Model theory of differential-henselian pre-H-fields

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A differential-henselian pre-H-field

Let $\mathbb{T}$ be the differential field of logarithmic-exponential transseries.

- Example series in $\mathbb{T}$:

$$7e^{e^{x}+e^{x/2}+e^{x/4}+\cdots}-3e^{x^2}+5x^{\sqrt{2}}-(\log x)^{\pi}+42+x^{-1}+x^{-2}+\cdots+e^{-x}$$

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  - Maximal Hardy field (Aschenbrenner–van den Dries–van der Hoeven, Boshernitzan).
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- Enlarge $\mathcal{O}$ to $\mathcal{O}^* := \{f : |f| \leqslant e^x \text{ or } |f| \leqslant e^{e^x} \text{ or } \ldots\}$.

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- $\mathrm{res}(K, \mathcal{O}^*) \models \mathrm{Th}(\mathbb{T})$.
- Question: What is the model theory of $(K, \mathcal{O}^*)$?

Pre-H-fields

- A **pre-H-field** is an ordered valued differential field K such that:
  1. $\mathcal{O}$ is convex with respect to $\leqslant$;
  2. for all $f \in K$, $f > \mathcal{O} \implies f' > 0$;
  3. for all $f, g \in K^{\times}$, $f \preccurlyeq g \prec 1 \implies \frac{f}{g} - \frac{f'}{g'} \prec 1$.

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- Examples: Hardy fields, transseries.
- Question: What are the completions of the theory of pre-H-fields?

Differential-henselianity

Let K be a pre-H-field.

- K is **differential-henselian** if:
  1. $\partial\mathcal{o} \subseteq \mathcal{o}$;
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- K is **d-Hensel-Liouville closed** if
  1. K is differential-henselian;
  2. K is real closed;
  3. K is **closed under exponential integration**: for every $a \in K$, there exists $f \in K^\times$ such that $f'/f = a$.

Ax–Kochen/Ershov theorem for pre-H-fields

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Theorem (PC)

Let K_1 and K_2 be d-*Hensel-Liouville closed pre-H-fields with ordered differential residue fields $\boldsymbol{k}_1$ and $\boldsymbol{k}_2$. Then*

$$K_1 \equiv K_2 \iff \boldsymbol{k}_1 \equiv \boldsymbol{k}_2.$$

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- Applies to $(K, \mathcal{O}^*)$ as above.
- Closure under exponential integration is a necessary assumption.

Two-sorted results

- Work with two-sorted structures $(K, \boldsymbol{k}; \pi)$, where:
  1. K is a d-Hensel-Liouville closed pre-H-field in $\{+, -, \cdot, 0, 1, \leqslant, \preccurlyeq, \partial\}$;
  2. $\boldsymbol{k}$ is an expansion of the ordered differential residue field of K;
  3. $\pi \colon K^2 \to \boldsymbol{k}$ is defined by $\pi(x, y) = \operatorname{res}(xy^{-1})$ if $x \preccurlyeq y \neq 0$ and 0 otherwise.

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Theorem (PC)

1. *If* $\operatorname{Th}(\boldsymbol{k})$ *is model complete, then so is* $\operatorname{Th}(K, \boldsymbol{k}; \pi)$.
2. *If* $\operatorname{Th}(\boldsymbol{k})$ *has quantifier elimination, then so does* $\operatorname{Th}(K, \boldsymbol{k}; \pi)$.

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- Even better, we eliminate quantifiers from the pre-H-field sort.

Corollary

1. *The structure* $\boldsymbol{k}$ *is stably embedded in* $(K, \boldsymbol{k}; \pi)$.
2. *If* $\boldsymbol{k}$ *has NIP, then* $(K, \boldsymbol{k}; \pi)$ *has NIP.*

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Quantifier reduction

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- Let x be an m-tuple of K-variables and y be a tuple of $\boldsymbol{k}$-variables. We call a formula $\psi(x, y)$ **special** if $\psi(x, y)$ is

$$\psi_{\mathrm{r}}\big(\pi(P_1(x), Q_1(x)), \ldots, \pi(P_k(x), Q_k(x)), y\big),$$

for some $\boldsymbol{k}$-formula $\psi_{\mathrm{r}}(v_1, \ldots, v_k, y)$ and some P_1, Q_1, …, P_k, Q_k in $\mathbb{Z}\{X_1, \ldots, X_m\}$.

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Theorem (PC)

Every formula $\phi(x, y)$ is equivalent to a disjunction of formulas of the form $\theta(x) \wedge \psi(x, y)$, where $\theta(x)$ is a quantifier-free K-formula and $\psi(x, y)$ is special.

Thank you!