Multiobjective Duality for Convex Ratios

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In this paper we present a duality approach for a multiobjective fractional programming problem. The components of the vector objective function are particular ratios involving the square of a convex function and a positive concave function.

Applying the Fenchel-Rockafellar duality theory for a scalar optimization problem associated to the multiobjective primal, a dual problem is derived. This scalar dual problem is formulated in terms of conjugate functions and its structure gives an idea about how to construct a multiobjective dual problem in a natural way. Weak and strong duality assertions are presented.

Key Words: multiobjective duality, fractional programming, conjugate duality, Pareto-efficiency

1. INTRODUCTION

C.H. Scott and T. R. Jefferson [11] have investigated the duality of a particular fractional programming problem having the objective function consisting of a sum of ratios, where the nominators are squared nonegative convex functions and the denominators are positive concave functions. This has to be minimized subject to linear inequality constraints. The method

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they used by the construction of the dual problem is based on geometric programming duality.

The aim of this paper is the study of duality for a multiobjective programming problem (P) with linear inequality constraints and a finite number of objective functions represented by ratios of the form described above. In order to formulate the multiobjective dual problem (D), we study first the duality for a scalar optimization problem obtained from (P) via linear scalarization. But, unlike [11], we use in our investigations the Fenchel-Rockafellar duality approach (cf. [5]). Moreover, we verify strong duality under some assumptions, concerning the required constraint qualification, weaker than the ones used in [11].

In the theory of fractional programming the study of duality is a well developed branch with many theoretical results. In general, these programs deal with ratios of a convex function and a positive concave function. In a large number of papers, for these functions various differentiability assumptions have been considered, like in [2], [10] for the scalar optimization problems and [3], [4] for the multiobjective optimization problems. Among the contributions devoted to duality for non-differentiable fractional programming problems we mention [16] for the scalar case and [7], [15], for the vector case.

The present paper is structured as follows. In section 2 we introduce the primal multiobjective problem (P) and remind the well-known definitions of Pareto-efficiency and proper efficiency.

To the problem (P) we associate, in section 3, a scalar optimization problem (P_λ), with $\lambda \in int\mathbb{R}^m_+$. Using the same transformations as in [11], we write (P_λ) in a form which is suitable for the investigations within the following sections.

Applying the Fenchel-Rockafellar concept based on conjugation and perturbation (cf. [12], [13], [14]), we obtain in section 4 (D_λ), a dual problem to (P_λ). We derive strong duality and optimality conditions which later are used to obtain duality assertions for (P) and its multiobjective dual (D). In comparison with the Lagrange dual problem, the structure of the scalar dual $(\tilde{D}_\lambda)$ has the advantage to yield an idea concerning the structure of (D).

The multiobjective dual problem is formulated in section 5 and results concerning weak and strong duality between the primal (P) and the dual (D) are proved.

Finally, in section 6, a special case which can be obtained from the general result is presented.

2. PROBLEM FORMULATION

We consider the following multiobjective fractional programming problem with linear inequality constraints

$$(P) \quad \operatorname*{v-min}_{x\in\mathcal{A}} \left(\frac{f_1^2(x)}{g_1(x)}, \ldots, \frac{f_m^2(x)}{g_m(x)} \right),$$

$$\mathcal{A} = \{x \in \mathbb{R}^n : Cx \leqq b\}.$$

The functions f_i and $g_i, i = 1, \ldots, m$, mapping from $\mathbb{R}^n$ into $\mathbb{R}$, are assumed to be convex and concave, respectively. For all $x \in \mathcal{A}$ and $i = 1, \ldots, m$, let $f_i(x) \geq 0$ and $g_i(x) > 0$ be fulfilled. By C is denoted a real $l \times n$ matrix and let be $b \in \mathbb{R}^l$.

The problem (P) is a multiobjective optimization problem with the components of the objective function being particular ratio functions. These ratio functions have the property that they are convex (cf. [1]). The solution concepts we will use in our paper for the problem (P) are the so-called Pareto minimal and properly minimal solutions. Now let us recall these notions.

Definition 2.1. An element $\bar{x} \in \mathcal{A}$ is said to be efficient (or minimal or Pareto-minimal) with respect to (P) if from $\frac{f_i^2(\bar{x})}{g_i(\bar{x})} \geq \frac{f_i^2(x)}{g_i(x)}$, for $x \in \mathcal{A}$, follows $\frac{f_i^2(\bar{x})}{g_i(\bar{x})} = \frac{f_i^2(x)}{g_i(x)}, i = 1, \ldots, m$.

