Tame pairs of transseries fields

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Transseries

Let $\mathbb{T}$ be the differential field of logarithmic-exponential transseries constructed by Van den Dries–Macintyre–Marker. Example series in $\mathbb{T}$:

$$7e^{e^x+e^{x/2}+e^{x/4}+\cdots} - 3e^{x^2} + 5x^{\sqrt{2}} - (\log x)^{\pi} + 42 + x^{-1} + x^{-2} + \cdots + e^{-x}$$

History:

1. nonstandard models of real exponentiation (Dahn–Göring), o-minimality and functions definable in $\mathbb{R}_{\text{an,exp}}$ (Van den Dries–Macintyre–Marker)
2. Dulac's problem & Hilbert's 16th problem (Écalle)

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Remarks:

1. Ring operations make $\mathbb{T}$ a real closed field, ordering given by coefficient of leading term.
2. Derivation $\partial\colon \mathbb{T} \to \mathbb{T}$ satisfies
   - $\partial x = 1$;
   - $\partial(f+g) = \partial f + \partial g$;
   - $\partial(fg) = g\partial f + f\partial g$;
   - Chain Rule, in particular $\partial(e^f) = e^f \partial f$.
3. $\mathcal{O}_{\mathbb{T}} := \mathrm{conv}_{\mathbb{T}}(\mathbb{R}) = \{f \in \mathbb{T} : |f| \leqslant r \text{ for some } r \in \mathbb{R}^{>0}\}$ is a convex valuation ring of $\mathbb{T}$ distinguishing "bounded" and "large" elements.
4. $\mathbb{T}$ is exponentially bounded.

Model completeness for transseries

Let $\mathcal{O}_{\mathbb{T}} := \mathrm{conv}_{\mathbb{T}}(\mathbb{R}) = \{f \in \mathbb{T} : |f| \leqslant r \text{ for some } r \in \mathbb{R}^{>0}\}$.

Theorem (Aschenbrenner–Van den Dries–Van der Hoeven '17)

$(\mathbb{T}, \mathcal{O}_{\mathbb{T}})$ *is model complete.*

More: effective axiomatization, quantifier elimination in mild expansion, ...

Remarks on the language:

- $\mathbb{T}$ is construed as an differential field and $(\mathbb{T}, \mathcal{O}_{\mathbb{T}})$ is its expansion to a valued differential field. Ordering is defined by:
  - $a \geqslant 0 \iff \exists b\ (a = b^2)$.
- Careful! $\mathbb{T}$ is not model complete without the valuation ring, even though it is existentially definable:
  - $\mathbb{R} = C_{\mathbb{T}} = \{a \in \mathbb{T} : \partial a = 0\}$;
  - $\mathcal{O}_{\mathbb{T}} = \mathrm{conv}_{\mathbb{T}}(\mathbb{R}) = \{f \in \mathbb{T} : \exists r \in \mathbb{R}\ (r > 0 \wedge |f| \leqslant r)\}$.

Beyond transseries

$E(x+1) = \exp E(x)$ has no solution in $\mathbb{T}$ but does have a real analytic solution on $\mathbb{R}^{\geqslant}$ (Kneser) that belongs to a Hardy field (Boshernitzan).

- Note: any solution is transexponential!

Solutions also exist in hyperseries $\mathbb{H}$ (Bagayoko–Van der Hoeven–Kaplan) and surreal numbers **No** (Bagayoko–Van der Hoeven–Mantova).

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Theorem

1. $\mathbb{H} \succcurlyeq \mathbb{T}$ *(BaHK '21+, Ba '22+)*
2. **No** $\succcurlyeq \mathbb{T}$ *(Berarducci–Mantova '18, ADH '19)*
3. *Each maximal Hardy field* $H \succcurlyeq \mathbb{T}$ *(ADH '24+, AD '24+)*

In particular, they are model complete as valued differential fields. But can we say more about the model theory of these nonstandard models?

