REMARKS ON DIMENSION IN ORDERED DIFFERENTIAL FIELDS

NIGEL PYNN-COATES

Abstract. First, this paper surveys results on differential-algebraic dimension in closed ordered differential fields due to Brihaye–Michaux–Rivière and in the differential field of (logarithmic-exponential) transseries due to Aschenbrenner–Van den Dries–Van der Hoeven. In particular, these results give topological characterizations of differential-algebraic dimension in such structures. We also explain connections to the model-theoretic notion of coanalyzability relative to the constants due to Eleftheriou–León Sánchez–Regnault for closed ordered differential fields and to Aschenbrenner–Van den Dries–Van der Hoeven for transseries.

Second, the paper announces new results concerning dimension in ordered valued differential fields that combine features of both examples, namely existentially closed pre-H-fields with gap 0. Further, we extend differential-algebraic dimension to a dimension on sets definable in pairs consisting of an existentially closed pre-H-field with gap 0 and a lift of its differential residue field, which is a closed ordered differential field.

1. Introduction

1.1. **Introduction.** The study of differential fields from the perspective of model theory has a rich history intertwined with differential algebra. Likewise, the model-theoretic study of ordered fields, particularly as developed in o-minimality, has had influential applications in number theory, complex geometry, and far beyond. This paper concerns *ordered differential fields*, which are simply ordered fields equipped with a derivation in the sense of differential algebra, and its goal is to examine differential-algebraic dimension in these fields. The behaviour of this dimension depends on how the derivation interacts with the ordering, which can be generic or satisfy some natural compatibility conditions reminiscent of differentiating functions definable in some o-minimal structure on the real numbers. We first survey results from [BMR09; ELR21; ADH17b] on differential-algebraic dimension in these distinct contexts. Then we announce new results to be shown in [Pyn25] concerning certain ordered differential fields combining aspects of both contexts.

The first result on the model theory of ordered differential fields is that this theory, in which no compatibility between the derivation and the ordering is assumed, has a model completion called the theory of *closed ordered differential fields* [Sin78], so called because this theory axiomatizes the class of ordered differential fields that are existentially closed. Informally, in a closed ordered differential field, any system of algebraic differential equations that could have a solution in an ordered field, does have a solution. Thus the derivation behaves generically with respect to the ordering, and is in particular discontinuous.

On the other hand, coming from o-minimality or asymptotic analysis, it makes sense to consider fields of real-valued functions as ordered differential fields. A natural example for model theorists is the set of germs at $+\infty$ of unary functions $\mathbb{R} \to \mathbb{R}$ definable in a fixed o-minimal expansion of the real closed field $\mathbb{R}$. By the monotonicity theorem, this set of

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germs forms a field that is closed under differentiation, and is thus a Hardy field. In this case, the derivation satisfies various compatibility properties with the ordering. For instance, it is continuous and satisfies a version of l'Hôpital's Rule at $+\infty$; if $f(x) \to +\infty$ as $x \to +\infty$, then $f'(x) > 0$ for sufficiently large x; if $f(x) \to 0$ as $x \to +\infty$, then similarly $f'(x) \to 0$. A *pre-H-field*, introduced in [AD02], is an ordered differential satisfying axioms meant to capture such behaviour in Hardy fields, and indeed the properties above can be abstractly formulated as sentences in the language of ordered differential fields. The theory of pre-H-fields has a model companion in the language of ordered valued differential fields [ADH17a], and models of this model companion are called *closed H-fields*, since it axiomatizes the class of existentially closed H-fields. This result has a long and difficult proof, and, together with the rest of the book, won the 2017 Karp Prize from the Association for Symbolic Logic. A natural closed H-field is the differential field of logarithmic-exponential transseries from [DMM97], which indeed motivated the model companion result. More recently, the same authors showed that maximal Hardy fields are closed H-fields. The proof of this result, spread across [ADH24b; ADH24a; ADH24d], is long and involves analysis as well as further algebraic elaborations of the machinery from [ADH17a]. We say a little more about transseries and Hardy fields in Section 4.1.

Finally, we consider existentially closed pre-H-fields with gap 0. Pre-H-fields with gap 0 can arise from enlarging the valuation ring of a large closed H-field like a maximal Hardy field (see [Pyn24a, Section 8] and [Pyn24b]). In [Pyn24a], we showed that this theory has a model completion, which axiomatizes the class of existentially closed pre-H-fields with gap 0. These pre-H-fields contain a closed ordered differential subfield that is bounded (in the sense of the ordering), so they combine features of closed H-fields and closed ordered differential fields. The rest of the paper announces without proof new results on dimensions in existentially closed pre-H-fields with gap 0. After studying differential-algebraic dimension, we extend it to a dimension $\dim_2$ on sets definable in the pair $(K, \boldsymbol{k})$ consisting of K, an existentially closed pre-H-field with gap 0, and $\boldsymbol{k}$, a closed ordered differential subfield that is tame in K. The idea for this dimension is adapted from a dimension for tame pairs of real closed fields from [ÁD16].

In all three of these structures: closed ordered differential fields, closed H-fields, and existentially closed pre-H-fields with gap 0, we compare dimension to order-topological properties and a topological notion of dimension, as well the model-theoretic notion of coanalyzability. The results are summarized below.

1.2. **Overview of results.** A key step of the investigation in each of the structures is to establish that differential-algebraic dimension is a *dimension function* in the sense of [Dri89]. This is defined in Definition 2.2 and means roughly that the dimension is definable in fibres in a way compatible with definable maps. Let dim refer to differential-algebraic dimension, which we define in Section 2. In [BMR09], they show that dim is a dimension function in closed ordered differential fields by proving a differential cell decomposition result based on the cell decomposition for the underlying real closed field, and then showing that the dimension coming from this differential cell decomposition coincides with dim. Since the derivation is not compatible with the order topology, they refine the order topology in such a way that:

Fact 1.1 ([BMR09])**.** *If $\boldsymbol{k}$ is a closed ordered differential field and $S \subseteq \boldsymbol{k}^n$ is nonempty and definable in $\boldsymbol{k}$, then the following are equivalent:*

(i) $\dim S = n$;
(ii) *S has nonempty interior in $\boldsymbol{k}^n$ in the topology refining the order topology.*

For a differential field K with derivation ∂, its constant field is $C_K := \{a \in K : \partial(a) = 0\}$. For a flavour of this refined topology, in a closed ordered differential field $\boldsymbol{k}$, $C_{\boldsymbol{k}}$ is dense and codense in $\boldsymbol{k}$ in the order topology, but becomes closed in the refined topology. This fact and others about closed ordered differential fields are in Section 3.

In the case of closed H-fields, showing that dim is a dimension function involves first connecting it to the order topology. Then as explained further in Section 4:

Fact 1.2 ([ADH17b])**.** *If K is a closed H-field and $S \subseteq K^n$ is nonempty and definable in K:*

(i) $\dim S = n \iff$ *S has nonempty interior in K^n.*
(ii) *The following are equivalent:*
  (a) $\dim S = 0$;
  (b) *S is discrete in K^n;*
  (c) *S is coanalyzable relative to C.*

The definition of the model-theoretic notion of coanalyzability in the last part of the fact above is technical, but can be viewed here as an analogue of the fact that the solutions of a linear differential equation form a finite-dimensional vector space over the constant field. That is, in a closed H-field, the solution set of a possibly nonlinear algebraic differential equation is still controlled by the constant field.