Definition 2.2. An element $\bar{x} \in \mathcal{A}$ is said to be properly efficient (or properly minimal) with respect to (P) if there exists $\lambda = (\lambda_1, \ldots, \lambda_m)^T \in intR_+^m$ (i.e. $\lambda_i > 0, i = 1, \ldots, m$) such that $\sum\limits_{i=1}^m \lambda_i \frac{f_i^2(\bar{x})}{g_i(\bar{x})} \leq \sum\limits_{i=1}^m \lambda_i \frac{f_i^2(x)}{g_i(x)}, \quad \forall x \in \mathcal{A}$.

By these definitions, a properly efficient element turns out to be also an efficient one.

Remark 2. 1. For the concept of proper efficiency there exist also other definitions, like those introduced by Benson, Borwein or Geoffrion (cf. [9]). But, for the problem (P), because of the convexity of $\mathcal{A}$ and of $\frac{f_i^2(x)}{g_i(x)}, i = 1, \ldots, m$, all these definitions are equivalent with Definition 2.2.

3. THE SCALAR OPTIMIZATION PROBLEM

In this paper we intend to study the duality for the multiobjective problem (P). In order to do this, first we will consider the scalarized problem

$$(P_\lambda) \quad \inf_{x\in\mathcal{A}} \sum_{i=1}^{m} \lambda_i \frac{f_i^2(x)}{g_i(x)},$$

where $\lambda = (\lambda_1, \ldots, \lambda_m)^T$ is a fixed vector in $int\mathbb{R}^m_+$.

Here, $\mathbb{R}^m_+$ denotes the ordering cone of the non-negative elements of $\mathbb{R}^m$ and it defines the partial ordering "$\geqq$" according to $x \geqq y$ if and only if $x - y \in \mathbb{R}^m_+$. We remark that $\inf(P_\lambda)$ (the infimum value of (P_λ)) is finite under the assumptions we have stated.

To (P_λ) we will associate now another scalar optimization problem $(\tilde{P}_\lambda)$ such that $\inf(P_\lambda) = \inf(\tilde{P}_\lambda)$. The dual problem of $(\tilde{P}_\lambda)$ will then suggest us how to construct a multiobjective dual problem to (P).

Therefore, let us consider for $s = (s_1, \ldots, s_m)^T, t = (t_1, \ldots, t_m)^T \in \mathbb{R}^m$, the following feasible set

$$\tilde{\mathcal{A}} = \{(x, s, t) : Cx \leqq b,\ t_i > 0,\ f_i(x) - s_i \leq 0,\ t_i - g_i(x) \leq 0,\ i = 1, \ldots, m\}.$$

For $i = 1, \ldots, m$, we consider the functions $\Phi_i : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m \to \overline{\mathbb{R}}$,

$$\Phi_i(x, s, t) = \begin{cases} \frac{s_i^2}{t_i}, & \text{if} \quad (x, s, t) \in \mathbb{R}^n \times \mathbb{R}^m \times int\mathbb{R}^m_+, \\ \\ +\infty, & \text{otherwise.} \end{cases}$$

Now, we can introduce the following scalar optimization problem

$$(\tilde{P}_\lambda) \quad \inf_{(x,s,t)\in\tilde{\mathcal{A}}} \sum_{i=1}^{m} \lambda_i \Phi_i(x, s, t).$$

Lemma 3.1. *It holds* $\inf(P_\lambda) = \inf(\tilde{P}_\lambda)$.

Proof. Let be $(x, s, t) \in \tilde{\mathcal{A}}$. This means that $x \in \mathcal{A}$ and, because of $f_i(x) \geq 0, \forall x \in \mathcal{A}$ it holds

$$\sum_{i=1}^{m} \lambda_i \Phi_i(x, s, t) = \sum_{i=1}^{m} \lambda_i \frac{s_i^2}{t_i} \geq \sum_{i=1}^{m} \lambda_i \frac{f_i^2(x)}{g_i(x)} \geq \inf_{x\in\mathcal{A}} \sum_{i=1}^{m} \lambda_i \frac{f_i^2(x)}{g_i(x)} = \inf(P_\lambda),$$

which implies that $\inf(\tilde{P}_\lambda) \geq \inf(P_\lambda)$.

