Monotone T-convex T-differential fields

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joint work with Elliot Kaplan

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Introduction: fields with analytic structure

- $\mathbb{R}_{\text{an}}$ is o-minimal (van den Dries, Gabrielov)
- $\mathbb{Q}_p$ with restricted analytic functions (Denef–van den Dries)
- Further work: Schoutens, van den Dries–Haskell–Macpherson, subsets of Cluckers–Lipshitz–Z. Robinson, van den Dries–Macintyre–Marker, Haskell–Cubides Kovacsics…

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One general framework:

- T-convex valued fields: nonstandard model of o-minimal theory T considered with the natural valuation ring given by the convex hull of a smaller model (van den Dries–Lewenberg)

Introduction: differential fields with analytic structure

- What about when the fields have natural derivations?
- $\mathbb{T}_{\text{an}}$ is a nonstandard model of the theory of $\mathbb{R}_{\text{an}}$ (van den Dries–Macintyre–Marker).
- Derivation should be *compatible* with the analytic structure: T-derivation (Fornasiero–Kaplan).

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Today:

- Monotone T-convex T-differential fields.

Antecedents:

- H_T-fields (E. Kaplan).
- Immediate extensions of T-convex T-differential fields (E. Kaplan).
- Related: analytic valued difference fields (Rideau, Scanlon).

Hahn fields

- Let $\boldsymbol{k}$ be a field and Γ be an ordered abelian group.

$$\boldsymbol{k}(\!(t^\Gamma)\!) \;:=\; \Big\{ f = \sum_\gamma f_\gamma t^\gamma : \operatorname{supp} f \text{ is well-ordered} \Big\}$$

is a **Hahn field**.

Example (Formal Laurent series over $\mathbb{R}$)

$\mathbb{R}(\!(t^{\mathbb{Z}})\!)$: take $\Gamma = \mathbb{Z}$ and $\boldsymbol{k} = \mathbb{R}$.

- $v \colon \boldsymbol{k}(\!(t^\Gamma)\!)^\times \to \Gamma$ defined by $v(f) = \min \operatorname{supp} f$ is a **valuation**.

Differential Hahn fields I

- Let $\boldsymbol{k}$ be a differential field and Γ be an ordered abelian group. The derivation ∂ of $\boldsymbol{k}$ can be extended to the Hahn field

$$\boldsymbol{k}(\!(t^{\Gamma})\!) := \Big\{ f = \sum_{\gamma} f_{\gamma} t^{\gamma} : \operatorname{supp} f \text{ is well-ordered} \Big\}$$

by

$$\partial\Big(\sum_{\gamma} f_{\gamma} t^{\gamma}\Big) := \sum_{\gamma} (\partial f_{\gamma}) t^{\gamma}.$$

- ∂ is **monotone**: $v(\partial f) \geqslant vf$.
- Differential Hahn fields were studied by Scanlon.

Differential Hahn fields II

- Let $\boldsymbol{k}$ be a differential field and Γ be an ordered abelian group. The derivation ∂ of $\boldsymbol{k}$ can be extended to $\boldsymbol{k}(\!(t^{\Gamma})\!)$ by

$$\partial\Big(\sum_{\gamma} f_{\gamma} t^{\gamma}\Big) \ := \ \sum_{\gamma} \big(\partial f_{\gamma} + f_{\gamma} c(\gamma)\big) t^{\gamma},$$

  where $c\colon \Gamma \to \boldsymbol{k}$ is an additive map.
- c satisfies $c(\gamma) = \partial(t^{\gamma})/t^{\gamma}$.
- ∂ is **monotone**: $v(\partial f) \geqslant vf$.

Example

In $\mathbb{R}(\!(t^{\mathbb{Z}})\!)$, define $\partial = t\frac{d}{dt}$, so $c(\ell) = \ell$ for $\ell \in \mathbb{Z}$.

- Differential Hahn fields were studied by Scanlon and Hakobyan; more generally, they studied monotone valued differential fields.

T_{an}-convex Hahn fields

- Let $\boldsymbol{k} \models T_{an}$ and Γ be a divisible ordered abelian group: $\boldsymbol{k}(\!(t^\Gamma)\!)$.
- Set $\mathcal{O} := \{f : vf \geqslant 0\} = \{\sum_{\gamma \geqslant 0} f_\gamma t^\gamma\}$ and $\scriptstyle\mathcal{O}$ $:= \{f : vf > 0\} = \{\sum_{\gamma > 0} f_\gamma t^\gamma\}$.
- Order $\boldsymbol{k}(\!(t^\Gamma)\!)$ by $f > 0$ if $f_{vf} > 0$.
- $\mathcal{O}$ is the convex hull of $\boldsymbol{k}$ with $\mathcal{O} = \boldsymbol{k} + \scriptstyle\mathcal{O}$.

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- Expand $\boldsymbol{k}((t^{\Gamma}))$ to a model of T_{an} by Taylor expansion.
- $\mathcal{O}$ is T_{an}-**convex**: every $\emptyset$-definable continuous $F \colon \boldsymbol{k}((t^{\Gamma})) \to \boldsymbol{k}((t^{\Gamma}))$ satisfies $F(\mathcal{O}) \subseteq \mathcal{O}$.