Combining the two parts when $n = 1$ implies that closed H-fields are d-minimal (we do not include definable completeness in the definition of d-minimal), and its proof relies on local o-minimality from [ADH17a, Proposition 16.6.8]. In [Fuj23; FKK22], it is shown that locally o-minimal structures have a nice dimension theory, provided they are definably complete: For example, a nonempty definable set has dimension 0 if and only if it is discrete. However, the locally o-minimal structures considered in this paper are not definably complete, since the convex hull of the constant field is definable and bounded but has no supremum. Therefore, different approaches are needed to establish the same result in ordered differential fields. Also note that no structure in this paper is weakly o-minimal, since the constant field is definable but not convex.

Showing that dim is a dimension function in existentially closed pre-H-fields with gap 0 is similar to the case of closed H-fields. Then as explained further in Section 5.2:

Theorem 1.3. *If K is an existentially closed pre-H-field with gap 0 and $S \subseteq K^n$ is nonempty and definable in K:*

(i) $\dim S = n \iff$ *S has nonempty interior in K^n.*
(ii) $\dim S = 0 \iff$ *S is discrete in K^n.*

Again, the case $n = 1$ means that existentially closed pre-H-fields with gap 0 are d-minimal, and some proofs go through local o-minimality. Here we lose the equivalence of having dimension 0 with coanalyzability relative to the constant field. This is essentially because in a closed ordered differential field, not every algebraic differential equation is coanalyzable relative to the constant field [ELR21], and an existentially closed pre-H-field with gap 0 contains a closed ordered differential subfield. We regain a modified form of this equivalence by extending dim to a dimension function $\dim_2$ that better captures behaviour in such pairs.

Theorem 1.4. *Let K be an existentially closed pre-H-field with gap 0 equipped with a lift $\boldsymbol{k}$ of its closed ordered differential residue field. If $S \subseteq K^n$ is nonempty and definable in $(K, \boldsymbol{k})$, then the following are equivalent:*

(i) $\dim_2 S = 0$;
(ii) *S is discrete in K^n;*
(iii) *S is coanalyzable relative to $\boldsymbol{k}$.*

In this case, we cannot show that having dimension n is equivalent to having empty interior in K^n, but can prove it in case $n = 1$, which shows d-minimality for such pairs $(K, \boldsymbol{k})$. The results for pairs are explained in Section 5.3.

1.3. **Notation and conventions.** Let m and n range over $\mathbb{N} = \{0, 1, 2, \dots\}$. We use standard model-theoretic terminology and notation. The logic here is always single-sorted. "Definable" means "definable with parameters"; when confusion may arise, we specify the ambient structure. Fix pairwise distinct variables $x_1, x_2, \dots,$ $y_1, y_2, \dots,$ and z, and let $x = (x_1, \dots, x_m)$ and $y = (y_1, \dots, y_n)$. Capital $X_1, \dots, X_m$, $Y, Y_1, \dots, Y_n$, and Z refer to differential indeterminates (in the sense explained in Section 2).

2. Preliminaries on differential fields and differential-algebraic dimension

A **differential field** is a field K of characteristic 0 equipped with a **derivation** $\partial\colon K \to K$, which satisfies for $a, b \in K$:

(D1) $\partial(a + b) = \partial(a) + \partial(b)$;
(D2) $\partial(ab) = a\partial(b) + b\partial(a)$.

Let K be a differential field. Attached to K is an important subfield $C \coloneqq \{a \in K : \partial(a) = 0\}$, called the **constant field** of K; to indicate the dependence on K, we write C_K. For $a \in K$, we often write a' for $\partial(a)$ if the derivation is clear from context, and similarly $a^{(n)}$ for $\partial^n(a)$. We let $K\{Y\}$ be the ring of differential polynomials over K in the differential indeterminate Y. That is, $K\{Y\}$ is the polynomial ring $K[Y, Y', Y'', \dots]$ over K in infinitely many distinct indeterminates $Y, Y', Y'', \dots$ with the evaluation map $a \mapsto P(a)$ given by substituting $a, a', a'', \dots$ for $Y, Y', Y'', \dots$ in P, where $a \in K$. If L is a differential field extension of K and $a \in L$, then $K\langle a\rangle$ denotes the differential subfield of L generated by a over K. Using differential polynomials instead of polynomials, one defines "differentially algebraic", "differentially transcendental", and "differential transcendence degree" in a similar fashion as the more familiar "algebraic", "transcendental", and "transcendence degree". See for example [ADH17a, Section 4], which uses the same notation as here. For a differential field extension L of K, let $\operatorname{tr.deg}_\partial(L|K)$ denote the differential transcendence degree of L over K. In particular, $\operatorname{tr.deg}_\partial(L|K) = 0$ if and only if every $a \in L$ is differentially algebraic over K.

Let $\mathcal{L}_\partial \coloneqq \{+, -, \cdot, 0, 1, \partial\}$ be the language of differential rings and let K be an expansion of a differential field in some language expanding $\mathcal{L}_\partial$. The fastest way to define the differential-algebraic dimension of definable sets in K uses elementary extensions, although this is not necessary. For nonempty $S \subseteq K^n$ definable in K, the **differential-algebraic dimension of** S is

$$\dim S \ \coloneqq\ \max\{\operatorname{tr.deg}_\partial(K\langle s\rangle|K) : s \in S^*\},$$

where $K^* \succcurlyeq K$ is $|K|^+$-saturated and $S^* \subseteq (K^*)^n$ is defined by the same formula as S. Of course, one must check that this is independent of the choice of $|K|^+$-saturated $K^* \succcurlyeq K$

and remains the same when passing to elementary extensions; for such details, see [ADH17b, Section 1] or [ÁD16, Section 2]. Differential-algebraic dimension has an alternate definition intrinsic to K, which makes no reference to elementary extensions; see [Dri89, Section 2] or [ADH17b, Section 1].

It is obvious from the definitions that for any nonempty definable $S \subseteq K^n$, we have $0 \leqslant \dim S \leqslant n$. We set $\dim \emptyset \coloneqq -\infty$. Here are some other desirable properties of differential-algebraic dimension, which are not hard to check using the definition above and some basic differential algebra (use for example [ADH17a, Lemma 4.2.1] for $\dim K = 1$).

Lemma 2.1. *Let $S_1 \subseteq K^m$ and $S_2 \subseteq K^n$ be definable. Then*

(i) $\dim K^n = n$;
(ii) *if S_1 is finite and nonempty, then* $\dim S_1 = 0$;
(iii) *if $m = n$ and $S_1 \subseteq S_2$, then* $\dim S_1 \leqslant \dim S_2$;
(iv) *if $m = n$, then* $\dim(S_1 \cup S_2) = \max\{\dim S_1, \dim S_2\}$;
(v) $\dim(S_1 \times S_2) = \dim S_1 + \dim S_2$.

It is also important for logical purposes that dimension be invariant under permutation of coordinates, which indeed holds for differential-algebraic dimension. For these properties, see for example [ADH17b, Lemmas 1.2, 1.3] and also the analogous [ÁD16, Lemmas 2.1, 2.4, Corollary 2.5] for a very general context in which these properties hold.

Definition 2.2. We say dim is a **dimension function** (**in** K) if for all definable $f\colon S \to K^n$ with $S \subseteq K^m$,

(1) $\dim S \geqslant \dim f(S)$ (in particular, $\dim S = \dim f(S)$ for injective f);
(2) $B_i \coloneqq \{b \in K^n : \dim f^{-1}(b) = i\}$ is definable and $\dim f^{-1}(B_i) = i + \dim B_i$ for $i = 0, \dots, m$.