Conversely, let be $x \in \mathcal{A}$. Considering $s_i = f_i(x)$ and $t_i = g_i(x)$, for $i = 1, \ldots, m$, one can observe that $(x, s, t) \in \tilde{\mathcal{A}}$. Moreover, we have the following relations

$$\sum_{i=1}^{m} \lambda_i \frac{f_i^2(x)}{g_i(x)} = \sum_{i=1}^{m} \lambda_i \Phi_i(x, s, t) \geq \inf_{(x,s,t) \in \tilde{\mathcal{A}}} \sum_{i=1}^{m} \lambda_i \Phi_i(x, s, t) = \inf(\tilde{P}_\lambda),$$

and this assures that the opposite inequality, $\inf(P_\lambda) \geq \inf(\tilde{P}_\lambda)$, also holds. In conclusion, $\inf(P_\lambda) = \inf(\tilde{P}_\lambda)$. ▮

4. DUALITY FOR THE SCALARIZED PROBLEM

In [11], the authors have used an approach based on the theory of geometric programming for finding the dual of a scalar optimization problem similar to $(\tilde{P}_\lambda)$. In this section we will obtain a dual for $(\tilde{P}_\lambda)$ using a completely different approach from that in [11]. Moreover, the regularity condition considered by us is "weaker" than the Slater condition used in the paper we mentioned above.

First let us introduce the general convex optimization problem

$$(PG) \quad \inf_{\substack{u \in V \\ \tilde{g}(u) \leqq 0}} \tilde{f}(u), \tag{1}$$

with $V \subseteq \mathbb{R}^v$ being a nonempty convex set and $\tilde{f} : \mathbb{R}^v \to \overline{\mathbb{R}}$, $\tilde{g} : \mathbb{R}^v \to \mathbb{R}^w$ convex functions such that $dom \tilde{f} = V$.

We can find a dual problem to (PG) using the Fenchel-Rockafellar approach (cf. [5], [12], [13], [14]), which requires a suitable perturbation of the original primal problem. Considering as a perturbation function

$$\Psi(u, \varphi, \gamma) = \begin{cases} \tilde{f}(u + \varphi), & \text{if } u \in V, \tilde{g}(u) \leqq \gamma, \\ \\ +\infty, & \text{otherwise}, \end{cases}$$

with the perturbation variables $\varphi \in \mathbb{R}^v$ and $\gamma \in \mathbb{R}^w$, we have the following perturbed problem to (PG)

$$(PG_{\varphi,\gamma}) \quad \inf_{u \in \mathbb{R}^v} \Psi(u, \varphi, \gamma).$$

Setting φ and γ equal to the zero vector of $\mathbb{R}^v$ and $\mathbb{R}^w$, respectively, one gets the original problem (PG). Now, the dual problem may be defined by

$$\sup_{\tilde{p} \in \mathbb{R}^v, \tilde{q} \in \mathbb{R}^w} \{-\Psi^*(0, \tilde{p}, \tilde{q})\},$$

where Ψ^* denotes the conjugate function to Ψ.

A detailed calculation (cf. [12]) yields the following dual problem to (PG)

$$(DG) \quad \sup_{\substack{\tilde{p}\in\mathbb{R}^v \\ \tilde{q}\in\mathbb{R}^w_+}} \left\{ -\tilde{f}^*(\tilde{p}) + \inf_{u\in V} [< \tilde{p}, u > + < \tilde{q}, \tilde{g}(u) >] \right\}. \tag{2}$$

Here, $\tilde{f}^*(\tilde{p}) = \sup_{u\in\mathbb{R}^v} \{< \tilde{p}, u > -\tilde{f}(u)\}$ represents the value of the conjugate function $\tilde{f}^*$ to $\tilde{f}$ at $\tilde{p}$. For any finite dimensional space $\mathbb{R}^k$ we write $< p, u >$ to denote the Euclidean scalar product by $p = (p_1, \ldots, p_k)^T \in \mathbb{R}^k$ and $u = (u_1, \ldots, u_k)^T \in \mathbb{R}^k$, i.e. $< p, u >= \sum_{i=1}^{k} p_i u_i$.

Let us point out that between the problems (PG) and (DG) the weak duality ($\sup(DG) \leq \inf(PG)$) always holds (cf. [5]). But, we are interested in the existence of strong duality ($\max(DG) = \inf(PG)$). One of the classical assumptions which assures the existence of strong duality is the fulfilment of a constraint qualification.

