 2.4.16] yields an embedding  $i: \Gamma + \mathbb{Q}\gamma_1 \rightarrow \Gamma_*$  of ordered vector spaces over  $\mathbb{Q}$  that is the identity on  $\Gamma$  and satisfies  $i(\beta_1) = \beta_*$ . This embedding is also an embedding of  $H$ -asymptotic couples.  $\square$

The next lemma is adapted from [ADH17c, Lemma 3.5].

**Lemma 6.14.** *Suppose that  $(\Gamma, \psi)$  is gap-closed,  $(\Gamma + \mathbb{Q}\gamma)^\dagger \neq \Gamma^\dagger$  for all  $\gamma \in \Gamma_1 \setminus \Gamma$ , and  $\Gamma^<$  is cofinal in  $\Gamma_1^<$ . Then  $(\Gamma\langle\gamma_1\rangle, \psi_1)$  can be embedded into  $(\Gamma_*, \psi_*)$  over  $\Gamma$ .*

PROOF. Take  $\alpha_1 \in \Gamma$  such that  $(\gamma_1 - \alpha_1)^\dagger \notin \Gamma^\dagger$ . Since  $(\gamma_1 - \alpha_1)^\dagger < 0$  and  $\Psi = \Gamma^<$ , we deduce that  $(\gamma_1 - \alpha_1)^\dagger \notin \Gamma$ . Let  $n \geq 1$ . We thus construct sequences  $\alpha_1, \alpha_2, \dots$  in  $\Gamma$  and  $\beta_1, \beta_2, \dots$  in  $\Gamma\langle\gamma_1\rangle \setminus \Gamma$  with  $\beta_1 = \gamma_1 - \alpha_1$ ,  $\beta_{n+1} = \beta_n^\dagger - \alpha_{n+1}$ , and  $\beta_n^\dagger \notin \Gamma$ . It follows that  $[\beta_n] \notin [\Gamma]$ , and by Lemma 6.10 we have  $\beta_n^\dagger < \beta_{n+1}^\dagger$  and thus  $[\beta_n] > [\beta_{n+1}]$ . In particular, the family  $(\beta_n)_{n \geq 1}$  is  $\mathbb{Q}$ -linearly independent over  $\Gamma$  and

$$\Gamma\langle\gamma_1\rangle = \Gamma \oplus \mathbb{Q}\beta_1 \oplus \mathbb{Q}\beta_2 \oplus \dots$$

By saturation, we take  $\gamma_* \in \Gamma_* \setminus \Gamma$  realizing the same cut in  $\Gamma$  as  $\gamma_1$  and define by recursion on  $n \geq 1$   $\beta_{*n} \in (\Gamma_*)_\infty$  by  $\beta_{*1} := \gamma_* - \alpha_1$  and  $\beta_{*(n+1)} := \beta_{*n}^\dagger - \alpha_{n+1}$ . We assume inductively that  $\beta_{*m} \in \Gamma_* \setminus \Gamma$  for  $m = 1, \dots, n$ , and that we have an embedding

$$i_n: \Gamma + \mathbb{Q}\beta_1 + \dots + \mathbb{Q}\beta_n \rightarrow \Gamma_*$$

of ordered vector spaces over  $\mathbb{Q}$  that is the identity on  $\Gamma$  and satisfies  $i_n(\beta_m) = \beta_{*m}$  for  $m = 1, \dots, n$ . Then  $\beta_n$  and  $\beta_{*n}$  realize the same cut in  $\Gamma$ , so  $\beta_{*n}^\dagger \notin \Gamma$  and  $\beta_{*n}^\dagger$  realizes the same cut in  $\Gamma$  as  $\beta_n^\dagger$ , by Lemma 6.9. Hence  $\beta_{*(n+1)} \in \Gamma_* \setminus \Gamma$  and  $\beta_{n+1}$  and  $\beta_{*(n+1)}$  realize the same cut in  $\Gamma$ . We have

$$[\Gamma + \mathbb{Q}\beta_1 + \dots + \mathbb{Q}\beta_n] = [\Gamma] \cup \{[\beta_1], \dots, [\beta_n]\} \quad \text{and} \quad [\beta_1] > \dots > [\beta_n] > [\beta_{n+1}].$$

Let  $D$  be the cut realized by  $[\beta_{n+1}]$  in  $[\Gamma + \mathbb{Q}\beta_1 + \dots + \mathbb{Q}\beta_n]$ . The comments above show that  $[\beta_{*(n+1)}]$  realizes the image under  $i_n$  of  $D$  in  $[i_n(\Gamma + \mathbb{Q}\beta_1 + \dots + \mathbb{Q}\beta_n)]$ , so we can extend  $i_n$  to an embedding

$$i_{n+1}: \Gamma + \mathbb{Q}\beta_1 + \dots + \mathbb{Q}\beta_n + \mathbb{Q}\beta_{n+1} \rightarrow \Gamma_*$$

of ordered vector spaces over  $\mathbb{Q}$  that is the identity on  $\Gamma$  and satisfies  $i_{n+1}(\beta_{n+1}) = \beta_{*(n+1)}$ . By induction, this yields a map  $i: \Gamma\langle\gamma_1\rangle \rightarrow \Gamma_*$  extending each  $i_n$ , so  $i$  is an embedding of  $H$ -asymptotic couples.  $\square$

The case considered in the next lemma is particular to the setting with gap 0.

**Lemma 6.15.** *Suppose that  $\Gamma^< < \gamma_1 < 0$ . Then  $(\Gamma\langle\gamma_1\rangle, \psi_1)$  can be embedded into  $(\Gamma_*, \psi_*)$  over  $\Gamma$ .*

PROOF. For this proof, set  $\gamma_1^{\dagger 0} := \gamma_1$  and  $\gamma_1^{\dagger(n+1)} := (\gamma_1^{\dagger n})^\dagger$ , so  $\gamma_1^{\dagger 1} = \gamma_1^\dagger$ ,  $\gamma_1^{\dagger 2} = \gamma_1^{\dagger\dagger}$ , etc. We have

$$[\gamma_1] > [\gamma_1^\dagger] > [\gamma_1^{\dagger\dagger}] > \dots$$

by [ADH17a, Lemma 9.2.10(iv)], and so

$$\Gamma^< < \gamma_1 < \gamma_1^\dagger < \gamma_1^{\dagger\dagger} < \dots < 0 \quad \text{and} \quad [\gamma_1^{\dagger n}] \notin [\Gamma] \text{ for all } n.$$

Hence the family  $(\gamma_1^{\dagger n})_{n \in \mathbb{N}}$  is  $\mathbb{Q}$ -linearly independent over  $\Gamma$  and

$$\Gamma\langle\gamma_1\rangle = \Gamma \oplus \mathbb{Q}\gamma_1 \oplus \mathbb{Q}\gamma_1^\dagger \oplus \mathbb{Q}\gamma_1^{\dagger\dagger} \oplus \dots$$

By saturation, we may take  $\gamma_* \in \Gamma_* \setminus \Gamma$  with  $\Gamma^< < \gamma_* < 0$ . The above holds in  $\Gamma_*$  with  $\gamma_*$  replacing  $\gamma_1$  (and  $\gamma_*^{\dagger n}$  defined analogously), so by induction and [ADH17a, Lemma 2.4.16] we construct an embedding of  $\Gamma\langle\gamma_1\rangle$  into  $\Gamma_*$  as ordered vector spaces over  $\mathbb{Q}$  that sends  $\gamma_1$  to  $\gamma_*$ . This is also an embedding of  $H$ -asymptotic couples.  $\square$

We can now complete the proof of the main theorem of this section. Recall from the introduction the language  $\mathcal{L}_{ac} = \{+, -, \leq, 0, \infty, \psi\}$  of asymptotic couples, though we first prove quantifier elimination in the expanded language  $\mathcal{L}_{ac, \text{div}} = \mathcal{L}_{ac} \cup \{\text{div}_n : n \geq 1\}$ , where each unary function symbol  $\text{div}_n$  is interpreted as division by  $n$  and  $\text{div}_n(\infty) := \infty$ .

**Theorem 6.7.** *The theory of gap-closed  $H$ -asymptotic couples has quantifier elimination, and it is the model completion of the theory of  $H$ -asymptotic couples with gap 0.*

PROOF. That the theory gap-closed  $H$ -asymptotic couples has quantifier elimination in  $\mathcal{L}_{ac, \text{div}}$  follows from Lemmas 6.13, 6.14, and 6.15, and Corollary 6.12 by a standard quantifier elimination test. (See for example [ADH17a, Corollary B.11.11].) To see that it has quantifier elimination in  $\mathcal{L}_{ac}$ , recall from the beginning of this section how, for an asymptotic couple  $(\Gamma, \psi)$ ,  $\psi$  extends uniquely to the divisible hull  $\mathbb{Q}\Gamma$  of  $\Gamma$ . The desired result then follows from [ADH17a, Corollary B.11.5].

The model completion statement follows from quantifier elimination combined with Corollary 6.12. (See for example [ADH17a, Corollary B.11.6].)  $\square$

**Corollary 6.16.** *The theory of gap-closed  $H$ -asymptotic couples is complete.*

PROOF. The  $H$ -asymptotic couple  $(\{0\}, \psi_\emptyset)$ , where  $\psi_\emptyset: \emptyset \rightarrow \{0\}$  is the empty function, embeds into every gap-closed  $H$ -asymptotic couple, yielding completeness. (See for example [ADH17a, Corollary B.11.7].) It also has gap 0, so its gap-closure is the prime model of this theory.  $\square$

**6.4.3. Quantifier elimination with max 0.** We derive similar quantifier elimination and model completion results in the setting allowing max 0. The proofs are as in the previous subsection,

except where indicated. This material is only used in one later theorem that itself is not used in the main results, but this subsection is naturally complementary to the previous one.

Suppose that  $(\Gamma, \psi)$  is a divisible  $H$ -asymptotic couple with  $\sup \Psi = 0$ . Let  $(\Gamma_1, \psi_1)$  and  $(\Gamma_*, \psi_*)$  be max-closed  $H$ -asymptotic couples extending  $(\Gamma, \psi)$  such that  $(\Gamma_*, \psi_*)$  is  $|\Gamma|^+$ -saturated. Let  $\gamma_1 \in \Gamma_1 \setminus \Gamma$  and  $(\Gamma\langle\gamma_1\rangle, \psi_1)$  be the divisible  $H$ -asymptotic couple generated by  $\Gamma \cup \{\gamma_1\}$  in  $(\Gamma_1, \psi_1)$ .

**Lemma 6.17.** *Suppose that  $\max \Psi = 0$  and  $(\Gamma + \mathbb{Q}\gamma_1)^\dagger = \Gamma^\dagger$ . Then  $(\Gamma\langle\gamma_1\rangle, \psi_1)$  can be embedded into  $(\Gamma_*, \psi_*)$  over  $\Gamma$ .*

PROOF. From  $(\Gamma + \mathbb{Q}\gamma_1)^\dagger = \Gamma^\dagger$ , we get  $\Gamma\langle\gamma_1\rangle = \Gamma + \mathbb{Q}\gamma_1$ .

*Case 1:*  $[\Gamma + \mathbb{Q}\gamma_1] = [\Gamma]$ . As in *Case 1* of Lemma 6.13.

*Case 2:*  $[\Gamma + \mathbb{Q}\gamma_1] \neq [\Gamma]$  but there does not exist  $\beta_1 \in \Gamma + \mathbb{Q}\gamma_1$  with  $0 < \beta_1 < \Gamma^>$ . As in *Case 2* of Lemma 6.13.

*Case 3:* there exists  $\beta_1 \in \Gamma + \mathbb{Q}\gamma_1$  with  $0 < \beta_1 < \Gamma^>$ . By saturation, take  $\beta_* \in \Gamma_*$  with  $0 < \beta_* < \Gamma^>$ , so  $\beta_*^\dagger = \beta_1^\dagger = 0$ . The proof continues as in *Case 2* of Lemma 6.13 after “ $\beta_*^\dagger = \beta_1^\dagger$ .”  $\square$

**Lemma 6.18.** *Suppose that  $(\Gamma, \psi)$  is max-closed and  $(\Gamma + \mathbb{Q}\gamma)^\dagger \neq \Gamma^\dagger$  for all  $\gamma \in \Gamma_1 \setminus \Gamma$ . Then  $(\Gamma\langle\gamma_1\rangle, \psi_1)$  can be embedded into  $(\Gamma_*, \psi_*)$  over  $\Gamma$ .*

PROOF. If  $\gamma \in \Gamma_1 \setminus \Gamma$  with  $0 < \gamma < \Gamma^>$ , then  $\gamma^\dagger = 0$  and so  $(\Gamma + \mathbb{Q}\gamma)^\dagger = \Gamma^\dagger$ , a contradiction. Hence there is no such  $\gamma$ , and thus  $\Gamma^<$  is cofinal in  $\Gamma_1^<$ . The rest of the proof is as in Lemma 6.14.  $\square$

Recall from the introduction to this section the language  $\mathcal{L}_{ac}$  of asymptotic couples.