In fact, [Dri89] shows how the first part follows from the second part and the properties in Lemma 2.1. This definition makes sense more generally, of course, although in general the properties of Lemma 2.1 (and invariance under permutation of coordinates) should be included in the definition of "dimension function".

If dim is a dimension function, then it is determined by the collection of definable $S \subseteq K$ with $\dim S = 0$. Thus characterizing the sets of dimension 0 is of particular interest. A paradigmatic example of a set with differential-algebraic dimension zero is C. More generally, a subset S of K is **thin** if $S \subseteq \{a \in K : P(a) = 0\}$ for some nonzero $P \in K\{Y\}$. Note that the thin subsets of K form an ideal, and for nonempty definable $S \subseteq K$, $\dim S = 0$ if and only if S is thin. As remarked above, K is not thin. More generally, $\dim S < n$ if and only if $S \subseteq \{a \in K^n : P(a) = 0\}$ for some nonzero $P \in K\{Y_1, \dots, Y_n\}$.

From this "bottom up perspective", the paper [ÁD16] studies dimension functions given by collections of formulas that define pregeometries. Essentially, identifying which definable sets have dimension 0 allows them to build a dimension from that collection. Although [ÁD16] does not explicitly consider the case of differential fields, the relevant collection here comprises formulas of the form $P(y, z) = 0 \wedge P(y, Z) \neq 0$ with parameter variables y, where $P \in \mathbb{Z}\{Y_1, \dots, Y_n, Z\}$. The expression $P(y, Z) \neq 0$ abbreviates a formula in the language of rings with free variables y expressing that $P(y, Z)$ is a nonzero differential polynomial in Z. The pregeometry in K associated to this collection takes $A \subseteq K$ to the set of elements of K that are differentially algebraic over $\mathbb{Q}\langle A\rangle$. This framework makes some of their results available here for free even for differential-algebraic dimension, but we only really need to exploit it when we introduce a new dimension function in Section 5.3. There is also overlap between [ÁD16] and [For11], but we use the former.

3. Closed ordered differential fields

The first theory of ordered differential fields we consider is just that—the theory in the language $\mathcal{L}_{\partial,\leqslant} = \mathcal{L}_\partial \cup \{\leqslant\}$ of ordered differential rings expressing that any model is an ordered field (in the sense that addition and multiplication are compatible with the ordering in the usual way) equipped with a derivation. Note that no interaction is assumed between the derivation and the ordering. Singer studied this theory and showed:

Fact 3.1 ([Sin78])**.** *The theory of ordered differential fields has a model completion, the theory of closed ordered differential fields; this theory is complete and admits quantifier elimination.*

Since the theory of ordered differential fields has a $\forall\exists$-axiomatization, the closed ordered differential fields are exactly the ordered differential fields that are existentially closed. Hence the derivation exhibits every behaviour possible that is consistent with the ordering. This is also apparent in the following axiomatization due to Singer, where $\boldsymbol{k}[T_0, \dots, T_n]$ denotes the ring of polynomials (not differential polynomials) over $\boldsymbol{k}$ in the $n+1$ indeterminates $T_0, \dots, T_n$. An ordered differential field $\boldsymbol{k}$ is a **closed ordered differential field** if

(1) $\boldsymbol{k}$ is a real closed field;
(2) given $P, Q_1, \dots, Q_m \in \boldsymbol{k}[T_0, \dots, T_n]$ with $P \notin \boldsymbol{k}[T_0, \dots, T_{n-1}]$, whenever there exists $(a_0, \dots, a_n) \in \boldsymbol{k}^{n+1}$ such that
   - $P(a_0, \dots, a_n) = 0$,
   - $Q_i(a_0, \dots, a_n) > 0$ for $i = 1, \dots, m$, and
   - $\frac{\partial P}{\partial T_n}(a_0, \dots, a_n) \neq 0$,

   then there exists $b \in \boldsymbol{k}$ such that
   - $P(b, b', \dots, b^{(n)}) = 0$ and
   - $Q_i(b, b', \dots, b^{(n)}) > 0$ for $i = 1, \dots, m$.

In the rest of this section, let $\boldsymbol{k}$ be a closed ordered differential field; since $\boldsymbol{k}$ is real closed, for definability purposes it does not matter whether we consider $\boldsymbol{k}$ as an $\mathcal{L}_\partial$-structure or as an $\mathcal{L}_{\partial,\leqslant}$-structure. For examples of the generic behaviour mentioned above, it follows easily from the given axiomatization that the constant field $C_{\boldsymbol{k}}$ of $\boldsymbol{k}$ is dense and codense in $\boldsymbol{k}$ (with respect to the order topology). In particular, ∂ is discontinuous and closed ordered differential fields are not locally o-minimal. More generally, Singer's closed ordered differential fields fit into the framework of o-minimal fields equipped with a generic derivation in [FK21].

In [BMR09, Theorem 4.9], Brihaye, Michaux, and Rivière prove a cell decomposition theorem for closed ordered differential fields, based on the o-minimal cell decomposition for the underlying real closed field. Since the derivation behaves generically with respect to the ordering, the idea is to exploit o-minimal cell decomposition in higher arities and then restrict to the jet space (i.e., the family of subsets $\{(a, a', a'', \dots, a^{(n)}) : a \in \boldsymbol{k}\} \subseteq \boldsymbol{k}^{n+1}$ as n varies). For example,

$$C_{\boldsymbol{k}} \;=\; \{a \in \boldsymbol{k} : a' = 0\} \;=\; \pi_1\Big(\{(a,b) \in \boldsymbol{k}^2 : b = 0\} \cap \{(a,b) \in \boldsymbol{k}^2 : b = a'\}\Big)$$

is a cell in the closed ordered differential $\boldsymbol{k}$ since $\{(a,b) \in \boldsymbol{k}^2 : b = 0\}$ is a cell in the real closed field $\boldsymbol{k}$, where $\pi_1 \colon \boldsymbol{k}^2 \to \boldsymbol{k}$ is projection onto the first coordinate. As in an o-minimal structure, this cell decomposition yields a notion of dimension, which they show in [BMR09, Corollary 5.27] coincides with differential-algebraic dimension; we thus continue to denote dimension in this context simply by dim.

Fact 3.2 ([BMR09, Theorem 5.19])**.** *The dimension* dim *is a dimension function.*

Coming to topological issues, since the derivation is not compatible with the order topology, it should therefore not be surprising that the differential-algebraic dimension does not agree with the topological dimension (in the sense made precise in Fact 3.3). For example, the subset $A \coloneqq \{a \in \boldsymbol{k} : a' > 0\}$ is dense and codense in $\boldsymbol{k}$ but has $\dim A = 1$. In particular, $\dim A = 1$ for $A \subseteq \boldsymbol{k}$ does not imply that A has nonempty interior in $\boldsymbol{k}$. Nevertheless, Brihaye, Michaux, and Rivière are able to give a topological characterization of dimension, but using a refined topology. The idea of this topology is similar to the idea for cell decomposition, in that it involves intersecting open sets (in the order topology) in higher arities with the jet space. For example, in the refined topology,

$$A \ = \ \{a \in \boldsymbol{k} : a' > 0\} \ = \ \pi_1\Big(\{(a,b) \in \boldsymbol{k}^2 : b > 0\} \cap \{(a,b) \in \boldsymbol{k}^2 : b = a'\}\Big)$$

is open since $\{(a,b) \in \boldsymbol{k}^2 : b > 0\}$ is open in $\boldsymbol{k}^2$ in the order topology (more precisely, the corresponding product topology on $\boldsymbol{k}^2$), but A is dense and codense in the order topology. Similarly, $C_{\boldsymbol{k}}$ is closed in the refined topology.