For $\tilde{g}(u) = (\tilde{g}_1(u), \ldots, \tilde{g}_w(u))^T$ consider the following sets

$$L = \{i \in \{1, \ldots, w\} : \tilde{g}_i \text{ is an affine function}\},$$

$$N = \{i \in \{1, \ldots, w\} : \tilde{g}_i \text{ is not an affine function}\}.$$

Let us consider the following constraint qualification (CQ) (cf. [6])

$$(CQ) \quad \left| \begin{array}{l} \text{There exists an element } u' \in rintV \text{ (the relative interior of } V) \\ \text{such that } \tilde{g}_i(u') < 0 \text{ for } i \in N \text{ and } \tilde{g}_i(u') \leq 0 \text{ for } i \in L. \end{array} \right.$$

In [12] we have proved that (CQ) is a sufficient condition to assure the existence of strong duality for the problems (PG) and (DG) (cf. Theorem 3, [12]). This result is now formulated by the following theorem.

THEOREM 4.1. *If* $\inf(PG)$ *is finite and the constraint qualification* (CQ) *is fulfilled, then the problem* (DG) *has a solution and strong duality holds*

$$\inf(PG) = \max(DG).$$

We will write now $(\tilde{P}_\lambda)$ in the form of (1). In order to do this, we will take $\mathbb{R}^v = \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m$, $\mathbb{R}^w = \mathbb{R}^l \times \mathbb{R}^m \times \mathbb{R}^m$, $V = \mathbb{R}^n \times \mathbb{R}^m \times int\mathbb{R}^m_+$

(i.e. $t_i > 0, i = 1, \ldots, m$),

$$\tilde{f}(x,s,t) = \sum_{i=1}^{m} \lambda_i \Phi_i(x,s,t)$$

and

$$\tilde{g}(x,s,t) = (Cx - b, f(x) - s, t - g(x)).$$

It is obvious that V is a nonempty convex set, $\tilde{f}$ is a convex function and $dom\tilde{f} = V$. From the convexity of f_i and the concavity of $g_i, i = 1, \ldots, m$, it follows that the function $\tilde{g}$ is also convex. This means that $(\tilde{P}_\lambda)$ is actually a particular case of the general convex optimization problem (PG).

By (2), (DG) yields the dual of the scalar problem $(\tilde{P}_\lambda)$, with $\tilde{p} = (p^x, p^s, p^t)$ and $\tilde{q} = (q^x, q^s, q^t)$ dual variables,

$$(\tilde{D}_\lambda) \qquad \sup_{\substack{\tilde{p} \in \mathbb{R}^v \\ \tilde{q} \in \mathbb{R}^w_+}} \left\{ - \sup_{(x,s,t) \in \mathbb{R}^v} \left[< \tilde{p}, (x,s,t) > - \sum_{i=1}^{m} \lambda_i \Phi_i(x,s,t) \right] + \right.$$
$$\left. \inf_{(x,s,t) \in V} [< \tilde{p}, (x,s,t) > + < \tilde{q}, (Cx - b, f(x) - s, t - g(x)) >] \right\},$$

or, equivalently,

$$(\tilde{D}_\lambda) \qquad \sup_{\substack{(p^x,p^s,p^t) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m \\ (q^x,q^s,q^t) \in \mathbb{R}^l_+ \times \mathbb{R}^m_+ \times \mathbb{R}^m_+}} \left\{ - \sup_{\substack{(x,s,t) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m \\ t_i > 0, i=1,\ldots,m}} [< p^x, x > + < p^s, s > \right.$$
$$+ < p^t, t > - \sum_{i=1}^{m} \lambda_i \frac{s_i^2}{t_i} \Bigg] + \inf_{s \in \mathbb{R}^m} < p^s - q^s, s > + \inf_{t \in int\mathbb{R}^m_+} < p^t + q^t, t >$$
$$\left. + \inf_{x \in \mathbb{R}^n} [< p^x, x > + < q^x, Cx - b > + < q^s, f(x) > - < q^t, g(x) >] \right\}.$$