**Theorem 6.19.** *The theory of max-closed  $H$ -asymptotic couples has quantifier elimination, and is the model completion of the theory of  $H$ -asymptotic couples  $(\Gamma, \psi)$  with  $\sup \Psi = 0$ .*

**Corollary 6.20.** *The theory of max-closed  $H$ -asymptotic couples is complete and has a prime model.*

## 6.5. Extensions controlled by the asymptotic couple

**6.5.1. A maximality theorem.** The results and proofs of this section are adapted from [ADH17a, §16.1]. This next lemma and its consequences are where we use the quantifier elimination for gap-closed asymptotic couples from §6.4. Note that if  $K$  is an  $H$ -asymptotic field with exponential integration and gap 0, then in fact  $\Psi = \Gamma^<$ , so if additionally  $\Gamma$  is divisible then  $(\Gamma, \psi)$  is a gap-closed  $H$ -asymptotic couple in the sense of the previous section.

**Lemma 6.21.** *Suppose that  $K$  is a d-henselian  $H$ -asymptotic field with exponential integration and gap 0 whose value group is divisible. Let  $L$  be an  $H$ -asymptotic extension of  $K$  with gap 0 and  $\mathbf{k}_L = \mathbf{k}$ , and suppose that there is no  $y \in L \setminus K$  such that  $K\langle y \rangle$  is an immediate extension of  $K$ . Let  $f \in L \setminus K$ . Then the vector space  $\mathbb{Q}\Gamma_{K\langle f \rangle}/\Gamma$  is infinite dimensional.*

PROOF. First, we argue that there is no divergent pc-sequence in  $K$  with a pseudolimit in  $L$ . Towards a contradiction, suppose that  $(a_\rho)$  is a divergent pc-sequence in  $K$  with pseudolimit  $\ell \in L$ . Since  $K$  is d-henselian and asymptotic, it is d-algebraically maximal by Theorem 4.19, so  $(a_\rho)$  is not

of d-algebraic type over  $K$ . Hence  $(a_\rho)$  is of d-transcendental type over  $K$ , so  $K\langle\ell\rangle$  is an immediate extension of  $K$  by Lemma 2.12, a contradiction.

Thus for all  $y \in L \setminus K$ , the set  $v_L(y - K) \subseteq \Gamma_L$  has a maximum by Lemma 2.6. If  $v_L(y - y_0) = \max v_L(y - K)$ , then  $v_L(y - y_0) \notin \Gamma$  since  $\mathbf{k}_L = \mathbf{k}$ . Otherwise, there would be  $y_1 \in K$  with  $y - y_0 \sim y_1$ , contradicting the maximality of  $v_L(y - y_0)$ . For convenience, assume below that  $L = K\langle f \rangle$ . Set  $f_0 := f$ , pick  $b_0 \in K$  with  $v_L(f_0 - b_0) = \max v_L(f_0 - K)$ , and set  $f_1 := (f_0 - b_0)^\dagger \in L$ . We claim that  $f_1 \notin K$ . Otherwise, there would be  $g \in K^\times$  with  $(f_0 - b_0)^\dagger = g^\dagger$ , so  $v_L(f_0 - b_0) = v(g)$ , contradicting that  $v_L(f_0 - b_0) \notin \Gamma$ . By induction we obtain sequences  $(f_n)$  in  $L \setminus K$  and  $(b_n)$  in  $K$  such that for all  $n$ :

- (i)  $v_L(f_n - b_n) = \max v_L(f_n - K)$ ;
- (ii)  $f_{n+1} = (f_n - b_n)^\dagger$ .

Hence  $v_L(f_n - b_n) \notin \Gamma$  for all  $n$ . The result follows from the next claim:

$$v_L(f_0 - b_0), v_L(f_1 - b_1), \dots \text{ are } \mathbb{Q}\text{-linearly independent over } \Gamma.$$

To see this, let  $n \geq 1$  and take  $a_n \in K^\times$  with  $a_n^\dagger = b_n$ , so

$$f_n - b_n = (f_{n-1} - b_{n-1})^\dagger - a_n^\dagger = \left( \frac{f_{n-1} - b_{n-1}}{a_n} \right)^\dagger,$$

and set  $\alpha_n := v(a_n) \in \Gamma$ . Recall the function  $\psi_{L, \alpha_1, \dots, \alpha_n}$  defined before Corollary 6.8, where the subscript  $L$  indicates that it is defined on  $\Gamma_L$ , not just  $\Gamma$ . Then we have

$$v_L(f_n - b_n) = \psi_L(v_L(f_{n-1} - b_{n-1}) - \alpha_n),$$

so by induction we get

$$v_L(f_n - b_n) = \psi_{L, \alpha_1, \dots, \alpha_n}(v_L(f_0 - b_0)).$$

Suppose towards a contradiction that  $v_L(f_0 - b_0), \dots, v_L(f_n - b_n)$  are  $\mathbb{Q}$ -linearly dependent over  $\Gamma$ , so we have  $m < n$  and  $q_1, \dots, q_{n-m} \in \mathbb{Q}$  such that

$$v_L(f_m - b_m) + q_1 v_L(f_{m+1} - b_{m+1}) + \dots + q_{n-m} v_L(f_n - b_n) \in \Gamma.$$

With  $\gamma := v_L(f_m - b_m) \in \Gamma_L \setminus \Gamma$ , this means

$$\gamma + q_1 \psi_{L, \alpha_{m+1}}(\gamma) + \dots + q_{n-m} \psi_{L, \alpha_{m+1}, \dots, \alpha_n}(\gamma) \in \Gamma,$$

so  $v_L(f_m - b_m) = \gamma \in \Gamma$  by Corollary 6.8, a contradiction.  $\square$

The previous lemma yields a maximality theorem that is used later to prove the minimality of differential-Hensel-Liouville closures, but is also of independent interest as a strengthening of the conclusion of Theorem 4.19 under additional hypotheses.

**Theorem 6.22.** *Suppose that  $K$  is a d-henselian  $H$ -asymptotic field with exponential integration and gap 0 whose value group is divisible. Then  $K$  has no proper d-algebraic  $H$ -asymptotic extension with gap 0 and the same residue field.*

PROOF. Let  $L$  be a proper d-algebraic  $H$ -asymptotic extension of  $K$  with gap 0 and  $\mathbf{k}_L = \mathbf{k}$ . By Theorem 4.19,  $K$  is d-algebraically maximal, so there is no  $y \in L \setminus K$  such that  $K\langle y \rangle$  is an immediate extension of  $K$ . But for  $f \in L \setminus K$ , the transcendence degree of  $K\langle f \rangle$  over  $K$  is finite,

so the vector space  $\mathbb{Q}\Gamma_{K\langle f \rangle}/\Gamma$  is finite dimensional by the Zariski–Abhyankar Inequality [ADH17a, Corollary 3.1.11], contradicting Lemma 6.21.  $\square$

By quantifier elimination for max-closed  $H$ -asymptotic couples and the same arguments, we also obtain the following, which is not used later. Here we say an asymptotic field  $K$  has  $\max 0$  if its asymptotic couple does.

**Theorem 6.23.** *If  $K$  is a d-henselian  $H$ -asymptotic field with exponential integration and  $\max 0$  whose value group is divisible, then  $K$  has no proper d-algebraic  $H$ -asymptotic extension with  $\max 0$  and the same residue field.*

We now extract more information from the proof of Lemma 6.21 for use in the next subsection.

**Lemma 6.24.** *Let  $K$ ,  $L$ , and  $f$  be as in Lemma 6.21, and let the sequences  $(f_n)$ ,  $(b_n)$ ,  $(a_n)_{n \geq 1}$ , and  $(\alpha_n)_{n \geq 1}$  be as in the proof of Lemma 6.21. Set  $\beta_n := v_L(f_n - b_n) - \alpha_{n+1}$ . The asymptotic couple  $(\Gamma_{K\langle f \rangle}, \psi_L)$  of  $K\langle f \rangle$  has the following properties:*

- (i)  $\Gamma_{K\langle f \rangle} = \Gamma \oplus \bigoplus_n \mathbb{Z}\beta_n$  (internal direct sum);
- (ii)  $\beta_n^\dagger \notin \Gamma$  for all  $n$ , and  $\beta_m^\dagger \neq \beta_n^\dagger$  for all  $m \neq n$ ;
- (iii)  $\psi_L(\Gamma_{K\langle f \rangle}^\neq) = \Psi \cup \{\beta_n^\dagger : n \in \mathbb{N}\}$ ;
- (iv)  $[\beta_n] \notin [\Gamma]$  for all  $n$ ,  $[\beta_m] \neq [\beta_n]$  for all  $m \neq n$ , and  $[\Gamma_{K\langle f \rangle}] = [\Gamma] \cup \{[\beta_n] : n \in \mathbb{N}\}$ ;
- (v) if  $\Gamma_K^<$  is cofinal in  $\Gamma_{K\langle f \rangle}^<$ , then  $\beta_0^\dagger < \beta_1^\dagger < \beta_2^\dagger < \dots$ .

PROOF. Set  $\mathbf{m}_n := (f_n - b_n)/a_{n+1}$ , so  $v_L(\mathbf{m}_n) = \beta_n$ . Then

$$\begin{aligned} \mathbf{m}_{n+1} &= \frac{f_{n+1} - b_{n+1}}{a_{n+2}} = \frac{(f_n - b_n)^\dagger - b_{n+1}}{a_{n+2}} = \frac{(a_{n+1}\mathbf{m}_n)^\dagger - b_{n+1}}{a_{n+2}} = \frac{a_{n+1}^\dagger + \mathbf{m}_n^\dagger - b_{n+1}}{a_{n+2}} \\ &= \frac{\mathbf{m}_n^\dagger}{a_{n+2}}. \end{aligned}$$

Hence  $\mathbf{m}'_n = a_{n+2}\mathbf{m}_n\mathbf{m}_{n+1}$ . From  $f = b_0 + a_1\mathbf{m}_0$  we get  $f' = b'_0 + a'_1\mathbf{m}_0 + a_1a_2\mathbf{m}_0\mathbf{m}_1$ , so induction yields  $F_n \in K[Y_0, \dots, Y_n]$  with  $\deg F_n \leq n+1$  and  $f^{(n)} = F_n(\mathbf{m}_0, \dots, \mathbf{m}_n)$ . Thus for  $P \in K\{Y\}^\neq$  of order at most  $r$  we have  $P(f) = \sum_{\mathbf{i} \in I} a_{\mathbf{i}}\mathbf{m}_0^{i_0} \dots \mathbf{m}_r^{i_r}$ , where  $I$  is a nonempty finite set of indices  $\mathbf{i} = (i_0, \dots, i_r) \in \mathbb{N}^{1+r}$ . Note that by the proof of Lemma 6.21, the family  $(\beta_n)$  is  $\mathbb{Q}$ -linearly independent over  $\Gamma$ . Hence  $v_L(P(f)) \in \Gamma + \sum_n \mathbb{N}\beta_n$ , which proves (i).

By the proof of Lemma 6.21, we also have

$$\beta_n^\dagger = \psi_L(v_L(f_n - b_n) - \alpha_{n+1}) = v_L(f_{n+1} - b_{n+1}) = \beta_{n+1} + \alpha_{n+2} \notin \Gamma.$$

Thus the family  $(\beta_n^\dagger)$  is  $\mathbb{Q}$ -linearly independent over  $\Gamma$ , since the family  $(\beta_n)$  is, proving (ii).

Note that (iii) follows from (i) and (ii). From (ii), we get  $[\beta_n] \notin [\Gamma]$  and  $[\beta_m] \neq [\beta_n]$  for all  $m \neq n$ , so (iv) now follows from (i). Finally, (v) follows from (ii) and Lemma 6.10.  $\square$

**6.5.2. Further consequences in the ordered setting.** Now we develop further the results of the previous subsection in the pre- $H$ -field setting. In this subsection,  $K$  and  $L$  are pre- $H$ -fields with small derivation. Suppose that  $K$  is d-henselian and has exponential integration, and that  $\Gamma$  is divisible. Suppose that  $L$  is an extension of  $K$  with  $\mathbf{k}_L = \mathbf{k}$ , and that there is no  $y \in L \setminus K$  such that  $K\langle y \rangle$  is an immediate extension of  $K$ . Let  $f \in L \setminus K$  with  $\Gamma^<$  cofinal in  $\Gamma_{K\langle f \rangle}^<$ , and let the

sequences  $(f_n)$ ,  $(b_n)$ ,  $(a_n)_{n \geq 1}$ , and  $(\alpha_n)_{n \geq 1}$  be as in the proof of Lemma 6.21. As before, we also set  $\beta_n := v_L(f_n - b_n) - \alpha_{n+1}$ . Note that since  $K$  is a pre- $H$ -field with small derivation and nontrivial induced derivation on  $\mathbf{k}$ , it has gap 0, and for the same reason so does  $L$ .