Fact 3.3 ([BMR09, Theorem 5.29])**.** *If $S \subseteq \boldsymbol{k}^n$ is nonempty and definable, then* $\dim S$ *is the largest $m \leqslant n$ such that some projection onto m of its coordinates has interior in $\boldsymbol{k}^m$ in the refined topology.*

Finally, we come to the issue of coanalyzability relative to $C_{\boldsymbol{k}}$ in $\boldsymbol{k}$, and its connection to dimension. Here is the easiest definition of the relevant notion of coanalyzability of a definable set in $\boldsymbol{k}$ (for an exposition of this notion, see [ADH17b, Section 6]). For $\varphi(x,y)$ and a as in Definition 3.4, the notation $\varphi(a,\boldsymbol{k})$ means $\{b \in \boldsymbol{k}^n : \boldsymbol{k} \models \varphi(a,b)\}$ and similarly in other instances.

Definition 3.4. Let $\varphi(x,y)$ be an $\mathcal{L}_\partial$-formula. Let $a \in \boldsymbol{k}^m$, and $\operatorname{Th}(\boldsymbol{k},a)$ denote the $\mathcal{L}_\partial(a)$-theory of $\boldsymbol{k}$ expanded by constants for a. Then $\varphi(a,\boldsymbol{k})$ is **coanalyzable relative to $C_{\boldsymbol{k}}$** if whenever $\boldsymbol{k}_1 \preccurlyeq \boldsymbol{k}_2 \models \operatorname{Th}(\boldsymbol{k},a)$ with $C_{\boldsymbol{k}_1} = C_{\boldsymbol{k}_2}$, we have $\varphi(a,\boldsymbol{k}_1) = \varphi(a,\boldsymbol{k}_2)$.

It is an easy consequence of dim being a dimension function and $\dim C_{\boldsymbol{k}} = 0$ that if $A \subseteq \boldsymbol{k}^n$ is coanalyzable relative to $C_{\boldsymbol{k}}$, then $\dim A = 0$ (see [ÁD16, Corollary 2.8] for a general statement of this kind). However, Eleftheriou, León-Sanchez, and Regnault show that the converse fails.

Fact 3.5 ([ELR21, Corollary 4.3])**.** *There exists a definable $S \subseteq \boldsymbol{k}$ such that* $\dim S = 0$ *but S is not coanalyzable relative to $C_{\boldsymbol{k}}$.*

In fact, using results of Rosenlicht from [Ros74], they give the explicit examples $z' = z^3 - z^2$ and $z'(z-1) = z$, formulas which define subsets of $\boldsymbol{k}$ that are not coanalyzable relative to $C_{\boldsymbol{k}}$ but clearly have differential-algebraic dimension 0.

4. Transseries and maximal Hardy fields

4.1. **Introduction.** The previous section dealt with closed ordered differential fields, in which the derivation interacts generically with respect to the order topology. Now we turn to ordered differential fields in which the derivation behaves nicely with respect to the order topology; in particular, it is continuous. We make this precise through the notion of pre-H-field in the next subsection, and just describe some motivation now. The elementary class of pre-H-fields is motivated by transseries and Hardy fields, and was introduced in [AD02] towards axiomatizing these structures, which have a rich mathematical history both inside and outside of logic.

Transseries were introduced independently by Dahn and Göring, motivated by Tarski's decidability problem for the real exponential field [DG87], and by Écalle in his solution of Dulac's problem, connected to Hilbert's 16th problem [Éca90; Éca92]. There are many fields of transseries, but the specific structure $\mathbb{T}$ considered here, the differential field of logarithmic-exponential transseries, was constructed in [DMM97] (it was denoted in that paper by $\mathbb{R}(\!(t)\!)^{\mathrm{LE}}$). This differential field has an intricate construction, but roughly it is built from real power series in x using exponentials and logarithms, allowing suitable infinite sums; some elements can be thought of as asymptotic expansions of solutions to real differential and functional equations (x is thought of as going to $+\infty$). Indeed, every unary function definable in the o-minimal field $\mathbb{R}_{\mathrm{an,exp}}$ can be asymptotically expanded at $+\infty$ as a series in $\mathbb{T}$, which was used in the proofs in [DMM97] that the real Gamma and Riemann zeta functions are not definable in $\mathbb{R}_{\mathrm{an,exp}}$. Of a more analytic flavour, Hardy fields are fields of germs at $+\infty$ of functions $f\colon \mathbb{R} \to \mathbb{R}$ that are closed under differentiation. For a fixed o-minimal expansion of the real field, the germs at $+\infty$ of the definable unary functions form a Hardy field.

These algebraic and analytic counterparts are closely related, and indeed there are various embeddings between some of these fields, of which one was already mentioned above (in one direction the embedding is seen as asymptotic expansion and in the other as summation). Their model theory too is closely related. For more details on the program of Aschenbrenner, Van den Dries, and Van der Hoeven on transseries and Hardy fields, see the survey paper [ADH18], which also concerns a third structure, the surreal numbers. The introduction to the book [ADH17a] is another good introduction to transseries and their model theory, as well as an earlier survey paper [ADH13]. Note that several conjectures from [ADH18] concerning Hardy fields have been established in [ADH24a; ADH24b; ADH24c; ADH24d]. In particular, maximal Hardy fields have the same elementary theory as $\mathbb{T}$ as differential fields.

4.2. **Pre-H-fields.** Based on the behaviour of differentiation in Hardy fields and transseries, we model nice interaction between the ordering and the derivation in ordered differential fields through the notion of pre-H-field defined below. First we need some notation for valued fields. Let K be an ordered differential field and $\mathcal{O}$ be a **valuation ring of** K, meaning that $\mathcal{O}$ is a subring of K such that $a \in \mathcal{O}$ or $a^{-1} \in \mathcal{O}$ for every $a \in K^\times$; if necessary, we indicate the dependence on K by $\mathcal{O}_K$. With $\mathcal{O}^\times = \{a \in K^\times : a, a^{-1} \in \mathcal{O}\}$, the ring $\mathcal{O}$ has a unique maximal ideal $\smallo \coloneqq \mathcal{O} \setminus \mathcal{O}^\times$. We introduce the following binary relations, for $a, b \in K$:

$$a \preccurlyeq b \iff a \in \mathcal{O}b, \qquad a \asymp b \iff a \preccurlyeq b \text{ and } b \preccurlyeq a,$$

$$a \prec b \iff a \in \smallo b \text{ and } b \neq 0, \qquad a \sim b \iff a - b \prec b.$$

The relation $\asymp$ is an equivalence relation on K, the relation $\sim$ is an equivalence relation on $K^\times$, and they satisfy: if $a \sim b$, then $a \asymp b$. The **residue field** of $(K, \mathcal{O})$ is $\operatorname{res}(K, \mathcal{O}) \coloneqq \mathcal{O}/\smallo$. Note that $\mathbb{Q} \subseteq \mathcal{O}$, so the characteristic of $\operatorname{res}(K, \mathcal{O})$ is 0.

We call $(K, \mathcal{O})$ a **pre-H-field** if

(PH1) $\mathcal{O}$ is convex (with respect to $\leqslant$);
(PH2) for all $a \in K$, if $a > \mathcal{O}$, then $a' > 0$;
(PH3) for all $a, b \in K^\times$ with $a \preccurlyeq 1$ and $b \prec 1$, we have $a' \prec b'/b$.

Condition (PH1) holds if and only if $\mathcal{o}$ is convex, in which case $\leqslant$ induces an ordering on $\operatorname{res}(K, \mathcal{O})$ making it an ordered field.