After some transformations we obtain the following dual problem

$$(\tilde{D}_\lambda) \qquad \sup_{\substack{p^x \in \mathbb{R}^n, p^s, p^t \in \mathbb{R}^m \\ q^x \in \mathbb{R}^l_+, q^s, q^t \in \mathbb{R}^m_+}} \left\{ - \sum_{i=1}^{m} \sup_{\substack{s_i \in \mathbb{R} \\ t_i > 0}} \left[< p_i^s, s_i > + < p_i^t, t_i > - \lambda_i \frac{s_i^2}{t_i} \right] \right.$$
$$- \sup_{x \in \mathbb{R}^n} < p^x, x > + \inf_{x \in \mathbb{R}^n} \left\{ < p^x - C^T q^x, x > + \sum_{i=1}^{m} [q_i^s f_i(x) - q_i^t g_i(x)] \right\}$$
$$\left. - < q^x, b > + \inf_{s \in \mathbb{R}^m} < p^s - q^s, s > + \inf_{t \in int\mathbb{R}^m_+} < p^t + q^t, t > \right\}.$$

Since

$$\sup_{x\in\mathbb{R}^n} < p^x, x > = \begin{cases} 0, & \text{if} \quad p^x = 0, \\ +\infty, & \text{otherwise,} \end{cases}$$

$$\inf_{s\in\mathbb{R}^m} < p^s - q^s, s > = \begin{cases} 0, & \text{if} \quad p^s = q^s, \\ -\infty, & \text{otherwise,} \end{cases}$$

and

$$\inf_{t\in int\mathbb{R}^m_+} < p^t + q^t, t > = \begin{cases} 0, & \text{if} \quad p^t + q^t \geqq 0, \\ -\infty, & \text{otherwise,} \end{cases}$$

in order to obtain supremum in $(\tilde{D}_\lambda)$, we have to take $p^x = 0$, $p^s = q^s$ and $p^t + q^t \geqq 0$.

Moreover, for $i = 1, \ldots, m$, we have

$$\sup_{\substack{s_i\in\mathbb{R} \\ t_i>0}} \left[< p_i^s, s_i > + < p_i^t, t_i > -\lambda_i \frac{s_i^2}{t_i} \right] = \begin{cases} 0, & \text{if} \quad \frac{(p_i^s)^2}{4\lambda_i} + p_i^t \leq 0, \\ +\infty, & \text{otherwise.} \end{cases}$$

After all these considerations, the dual problem of $(\tilde{P}_\lambda)$ becomes

$$(\tilde{D}_\lambda) \qquad \sup_{\substack{q^x\in\mathbb{R}^l_+, q^s, q^t\in\mathbb{R}^m_+ \\ p^s = q^s, p^t + q^t \geqq 0 \\ \frac{(p_i^s)^2}{4\lambda_i} + p_i^t \leq 0, i=1,\ldots,m}} \left\{ - < q^x, b > - \sup_{x\in\mathbb{R}^n} \left[< -C^T q^x, x > - \left(\sum_{i=1}^m (q_i^s f_i - q_i^t g_i) \right)(x) \right] \right\},$$

or, equivalently, using the definition of the conjugate function,

$$(\tilde{D}_\lambda) \quad \sup \left\{ - < q^x, b > - \left(\sum_{i=1}^m (q_i^s f_i - q_i^t g_i) \right)^* (-C^T q^x) \right\}. \tag{3}$$
$$\text{s.t.} \quad (q^x, q^s, q^t) \geqq 0, \quad \frac{(q_i^s)^2}{4\lambda_i} \leq q_i^t, \; i = 1, \ldots, m$$

Remark 4. 1.

(a) In (3) the conjugate of the sum can be written in the following form (cf.[8])

$$\left(\sum_{i=1}^{m}(q_i^s f_i - q_i^t g_i)\right)^*(-C^T q^x) = \inf \left\{\sum_{i=1}^{m}(q_i^s f_i)^*(u_i) + \sum_{i=1}^{m}(-q_i^t g_i)^*(v_i) : \sum_{i=1}^{m}(u_i + v_i) = -C^T q^x \right\}.$$