**Lemma 6.25.** *Suppose  $M$  is a pre- $H$ -field extension of  $K$  and  $g \in M$  realizes the same cut in  $K$  as  $f$ . Then  $v_M(g - b_0) = \max v_M(g - K) \notin \Gamma$  and  $g_1 := (g - b_0)^\dagger$  realizes the same cut in  $K$  as  $f_1$ .*

PROOF. Let  $\alpha \in \Gamma$  and  $b \in K$ . We first claim that

$$v_L(f - b) < \alpha \iff v_M(g - b) < \alpha \quad \text{and} \quad v_L(f - b) > \alpha \iff v_M(g - b) > \alpha.$$

To see this, take  $a \in K^>$  with  $va = \alpha$ . Suppose that  $v_L(f - b) < \alpha$ , so  $|f - b| > a$ . Hence  $|g - b| > a$  and thus  $v_M(g - b) \leq \alpha$ . By the cofinality assumption, take  $\delta \in \Gamma$  with  $v_L(f - b) < \delta < \alpha$ , and then the same argument yields  $v_M(g - b) \leq \delta < \alpha$ . One proves similarly that  $v_L(f - b) > \alpha \implies v_M(g - b) > \alpha$ . Finally, consider the case that  $v_L(f - b) = \alpha$ . This yields  $f - b \sim ua$  for  $u \in K$  with  $u \asymp 1$ , since  $\mathbf{k} = \mathbf{k}_L$ . From the convexity of  $\mathcal{O}_{K\langle f \rangle}$  we obtain  $|u|a/2 < |f - b| < 2|u|a$ , so  $|u|a/2 < |g - b| < 2|u|a$ , and thus  $v_M(g - b) = \alpha$ , completing the proof of the claim.

By the claim above and the fact that  $v_L(f - b_0) \notin \Gamma$ , we get  $v_M(g - b_0) \notin \Gamma$ . This yields  $v_M(g - b_0) = \max v_M(g - K)$ , as otherwise we would have  $b \in K$  with  $v_M(g - b) > v_M(g - b_0)$ , so  $v_M(g - b_0) = v(b - b_0) \in \Gamma$ . It also follows that  $(g - b_0)^\dagger \notin K$ , as otherwise  $(g - b_0)^\dagger = b^\dagger$  for some  $b \in K^\times$ , so  $v_M(g - b_0) = vb \in \Gamma$ .

Finally, we show that  $(g - b_0)^\dagger$  realizes the same cut in  $K$  as  $(f - b_0)^\dagger$ . By replacing  $f$ ,  $g$ , and  $b_0$  with  $-f$ ,  $-g$ , and  $-b_0$  if necessary, we may assume that  $f > b_0$ , so  $g > b_0$ . First, suppose that we have  $h \in K$  with  $(f - b_0)^\dagger < h$  and  $h < (g - b_0)^\dagger$ . Take  $\phi \in K^>$  with  $h = \phi^\dagger$  and set  $s := (f - b_0)/\phi$ . Then we have  $s > 0$  and  $s^\dagger = (f - b_0)^\dagger - h < 0$ . By Lemma 6.1(i),  $v_L(s) \geq 0$ , but since  $v_L(f - b_0) \notin \Gamma$ , we get  $v_L(s) > 0$ ; in particular,  $0 < s < 1$  (see [ADH17a, Lemma 3.5.11]). Similarly,  $h < (g - b_0)^\dagger$  gives  $t := (g - b_0)/\phi > 0$  and  $t^\dagger > 0$ , so  $v_M(t) < 0$ ; in particular,  $t > 1$ . Putting this together yields

$$f = b_0 + \phi s < b_0 + \phi \quad \text{and} \quad b_0 + \phi < b_0 + \phi t = g,$$

contradicting that  $f$  and  $g$  realize the same cut in  $K$ . The other case, that there is  $h \in K$  with  $(f - b_0)^\dagger > h$  and  $h > (g - b_0)^\dagger$ , is handled in the same fashion.  $\square$

**Proposition 6.26.** *Suppose that  $M$  is a pre- $H$ -field extension of  $K$  with gap 0 and  $g \in M$  realizes the same cut in  $K$  as  $f$ . Then there exists an embedding  $K\langle f \rangle \rightarrow M$  over  $K$  with  $f \mapsto g$ .*

PROOF. Define  $g_0 := g$  and  $g_{n+1} := (g_n - b_n)^\dagger$  for all  $n$ , so by the previous lemma  $g_n \in M \setminus K$  realizes the same cut in  $K$  as  $f_n$ , and in particular  $v_M(g_n - b_n) \notin \Gamma$  for all  $n$ . Then using the same argument as in the proof of Lemma 6.21, we have that  $v_M(g_0 - b_0), v_M(g_1 - b_1), \dots$  are  $\mathbb{Q}$ -linearly independent over  $\Gamma$ . Set  $\beta_n^* := v_M(g_n - b_n) - \alpha_{n+1}$  and  $\mathbf{m}_n^* := (g_n - b_n)/a_{n+1}$ , so  $v_M(\mathbf{m}_n^*) = \beta_n^*$  and the family  $(\beta_n^*)$  is  $\mathbb{Q}$ -linearly independent over  $\Gamma$ . Note that since  $f_n$  and  $g_n$  realize the same cut in  $K$ , so do  $\mathbf{m}_n$  and  $\mathbf{m}_n^*$ , and hence  $\beta_n$  and  $\beta_n^*$  realize the same cut in  $\Gamma$ . From the proof of Lemma 6.24 we have  $F_n(Y_0, \dots, Y_n) \in K[Y_0, \dots, Y_n]$  with  $\deg F_n \leq n+1$  and  $g^{(n)} = F_n(\mathbf{m}_0^*, \dots, \mathbf{m}_n^*)$ . For  $P \in K\{Y\}^\neq$  of order at most  $r$  we thus get  $P(g) = \sum_{i \in I} a_i \mathbf{m}_0^{*i_0} \dots \mathbf{m}_r^{*i_r}$ , where  $I$  is the same nonempty finite index set and  $a_i$  are the same coefficients as in the proof of Lemma 6.24. Since the



family  $(\beta_n^*)$  is  $\mathbb{Q}$ -linearly independent over  $\Gamma$ , we have that  $v_M(P(g)) \in \Gamma + \sum_n \mathbb{N}\beta_n^*$ . The rest of the proof of Lemma 6.24 now goes through replacing  $f_n$  with  $g_n$  and  $\beta_n$  with  $\beta_n^*$ .

From this we obtain an ordered abelian group isomorphism  $j: \Gamma_{K\langle f \rangle} \rightarrow \Gamma_{K\langle g \rangle}$  over  $\Gamma$  with  $\beta_n \mapsto \beta_n^*$ . Using the expressions for  $P(f)$  and  $P(g)$ , we get  $j(v_L(P(f))) = v_M(P(g))$  for all  $P \in K\{Y\}^\neq$ , so we have a valued differential field embedding  $K\langle f \rangle \rightarrow M$  over  $K$  with  $f \mapsto g$ . By the above and since  $\mathfrak{m}_n$  and  $\mathfrak{m}_n^*$  have the same sign,  $P(f) > 0 \iff P(g) > 0$  for all  $P \in K\{Y\}^\neq$ , so this is in fact an ordered valued differential field embedding, as desired.  $\square$

**6.5.3. The non-cofinal case.** In the previous subsection we assumed that  $\Gamma^<$  was cofinal in  $\Gamma_{K\langle f \rangle}^<$ , and now we turn to the other case. In this subsection,  $K$  and  $L$  are pre- $H$ -fields with gap 0 and  $L$  is an extension of  $K$ .

**Lemma 6.27.** *Let  $f \in L^>$  with  $\Gamma^< < v_L(f) < 0$ . Suppose that  $M$  is a pre- $H$ -field extension of  $K$  with gap 0 and  $g \in M^>$  satisfies  $\Gamma^< < v_M(g) < 0$ . Then there is an embedding  $K\langle f \rangle \rightarrow M$  over  $K$  with  $f \mapsto g$ .*

PROOF. Set  $f_0 := f$  and  $f_{n+1} := f_n^\dagger$ , and let  $\beta_n := v_L(f_n) \in \Gamma_L$ . By [ADH17a, Lemma 9.2.10(iv)],

$$[\Gamma^\neq] > [\beta_0] > [\beta_1] > [\beta_2] > \dots > [0].$$

In particular,  $[\beta_n] \notin [\Gamma]$  for all  $n$  and the family  $(\beta_n)$  is  $\mathbb{Q}$ -linearly independent over  $\Gamma$ . Hence the vector space  $\mathbb{Q}\Gamma_{K\langle f \rangle}/\Gamma$  is infinite dimensional, so  $f$  is d-transcendental over  $K$  [ADH17a, Corollary 3.1.11]. By the same argument as in Lemma 6.24 with  $f_n$  in place of  $\mathfrak{m}_n$  (i.e., with  $b_n = 0$  and  $a_n = 1$ ), one shows that for any  $P \in K\{Y\}^\neq$  of order at most  $r$ , we have  $P(f) = \sum_{\mathbf{i} \in I} a_{\mathbf{i}} f_0^{i_0} \dots f_r^{i_r}$ , where  $I$  is a nonempty finite set of indices  $\mathbf{i} = (i_0, \dots, i_r) \in \mathbb{N}^{1+r}$ . In particular,  $\Gamma_{K\langle f \rangle} = \Gamma \oplus \bigoplus_n \mathbb{Z}\beta_n$ .

Set  $g_0 := g$ ,  $g_{n+1} := g_n^\dagger$ , and  $\beta_n^* := v_M(g_n) \in \Gamma_M$ . The same argument yields that  $g$  is d-transcendental over  $K$  and  $P(g) = \sum_{\mathbf{i} \in I} a_{\mathbf{i}} g_0^{i_0} \dots g_r^{i_r}$ , where  $I$  is the same set of indices as in  $P(f)$  and  $a_{\mathbf{i}}$  are the same coefficients. Hence  $\Gamma_{K\langle g \rangle} = \Gamma \oplus \bigoplus_n \mathbb{Z}\beta_n^*$ . Thus we have an isomorphism of ordered abelian groups  $j: \Gamma_{K\langle f \rangle} \rightarrow \Gamma_{K\langle g \rangle}$  with  $\beta_n \mapsto \beta_n^*$ . By the expressions for  $P(f)$  and  $P(g)$ ,  $j(v_L(P(f))) = v_M(P(g))$ , which yields a valued differential field embedding from  $K\langle f \rangle \rightarrow M$  over  $K$  with  $f \mapsto g$ . To see that this is an ordered valued differential field embedding, note that by Lemma 6.1(i),  $f_n > 0$  and  $g_n > 0$  for all  $n$ , so  $P(f) > 0 \iff P(g) > 0$ .  $\square$

## 6.6. Extending the constant field

**Assumption.** In this section,  $K$  is asymptotic with small derivation.

Since  $C \subseteq \mathcal{O}$ ,  $C$  maps injectively into  $\mathbf{k}$  under the residue field map, and hence into  $C_{\mathbf{k}}$ . We say that  $K$  is *residue constant closed* if  $K$  is henselian and  $C$  maps onto  $C_{\mathbf{k}}$ , that is,  $\text{res}(C) = C_{\mathbf{k}}$ . We say that  $L$  is a *residue constant closure* of  $K$  if it is a residue constant closed  $H$ -asymptotic extension of  $K$  with small derivation that embeds into every residue constant closed  $H$ -asymptotic extension  $M$  of  $K$  with small derivation.

**Proposition 6.28.** *Suppose that  $K$  is pre-d-valued of  $H$ -type with  $\sup \Psi = 0$ . Then  $K$  has a residue constant closure that is an immediate extension of  $K$ .*

PROOF. Recall from §2.4 that  $\sup \Psi = 0$  is equivalent to  $(\Gamma^>)' = \Gamma^>$ . Also note that if  $L$  is an immediate asymptotic extension of  $K$ , then it is  $H$ -asymptotic, satisfies  $\Psi = \Psi_L$ , so  $\sup \Psi_L = 0$ , and is pre-d-valued by [ADH17a, Corollary 10.1.17].