In the definition of pre-H-field, $\mathcal{O}$ may be any valuation ring of K, but (PH2) forces $C \subseteq \mathcal{O}$. For this paper, the reader can safely assume that $\mathcal{O}$ is the convex hull of C in K, i.e., the set $\{a \in K : c_1 \leqslant a \leqslant c_2 \text{ for some } c_1, c_2 \in C\}$. In this case, $\mathcal{O}$ is existentially definable without parameters in K as an ordered differential field, but it is nevertheless important to include $\mathcal{O}$ in some way as a primitive for model completeness and quantifier elimination results. We thus usually write K instead of $(K, \mathcal{O})$ and say, for example, "K is a pre-H-field". Equipped with the convex hull of $\mathbb{Q}$ as a valuation ring, $\mathbb{T}$ is a pre-H-field and every Hardy field is a pre-H-field.

4.3. **Dimension in closed H-fields.**

Fact 4.1 ([ADH17a, Corollaries 16.2.5, 16.6.3, Proposition 16.6.8])**.** *The theory of pre-H-fields has a model companion, the theory of closed H-fields; this theory has two completions and it is locally o-minimal.*

The proof of this landmark result from [ADH17a], which won the 2017 Karp Prize from the Association for Symbolic Logic, requires most of the long book. Moreover, the theory of closed H-fields, denoted in [ADH17a] by T^{nl}, has an effective axiomatization. Since it is rather technical and plays no role in this summary, we omit it, but the two technical conditions in the axiomatization were shown in [ADH22] to be equivalent modulo the others to the more natural *Differential Intermediate Value Property*. Every closed H-field is in particular a real closed field. Note that since the theory of pre-H-fields has a $\forall\exists$-axiomatization, a pre-H-field is existentially closed if and only if it is a closed H-field.

In the remainder of this subsection, K is a closed H-field and dim continues to refer to differential-algebraic dimension, now on definable subsets of K^n; for definability purposes, it does not matter whether we consider K as an $\mathcal{L}_\partial$-structure, an $\mathcal{L}_{\partial,\leqslant}$-structure, or an $\mathcal{L}_{\partial,\leqslant} \cup \{\preccurlyeq\}$-structure, since the ordering and valuation ring are existentially $\mathcal{L}_\partial$-definable without parameters. One of the two completions of T^{nl} completely axiomatizes the theory of $\mathbb{T}$ and maximal Hardy fields. In particular, the results from [ADH17b] summarized below apply to $K = \mathbb{T}$ or K a maximal Hardy field.

In contrast with closed ordered differential fields, differential-algebraic dimension behaves well with respect to the order topology in closed H-fields. In this section, we always construe K as equipped with its order topology (equivalently, its valuation topology) and K^n as equipped with the corresponding product topology.

Fact 4.2 ([ADH17b, Corollary 3.1])**.** *If $S \subseteq K^n$ is definable, then*

$$\dim S = n \iff S \text{ has nonempty interior in } K^n.$$

This yields the following topological characterization of dim, which in particular shows that the differential-algebraic dimension of a set definable in the real closed field K is the same as its semialgebraic dimension (if you like, coming from the o-minimal cell decomposition for real closed fields).

Fact 4.3 ([ADH17b, Corollary 3.2])**.** *If $S \subseteq K^n$ is nonempty and definable, then* $\dim S$ *is the largest $m \leqslant n$ such that some projection onto m of its coordinates has interior in K^m.*

The proof of Fact 4.2 is combined with a criterion from [Dri89] to obtain:

Fact 4.4 ([ADH17b, Corollary 3.3])**.** *The dimension* dim *is a dimension function.*

Additionally, there is a more precise topological characterization of dimension 0 sets.

Fact 4.5 ([ADH17b, Proposition 4.1])**.** *If $S \subseteq K^n$ is nonempty and definable, then*

$$\dim S = 0 \iff S \text{ is discrete in } K^n.$$

The key step in the proof of Fact 4.5 is establishing it for $n = 1$ in [ADH17a, Corollary 16.6.11], which relies on the local o-minimality of K shown in [ADH17a, Proposition 16.6.8] and also shows that K is d-minimal (although K is not definably complete). In turn, this relies on the quantifier elimination result [ADH17a, Theorem 16.0.1], which refines the model companion result stated above. The language for quantifier elimination is the language $\mathcal{L}_{\partial,\leqslant,\preccurlyeq} \coloneqq \mathcal{L}_{\partial,\leqslant} \cup \{\preccurlyeq\}$ of ordered valued differential rings expanded by two binary predicates concerning certain second order differential equations.

Here is another characterization of having dimension 0, showing that three notions of "smallness", namely differential-algebraic, topological, and model-theoretic coanalyzability, coincide. Coanalyzability here means the same thing as in Definition 3.4, but to be clear:

Definition 4.6. Let $\varphi(x, y)$ be an $\mathcal{L}_\partial$-formula. Let $a \in K^m$ and $\mathrm{Th}(K, a)$ denote the $\mathcal{L}_\partial(a)$-theory of K expanded by constants for a. Then $\varphi(a, K)$ is **coanalyzable relative to** C if whenever $K_1 \preccurlyeq K_2 \models \mathrm{Th}(K, a)$ with $C_{K_1} = C_{K_2}$, we have $\varphi(a, K_1) = \varphi(a, K_2)$.

Fact 4.7 ([ADH17b, Proposition 6.2])**.** *If $S \subseteq K^n$ is nonempty and definable, then*

$$\dim S = 0 \iff S \text{ is coanalyzable relative to } C.$$

Again, the key case is when $n = 1$ and S is the zero set of a differential polynomial. Establishing coanalyzability relative to C for such S is essentially [ADH17a, Theorem 16.0.3].

5. Existentially closed pre-H-fields with gap 0

5.1. **Introduction.** In the previous section, we explained how differential-algebraic dimension behaves in the model companion of the theory of pre-H-fields, a theory motivated by transseries and Hardy fields. Now we consider instead the theory of pre-H-fields with gap 0, where a pre-H-field $(K, \mathcal{O})$ has **gap** 0 if every $a \in K^\times$ with $a \prec 1$ satisfies $a \prec a' \prec 1$. This theory is harder to motivate than before, but consider the situation of a closed H-field K and an elementary submodel L that is bounded inside K in the sense that there is $a \in K$ with $a > L$. This pair (K, L) gives rise to a *transserial tame pair* in the sense of [Pyn24b]. This may look like model-theoretic abstraction, but there are indeed such pairs. For instance, $\mathbb{T}$ can be elementarily embedded into any maximal Hardy field by [AD23, Corollary 7.10]. Given such a pair, K together with the convex hull $\mathcal{O}$ of L inside K form a pre-H-field $(K, \mathcal{O})$ with gap 0. The idea is that L is some kind of exponentially bounded part, while $\mathcal{O}$ is a purely transexponential scale in K; in this case, $\mathcal{O}$ will be much larger than the convex hull of C in K. For more precision, see [Pyn24b].