(b) For the positive components of the vectors q^s and q^t it holds for $i = 1, \ldots, m$,

$$(q_i^s f_i)^*(u_i) = q_i^s f_i^* \left(\frac{1}{q_i^s} u_i\right)$$

and

$$(-q_i^t g_i)^*(v_i) = q_i^t (-g_i)^* \left(\frac{1}{q_i^t} v_i\right).$$

Here it is important to remark that these formulas can be applied even if $q_i^s = 0$ or $q_i^t = 0$. In this case, in order to obtain supremum in $(\tilde{D}_\lambda)$, we must consider $u_i = 0$, $(q_i^s f_i)^*(u_i) = 0$ and $v_i = 0$, $(-q_i^t g_i)^*(v_i) = 0$, respectively. This means that if $q_i^s = 0$ or $q_i^t = 0$, then we have to take in the objective function of the dual $(\tilde{D}_\lambda)$ instead of $q_i^s f_i^* \left(\frac{1}{q_i^s} u_i\right)$ or, respectively, $q_i^t(-g_i)^* \left(\frac{1}{q_i^t} v_i\right)$, the value 0. Also in the feasible set of the dual problem we have to consider the additional conditions $u_i = 0$ and $v_i = 0$, respectively.

By Remark 4.1 ((a) and (b)), we obtain the following final form of the scalar dual problem

$$(\tilde{D}_\lambda) \quad \sup \left\{ - < q^x, b > - \sum_{i=1}^{m} q_i^s f_i^* \left(\tfrac{1}{q_i^s} u_i\right) - \sum_{i=1}^{m} q_i^t (-g_i)^* \left(\tfrac{1}{q_i^t} v_i\right) \right\}.$$

$$\text{s.t.} \quad \begin{array}{l} (q^x, q^s, q^t) \geqq 0 \\ \frac{(q_i^s)^2}{4\lambda_i} \leq q_i^t, i = 1, \ldots, m \\ \sum\limits_{i=1}^{m}(u_i + v_i) + C^T q^x = 0 \end{array}$$

Now, according to Theorem 4.1, we can present the strong duality theorem for the problems $(\tilde{P}_\lambda)$ and $(\tilde{D}_\lambda)$.

THEOREM 4.2. *Let be $\mathcal{A} \neq \emptyset$. Then the dual problem $(\tilde{D}_\lambda)$ has a solution and strong duality holds*

$$\inf(P_\lambda) = \inf(\tilde{P}_\lambda) = \max(\tilde{D}_\lambda).$$

Proof. The set $\mathcal{A}$ being nonempty, by Lemma 3.1 we obtain that $\inf(P_\lambda) = \inf(\tilde{P}_\lambda) \in \mathbb{R}$. If $x' \in \mathcal{A}$ (i.e. $Cx' \leqq b$), then let us consider for $i = 1, \ldots, m$, $t_i' = \frac{1}{2} g_i(x') > 0$ and $s_i' = f_i(x') + c_i, (c_i > 0)$. The element $u' = (x', s', t')$ belongs to the relative interior of $V = \mathbb{R}^n \times \mathbb{R}^m \times int\mathbb{R}_+^m$. Moreover, it satisfies the constraint qualification (CQ).

So, the hypotheses of Theorem 4.1 are verified. In conclusion, $(\tilde{D}_\lambda)$ has a solution and the equality $\inf(P_\lambda) = \inf(\tilde{P}_\lambda) = \max(\tilde{D}_\lambda)$ is true. ▌

In order to investigate the duality for the multiobjective problem (P), we will use the optimality conditions which result from the equality of the optimal values in Theorem 4.2. The following theorem gives us these conditions.

THEOREM 4.3.

(1)Let $\hat{x}$ be a solution to (P_λ). Then there exists $(\hat{u}, \hat{v}, \hat{q}^x, \hat{q}^s, \hat{q}^t)$, a solution to $(\tilde{D}_\lambda)$, such that the following optimality conditions are satisfied

$$(i)\quad \hat{q}_i^s f_i^*\left(\frac{1}{\hat{q}_i^s}\hat{u}_i\right) + \hat{q}_i^s f_i(\hat{x}) = <\hat{u}_i, \hat{x}>, \quad i = 1, \ldots, m,$$

$$(ii)\quad \hat{q}_i^t (-g_i)^*\left(\frac{1}{\hat{q}_i^t}\hat{v}_i\right) - \hat{q}_i^t g_i(\hat{x}) = <\hat{v}_i, \hat{x}>, \quad i = 1, \ldots, m,$$

$$(iii)\quad <\hat{q}^x, b - C\hat{x}> = 0,$$

$$(iv)\quad \sum_{i=1}^m (\hat{u}_i + \hat{v}_i) + C^T \hat{q}^x = 0,$$

$$(v)\quad \hat{q}_i^s = 2\lambda_i \frac{f_i(\hat{x})}{g_i(\hat{x})}, \quad i = 1, \ldots, m,$$

$$(vi)\quad \hat{q}_i^t = \lambda_i \frac{f_i^2(\hat{x})}{g_i^2(\hat{x})}, \quad i = 1, \ldots, m.$$