Build a tower of immediate asymptotic extensions of  $K$  as follows. Set  $K_0 := K$ . If  $K_\lambda$  is not henselian, set  $K_{\lambda+1} := K_\lambda^h$ , the henselization of  $K_\lambda$ , which as an algebraic extension of  $K_\lambda$  is asymptotic by Proposition 2.17. If  $K_\lambda$  is residue constant closed, we are done. So suppose that  $K_\lambda$  is henselian but not residue constant closed and take  $u \in K_\lambda$  with  $u \asymp 1$ ,  $u' \prec 1$ , and  $u' \notin \partial \mathcal{O}_{K_\lambda}$ . Let  $y$  be transcendental over  $K_\lambda$  and equip  $K_{\lambda+1} := K_\lambda(y)$  with the unique derivation extending that of  $K_\lambda$  such that  $y' = u'$ . Then by [ADH17a, Lemma 10.2.5(iii)]  $\{v(u' - a') : a \in \mathcal{O}_{K_\lambda}\}$  has no maximum, so by [ADH17a, Lemma 10.2.4] we can equip  $K_{\lambda+1}$  with the unique valuation making it an  $H$ -asymptotic extension of  $K_\lambda$  with  $y \not\asymp 1$ ; with this valuation,  $y \prec 1$  and  $K_{\lambda+1}$  is an immediate extension of  $K_\lambda$ . If  $\lambda$  is a limit ordinal, set  $K_\lambda := \bigcup_{\rho < \lambda} K_\rho$ . Since each extension is immediate, by Zorn's lemma we may take a maximal such tower  $(K_\lambda)_{\lambda \leq \mu}$ .

It is clear that  $K_\mu$  is residue constant closed, and we show that it also has the desired semi-universal property. Let  $M$  be an  $H$ -asymptotic extension of  $K$  with small derivation that is residue constant closed, and let  $\lambda < \mu$  and  $i: K_\lambda \rightarrow M$  be an embedding. It suffices by induction to extend  $i$  to an embedding  $K_{\lambda+1} \rightarrow M$ . If  $K_{\lambda+1} = K_\lambda^h$ , then we use the universal property of henselizations. Now suppose that  $K_{\lambda+1} = K_\lambda(y)$  with  $y$  and  $u$  as above. Take  $c \in C_M$  with  $c \sim i(u)$  and set  $z := i(u) - c$ . Then  $z' = i(u)'$  and  $z \prec 1$ , so by the remarks after [ADH17a, Lemma 10.2.4],  $z$  is transcendental over  $i(K_\lambda)$ , and thus mapping  $y \mapsto z$  yields a differential field embedding  $K_{\lambda+1} \rightarrow M$  extending  $i$ . By the uniqueness of [ADH17a, Lemma 10.2.4], this is a valued differential field embedding.  $\square$

Note that if  $K$  is a pre- $H$ -field with  $\sup \Psi = 0$ , then as an immediate extension of  $K$  any residue constant closure of  $K$  embeds (as an *ordered* valued differential field) into every residue constant closed pre- $H$ -field extension of  $K$  with small derivation by Lemma 6.2.

In fact, residue constant closures are unique. To see this, we say that an asymptotic extension  $L$  of  $K$  with small derivation is a *residue constant extension* (*rc-extension* for short) if, for every  $a \in L$ , there are  $t_1, \dots, t_n \in L^\times$  such that  $a \in K(t_1, \dots, t_n)$  and, for  $i = 1, \dots, n$ , either  $t_i$  is algebraic over  $K(t_1, \dots, t_{i-1})$  or  $t_i' = 0$  and  $t_i \sim u$  for some  $u \in K(t_1, \dots, t_{i-1})$  with  $u \asymp 1$ . It is routine to prove the following.

**Lemma 6.29.** *Let  $K \subseteq L \subseteq M$  be a chain of asymptotic extensions with small derivation.*

- (i) *If  $M$  is an rc-extension of  $L$  and  $L$  is an rc-extension of  $K$ , then  $M$  is an rc-extension of  $K$ .*
- (ii) *If  $M$  is an rc-extension of  $K$ , then  $M$  is an rc-extension of  $L$ .*

**Lemma 6.30.** *If  $K$  is pre-d-valued of  $H$ -type with  $\sup \Psi = 0$ , then the residue constant closure from the proof of Proposition 6.28 is an rc-extension of  $K$ .*

**Lemma 6.31.** *Suppose that  $K$  is residue constant closed. Then  $K$  has no proper immediate rc-extension.*

PROOF. Let  $L$  be an immediate rc-extension of  $K$  and  $t \in L$ . Since  $K$  is henselian, it is algebraically maximal, and thus if  $t$  is algebraic over  $K$ , then  $t \in K$ . Suppose instead that  $t' = 0$  and  $t \sim u$  with  $u \in K$ ,  $u \succ 1$ . Thus  $u' \prec 1$ , so we have  $c \in C$  with  $c \sim u$ . But then  $t \sim c$ , so  $t = c \in K$ .  $\square$

**Corollary 6.32.** *Suppose that  $K$  is pre-d-valued of  $H$ -type with  $\sup \Psi = 0$ . Then any two residue constant closures of  $K$  are isomorphic over  $K$ .*

PROOF. Let  $L_0$  be the residue constant closure of  $K$  from the proof of Proposition 6.28. Let  $L_1$  be another residue constant closure of  $K$  and take an embedding  $i: L_1 \rightarrow L_0$  over  $K$ . Since  $L_0$  is an immediate rc-extension of  $K$ , and thus of  $i(L_1)$ , we have  $L_0 = i(L_1)$  and hence  $L_0 \cong L_1$  over  $K$ .  $\square$

**Lemma 6.33.** *Suppose that  $K$  is residue constant closed. Then the algebraic closure  $K^{\text{ac}}$  of  $K$  is residue constant closed. If  $K$  is additionally an ordered field with convex valuation ring, then  $K^{\text{rc}}$  is residue constant closed.*

PROOF. First, note that an algebraic extension of a henselian valued field is henselian [ADH17a, Corollary 3.3.12]. Let  $u \in K^{\text{ac}}$  with  $u \succ 1$  and  $u' \prec 1$ ; we need to show that there is  $c \in C_{K^{\text{ac}}} = C^{\text{ac}}$  with  $c \sim u$ . We have that  $\bar{u} \in C_{\text{res}(K^{\text{ac}})}$  is algebraic over  $\text{res}(K)$ , so it is algebraic over  $C_{\text{res}(K)}$  [ADH17a, Lemma 4.1.2]. Take a monic  $P \in C[X]$ , say of degree  $n$ , such that  $\bar{P} \in C_{\text{res}(K)}[X]$  is the minimum polynomial of  $\bar{u}$  over  $C_{\text{res}(K)}$ . Then  $P = \prod_{i=1}^n (X - c_i)$  with  $c_1, \dots, c_n \in C^{\text{ac}}$ , hence  $\bar{P} = \prod_{i=1}^n (X - \bar{c}_i)$ , so we have  $i$  with  $1 \leq i \leq n$  and  $\bar{u} = \bar{c}_i$ , and thus  $u \sim c_i$ . The second statement is proved similarly.  $\square$

## 6.7. Differential-Hensel-Liouville closures

In this section we construct differential-Hensel-Liouville closures (Theorem 6.49) in analogy with the Newton-Liouville closures of [ADH17a, §14.5] and prove that they are unique (Corollary 6.51). First we construct extensions that are real closed, have exponential integration, and satisfy an embedding property (Corollary 6.44), in analogy with the Liouville closures of [ADH17a, §10.6]; some preliminaries are adapted from [ADH17a, §10.4–10.6]. Combining this with the residue constant closures from the previous section, we construct extensions that are residue constant closed, are real closed, have exponential integration, and satisfy an embedding property (Corollary 6.48); such extensions are unique.

**Assumption.** In this section,  $K$  is a pre- $H$ -field.

**6.7.1. Adjoining exponential integrals.** Suppose that  $s \in K \setminus (K^\times)^\dagger$  and  $f$  is transcendental over  $K$ . We give  $K(f)$  the unique derivation extending that of  $K$  with  $f^\dagger = s$ . In the first lemma,  $K$  need only be an ordered differential field.

**Lemma 6.34.** *If  $K$  is real closed and  $K(f)$  can be ordered making it an ordered field extension of  $K$ , then  $C_{K(f)} = C$ .*

PROOF. This follows from [ADH17a, Lemma 4.6.11 and Corollary 4.6.12].  $\square$

In the next two lemmas,  $K$  is just a valued differential field, and need not be ordered. The first is based on [ADH17a, Lemma 10.4.2].

**Lemma 6.35.** *Suppose that  $K$  has small derivation and  $\mathbf{k} = (\mathbf{k}^\times)^\dagger$ . Let  $K(f)$  have a valuation that makes it an extension of  $K$  with  $\Gamma_{K(f)} = \Gamma$  and  $\partial\mathcal{O}_{K(f)} \subseteq \mathcal{O}_{K(f)}$ . Then  $s - a^\dagger \prec 1$  for some  $a \in K^\times$ .*

PROOF. Since  $vf \in \Gamma$ , there is  $b \in K^\times$  with  $g := f/b \asymp 1$ . Then  $s - b^\dagger = g^\dagger \asymp g' \asymp 1$ . If  $s - b^\dagger \prec 1$ , set  $a := b$ . If  $s - b^\dagger \asymp 1$ , since  $\mathbf{k} = (\mathbf{k}^\times)^\dagger$ , we have  $u \in K^\times$  with  $s - b^\dagger \sim u^\dagger$ . Then set  $a := bu$ .  $\square$

The last part of the argument also yields the following useful fact.

**Lemma 6.36.** *Suppose that  $K$  has small derivation and  $\mathbf{k} = (\mathbf{k}^\times)^\dagger$ . If  $s - a^\dagger \asymp 1$  for all  $a \in K^\times$ , then  $s - a^\dagger \succ 1$  for all  $a \in K^\times$ .*

Now we return to the situation that  $K$  is a pre- $H$ -field.

**Lemma 6.37** ([ADH17a, Lemma 10.5.18]). *Suppose that  $K$  is henselian and  $vs \in (\Gamma^>)'$ . Then there is a unique valuation on  $K(f)$  making it an  $H$ -asymptotic extension of  $K$  with  $f \sim 1$ . With this valuation,  $K(f)$  is an immediate extension of  $K$ , so there is a unique ordering of  $K(f)$  making it a pre- $H$ -field extension of  $K$  by Lemma 6.2.*

Here is a pre- $H$ -field version of [ADH17a, Lemma 10.5.20] with the same proof.

**Lemma 6.38.** *Suppose that  $K$  is real closed,  $s < 0$ , and  $v(s - a^\dagger) \in \Psi^\downarrow$  for all  $a \in K^\times$ . Then there is a unique pair of a field ordering and a valuation on  $L := K(f)$  making it a pre- $H$ -field extension of  $K$  with  $f > 0$ . Moreover, we have:*

- (i)  $vf \notin \Gamma$ ,  $\Gamma_L = \Gamma \oplus \mathbb{Z}vf$ ,  $f \prec 1$ ;
- (ii)  $\Psi$  is cofinal in  $\Psi_L := \psi_L(\Gamma_L^\neq)$ ;
- (iii) a gap in  $K$  remains a gap in  $L$ ;
- (iv) if  $L$  has a gap not in  $\Gamma$ , then  $[\Gamma_L] = [\Gamma]$ ;
- (v)  $\mathbf{k}_L = \mathbf{k}$ .

**6.7.2. Exponential integration extensions.** Let  $E$  be a differential field. We call a differential field extension  $F$  of  $E$  an *exponential integration extension* of  $E$  (*expint-extension* for short) if  $C_F$  is algebraic over  $C_E$  and for every  $a \in F$  there are  $t_1, \dots, t_n \in F^\times$  with  $a \in E(t_1, \dots, t_n)$  such that for  $i = 1, \dots, n$ , either  $t_i$  is algebraic over  $E(t_1, \dots, t_{i-1})$  or  $t_i^\dagger \in E(t_1, \dots, t_{i-1})$ . In particular, any expint-extension is d-algebraic. The following is routine.

**Lemma 6.39.** *Let  $E \subseteq F \subseteq M$  be a chain of differential field extensions.*

- (i) *If  $M$  is an expint-extension of  $E$ , then  $M$  is an expint-extension of  $F$ .*
- (ii) *If  $M$  is an expint-extension of  $F$  and  $F$  is an expint-extension of  $E$ , then  $M$  is an expint-extension of  $E$ .*

Minor modifications to the proof of [ADH17a, Lemma 10.6.8] yield the following.