Forgetting these pairs for now, the theory of pre-H-fields with gap 0 has a model completion in the language $\mathcal{L}_{\partial,\leqslant,\preccurlyeq}$ of ordered valued differential rings; we denote the theory of the model completion of pre-H-fields with gap 0 by T^{dhl}. This model completion has some parallels with the model companion of the theory of pre-H-fields, but many differences. For example, the derivation of a pre-H-field $(K, \mathcal{O})$ with gap 0 induces a derivation on its residue field $\mathrm{res}(K, \mathcal{O})$ by [ADH17a, Lemma 4.4.2], and if $(K, \mathcal{O}) \models T^{\mathrm{dhl}}$, then $\mathrm{res}(K, \mathcal{O})$ is a closed ordered differential field. In contrast, the residue field of a closed H-field is a pure real closed field. Note that if $(K, \mathcal{O}) \models T^{\mathrm{dhl}}$, then $\mathcal{O}$ is the convex hull of C in K; again this justifies writing K instead of $(K, \mathcal{O})$.[1] The theory T^{dhl} also has an effective axiomatization, but the details of these axioms may not be illuminating, so we refer to [Pyn24a] (T^{dhl} was there denoted by $T^{\mathrm{dhl}}_{\mathrm{codf}}$). As before, a pre-$H$-field with gap 0 is existentially closed if and only if it models T^{dhl}. If $K \models T^{\mathrm{dhl}}$, then K is a real closed field. Now we summarize what we need about T^{dhl}.

Fact 5.1 ([Pyn24a, Theorem 7.9])**.** *The theory of pre-H-fields with gap 0 has a model completion, T^{dhl}; this theory is complete, admits quantifier elimination, and is locally o-minimal.*

As in the previous section, it follows from quantifier elimination that the differential-algebraic dimension has good properties in models of T^{dhl}. In particular, it is a dimension function. Additionally, it behaves well topologically, where $K \models T^{\mathrm{dhl}}$ is equipped with its order topology, which coincides with its valuation topology, and K^n is equipped with the corresponding product topology. In particular, for nonempty $S \subseteq K^n$ definable in K, we have $\dim S = n$ if and only if S has nonempty interior (cf. Fact 4.2) and $\dim S = 0$ if and only if S is discrete (cf. Fact 4.5). Full statements are in the next subsection.

5.2. **Dimension in existentially closed pre-H-fields with gap 0.** Let $K \models T^{\mathrm{dhl}}$. Quantifier elimination yields the same topological characterization of differential-algebraic dimension in models of T^{dhl} as for T^{nl} (cf. Fact 4.2). As before, definability in this subsection always means in K as a differential field, since the ordering and valuation ring are existentially definable in $\mathcal{L}_\partial$ without parameters.

Lemma 5.2. *If $S \subseteq K^n$ is definable, then*

$$\dim S = n \iff S \text{ has nonempty interior in } K^n.$$

It follows that the differential-algebraic dimension of a set definable in the real closed field K is the same as its semialgebraic dimension. More generally (cf. Fact 4.3):

Corollary 5.3. *If $S \subseteq K^n$ is nonempty and definable, then* $\dim S$ *is the largest $m \leqslant n$ such that some projection onto m of its coordinates has interior in K^m.*

Using the criterion [Dri89, Proposition 2.15], or more precisely its differential analogue on [Dri89, p. 203], and the proof of Lemma 5.2, we obtain (cf. Fact 4.4):

Corollary 5.4. *The dimension* dim *is a dimension function.*

Using local o-minimality for the first time together with Lemma 5.2 yields the following analogue of [ADH17a, Corollary 16.6.11], which in particular shows that K is d-minimal (although K is not definably complete).

Corollary 5.5. *Let $S \subseteq K$ be definable. Then S is the disjoint union of an open definable set and a discrete definable set. Moreover, S is discrete if and only if S is thin.*

In particular, the previous corollary states that discreteness is equivalent to having differential-algebraic dimension 0 for subsets of K. That holds moreover for subsets of K^n, for which the key step is the case $n = 1$ above. The proof is similar to that of Fact 4.5.

[1] By passing to the model companion, we have left the realm of the pairs of the previous paragraph.

Proposition 5.6. *If $S \subseteq K^n$ is definable and nonempty, then*

$$\dim S = 0 \iff S \text{ is discrete.}$$

The analogue of [ADH17b, Corollary 4.2] follows in the same way. Thus:

Corollary 5.7. *Every discrete definable subset of K^n is closed in K^n.*

5.3. **Dimension in the pair.** Let $K \models T^{\mathrm{dhl}}$. By [ADH17a, Proposition 7.1.3], we can equip K with a lift $\boldsymbol{k}$ of its differential residue field $\operatorname{res}(K)$. That is, $\boldsymbol{k} \subseteq \mathcal{O}$ and $\boldsymbol{k}$ maps isomorphically as a differential field onto $\operatorname{res}(K)$ under the residue map $\mathcal{O} \to \operatorname{res}(K)$; note that the restriction $\boldsymbol{k} \to \operatorname{res}(K)$ is automatically an isomorphism of *ordered* differential fields. It follows from basic properties of the valuation and derivation that $C = C_{\boldsymbol{k}} \cong C_{\operatorname{res}(K)}$. Since $\boldsymbol{k}$ is a closed ordered differential field, generic behaviour of the derivation is present in the pair $(K, \boldsymbol{k})$, but only on the bounded discrete set $\boldsymbol{k}$. As summarized in the previous subsection, differential-algebraic dimension behaves well in K, but it does not behave well in $(K, \boldsymbol{k})$. For example, since the closed discrete subset $\boldsymbol{k}$ of K is a nontrivial differential subfield of K, we have $\dim \boldsymbol{k} = 1$ (see [ADH17a, Lemmas 4.2.1, 4.2.2]). Hence, Proposition 5.6 fails for dim in $(K, \boldsymbol{k})$ (i.e., with "definable in $(K, \boldsymbol{k})$" replacing "definable in K"). In particular, this also shows:

Corollary 5.8. *No lift of* $\operatorname{res}(K)$ *is definable in K.*

To find a dimension function on definable sets in $(K, \boldsymbol{k})$ compatible with the order topology, we adapt a dimension for pairs of real closed fields from [ÁD16] to this setting of pairs of real closed *differential* fields. The dimension $\dim_2$ we define in the next subsection extends dim in the sense that $\dim S = \dim_2 S$ for $S \subseteq K^n$ definable in K, but it also gives $\dim_2 \boldsymbol{k} = 0$, thereby rectifying the defect above.

Model-theoretically, we expand K by a unary relation U interpreted as $\boldsymbol{k}$ and axiomatize the theory $T^{\mathrm{dhl}}_{\mathrm{lift}}$ by adding to T^{dhl} axioms expressing that $\boldsymbol{k}$ is a subfield of $\mathcal{O}$ and for every $a \asymp 1$ in K, there exists $b \in \boldsymbol{k}$ such that $a \sim b$. Set $\mathcal{L} \coloneqq \mathcal{L}_{\partial,\leqslant,\preccurlyeq}$ and $\mathcal{L}_{\mathrm{lift}} \coloneqq \mathcal{L} \cup \{U\}$. Then as an $\mathcal{L}_{\mathrm{lift}}$-structure, $(K, \boldsymbol{k}) \models T^{\mathrm{dhl}}_{\mathrm{lift}}$. In this language, $T^{\mathrm{dhl}}_{\mathrm{lift}}$ does not have quantifier elimination, but we give instead in Theorem 5.10 a reduction to certain special formulas. In [Pyn24b, Section 5.3], we give a general quantifier elimination statement that applies to such structures in $\mathcal{L}_{\mathrm{lift}}$ expanded by a "standard part" map $\pi \colon \mathcal{O} \to \boldsymbol{k}$ satisfying $\pi(a) = b$ for a and b as above. Omitting π from the language used here simplifies the terms of the language, but at the cost of requiring considerable care in the proof of Theorem 5.10 and forgoing full quantifier elimination.

Definition 5.9. An $\mathcal{L}_{\mathrm{lift}}$-formula is **special** if it is of the form $\exists x \in U\ \varphi(x, y)$, where $\varphi(x, y)$ is a quantifier-free $\mathcal{L}$-formula.