T_{an}-differential Hahn fields

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- Expand $\boldsymbol{k}(\!(t^{\Gamma})\!)$ to a model of T_{an} by Taylor expansion.
- Let $\boldsymbol{k}$ be equipped with a T_{an}-**derivation** ∂: every $\emptyset$-definable $\mathcal{C}^1$-function $F\colon U \to \boldsymbol{k}$ satisfies $\partial F(u) = \nabla F(u) \cdot \partial u$ for $u \in U \subseteq \boldsymbol{k}^n$ open (Fornasiero–Kaplan).

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- Extend ∂ to $\boldsymbol{k}(\!(t^\Gamma)\!)$ as before, for an additive $c\colon \Gamma \to \boldsymbol{k}$:

$$\partial\Big(\sum_{\gamma} f_\gamma t^\gamma\Big) \;:=\; \sum_{\gamma} \big(\partial f_\gamma + f_\gamma c(\gamma)\big)\, t^\gamma,$$

- Then ∂ is a T_{an}-derivation on $\boldsymbol{k}(\!(t^\Gamma)\!)$.

Monotone T_{an}-convex T_{an}-differential fields

Putting these together, $\boldsymbol{k}((t^\Gamma))$ is a T_{an}-convex T_{an}-differential field that is **monotone**: $v(\partial f) \geqslant vf$.

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Equip $K \models T_{an}$ with a T_{an}-convex subring $\mathcal{O} \subseteq K$ and a T_{an}-derivation ∂.

- $\mathcal{O}$ induces a valuation $v\colon K^{\times} \to \Gamma$.
- $\boldsymbol{k} = \mathcal{O}/\mathcal{o}$, where $\mathcal{o} = \{f : vf > 0\}$.
- Suppose that ∂ is monotone.

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- $\boldsymbol{k} = \mathcal{O}/\smallo$, where $\smallo = \{f : vf > 0\}$.
- Suppose that ∂ is monotone.

Theorem (Kaplan–PC)

If K is T_{an}^{∂}-henselian, then K is elementarily equivalent to $\boldsymbol{k}(\!(t^{\Gamma})\!)$, for some additive $c\colon \Gamma \to \boldsymbol{k}$ associated to K.

- T_{an}^{∂}-henselianity is an analogue of henselianity of a valued field.

An Ax–Kochen/Ershov theorem

More generally:

- Suppose that T is o-minimal, complete, model complete, and power bounded with field of exponents Λ.
- Equip $K \models T$ with a T-convex subring and a T-derivation that is monotone.
- Associate to K a Λ-linear $c\colon \Gamma \to \boldsymbol{k}$.

Define K^* likewise.

Theorem (Kaplan–PC)

If K and K^* are T^∂-henselian, then

$$K \equiv K^* \iff (\boldsymbol{k}, \Gamma; c) \equiv (\boldsymbol{k}^*, \Gamma^*; c^*).$$

Monotonicity and power boundedness

Lemma

If K is a monotone T-convex T-differential field and $\boldsymbol{k}$ has nontrivial derivation (in particular, if K is T^{∂}-henselian), then T is power bounded.

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Related:

Theorem (Kuhlmann–Kuhlmann–Shelah)

If T is exponential and $K \models T$ is equipped with a T-convex subring, then K has no spherically complete immediate T-convex extension.

In contrast, Kaplan proves that such T-convex T-differential extensions exist when T is power bounded.

Further results

Theorem (Kaplan–PC)

The theory of monotone T-convex T-differential fields has a model completion, which is distal.

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Working in a three-sorted setting $\mathcal{K} = (K, \boldsymbol{k}, \Gamma; \pi, v, c)$ sharpens the AKE theorem (relative completeness) and yields relative model completeness.

Theorem (Kaplan–PC)

Expanding $\mathcal{K}$ by an angular component map yields elimination of quantifiers from K.

Corollary

Any subset of $\boldsymbol{k}^m \times \Gamma^n$ definable in $\mathcal{K}$ is definable in $(\boldsymbol{k}, \Gamma; c)$.

T^∂-henselianity

Let K be a T-convex T-differential field with $\partial\mathcal{O} \subseteq \mathcal{O}$ and let:

- $F\colon K^{1+r} \to K$ be a definable function;
- $A(Y) = a_0 Y + a_1 Y' + \cdots + a_r Y^{(r)}$ be a linear differential polynomial over K of order r.

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Definition

K is T^∂-**henselian** if:

1. Every linear differential equation over $\boldsymbol{k}$ has a solution in $\boldsymbol{k}$;
2. Whenever:
   (i) A linearly approximates F on $B(a, vg) = \{b : v(b-a) > vg\}$, and
   (ii) $vF(a, a', \ldots, a^{(r)}) > vA(gY)$,

   there is $b \in B(a, vg)$ such that $F(b, b', \ldots, b^{(r)}) = 0$ and $vA\big((b-a)Y\big) \geqslant vF(a, a', \ldots, a^{(r)})$.

Thank you!

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