**Lemma 6.40.** *If  $F$  is an expint-extension of  $E$ , then  $|F| = |E|$ .*

**6.7.3. Exponential integration closures.** We call  $K$  *exponential integration closed* (*expint-closed* for short) if it is real closed and it has exponential integration. We say a pre- $H$ -field extension  $L$  of  $K$  is an *exponential integration closure* (*expint-closure* for short) of  $K$  if it is an expint-extension of  $K$  that is expint-closed. In particular, an expint-closure is a d-algebraic extension.

The next observation has the same proof as [ADH17a, Lemma 10.6.9].

**Lemma 6.41.** *If  $K$  is expint-closed, then  $K$  has no proper expint-extension with the same constants.*

**Assumption.** For the rest of this subsection, suppose that  $K$  has gap 0.

From this assumption it follows that  $(\Gamma^>)' = \Gamma^>$  and  $\Psi^\downarrow = \Gamma^<$ . Recall how we construe the real closure of  $K$  as a pre- $H$ -field extension of  $K$  with gap 0.

**Definition.** We call a strictly increasing chain  $(K_\lambda)_{\lambda \leq \mu}$  of pre- $H$ -fields with gap 0 an *expint-tower* on  $K$  if:

- (i)  $K_0 = K$ ;
- (ii) if  $\lambda$  is a limit ordinal, then  $K_\lambda = \bigcup_{\rho < \lambda} K_\rho$ ;
- (iii) if  $\lambda < \lambda + 1 \leq \mu$ , then either:
  - (a)  $K_\lambda$  is not real closed and  $K_{\lambda+1}$  is the real closure of  $K_\lambda$ ; or
  - (b)  $K_\lambda$  is real closed and  $K_{\lambda+1} = K_\lambda(y_\lambda)$  with  $y_\lambda \notin K_\lambda$  satisfying either:
    - (b1)  $y_\lambda^\dagger = s_\lambda \in K_\lambda$  with  $y_\lambda \sim 1$ ,  $s_\lambda \prec 1$ , and  $s_\lambda \neq a^\dagger$  for all  $a \in K_\lambda^\times$ ; or
    - (b2)  $y_\lambda^\dagger = s_\lambda \in K_\lambda$  with  $s_\lambda < 0$ ,  $y_\lambda > 0$ , and  $s_\lambda - a^\dagger \succ 1$  for all  $a \in K_\lambda^\times$ .

We call  $K_\mu$  the *top* of such a tower.

For notational convenience in the next lemma, we set  $C_\lambda := C_{K_\lambda}$  and  $\mathbf{k}_\lambda := \mathbf{k}_{K_\lambda}$ .

**Lemma 6.42.** *Let an expint-tower  $(K_\lambda)_{\lambda \leq \mu}$  on  $K$  be given. Then:*

- (i)  $K_\mu$  is an expint-extension of  $K$ ;
- (ii)  $C_\mu$  is the real closure of  $C$  if  $\mu > 0$ ;
- (iii)  $\mathbf{k}_\mu$  is the real closure of  $\mathbf{k}$  if  $\mu > 0$ ;
- (iv)  $|K_\lambda| = |K|$ , hence  $\mu < |K|^+$ .

PROOF. For (i), go by induction on  $\lambda \leq \mu$ . The main thing to check is the condition on the constant fields. If  $\lambda = 0$  or  $\lambda$  is a limit ordinal, this is clear. If  $K_{\lambda+1}$  is the real closure of  $K_\lambda$ , then  $C_{\lambda+1}$  is the real closure of  $C_\lambda$ . If  $K_\lambda$  is real closed and  $K_{\lambda+1}$  is as in (b) above, then  $C_{\lambda+1} = C_\lambda$  by Lemma 6.34.

For (ii),  $C_1$  is the real closure of  $C$ , and then  $C_\lambda = C_1$  for all  $\lambda \geq 1$  as in the proof of (i).

For (iii),  $\mathbf{k}_1$  is the real closure of  $\mathbf{k}$ , and then  $\mathbf{k}_\lambda = \mathbf{k}_1$  for all  $\lambda \geq 1$  by the uniqueness of Lemma 6.37 and Lemma 6.38.

Finally, (iv) follows from (i) and Lemma 6.40. □

**Lemma 6.43.** *Suppose that  $\mathbf{k}$  is real closed and has exponential integration. Let  $L$  be the top of a maximal expint-tower on  $K$ . Then  $L$  is expint-closed, and hence an expint-closure of  $K$ .*

PROOF. Suppose that  $L$  is not expint-closed. If  $L$  is not real closed, then its real closure is a proper pre- $H$ -field extension of  $L$  with gap 0, so we could extend the expint-tower. We are left with the

case that  $L$  is real closed and we have  $s \in L \setminus (L^\times)^\dagger$ . In particular,  $L$  is henselian and  $\Gamma$  is divisible. Take  $f$  transcendental over  $L$  with  $f^\dagger = s$ . By replacing  $f$  with  $f^{-1}$  if necessary, we may assume that  $s < 0$ .

First suppose that  $s - a^\dagger < 1$  for some  $a \in L^\times$ . Then taking such an  $a$  and replacing  $f$  and  $s$  by  $f/a$  and  $s - a^\dagger$ , we arrange that  $s < 1$ . Giving  $L(f)$  the valuation and ordering from Lemma 6.37 makes it a pre- $H$ -field extension of  $L$  with gap 0 of type (b1).

Now suppose that  $s - a^\dagger \succ 1$  for all  $a \in L^\times$ . By Lemma 6.36,  $s - a^\dagger \succ 1$  for all  $a \in L^\times$ . Then giving  $L(f)$  the ordering and valuation from Lemma 6.38 makes it a pre- $H$ -field extension of  $L$  with gap 0 of type (b2).

Thus  $L$  is expint-closed, and hence an expint-closure of  $K$  by Lemma 6.42(i).  $\square$

**Corollary 6.44.** *Suppose that  $\mathbf{k}$  is real closed and has exponential integration. Then  $K$  has an expint-closure with gap 0. Moreover, if  $K$  is residue constant closed, then  $K$  has a residue constant closed expint-closure with gap 0.*

PROOF. By Lemma 6.42(iv), Zorn gives a maximal expint-tower on  $K$ . The first statement follows from Lemma 6.43, the second from Lemmas 6.33 and 6.42.  $\square$

**Lemma 6.45.** *Suppose that  $\mathbf{k}$  is real closed and has exponential integration, and let  $M$  be a residue constant closed, expint-closed pre- $H$ -field extension of  $K$  with gap 0. Suppose that  $(K_\lambda)_{\lambda \leq \mu}$  is an expint-tower on  $K$  in  $M$  (i.e., each  $K_\lambda$  is a pre- $H$ -subfield of  $M$ ) and maximal in  $M$  (i.e., it cannot be extended to an expint-tower  $(K_\lambda)_{\lambda \leq \mu+1}$  on  $K$  in  $M$ ). Then  $(K_\lambda)_{\lambda \leq \mu}$  is a maximal expint-tower on  $K$ .*

PROOF. Note that since  $\mathbf{k}$  is real closed,  $\mathbf{k}_\mu = \mathbf{k}$ , and hence has exponential integration. Since  $M$  is real closed,  $K_\mu$  must be real closed by maximality in  $M$ . So supposing  $(K_\lambda)_{\lambda \leq \mu}$  is not a maximal expint-tower on  $K$ , there is  $s_\mu \in K_\mu$  such that  $s_\mu \neq a^\dagger$  for all  $a \in K_\mu^\times$ ; we may assume that  $s_\mu < 0$ . Since  $M$  is expint-closed, there is  $y_\mu \in M$  with  $y_\mu^\dagger = s_\mu$ ; we may assume that  $y_\mu > 0$ .

First suppose that  $s_\mu - a^\dagger \succ 1$  for all  $a \in K_\mu^\times$ , so actually  $s_\mu - a^\dagger \succ 1$  for all  $a \in K_\mu^\times$  by Lemma 6.36. Thus setting  $K_{\mu+1} := K_\mu(y_\mu)$  yields an extension of  $(K_\lambda)_{\lambda \leq \mu}$  in  $M$  of type (b2).

Now suppose that  $s_\mu - a^\dagger < 1$  for some  $a \in K_\mu^\times$ . Taking such an  $a$  and replacing  $s_\mu$  and  $y_\mu$  by  $s_\mu - a^\dagger$  and  $y_\mu/a$ , we may assume that  $s_\mu < 1$ . Since  $M$  has gap 0, we have  $y_\mu \asymp 1$  and so  $y'_\mu \asymp s_\mu < 1$ . That is,  $\overline{y_\mu} \in C_{\text{res}(M)}$ , so we have  $c \in C_M$  with  $y_\mu \sim c$ . Replacing  $y_\mu$  by  $y_\mu/c$ , we obtain the desired extension of  $(K_\lambda)_{\lambda \leq \mu}$  in  $M$  of type (b1).  $\square$

This is not used later, but in the above lemma, we can replace the assumption that  $M$  is residue constant closed (so  $C_{\text{res}(M)} = \text{res}(C_M)$ ) with  $C_{\text{res}(M)} = C_{\text{res}(K)}$ . In the final argument, instead of  $c \in C_M$  we have  $u \in K$  with  $u \asymp 1$  and  $u' < 1$ , so we replace  $s_\mu$  with  $s_\mu - u^\dagger$ .

**Corollary 6.46.** *Suppose that  $L$  is an expint-closed pre- $H$ -field extension  $L$  of  $K$ .*

- (i) *If  $L$  is an expint-closure of  $K$ , then no proper differential subfield of  $L$  containing  $K$  is expint-closed.*
- (ii) *Suppose that  $\mathbf{k}$  is real closed and has exponential integration, and that  $L$  has gap 0 and is residue constant closed. If no proper differential subfield of  $L$  containing  $K$  is expint-closed, then  $L$  is an expint-closure of  $K$ .*

PROOF. For (i), if  $L$  is an expint-closure of  $K$ , then no proper differential subfield of  $L$  containing  $K$  is expint-closed by Lemmas 6.39 and 6.41.

For (ii), suppose that no proper differential subfield of  $L$  containing  $K$  is expint-closed. Take an expint-tower on  $K$  in  $L$  that is maximal in  $L$ . By Lemma 6.45, it is a maximal expint-tower on  $K$ . By Lemma 6.43, the top of this tower is an expint-closure of  $K$ , and hence equal to  $L$ .  $\square$

**Theorem 6.47.** *Suppose that  $\mathbf{k}$  is real closed and has exponential integration. Let  $(K_\lambda)_{\lambda \leq \mu}$  be an expint-tower on  $K$ . Then any embedding of  $K$  into a residue constant closed, expint-closed pre- $H$ -field extension  $M$  of  $K$  with gap 0 extends to an embedding of  $K_\mu$ . Moreover, if  $K$  is residue constant closed, then any two residue constant closed expint-closures of  $K$  with gap 0 are isomorphic over  $K$ .*

PROOF. Note that the second statement follows from the first by Corollary 6.46.

Let  $M$  be a residue constant closed, expint-closed pre- $H$ -field with gap 0 and suppose that we have an embedding  $K \rightarrow M$ . We prove that for  $\lambda < \mu$  any embedding  $K_\lambda \rightarrow M$  extends to an embedding  $K_{\lambda+1} \rightarrow M$ , which yields the result by induction. Suppose that  $i: K_\lambda \rightarrow M$  is an embedding. If  $K_{\lambda+1}$  is the real closure of  $K_\lambda$ , then we may extend  $i$  to  $K_{\lambda+1}$ .

So suppose that  $K_\lambda$  is real closed and we have  $s_\lambda \in K_\lambda$  and  $y_\lambda \in K_{\lambda+1} \setminus K_\lambda$  with  $K_{\lambda+1} = K_\lambda(y_\lambda)$ ,  $y_\lambda^\dagger = s_\lambda$ ,  $y_\lambda \sim 1$ ,  $s_\lambda < 1$ , and  $s_\lambda \neq a^\dagger$  for all  $a \in K_\lambda^\times$ . Take  $z \in M$  with  $z^\dagger = i(s_\lambda)$ . Hence  $z \asymp 1$  and  $\bar{z} \in C_{\text{res}(M)}$ , so we have  $c \in C_M$  with  $z \sim c$ . By the uniqueness of Lemma 6.37, we may extend  $i$  to an embedding of  $K_\lambda(y_\lambda)$  into  $M$  sending  $y_\lambda$  to  $z/c$ .

