Differential-algebraic dimension in transserial tame pairs

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Mini-workshop: Recent developments in the model theory of fields
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Closed H-fields

Definition

A differential field K is a **closed H-field** if $K \equiv \mathbb{T}$, where $\mathbb{T}$ is the differential field of logarithmic-exponential transseries.

Example

1. $\mathbb{T}$ (Aschenbrenner–van den Dries–van der Hoeven)
2. Conway's surreal numbers **No** (Berarducci–Mantova, ADH)
3. hyperseries $\mathbb{H}$ (Bagayoko–van der Hoeven–Kaplan, Ba.)
4. Each maximal Hardy field (ADH)

Basic properties

Remark

Every closed H-field K satisfies:

1. *K is real closed, hence an ordered field;*
2. *K is a valued field with convex valuation ring $\mathcal{O} = \mathrm{conv}_K(C) = \{a \in K : |a| \leqslant c$ for some $c \in C^{>0}\}$, where $C = \{c \in K : \partial c = 0\}$;*
3. *C is* **tame in** *K, i.e., $\mathcal{O} = C + \scriptstyle\mathcal{O}$, where $\scriptstyle\mathcal{O}$ $= \{a \in K : |a| < C^{>0}\}$*
   - *i.e., for every $f \in \mathcal{O}$, there is $c \in C$ with $f - c \in \scriptstyle\mathcal{O}$;*
4. *$\partial \scriptstyle\mathcal{O} \subseteq \mathcal{O}$.*

Example

In a Hardy field H:

1. $C \subseteq \mathbb{R}$;
2. $\mathcal{O} = \{f \in H : \lim_{x \to +\infty} f(x) \in \mathbb{R}\}$;
3. $\scriptstyle\mathcal{O}$ $= \{f \in H : \lim_{x \to +\infty} f(x) = 0\}$.

Pairs of closed H-fields

$\mathbb{T}$, **No**, $\mathbb{H}$, and maximal Hardy fields are closed H-fields. Moreover:

Theorem

1. $\mathbb{H} \succcurlyeq \mathbb{T}$ *(BaHK, Ba.).*
2. $\mathbf{No} \succcurlyeq \mathbb{T}$ *(BeM, ADH).*
3. *Each maximal Hardy field* $H \succcurlyeq \mathbb{T}$ *(ADH, AD).*

Question: What is the model theory of $(K, \mathbb{T})$, for $K \in \{\mathbb{H}, \mathbf{No}, \text{maximal Hardy fields}\}$?

Tame pairs of closed H-fields

Definition

A **transserial tame pair** is a pair (K, L) of differential fields such that:

1. K and L are closed H-fields;
2. $L \subsetneq K$ is a proper differential subfield;
3. L is tame in K, i.e., $\dot{\mathcal{O}} = L + \dot{o}$, where $\dot{\mathcal{O}} := \operatorname{conv}_K(L)$ and $\dot{o} := \{a \in K : |a| < L^{>0}\}$.

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Example

For $K \in \{\mathbb{H}, \mathbf{No}, \text{maximal Hardy fields}\}$, $\mathbb{T}$ can be extended to $\mathbb{T}^* \subseteq \dot{\mathcal{O}} = \mathrm{conv}_K(\mathbb{T})$ so that $(K, \mathbb{T}^*)$ is a transserial tame pair.

Differential-algebraic dimension

Definition

Let $S \subseteq K^n$ be nonempty and definable in an expansion of a differential field K. Let $K^* \succcurlyeq K$ be $|K|^+$-saturated and define

$$\dim S \;=\; \max\{\mathrm{tr.deg}_\partial(K\langle s\rangle|K) : s \in S^*\}.$$

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For S as above,

$$\dim S < n \iff S \subseteq \{a \in K^n : P(a) = 0\} \text{ for some } P \in K\{Y\}^{\neq 0}.$$

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Theorem (Aschenbrenner–van den Dries–van der Hoeven)

If $S \subseteq K^n$ is nonempty and definable in the closed H-field K, then:

1. $\dim S = 0 \iff$ *S is discrete in K^n;*
2. $\dim S < n \iff$ *S has empty interior in K^n.*

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Note: $\dim_2 L = 0$.

For S as above, the following are equivalent:

- $\dim_2 S < n$.
- $S \subseteq \bigcup_{i=1}^{k}\{b \in K^n : \exists a \in L^m\ (P_i(a, b) = 0 \wedge P_i(a, Y) \neq 0)\}$, for some $P_i \in K\{X, Y\}$.

Main theorem

Definition

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Theorem (PC)

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Key steps:

- case $n = 1$
- local o-minimality for transserial tame pairs and uniformity in its proof
- eliminating quantifiers from K

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Question: $\dim_2 S < n \iff S$ has empty interior in K^n?

Thank you!

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