Monotone T-convex T-differential fields

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joint work with Elliot Kaplan

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Hahn fields

- Let $\boldsymbol{k}$ be a field and Γ be an ordered abelian group.

$$\boldsymbol{k}(\!(t^{\Gamma})\!) := \Big\{ f = \sum_{\gamma} f_{\gamma} t^{\gamma} : \operatorname{supp} f \text{ is well-ordered} \Big\}$$

  is a **Hahn field**.

Example (Formal Laurent series over $\mathbb{R}$)

$\mathbb{R}(\!(t^{\mathbb{Z}})\!)$: take $\Gamma = \mathbb{Z}$ and $\boldsymbol{k} = \mathbb{R}$.

- $v\colon \boldsymbol{k}(\!(t^{\Gamma})\!)^{\times} \to \Gamma$ defined by $v(f) = \min \operatorname{supp} f$ is a **valuation**.

Differential Hahn fields I

- Let $\boldsymbol{k}$ be a differential field and Γ be an ordered abelian group. The derivation ∂ of $\boldsymbol{k}$ can be extended to the Hahn field

$$\boldsymbol{k}(\!(t^{\Gamma})\!) \ :=\ \Big\{ f = \sum_{\gamma} f_\gamma t^\gamma : \operatorname{supp} f \text{ is well-ordered} \Big\}$$

  by

$$\partial\Big(\sum_{\gamma} f_\gamma t^\gamma\Big) \ :=\ \sum_{\gamma} (\partial f_\gamma) t^\gamma.$$

- ∂ is **monotone**: $v(\partial f) \geqslant vf$.
- Differential Hahn fields were studied by Scanlon.

Differential Hahn fields II

- Let $\boldsymbol{k}$ be a differential field and Γ be an ordered abelian group. The derivation ∂ of $\boldsymbol{k}$ can be extended to $\boldsymbol{k}(\!(t^\Gamma)\!)$ by

$$\partial\Big(\sum_\gamma f_\gamma t^\gamma\Big) := \sum_\gamma \big(\partial f_\gamma + f_\gamma c(\gamma)\big)t^\gamma,$$

  where $c\colon \Gamma \to \boldsymbol{k}$ is an additive map.
- c satisfies $c(\gamma) = \partial(t^\gamma)/t^\gamma$.
- ∂ is monotone: $v(\partial f) \geqslant vf$.

Example

In $\mathbb{R}(\!(t^{\mathbb{Z}})\!)$, define $\partial = t\frac{d}{dt}$, so $c(\ell) = \ell$ for $\ell \in \mathbb{Z}$.

- Differential Hahn fields were studied by Scanlon and Hakobyan; more generally, they studied monotone valued differential fields.

T_{an}-convex T_{an}-differential Hahn fields

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- Expand $\boldsymbol{k}(\!(t^\Gamma)\!)$ to a model of T_{an} by Taylor expansion.
- The subring $\mathcal{O} := \{f : vf \geqslant 0\} = \mathrm{conv}(\boldsymbol{k})$ is T_{an}-**convex**: every $\emptyset$-definable continuous $F\colon \boldsymbol{k}(\!(t^\Gamma)\!) \to \boldsymbol{k}(\!(t^\Gamma)\!)$ satisfies $F(\mathcal{O}) \subseteq \mathcal{O}$.

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- Assume that $\boldsymbol{k}$ is equipped with a T_{an}-**derivation** ∂: every $\emptyset$-definable $\mathcal{C}^1$-function $F\colon U \to \boldsymbol{k}$ satisfies $\partial F(u) = \nabla F(u) \cdot \partial u$ for $u \in U$. Then ∂ extended to $\boldsymbol{k}((t^{\Gamma}))$ is a T_{an}-derivation.

Monotone T_{an}-convex T_{an}-differential fields

Equip $K \models T_{\mathrm{an}}$ with a T_{an}-convex subring $\mathcal{O} \subseteq K$ and a T_{an}-derivation ∂.

- $\mathcal{O}$ induces a valuation $v\colon K^\times \to \Gamma$.
- $\boldsymbol{k} = \mathcal{O}/\smallo$, where $\smallo = \{f : vf > 0\}$.
- Suppose that ∂ is monotone: $v(\partial f) \geqslant vf$.

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Theorem (Kaplan–PC)

If K is $T^{\partial}_{\mathrm{an}}$-henselian, then K is elementarily equivalent to $\boldsymbol{k}(\!(t^\Gamma)\!)$, for an additive $c\colon \Gamma \to \boldsymbol{k}$ associated to K.

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If K is $T^{\partial}_{\mathrm{an}}$-henselian, then K is elementarily equivalent to $\boldsymbol{k}((t^\Gamma))$, for an additive $c\colon \Gamma \to \boldsymbol{k}$ associated to K.

- $T^{\partial}_{\mathrm{an}}$-henselianity is an analogue of henselianity of a valued field.

An Ax–Kochen/Ershov theorem

More generally:

- Suppose that T is o-minimal and power bounded.
- Equip $K \models T$ with a T-convex subring and a T-derivation.
- Associate to K a Λ-linear $c\colon \Gamma \to \boldsymbol{k}$, where Λ is the field of exponents.

Define K^* likewise.

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- Suppose that T is o-minimal and power bounded.
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Theorem (Kaplan–PC)

If K and K^* are T^∂-henselian, then

$$K \equiv K^* \iff (\boldsymbol{k}, \Gamma; c) \equiv (\boldsymbol{k}^*, \Gamma^*; c^*).$$

Thank you!