Now suppose that  $K_\lambda$  is real closed and we have  $s_\lambda \in K_\lambda$  and  $y_\lambda \in K_{\lambda+1} \setminus K_\lambda$  with  $K_{\lambda+1} = K_\lambda(y_\lambda)$ ,  $y_\lambda^\dagger = s_\lambda$ ,  $s_\lambda < 0$ ,  $y_\lambda > 0$ , and  $s_\lambda - a^\dagger \succ 1$  for all  $a \in K_\lambda^\times$ . Take  $z \in M$  with  $z^\dagger = i(s_\lambda)$ ; we may assume that  $z > 0$ . Then by the uniqueness of Lemma 6.38, we can extend  $i$  to an embedding of  $K_\lambda(y_\lambda)$  into  $M$  sending  $y_\lambda$  to  $z$ .  $\square$

Putting these results together yields the following.

**Corollary 6.48.** *Suppose that  $\mathbf{k}$  is real closed and has exponential integration. Then  $K$  has a pre- $H$ -field extension  $L$  with gap 0 such that:*

- (i)  *$L$  is a residue constant closed, expint-closed extension of  $K$ ;*
- (ii)  *$L$  embeds over  $K$  into any residue constant closed, expint-closed pre- $H$ -field extension of  $K$  with gap 0.*

*Moreover, any two such extensions of  $K$  are isomorphic over  $K$ .*

PROOF. By Proposition 6.28, let  $K_0$  be the residue constant closure of  $K$ . Taking the top of a maximal expint-tower on  $K_0$  yields an expint-closure  $L$  of  $K_0$ , which is residue constant closed by Lemma 6.33. Let  $M$  be a pre- $H$ -field extension of  $K$  with gap 0 that is residue constant closed and expint-closed. Then  $K_0$  embeds into  $M$  over  $K$ , so by Theorem 6.47 we can extend this to an embedding of  $L$ . The uniqueness of  $L$  follows from the embedding property by Corollary 6.46.  $\square$

Note that both rc-extensions and expint-extensions are Liouville extensions in the sense of [ADH17a, §10.6],<sup>1</sup> so the  $L$  from the result above is a Liouville extension of  $K$ .

<sup>1</sup>The definition is similar to that of expint-extensions in §6.7.2 except that we also allow  $t'_i \in E(t_1, \dots, t_{i-1})$ .

#### 6.7.4. Differential-Hensel-Liouville closures.

**Assumption.** We continue to assume in this subsection that  $K$  has gap 0.

**Definition.** We call a pre- $H$ -field extension  $L$  of  $K$  a *differential-Hensel-Liouville closure* (slightly shorter: *d-Hensel-Liouville closure*) of  $K$  if it is d-henselian and expint-closed, and embeds over  $K$  into every pre- $H$ -field extension  $M$  of  $K$  that is d-henselian and expint-closed.

To build them, we use the fact that if  $F$  is an asymptotic valued differential field with small derivation and linearly surjective differential residue field, then it has a (unique) d-henselization  $F^{\text{dh}}$  by Theorem 4.20. If  $F$  is a pre- $H$ -field, then  $F^{\text{dh}}$  is too and embeds (as an *ordered* valued differential field) into every d-henselian pre- $H$ -field extension of  $F$  by Lemma 6.2.

**Theorem 6.49.** *Suppose that  $\mathbf{k}$  is real closed, linearly surjective, and has exponential integration. Then  $K$  has a d-Hensel-Liouville closure.*

PROOF. We use below that any d-henselian asymptotic field is residue constant closed by [ADH17a, Lemma 9.4.10]. Define a sequence of pre- $H$ -field extensions of  $K$  with gap 0 as follows. Set  $K_0 := K$ . For  $n \geq 1$ , if  $n$  is odd, let  $K_n$  be the d-henselization of  $K_{n-1}$ , and if  $n$  is even, let  $K_n$  be the expint-closure of  $K_{n-1}$  from Corollary 6.44. Note that  $\mathbf{k}_{K_n} = \mathbf{k}$  for all  $n$ . We set  $L := \bigcup_n K_n$  and show that  $L$  is a d-Hensel-Liouville closure of  $K$ .

Let  $M$  be a pre- $H$ -field extension of  $K$  that is d-henselian and expint-closed. We show by induction on  $n$  that we can extend any embedding  $K_n \rightarrow M$  to an embedding  $K_{n+1} \rightarrow M$ . Suppose we have an embedding  $i: K_n \rightarrow M$ . If  $n$  is even, then  $K_{n+1}$  is the d-henselization of  $K_n$ , so we may extend  $i$  to  $K_{n+1}$ . If  $n$  is odd, then  $K_n$  is d-henselian and  $K_{n+1}$  is the expint-closure of  $K_n$ , so we can extend  $i$  to an embedding  $K_{n+1} \rightarrow M$  by Theorem 6.47.  $\square$

In the next two results, adapted from [ADH17a, §16.2], let  $K^{\text{dhl}}$  be the d-Hensel-Liouville closure of  $K$  from the previous theorem. Note that  $K^{\text{dhl}}$  is a d-algebraic extension of  $K$  with the same residue field. We show that  $K^{\text{dhl}}$  is the unique, up to isomorphism over  $K$ , d-Hensel-Liouville closure of  $K$ .

**Lemma 6.50.** *Suppose  $\mathbf{k}$  is real closed, linearly surjective, and has exponential integration. Let  $i: K^{\text{dhl}} \rightarrow L$  be an embedding into a pre- $H$ -field  $L$  with gap 0 such that  $\text{res}(i(K^{\text{dhl}})) = \text{res}(L)$ . Then*

$$i(K^{\text{dhl}}) = i(K)^{\text{dalg}} := \{f \in L : f \text{ is d-algebraic over } i(K)\}.$$

PROOF. That  $i(K^{\text{dhl}}) \subseteq i(K)^{\text{dalg}}$  is clear, since  $K^{\text{dhl}}$  is a d-algebraic extension of  $K$ . For the other direction, note that  $i(K^{\text{dhl}})$  is a d-henselian, expint-closed pre- $H$ -subfield of  $i(K)^{\text{dalg}}$ , so  $i(K^{\text{dhl}}) = i(K)^{\text{dalg}}$  by Theorem 6.22.  $\square$

Hence for  $K$  as in the lemma above, any d-algebraic extension of  $K$  that is a d-henselian, expint-closed pre- $H$ -field with the same residue field is isomorphic to  $K^{\text{dhl}}$  over  $K$ , and is thus a d-Hensel-Liouville closure of  $K$ .

**Corollary 6.51.** *Suppose that  $\mathbf{k}$  is real closed, linearly surjective, and has exponential integration. Then  $K^{\text{dhl}}$  does not have any proper differential subfield containing  $K$  that is d-henselian and expint-closed. Thus any d-Hensel-Liouville closure of  $K$  is isomorphic to  $K^{\text{dhl}}$  over  $K$ .*



PROOF. If  $L \supseteq K$  is a d-henselian, expint-closed differential subfield of  $K^{\text{dhl}}$ , then we have an embedding  $i: K^{\text{dhl}} \rightarrow L$  over  $K$ . Viewing this as an embedding into  $K^{\text{dhl}}$ , by Lemma 6.50 we have  $K^{\text{dhl}} = i(K^{\text{dhl}})$ , so  $K^{\text{dhl}} = L$ .

If instead  $L$  is any d-Hensel-Liouville closure of  $K$ , then by embedding it into  $K^{\text{dhl}}$  and using the minimality property just proved, we obtain an isomorphism  $L \cong K^{\text{dhl}}$  over  $K$ .  $\square$

## 6.8. Main results

**6.8.1. Quantifier elimination.** We now turn to the proof of quantifier elimination for the theory of d-henselian, real closed pre- $H$ -fields that have exponential integration and closed ordered differential residue field. The language for this and the other model-theoretic results of this subsection is the language  $\{+, -, \cdot, 0, 1, \partial, \preceq, \leq\}$  of ordered valued differential fields.

**Assumption.** In this section,  $K$  and  $L$  are pre- $H$ -fields with small derivation.

In the next results, for an ordered set  $S$  we denote the cofinality of  $S$  by  $\text{cf}(S)$ .

**Lemma 6.52.** *Suppose that  $K$  is d-henselian, real closed, and has exponential integration, and let  $E$  be a differential subfield of  $K$  with  $\mathbf{k}_E = \mathbf{k}$ . Suppose that  $L$  is d-henselian, real closed, and has exponential integration. Assume that  $L$  is  $|K|^+$ -saturated as an ordered set and  $\text{cf}(\Gamma_L^<) > |\Gamma|$ . Then any embedding  $i: E \rightarrow L$  can be extended to an embedding  $K \rightarrow L$ .*

PROOF. Let  $i: E \rightarrow L$  be an embedding. We may assume that  $E \neq K$ . It suffices to show that  $i$  can be extended to an embedding  $F \rightarrow L$  for some differential subfield  $F$  of  $K$  properly containing  $E$ .

First, suppose that  $\Gamma_E^<$  is not cofinal in  $\Gamma^<$  and let  $f \in K^>$  with  $\Gamma_E^< < vf < 0$ . By the cofinality assumption on  $\Gamma_L^<$ , take  $g \in L^>$  with  $\Gamma_{i(E)}^< < v_L(g) < 0$ . Then we extend  $i$  to an embedding  $E\langle f \rangle \rightarrow L$  sending  $f \mapsto g$  by Lemma 6.27.

Now suppose that  $\Gamma_E^<$  is cofinal in  $\Gamma^<$  and consider the following three cases.

*Case 1:  $E$  is not d-henselian and expint-closed.* From the assumptions on  $K$ ,  $\mathbf{k}$  is real closed, linearly surjective, and has exponential integration. Since  $\mathbf{k}_E = \mathbf{k}$ , we may extend  $i$  to an embedding of the d-Hensel-Liouville closure of  $E$  into  $L$  by Theorem 6.49.

*Case 2:  $E$  is d-henselian and expint-closed, and  $E\langle y \rangle$  is an immediate extension of  $E$  for some  $y \in K \setminus E$ .* Take such a  $y$  and let  $(a_\rho)$  be a divergent pc-sequence in  $K$  with  $a_\rho \rightsquigarrow y$ . Since  $E$  is d-henselian, it is d-algebraically maximal by Theorem 4.19, and so  $(a_\rho)$  is of d-transcendental type over  $E$ . By the saturation assumption on  $L$  and [ADH17a, Lemma 2.4.2], we have  $z \in L$  with  $i(a_\rho) \rightsquigarrow z$ . Then Lemma 2.12 yields a valued differential field embedding  $E\langle y \rangle \rightarrow L$  sending  $y \mapsto z$ ; by Lemma 6.2, this is also an ordered field embedding.

*Case 3:  $E$  is d-henselian and expint-closed, and there is no  $y \in K \setminus E$  making  $E\langle y \rangle$  an immediate extension of  $E$ .* Take any  $f \in K \setminus E$ . By saturation, take  $g \in L$  such that for all  $a \in E$ , we have

$$a < f \implies i(a) < g \quad \text{and} \quad f < a \implies g < i(a).$$

Then we can extend  $i$  to an embedding  $E\langle f \rangle \rightarrow L$  with  $f \mapsto g$  by Proposition 6.26.  $\square$

Recall from [Sin78] the theory of closed ordered differential fields, which has quantifier elimination and is the model completion of the theory of ordered differential fields (where no assumption is made on the interaction between the ordering and the derivation).

**Theorem 6.53.** *The theory of d-henselian, real closed pre- $H$ -fields with exponential integration and closed ordered differential residue field has quantifier elimination.*

PROOF. Suppose that  $K$  and  $L$  are d-henselian and real closed, have exponential integration, and have closed ordered differential residue field. Suppose further that  $L$  is  $|K|^+$ -saturated as an ordered set,  $\text{cf}(\Gamma_L^<) > |\Gamma|$ , and  $\mathbf{k}_L$  is  $|\mathbf{k}|^+$ -saturated as an ordered differential field. Let  $E$  be a substructure of  $K$ , so  $E$  is a differential subring of  $K$  with the induced dominance relation and ordering. By a standard quantifier elimination test (see for example [ADH17a, Corollary B.11.9]), it suffices to show that any embedding  $i: E \rightarrow L$  can be extended to an embedding  $K \rightarrow L$ , so let  $i: E \rightarrow L$  be an embedding.