Nonstandard models: two valuations

Let $K \succcurlyeq \mathbb{T}$ with $f > \mathbb{T}$ for some $f \in K$ and

- $\mathcal{O}_K := \mathrm{conv}_K(\mathbb{R}) = \{f \in K : |f| \leqslant r \text{ for some } r \in \mathbb{R}^{>0}\}$;
- $\dot{\mathcal{O}} := \mathrm{conv}_K(\mathbb{T}) = \{f \in K : |f| \leqslant g \text{ for some } g \in \mathbb{T}^{>0}\}$.

Comparing the valuations:

- $a \prec b \iff |a/b| < r$ for every $r \in \mathbb{R}^{>}$
- $a \mathrel{\dot{\prec}} b \iff |a/b| < 1/\exp_n(x)$ for every $n \in \mathbb{N}$

For example:

- $x \prec \exp(x)$
- $x \mathrel{\dot{\not\prec}} \exp(x)$
- $x \mathrel{\dot{\prec}} \exp_\omega(x)$

So with both $\mathcal{O}_K$ and $\dot{\mathcal{O}}$ we can express more about the nonstandard K.

Model completeness for nonstandard models

Let $K \succcurlyeq \mathbb{T}$ with $f > \mathbb{T}$ for some $f \in K$ (e.g., hyperseries, surreal numbers, maximal Hardy fields) and $\mathcal{O}_K \coloneqq \mathrm{conv}_K(\mathbb{R})$ and $\dot{\mathcal{O}} \coloneqq \mathrm{conv}_K(\mathbb{T})$.

Theorem (PC '24+)

$(K, \mathcal{O}_K, \dot{\mathcal{O}})$ *is model complete.*

Transserial tame pairs

Definition

A **transserial tame pair** is a pair (K, L) of differential fields such that:

1. $(K, \mathcal{O}_K), (L, \mathcal{O}_L) \models \mathrm{Th}(\mathbb{T})$, where $\mathcal{O}_K := \mathrm{conv}_K(C_K)$ and $C_K := \{c \in K : \partial c = 0\}$ and $\mathcal{O}_L$ is defined likewise;
2. $L \subsetneq K$ is a proper differential subfield;
3. L is tame in K, i.e., $\dot{\mathcal{O}} = L + \dot{o}$, where $\dot{\mathcal{O}} := \mathrm{conv}_K(L)$ and $\dot{o} := \{a \in K : |a| < L^{>0}\}$.

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Example (tame pairs of RCFs)

- $(\bigcup_{n\geqslant 1} \mathbb{R}((t^{1/n})), \mathbb{R})$
- $(\mathbb{T}, \mathbb{R})$
- but not $(\mathbb{R}, \mathbb{Q}^{\mathrm{rc}})$!

Tame pairs of RCFs go back to Macintyre and Cherlin–Dickmann; o-minimal generalizations by Van den Dries–Lewenberg.

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Theorem (PC '24+)

Every transserial tame pair $(K, \dot{\mathcal{O}}, L, \mathcal{O}_L)$ is model complete.

Case study: Hyperseries

BHK/B construct a differential field $\mathbb{H} \succcurlyeq \mathbb{T}$ containing $\exp_\alpha(x)$ and $\log_\alpha(x)$ for every ordinal α; e.g., $\exp_\omega(x)$ solves $E(x+1) = \exp E(x)$.

Let:

1. $\mathcal{O}_{\mathbb{H}} := \operatorname{conv}_{\mathbb{H}}(\mathbb{R})$ and $\dot{\mathcal{O}} := \operatorname{conv}_{\mathbb{H}}(\mathbb{T})$;
2. $\mathfrak{M}$ be the monomial group of $\mathbb{H}$ and $\mathfrak{B} := \mathfrak{M} \cap \dot{\mathcal{O}}^\times$;
3. $\mathbb{T}^* := \{f \in \mathbb{H} : \operatorname{supp} f \subseteq \mathfrak{B}\}$ and $\mathcal{O}_{\mathbb{T}^*} := \operatorname{conv}_{\mathbb{T}^*}(\mathbb{R})$.