In the definition above and later, $x \in U$ abbreviates $\bigwedge_{i=1}^{m} x_i \in U$ and $\exists x \in U\ \varphi(x, y)$ abbreviates $\exists x (x \in U \wedge \varphi(x, y))$.

Theorem 5.10. *Every $\mathcal{L}_{\mathrm{lift}}$-formula is $T^{\mathrm{dhl}}_{\mathrm{lift}}$-equivalent to a boolean combination of special formulas.*

As shown earlier, we have closed discrete subsets of K definable in $(K, \boldsymbol{k})$ that are not thin, i.e., have differential-algebraic dimension 1 (like $\boldsymbol{k}$ itself). We thus broaden the notion of thinness to incorporate $\boldsymbol{k}$, thereby singling out the sets which will have dimension 0 as measured by the corresponding dimension.

Definition 5.11. We call $S \subseteq K$ **lean** if S is contained in a finite union of sets of the form
$$\{a \in K : P(u,a) = 0 \text{ for some } u \in \boldsymbol{k}^m \text{ with } P(u,Z) \neq 0\},$$
where $P \in K\{X_1, \dots, X_m, Z\}$.

Regarding terminology, "lean" here has a different meaning and was defined independently of its other usage in this volume.

Note that every thin set is lean, but so is $\boldsymbol{k}$, which is not thin. On the other hand, we show in Lemma 5.15 that K is not lean, which is more subtle than showing it is not thin. The lean subsets of K thus form an ideal properly containing the ideal of thin sets. Additionally, they give rise to a pregeometry in $(K, \boldsymbol{k})$ in which the closure of $A \subseteq K$ is the set of elements of K that are differentially algebraic over $\boldsymbol{k}\langle A \rangle$. The collection of formulas defining this pregeometry comprises those of the form $\exists x \in U\ (P(x,y,z) = 0 \wedge P(x,y,Z) \neq 0)$ with parameter variables y, where $P \in \mathbb{Z}\{X_1, \dots, X_m, Y_1, \dots, Y_n, Z\}$. As in [ÁD16], this yields a dimension notion: For nonempty $S \subseteq K^n$ definable in $(K, \boldsymbol{k})$, set
$$\dim_2 S \ := \ \max\{\operatorname{tr.deg}_\partial(K\boldsymbol{k}^*\langle s \rangle | K\boldsymbol{k}^*) : s \in S^*\},$$
where $(K^*, \boldsymbol{k}^*) \succcurlyeq (K, \boldsymbol{k})$ is $|K|^+$-saturated, $S^* \subseteq (K^*)^n$ is defined by the same formula as S, and $K\boldsymbol{k}^*$ denotes the subfield of K^* generated by K and $\boldsymbol{k}^*$ (which is a differential subfield of K^*). As explained in [ÁD16, Section 2], $\dim_2 S$ does not depend on the choice of the $|K|^+$-saturated $(K^*, \boldsymbol{k}^*) \succcurlyeq (K, \boldsymbol{k})$ and remains the same in an elementary extension. We set $\dim_2 \emptyset := -\infty$. In [ÁD16], $\dim_2$ denotes a dimension function in a pair of real closed fields, and we use the same notation for the relevant dimension in our expansions of such pairs. Our definition of lean sets and the corresponding definition of $\dim_2$ singles out the sets $S \subseteq K$ with $\dim_2 S = 0$, by [ÁD16, Lemma 2.2]:

Lemma 5.12. *If $S \subseteq K$ is definable in $(K, \boldsymbol{k})$ and nonempty, then $\dim_2 S = 0$ if and only if S is lean.*

In particular, if $S \subseteq K$ is definable in K with $\dim S = 0$, then $\dim_2 S = 0$.

As in Lemma 2.1, various natural basic properties of $\dim_2$ follow from this definition by [ÁD16, Lemmas 2.1, 2.4].

Lemma 5.13. *Let $S_1 \subseteq K^m$ and $S_2 \subseteq K^n$ be definable in $(K, \boldsymbol{k})$. Then*
- (i) *if S_1 is finite and nonempty, then $\dim_2 S_1 = 0$;*
- (ii) *if $m = n$ and $S_1 \subseteq S_2$, then $\dim_2 S_1 \leqslant \dim_2 S_2$;*
- (iii) *if $m = n$, then $\dim_2(S_1 \cup S_2) = \max\{\dim_2 S_1, \dim_2 S_2\}$;*
- (iv) *$\dim_2(S_1 \times S_2) = \dim_2 S_1 + \dim_2 S_2$.*

As well, $\dim_2$ is invariant under permutation of coordinates. Note that the statement $\dim_2 K^n = n$ comes later in Corollary 5.16, since its proof is nontrivial. After that, we show that $\dim_2$ is a dimension function in the sense of Definition 2.2. Towards these results, we adapt [ADH17a, Lemmas 16.6.9 and 16.6.10] to establish a technical result enabling us also to prove local o-minimality of $T^{\mathrm{dhl}}_{\mathrm{lift}}$. Along the way, we get the following consequence:

Corollary 5.14. *If $(E, \mathcal{O}_E)$ is a nontrivially valued pre-H-field with gap 0 and $\boldsymbol{k}_E$ is a lift of its differential residue field, then E is differentially transcendental over $\boldsymbol{k}_E$.*

Some years ago, James Freitag asked whether an existentially closed pre-H-field with gap 0 could be differentially algebraic over a lift of its differential residue field, which the above corollary answers negatively in a general way. Using the same technical result:

Proposition 5.15. *The set K is not lean in $(K, \boldsymbol{k})$.*

If K were lean in $(K, \boldsymbol{k})$, then every definable subset $S \subseteq K$ would have $\dim_2 S = 0$. Proposition 5.15 thus shows that $\dim_2$ is a nontrivial notion of dimension. By Lemma 5.13:

Corollary 5.16. *For all n, we have $\dim_2 K^n = n$.*

Next, we use the technical result alluded to above to establish local o-minimality of $T^{\mathrm{dhl}}_{\mathrm{lift}}$, which underlies the connection between dimension and the order topology.

Proposition 5.17. *The theory $T^{\mathrm{dhl}}_{\mathrm{lift}}$ is locally o-minimal.*

In a somewhat similar way to [ADH17a, Lemma 4.4.10], we can show:

Lemma 5.18. *Let $S \subseteq K^n$ be definable in $(K, \boldsymbol{k})$. If*

$$S \ \subseteq \ \bigcup_{i=1}^{r} \{a \in K^n : \exists x \in U^m \ (P_i(x, a) = 0 \wedge P_i(x, Y_1, \dots, Y_n) \neq 0)\},$$

for some $r \in \mathbb{N}$ and $P_i \in K\{X_1, \dots, X_m, Y_1, \dots, Y_n\}$, $i = 1, \dots, r$, then S has empty interior in K^n.

We have not been able to establish the converse, which would yield the equivalence of $\dim_2$ with topological dimension in the sense of Lemma 5.2 and Corollary 5.3. However, we are able to establish it in case $n = 1$, which is an analogue of [ADH17a, Corollary 16.6.11]. Its proof requires considerably more care, and relies heavily on local o-minimality and its proof.