By extending  $i$  to the fraction field of  $E$ , we may assume that  $E$  is a field. The embedding  $i$  induces an embedding  $i_{\text{res}}: \mathbf{k}_E \rightarrow \mathbf{k}_L$  of ordered differential fields. Since  $\mathbf{k}_L$  is  $|\mathbf{k}|^+$ -saturated, by the proof of quantifier elimination for closed ordered differential fields [Sin78] and Zorn's lemma we may extend  $i_{\text{res}}$  to an embedding  $\mathbf{k} \rightarrow \mathbf{k}_L$ . By Corollary 6.6, we can now extend  $i$  to an embedding  $F \rightarrow L$  for a differential subfield  $F$  of  $K$  with  $\mathbf{k}_F = \mathbf{k}$ . It remains to apply Lemma 6.52.  $\square$

**Lemma 6.54.** *Any pre- $H$ -field with gap 0 can be extended to a d-henselian, real closed pre- $H$ -field with exponential integration and closed ordered differential residue field.*

PROOF. Suppose we have a pre- $H$ -field  $K_0$  with gap 0. We first extend its residue field to a closed ordered differential field, since the theory of closed ordered differential fields is the model completion of the theory of ordered differential fields, and apply Corollary 6.6 to obtain a pre- $H$ -field extension  $K_1$  of  $K_0$  with gap 0 whose residue field is a closed ordered differential field. It follows from the definition that closed ordered differential fields are real closed, linearly surjective, and have exponential integration, so we can extend  $K_1$  to a pre- $H$ -field  $K_2$  with the same residue field that is d-henselian, real closed, and has exponential integration by Theorem 6.49.  $\square$

**Corollary 6.55.** *The theory of d-henselian, real closed pre- $H$ -fields with exponential integration and closed ordered differential residue field is the model completion of the theory of pre- $H$ -fields with gap 0.*

PROOF. This follows from Theorem 6.53 and Lemma 6.54 by standard model-theoretic facts (see for example [ADH17a, Corollary B.11.6]).  $\square$

**Corollary 6.56.** *The theory of d-henselian, real closed pre- $H$ -fields with exponential integration and closed ordered differential residue field is complete.*

PROOF. The structure  $(\mathbb{Z}; +, -, \cdot, 0, 1, \partial_0, \preceq_0, \leq)$ , where  $\partial_0$  is the trivial derivation ( $\partial_0(\mathbb{Z}) = \{0\}$ ) and  $\preceq_0$  is the trivial dominance relation ( $k \preceq_0 l$  for all  $k, l \in \mathbb{Z}^\neq$ ), embeds into every model of the theory in the statement, so the theory is complete (see for example [ADH17a, Corollary B.11.7]).  $\square$

We now use quantifier elimination to show that this theory is distal, a notion of model-theoretic tameness introduced by P. Simon to isolate those NIP theories that are “purely unstable” [Sim13]. The definition used here, one of several equivalent formulations, is in terms of indiscernible sequences; first, some conventions. We use the term “indiscernible” to mean “indiscernible over  $\emptyset$ ,” and if  $B$  is a parameter set, “ $B$ -indiscernible” to mean “indiscernible over  $B$ .” If  $I_0$  and  $I_1$  are linearly ordered

sets, then  $I_0 + I_1$  denotes the natural concatenation. The singleton  $\{\ell\}$  viewed as a linearly ordered set is denoted by  $(\ell)$ .

**Definition.** Let  $T$  be a complete theory in a language  $\mathcal{L}$ . Then  $T$  is *distal* if in every model  $\mathbf{M}$  of  $T$ , for any  $B \subseteq M$  and infinite linearly ordered sets  $I_0, I_1$ , whenever

- (i)  $(a_i)_{i \in I_0 + (\ell) + I_1}$  is indiscernible, and
- (ii)  $(a_i)_{i \in I_0 + I_1}$  is  $B$ -indiscernible,

$(a_i)_{i \in I_0 + (\ell) + I_1}$  is also  $B$ -indiscernible.

Recall that the theory RCVF of real closed fields with a nontrivial valuation whose valuation ring is convex, in the language  $\{+, -, \cdot, 0, 1, \preceq, \leq\}$ , is distal: It is weakly o-minimal by quantifier elimination [CD83], hence dp-minimal by [DGL11, Corollary 4.3], and thus distal by [Sim13, Lemma 2.10]. We reduce Theorem 6.57 to the distality of RCVF by “forgetting” the derivation.

**Theorem 6.57.** *The theory of d-henselian, real closed pre- $H$ -fields with exponential integration and closed ordered differential residue field is distal, and hence has NIP.*

PROOF. Let  $K$  be a d-henselian, real closed pre- $H$ -field with exponential integration and closed ordered differential residue field. Let  $I_0$  and  $I_1$  be infinite linearly ordered sets and  $B \subseteq K$ . Suppose that  $(a_i)_{i \in I_0 + (\ell) + I_1}$  is indiscernible with  $a_i \in K^d$ ,  $d \geq 1$ , and  $(a_i)_{i \in I_0 + I_1}$  is  $B$ -indiscernible. Let  $\varphi(x_1, \dots, x_k, y)$ ,  $k \in \mathbb{N}$ , be a formula with  $|x_1| = \dots = |x_k| = m$  and  $|y| = n$ . We need to show that for all  $b \in B^n$  and  $i_1, \dots, i_k, j_1, \dots, j_k \in I_0 + (\ell) + I_1$  with  $i_1 < \dots < i_k$  and  $j_1 < \dots < j_k$ ,

$$K \models \varphi(a_{i_1}, \dots, a_{i_k}, b) \leftrightarrow \varphi(a_{j_1}, \dots, a_{j_k}, b).$$

For simplicity of notation, we assume that  $d = k = m = n = 1$ . By quantifier elimination, any formula in the variable  $z = (z_1, \dots, z_t)$ ,  $t \in \mathbb{N}$ , is equivalent in  $K$  to a boolean combination of formulas of one of the following forms:

$$F(z) = 0, \quad F(z) > 0, \quad F(z) \preceq G(z) \quad \text{where } F, G \in \mathbb{Z}\{Z_1, \dots, Z_t\}.$$

In particular, there is a formula  $\psi(x_0, \dots, x_r, y_0, \dots, y_r)$  in the language  $\mathcal{L}_{\text{OR}, \preceq} := \{+, -, \cdot, 0, 1, \leq, \preceq\}$  such that, for all  $a, b \in K$ ,

$$K \models \varphi(a, b) \leftrightarrow \psi(a, a', \dots, a^{(r)}, b, b', \dots, b^{(r)}).$$

Let  $B_r := B \cup \partial(B) \cup \dots \cup \partial^r(B)$ . With respect to  $\mathcal{L}_{\text{OR}, \preceq}$ , the sequences  $(a_i, a'_i, \dots, a_i^{(r)})_{i \in I_0 + (\ell) + I_1}$  and  $(a_i, a'_i, \dots, a_i^{(r)})_{i \in I_0 + I_1}$  are indiscernible and  $B_r$ -indiscernible, respectively. As a structure in this language,  $K \models \text{RCVF}$ , and RCVF is distal. Thus for any  $b \in B$  and  $i, j \in I_0 + (\ell) + I_1$ ,

$$K \models \psi(a_i, a'_i, \dots, a_i^{(r)}, b, b', \dots, b^{(r)}) \leftrightarrow \psi(a_j, a'_j, \dots, a_j^{(r)}, b, b', \dots, b^{(r)}). \quad \square$$

**6.8.2. Model completeness with extra structure on the residue field.** In this final subsection, we consider a theory of pre- $H$ -fields with gap 0 where extra structure is allowed on the residue field and prove model completeness and model companion results similar to those in the previous subsection. More precisely, consider the two-sorted structure  $(K, \mathbf{k}; \pi)$ , where the language on the sort of  $K$  is  $\{+, -, \cdot, 0, 1, \partial, \preceq, \leq\}$ , the language on the sort of  $\mathbf{k}$  is  $\mathcal{L}_{\text{res}} \supseteq \{+, -, \cdot, 0, 1, \partial, \leq\}$ , and  $\pi$

is a map  $\pi: K \rightarrow \mathbf{k}$ . We fix an  $\mathcal{L}_{\text{res}}$ -theory  $T_{\text{res}}$  of ordered differential fields and let  $T$  be the theory asserting that:

- (i)  $K$  is a pre- $H$ -field with gap 0;
- (ii)  $\mathbf{k} \models T_{\text{res}}$ ;
- (iii)  $\pi|_{\mathcal{O}}$  is a surjective ordered differential ring homomorphism with kernel  $\mathcal{o}$  and  $\pi(K \setminus \mathcal{O}) = \{0\}$ .

Thus  $\pi$  induces an isomorphism of ordered differential fields  $\text{res}(K) \cong \mathbf{k}$ ; conversely, an isomorphism  $\text{res}(K) \cong \mathbf{k}$  lifts to a surjective ordered differential ring homomorphism  $\mathcal{O} \rightarrow \mathbf{k}$  with kernel  $\mathcal{o}$ .

Suppose that  $T_{\text{res}}^*$  is a model complete theory extending the theory of real closed fields and the theory of linearly surjective differential fields with exponential integration; these conditions are necessary if  $T_{\text{res}}^*$  is to be the theory of a residue field of a d-henselian, real closed pre- $H$ -field with exponential integration. Consider the two-sorted structure  $(K^*, \mathbf{k}^*; \pi^*)$  in the same language and let  $T^*$  be the theory asserting that:

- (i)  $K^*$  is a pre- $H$ -field that is d-henselian, real closed, and has exponential integration;
- (ii)  $\mathbf{k}^* \models T_{\text{res}}^*$ ;
- (iii)  $\pi^*: K^* \rightarrow \mathbf{k}^*$  is as in  $T$ .

**Theorem 6.58.** *The theory  $T^*$  is model complete.*

PROOF. Let  $(K, \mathbf{k}; \pi)$ ,  $(L, \mathbf{k}_L; \pi_L)$ , and  $(K^*, \mathbf{k}^*; \pi^*)$  be models of  $T^*$  such that  $(K, \mathbf{k}; \pi) \subseteq (L, \mathbf{k}_L; \pi_L)$  and  $(K^*, \mathbf{k}^*; \pi^*) \succ (K, \mathbf{k}; \pi)$  is  $|L|^+$ -saturated. Let  $i: (K, \mathbf{k}; \pi) \rightarrow (K^*, \mathbf{k}^*; \pi^*)$  be the natural inclusion map; it suffices to extend  $i$  to an embedding  $i^*: (L, \mathbf{k}_L; \pi_L) \rightarrow (K^*, \mathbf{k}^*; \pi^*)$  (see for example [ADH17a, Corollary B.10.4]).

Let  $i_{\text{res}}: \mathbf{k} \rightarrow \mathbf{k}^*$  be the restriction of  $i$  to  $\mathbf{k}$ . Since  $T_{\text{res}}^*$  is model complete and  $\mathbf{k}^* \succ \mathbf{k}$  is  $|\mathbf{k}_L|^+$ -saturated, we may extend  $i_{\text{res}}$  to an embedding  $i_{\text{res}}^*: \mathbf{k}_L \rightarrow \mathbf{k}^*$ . By pulling back  $i_{\text{res}}^*$  via  $\pi$  and  $\pi^*$  we obtain an embedding  $\text{res}(L) \rightarrow \text{res}(K^*)$ , so by Corollary 6.6 with  $\text{res}(L)$  instead of  $\mathbf{k}_L$  we have a pre- $H$ -field  $F \subseteq L$  with gap 0 extending  $K$  with residue field  $\text{res}(F) = \text{res}(L)$  that embeds into  $K^*$  over  $K$ . Now by Lemma 6.52, this embedding extends further to an embedding  $j: L \rightarrow K^*$ . Then the map  $i^*$  that is  $j$  on  $L$  and  $i_{\text{res}}^*$  on  $\mathbf{k}_L$  is an embedding  $(L, \mathbf{k}_L; \pi_L) \rightarrow (K^*, \mathbf{k}^*; \pi^*)$  extending  $i$ .  $\square$

In the next two results, we suppose that  $T_{\text{res}}^*$  is the model companion of  $T_{\text{res}}$ , so  $T^* \supseteq T$ .