Proposition

$(\mathbb{H}, \mathbb{T}^*)$ *is a transserial tame pair.*

Corollary

$(\mathbb{H}, \mathcal{O}_{\mathbb{H}}, \dot{\mathcal{O}})$ *and* $(\mathbb{H}, \dot{\mathcal{O}}, \mathbb{T}^*, \mathcal{O}_{\mathbb{T}^*})$ *are model complete.*

Three perspectives: coarsening and uncoarsening

Proposition

There are theories

- T_1 *in the language* $\{+, -, \cdot, 0, 1, \partial, \leqslant, \dot{\mathcal{O}}\}$ *and*
- T_2 *in the language* $\{+, -, \cdot, 0, 1, \partial, \leqslant, \mathcal{O}, \dot{\mathcal{O}}\}$

such that the following are equivalent:

1. $(K, \dot{\mathcal{O}}) \models T_1$ *and* $\dot{K} = \dot{\mathcal{O}}/\dot{o} \models \mathrm{Th}(\mathbb{T})$*;*
2. $(K, \dot{\mathcal{O}}, \dot{K}, \mathcal{O}_{\dot{K}})$ *is a transserial tame pair;*
3. $(K, \mathcal{O}, \dot{\mathcal{O}}) \models T_2$*.*

Essentially, the proof of the main results go through this decomposition.

Relative results

More precisely, let:

- $(K, \boldsymbol{k})$ be a pair of differential fields, possibly extra structure on $\boldsymbol{k}$;
- $\dot{\mathcal{O}} := \mathrm{conv}_K(\boldsymbol{k})$;
- $(K, \dot{\mathcal{O}}) \models T_1$;
- $\boldsymbol{k}$ is a lift of $\dot{\mathcal{O}}/\dot{o}$, i.e., $\boldsymbol{k}$ is a differential subfield of $\dot{\mathcal{O}}$ that is tame.

Theorem (PC '24+)

1. $(K, \boldsymbol{k}) \equiv (K^*, \boldsymbol{k}^*) \iff \boldsymbol{k} \equiv \boldsymbol{k}^*$.
2. *If $\boldsymbol{k}$ is model complete, then so is $(K, \dot{\mathcal{O}}, \boldsymbol{k})$.*
3. *Any subset of $\boldsymbol{k}^n$ definable in $(K, \boldsymbol{k})$ is definable in $\boldsymbol{k}$.*

Also, relative quantifier elimination with standard part map $\dot{\mathcal{O}} \to \boldsymbol{k}$.

Relative results II

Theorem (PC '24+)

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2. *If $\boldsymbol{k}$ is model complete, then so is $(K, \dot{\mathcal{O}}, \boldsymbol{k})$.*
3. *Any subset of $\boldsymbol{k}^n$ definable in $(K, \boldsymbol{k})$ is definable in $\boldsymbol{k}$.*

Corollary (PC '24+)

1. *The theory of transserial tame pairs is complete.*
2. *The theory of transserial tame pairs is model complete.*
3. *Any subset of L^n definable in (K, L) is definable in L and any subset of C^n definable in (K, L) is definable in C.*

Also, QE with standard part map and mild expansion on L.

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These extensions decompose into a "transexponential part" and an "exponentially bounded part", e.g., $(\mathbb{H}, \dot{\mathcal{O}})$ and $(\mathbb{T}^*, \mathcal{O}_{\mathbb{T}^*})$.

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The valued differential field $\mathbb{T}$ of transseries is model complete (ADH).

Transserial tame pairs refine our understanding of nonstandard extensions of $\mathbb{T}$, namely hyperseries, surreal numbers, and maximal Hardy fields.

These extensions decompose into a "transexponential part" and an "exponentially bounded part", e.g., $(\mathbb{H}, \dot{\mathcal{O}})$ and $(\mathbb{T}^*, \mathcal{O}_{\mathbb{T}^*})$.

Considering $\mathbb{T}^*$ within $(\mathbb{H}, \dot{\mathcal{O}})$ does not define any new subsets of $\mathbb{T}^*$.

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