Proposition 5.19. *Let $S \subseteq K$ be definable in $(K, \boldsymbol{k})$. Then S is the disjoint union of an open definable set and a discrete definable set. Moreover, S is discrete if and only if S is lean.*

Recall from Definition 2.2 that $\dim_2$ is a **dimension function (in $(K, \boldsymbol{k})$)** if for all $f\colon S \to K^n$ definable in $(K, \boldsymbol{k})$ with $S \subseteq K^m$,

(1) $\dim_2 S \geqslant \dim_2 f(S)$ (in particular, $\dim_2 S = \dim_2 f(S)$ for injective f);
(2) $B_i := \{b \in K^n : \dim_2 f^{-1}(b) = i\}$ is definable in $(K, \boldsymbol{k})$ and for $i = 0, \dots, m$, $\dim_2 f^{-1}(B_i) = i + \dim_2 B_i$.

Now using [ÁD16, Proposition 2.7], we obtain from Proposition 5.19:

Corollary 5.20. *The dimension $\dim_2$ is a dimension function.*

Next we extend the last part of Proposition 5.19 to subsets of K^n, following Fact 4.5.

Theorem 5.21. *Let $S \subseteq K^n$ be definable in $(K, \boldsymbol{k})$ and nonempty. Then*

$$\dim_2 S = 0 \iff S \text{ is discrete.}$$

The proof of [ADH17b, Corollary 4.2] also goes through unchanged, which has the following consequence worth recording:

Corollary 5.22. *Every discrete subset of K^n definable in $(K, \boldsymbol{k})$ is closed in K^n.*

Finally, we observe that $\dim_2$ agrees with dim on sets definable in K, for which the key step is that $S \subseteq K$ definable in K is thin if and only if it lean, by Corollary 5.5 and Proposition 5.19.

Corollary 5.23. *Let $S \subseteq K^n$ be definable in K. Then $\dim S = \dim_2 S$.*

5.4. **Coanalyzability.** In this final section, we explain how the dimensions considered here are connected to the model-theoretic notion of coanalyzability. In the previous sections, coanalyzability is relative to the constant field. Here we will consider that notion as well as coanalyzability in $(K, \boldsymbol{k})$ relative to $\boldsymbol{k}$, and contrast the two. The contrast is connected to and indeed suggested by the difference between [Pyn24a, Theorem 4.12] and [ADH17a, Theorem 16.0.3]. Let $K \models T^{\mathrm{dhl}}$. As before, we equip K with a lift $\boldsymbol{k} \subseteq \mathcal{O}$ of its differential residue field. Here are the two relevant notions of coanalyzability of a definable set in $(K, \boldsymbol{k})$:

Definition 5.24. Let $\varphi(x, y)$ be an $\mathcal{L}_{\mathrm{lift}}$-formula. Let $a \in K^m$, and $\mathrm{Th}(K, \boldsymbol{k}, a)$ denote the $\mathcal{L}_{\mathrm{lift}}(a)$-theory of $(K, \boldsymbol{k})$ expanded by constants for a. Then $\varphi(a, K)$ is

(1) **coanalyzable relative to** C if whenever $(L, \boldsymbol{k}_L) \preccurlyeq (L^*, \boldsymbol{k}_{L^*}) \models \mathrm{Th}(K, \boldsymbol{k}, a)$ with $C_L = C_{L^*}$, we have $\varphi(a, L) = \varphi(a, L^*)$;

(2) **coanalyzable relative to** $\boldsymbol{k}$ if whenever $(L, \boldsymbol{k}_L) \preccurlyeq (L^*, \boldsymbol{k}_{L^*}) \models \mathrm{Th}(K, \boldsymbol{k}, a)$ with $\boldsymbol{k}_L = \boldsymbol{k}_{L^*}$, we have $\varphi(a, L) = \varphi(a, L^*)$.

It follows from [Pyn24b, Theorem 4.3] and [ADH17a, Proposition 7.1.3] that for an $\mathcal{L}$-formula $\varphi(x, y)$ and $a \in K^m$, $\varphi(a, K)$ is coanalyzable relative to C in $(K, \boldsymbol{k})$ if and only if $\varphi(a, K)$ is coanalyzable relative to C in K (i.e., whenever $L \preccurlyeq L^* \models \mathrm{Th}(K, a)$ with $C_L = C_{L^*}$, we have $\varphi(a, L) = \varphi(a, L^*)$). Also, since $C = C_{\boldsymbol{k}} \subseteq \boldsymbol{k}$, if $S \subseteq K^n$ is coanalyzable relative to C, then S is coanalyzable relative to $\boldsymbol{k}$.

In a closed H-field, Fact 4.7 shows that coanalyzability relative to C is equivalent to having differential-algebraic dimension 0. In contrast, this fails here, essentially due to Fact 3.5.

Lemma 5.25. *There exists $S \subseteq K$ definable in K such that $\dim S = 0$ but S is not coanalyzable relative to C.*

Although having differential-algebraic dimension 0 does not imply being coanalyzable relative to C, the converse remains true in K, as an easy consequence of $\dim_2$ being a dimension function.

Lemma 5.26. *If $S \subseteq K^n$ is definable in K and coanalyzable relative to C, then $\dim S = 0$.*

Additionally, the proof of Lemma 5.25 shows that $\boldsymbol{k}$ itself is not coanalyzable relative to C in $(K, \boldsymbol{k})$. On the other hand, $\boldsymbol{k}$ is obviously coanalyzable relative to $\boldsymbol{k}$ in $(K, \boldsymbol{k})$, so the two variants of coanalyzability are indeed different. Since $\dim \boldsymbol{k} = 1$, it also shows that having differential-algebraic dimension 0 does not coincide with coanalyzability relative to $\boldsymbol{k}$ in the pair $(K, \boldsymbol{k})$. This contrasts with Fact 4.7 for closed H-fields, since in a closed H-field the constant field is a lift of the residue field. This defect is rectified by $\dim_2$, where of course $\dim_2 \boldsymbol{k} = 0$. Towards that, we first use [Pyn24a, Theorem 4.12] to show:

Lemma 5.27. *Every lean subset of K is coanalyzable relative to $\boldsymbol{k}$.*

As a side note, Lemma 5.25 explains why the right analogue of [ADH17a, Theorem 16.0.3] for pre-H-fields with gap 0, namely [ADH17a, Theorem 4.12] used above, involves the differential residue field rather than the constant field. Indeed, the direct analogue of [ADH17a, Theorem 16.0.3] (without changing C to $\boldsymbol{k}$ in the statement) fails.

Corollary 5.28. *There exist $K \preccurlyeq K^* \models T^{\mathrm{dhl}}$ such that K^* is a proper differentially algebraic pre-H-field extension of K with $C = C_{K^*}$.*

On the other hand, we regain the equivalence of coanalyzability and dimension 0 as soon as we pass to the appropriate notion of dimension in this situation, namely $\dim_2$.

Theorem 5.29. *Let $S \subseteq K^n$ be definable in $(K, \boldsymbol{k})$ and nonempty. Then*

$$\dim_2 S = 0 \iff S \text{ is coanalyzable relative to } \boldsymbol{k}.$$

Hence, the three notions of smallness, i.e., having $\dim_2 S = 0$, being discrete, and being coanalyzable relative to $\boldsymbol{k}$, all agree for nonempty sets $S \subseteq K^n$ definable in $(K, \boldsymbol{k})$. This result should remain true for transserial tame pairs from [Pyn24b] (although the proofs require more care). Maybe $\dim_2$ can be combined in some way with dim, which was well-studied in the context of closed H-fields in [ADH17b], to give a finer understanding of the sets definable in such transserial tame pairs.

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Kurt Gödel Research Center, Institute of Mathematics, University of Vienna, Austria
Email address: nigel.pynn-coates@univie.ac.at
URL: https://www.mat.univie.ac.at/~pnigel/