**Lemma 6.59.** *Every model of  $T$  can be extended to a model of  $T^*$ .*

PROOF. Let  $(K, \mathbf{k}; \pi) \models T$ . Since  $T_{\text{res}}^*$  is the model companion of  $T_{\text{res}}$ , we can extend  $\mathbf{k}$  to a model  $\mathbf{k}^* \models T_{\text{res}}^*$ . Let  $\mathbf{k}_L$  be an ordered differential field extension of  $\text{res}(K)$  such that we have an isomorphism  $i: \mathbf{k}_L \rightarrow \mathbf{k}^*$  of ordered differential fields extending the isomorphism  $\text{res}(K) \cong \mathbf{k}$  induced by  $\pi$ . Then by applying Corollary 6.6 with  $\text{res}(K)$  instead of  $\mathbf{k}$ , we obtain an extension  $L$  of  $K$  that is a pre- $H$ -field with gap 0 and has ordered differential residue field isomorphic to  $\mathbf{k}_L$  over  $\text{res}(K)$ . By composing this isomorphism with  $i$ , we may assume that  $i$  is an isomorphism  $i: \text{res}(L) \rightarrow \mathbf{k}^*$ . By Theorem 6.49, we extend  $L$  to its d-Hensel-Liouville closure  $L^{\text{dhl}}$  with residue field  $\text{res}(L^{\text{dhl}}) = \text{res}(L)$ . Defining  $\pi^*: L^{\text{dhl}} \rightarrow \mathbf{k}^*$  by  $\pi^*(f) := i(\text{res } f)$  for  $f \in \mathcal{O}_{L^{\text{dhl}}}$  and  $\pi^*(f) = 0$  otherwise, we have that  $(L^{\text{dhl}}, \mathbf{k}^*; \pi^*) \models T^*$ .  $\square$

**Corollary 6.60.** *The theory  $T^*$  is the model companion of  $T$ .*

## CHAPTER 7

### Concluding remarks and questions

In the previous chapter, we studied the theory  $T^*$  of pre- $H$ -fields that are d-henselian, real closed, have exponential integration, and that have closed ordered differential residue field in parallel with the way that the theory  $T^{\text{nl}}$  is studied in [ADH17a]. But the main reason for isolating and studying that theory is because it has  $\mathbb{T}$  as a model. We know that models of  $T^*$  exist, as we can take any pre- $H$ -field with gap 0 and extend it to an existentially closed pre- $H$ -field with gap 0, which is a model of  $T^*$  by Corollary 6.55, but it is desirable to have a natural model of  $T^*$ . Since there is no known natural model of the theory of closed ordered differential fields, it is impossible to expect to find a natural model of  $T^*$ , but perhaps, given a closed ordered differential field  $\mathbf{k}$ , there is something like a transseries model of  $T^*$  with  $\mathbf{k}$  as its ordered differential residue field.

Theorem 6.53 shows that  $T^*$  has quantifier elimination, but of course this is just a starting point for studying this theory and its complexity. In particular, natural questions about the theories of the asymptotic couples and ordered differential residue fields of models of  $T^*$ , and their relationship to  $T^*$ , remain open. In Theorem 6.57, we showed that  $T^*$  is combinatorially tame in the sense that it is distal, and hence has NIP. It follows that since the asymptotic couple of a model  $K$  of  $T^*$  is interpretable in  $K$ , its theory (that is, the theory of gap-closed  $H$ -asymptotic couples from §6.4) has NIP. However, this theory should also be distal, or even weakly o-minimal.

In every model  $K$  of  $T^{\text{nl}}$ , the constant field  $C$  is stably embedded in the sense that any subset of  $C$  definable in  $K$  is actually definable in  $C$  as a real closed field. We believe that similarly the ordered differential residue fields of models of  $T^*$  are stably embedded. To prove this, it may be useful to strengthen the multi-sorted model completeness result Theorem 6.58, such as to a reduction of quantifiers down to the ordered differential residue field. Moreover, we hope to use such a theorem, or a similar result in a three-sorted setting including the asymptotic couple, to deduce an Ax–Kochen/Ershov principle for d-henselian pre- $H$ -fields with exponential integration. It may also be possible to undertake an analysis of dimension in models of  $T^*$  similar to that done for models of  $T^{\text{nl}}$  in [ADH17b], with the ordered differential residue field likely playing the role of the constant field.

The only pre- $H$ -fields with small derivation studied in this thesis are those with gap 0. Another condition that an asymptotic field  $K$  with small derivation can satisfy instead is  $\max 0$ , which means that  $\max \Psi = 0$ ; equivalently, for all  $f \succ 1$  in  $K$ ,  $f' \not\asymp f$ , and there exists  $g \succ 1$  in  $K$  with  $g' \asymp g$ . Does the theory of pre- $H$ -fields with  $\max 0$  have a model companion? The derivation induced on the residue field of a pre- $H$ -field with  $\max 0$  is necessarily trivial, so pre- $H$ -fields with  $\max 0$  cannot be d-henselian. But they also cannot be newtonian, and thus if this theory has a model companion, it would be quite different from both  $T^*$  and  $T^{\text{nl}}$ . In this direction, Theorem 6.19 isolates the model

completion of the theory of  $H$ -asymptotic couples  $(\Gamma, \psi)$  satisfying  $\max \Psi = 0$  and shows that it has quantifier elimination. Theorem 6.23 could also be useful.

Outside of the setting of pre- $H$ -fields, another direction of research is to consider the model theory of unordered asymptotic fields. Do the theories of  $H$ -asymptotic fields with either gap 0 or  $\max 0$ , respectively, have model companions? Theorems 6.7 and 6.19 and their consequences Theorems 6.22 and 6.23 gave us some hope that model companion results could be obtained. Using similar arguments to those in §6.7.3 (with [ADH17a, Lemmas 10.4.3, 10.4.5, and 10.4.6] replacing Lemmas 6.37 and 6.38), one can show an analogue of Corollary 6.44. More precisely, if  $K$  is an  $H$ -asymptotic field with gap 0 and  $\mathbf{k}$  is algebraically closed and has exponential integration, then  $K$  has an  $H$ -asymptotic expint-extension with gap 0 that is algebraically closed and has exponential integration; the same is true with “ $\max 0$ ” replacing “gap 0.” However, the argument used in Theorem 6.47 to prove an embedding property for such extensions fails here due to the lack of uniqueness in [ADH17a, Lemma 10.4.6], in contrast to Lemma 6.38. Hence a different approach is needed to prove model completeness for such unordered  $H$ -asymptotic fields, if it is possible.

## Bibliography

- [AD00] M. Aschenbrenner and L. van den Dries. “Closed asymptotic couples”. *J. Algebra* 225.1 (2000), pp. 309–358. URL: <https://doi.org/10.1006/jabr.1999.8128>.
- [AD02] M. Aschenbrenner and L. van den Dries. “ $H$ -fields and their Liouville extensions”. *Math. Z.* 242.3 (2002), pp. 543–588. URL: <https://doi.org/10.1007/s002090000358>.
- [ADH13] M. Aschenbrenner, L. van den Dries, and J. van der Hoeven. “Toward a model theory for transseries”. *Notre Dame J. Form. Log.* 54.3-4 (2013), pp. 279–310. URL: <https://doi.org/10.1215/00294527-2143898>.
- [ADH17a] M. Aschenbrenner, L. van den Dries, and J. van der Hoeven. *Asymptotic differential algebra and model theory of transseries*. Vol. 195. Annals of Mathematics Studies. Princeton University Press, Princeton, NJ, 2017, pp. xxi+849. URL: <https://doi.org/10.1515/9781400885411>.
- [ADH17b] M. Aschenbrenner, L. van den Dries, and J. van der Hoeven. “Dimension in the realm of transseries”. In: *Ordered algebraic structures and related topics*. Vol. 697. Contemp. Math. Amer. Math. Soc., Providence, RI, 2017, pp. 23–39.
- [ADH17c] M. Aschenbrenner, L. van den Dries, and J. van der Hoeven. “Revisiting closed asymptotic couples”. In preparation. 2017.
- [ADH18] M. Aschenbrenner, L. van den Dries, and J. van der Hoeven. “Maximal immediate extensions of valued differential fields”. *Proc. Lond. Math. Soc. (3)* 117.2 (2018), pp. 376–406. URL: <https://doi.org/10.1112/plms.12128>.
- [AK66] J. Ax and S. Kochen. “Diophantine problems over local fields: III: Decidable fields”. *Ann. of Math. (2)* 83 (1966), pp. 437–456. URL: <https://doi.org/10.2307/1970476>.
- [Bos86] M. Boshernitzan. “Hardy fields and existence of transexponential functions”. *Aequationes Math.* 30.2-3 (1986), pp. 258–280. URL: <https://doi.org/10.1007/BF02189932>.
- [CD83] G. Cherlin and M. A. Dickmann. “Real closed rings II. Model theory”. *Ann. Pure Appl. Logic* 25.3 (1983), pp. 213–231. URL: [https://doi.org/10.1016/0168-0072\(83\)90019-2](https://doi.org/10.1016/0168-0072(83)90019-2).
- [DG87] B. Dahn and P. Göring. “Notes on exponential-logarithmic terms”. *Fund. Math.* 127.1 (1987), pp. 45–50. URL: <https://doi.org/10.4064/fm-127-1-45-50>.
- [DGL11] A. Dolich, J. Goodrick, and D. Lippel. “Dp-minimality: basic facts and examples”. *Notre Dame J. Form. Log.* 52.3 (2011), pp. 267–288. URL: <https://doi.org/10.1215/00294527-1435456>.
- [DMM97] L. van den Dries, A. Macintyre, and D. Marker. “Logarithmic-exponential power series”. *J. London Math. Soc. (2)* 56.3 (1997), pp. 417–434. URL: <https://doi.org/10.1112/S0024610797005437>.

- [DMM01] L. van den Dries, A. Macintyre, and D. Marker. “Logarithmic-exponential series”. In: *Proceedings of the International Conference “Analyse & Logique” (Mons, 1997)*. Vol. 111. 1-2. 2001, pp. 61–113. URL: [https://doi.org/10.1016/S0168-0072\(01\)00035-5](https://doi.org/10.1016/S0168-0072(01)00035-5).
- [DP19] L. van den Dries and N. Pynn-Coates. “On the uniqueness of maximal immediate extensions of valued differential fields”. *J. Algebra* 519 (2019), pp. 87–100. URL: <https://doi.org/10.1016/j.jalgebra.2018.10.025>.
- [Éca90] J. Écalle. “Finitude des cycles-limites et accéléro-sommation de l’application de retour”. In: *Bifurcations of planar vector fields (Luminy, 1989)*. Vol. 1455. Lecture Notes in Math. Springer, Berlin, 1990, pp. 74–159. URL: <https://doi.org/10.1007/BFb0085391>.
- [Éca92] J. Écalle. *Introduction aux fonctions analysables et preuve constructive de la conjecture de Dulac*. Actualités Mathématiques. [Current Mathematical Topics]. Hermann, Paris, 1992.
- [Ers65] Yu. L. Ershov. “On the elementary theory of maximal normed fields”. *Dokl. Akad. Nauk SSSR* 165 (1965), pp. 21–23.
- [Har12] G. H. Hardy. “Properties of Logarithmico-Exponential Functions”. *Proc. London Math. Soc. (2)* 10 (1912), pp. 54–90. URL: <https://doi.org/10.1112/plms/s2-10.1.54>.
- [Kap42] I. Kaplansky. “Maximal fields with valuations”. *Duke Math. J.* 9 (1942), pp. 303–321. URL: <http://projecteuclid.org/euclid.dmj/1077493226>.
- [Pyn19] N. Pynn-Coates. “Newtonian valued differential fields with arbitrary value group”. *Comm. Algebra* 47.7 (2019), pp. 2766–2776. URL: <https://doi.org/10.1080/00927872.2018.1539173>.
- [Pyn20a] N. Pynn-Coates. “A model complete theory of transexponential pre- $H$ -fields”. Preprint. 2020. URL: <https://arxiv.org/abs/1910.12171>.
- [Pyn20b] N. Pynn-Coates. “Differential-henselianity and maximality of asymptotic valued differential fields”. Preprint. Accepted to *Pacific J. Math.* 2020. URL: <https://arxiv.org/abs/1805.05890>.
- [Ros80] M. Rosenlicht. “Differential valuations”. *Pacific J. Math.* 86.1 (1980), pp. 301–319. URL: <http://projecteuclid.org/euclid.pjm/1102780625>.
- [Ros83] M. Rosenlicht. “Hardy fields”. *J. Math. Anal. Appl.* 93.2 (1983), pp. 297–311. URL: [https://doi.org/10.1016/0022-247X\(83\)90175-0](https://doi.org/10.1016/0022-247X(83)90175-0).
- [Sca00] T. Scanlon. “A model complete theory of valued  $D$ -fields”. *J. Symbolic Logic* 65.4 (2000), pp. 1758–1784. URL: <https://doi.org/10.2307/2695074>.
- [Sca03] T. Scanlon. “Quantifier elimination for the relative Frobenius”. In: *Valuation theory and its applications, Vol. II (Saskatoon, SK, 1999)*. Vol. 33. Fields Inst. Commun. Amer. Math. Soc., Providence, RI, 2003, pp. 323–352.
- [Sim13] P. Simon. “Distal and non-distal NIP theories”. *Ann. Pure Appl. Logic* 164.3 (2013), pp. 294–318. URL: <https://doi.org/10.1016/j.apal.2012.10.015>.
- [Sin78] M. F. Singer. “The model theory of ordered differential fields”. *J. Symbolic Logic* 43.1 (1978), pp. 82–91. URL: <https://doi.org/10.2307/2271951>.