(2)Let $\hat{x}$ be admissible to (P_λ) and $(\hat{u}, \hat{v}, \hat{q}^x, \hat{q}^s, \hat{q}^t)$ be admissible to $(\tilde{D}_\lambda)$, satisfying (i)-(vi). Then $\hat{x}$ is a solution to (P_λ), $(\hat{u}, \hat{v}, \hat{q}^x, \hat{q}^s, \hat{q}^t)$ is a solution to $(\tilde{D}_\lambda)$ and strong duality holds.

Proof.

(1) Assume that $\hat{x}$ is a solution to (P_λ). By Theorem 4.2, a solution $(\hat{u}, \hat{v}, \hat{q}^x, \hat{q}^s, \hat{q}^t)$ to $(\tilde{D}_\lambda)$ exists such that $\inf(P_\lambda) = \inf(\tilde{P}_\lambda) = \max(\tilde{D}_\lambda)$ or,

equivalently,

$$\begin{aligned}
0 &= \sum_{i=1}^{m} \lambda_i \frac{f_i^2(\hat{x})}{g_i(\hat{x})} + < \hat{q}^x, b > + \sum_{i=1}^{m} \hat{q}_i^s f_i^* \left(\frac{1}{\hat{q}_i^s} \hat{u}_i \right) + \sum_{i=1}^{m} \hat{q}_i^t (-g_i)^* \left(\frac{1}{\hat{q}_i^t} \hat{v}_i \right) \\
&= \sum_{i=1}^{m} \left[\hat{q}_i^s f_i^* \left(\frac{1}{\hat{q}_i^s} \hat{u}_i \right) + \hat{q}_i^s f_i(\hat{x}) - < \hat{u}_i, \hat{x} > \right] + \sum_{i=1}^{m} g_i(\hat{x}) \left[\hat{q}_i^t - \frac{(\hat{q}_i^s)^2}{4\lambda_i} \right] \\
&+ \sum_{i=1}^{m} \left[\hat{q}_i^t (-g_i)^* \left(\frac{1}{\hat{q}_i^t} \hat{v}_i \right) - \hat{q}_i^t g_i(\hat{x}) - < \hat{v}_i, \hat{x} > \right] + < \hat{q}^x, b - C\hat{x} > \\
&+ \sum_{i=1}^{m} \lambda_i g_i(\hat{x}) \left(\frac{f_i(\hat{x})}{g_i(\hat{x})} - \frac{\hat{q}_i^s}{2\lambda_i} \right)^2 + < \sum_{i=1}^{m} (\hat{u}_i + \hat{v}_i) + C^T \hat{q}^x, \hat{x} > . \qquad (4)
\end{aligned}$$

By the definition of the conjugate function and Remark 4.1 (b), for $i = 1, \ldots, m$, the so-called Young inequalities

$$\hat{q}_i^s f_i^* \left(\frac{1}{\hat{q}_i^s} \hat{u}_i \right) + \hat{q}_i^s f_i(\hat{x}) \geq < \hat{u}_i, \hat{x} > \qquad (5)$$

and

$$\hat{q}_i^t (-g_i)^* \left(\frac{1}{\hat{q}_i^t} \hat{v}_i \right) - \hat{q}_i^t g_i(\hat{x}) \geq < \hat{v}_i, \hat{x} > \qquad (6)$$

are true.

By the inequalities (5), (6), the feasibility of $\hat{x}$ to (P_λ) and the feasibility of $(\hat{u}, \hat{v}, \hat{q}^x, \hat{q}^s, \hat{q}^t)$ to $(\tilde{D}_\lambda)$, it follows that the terms of the sum in (4) are greater or equal to zero. This means that all of them must be equal to zero and, in conclusion, the optimality conditions (i)-(vi) must be fulfilled.

(2) All the calculations and transformations done before may be carried out in the reverse direction starting from the relations (i)-(vi).

∎

5. THE MULTIOBJECTIVE DUAL PROBLEM

With the above preparation, we are able now to formulate a multiobjective dual problem to (P). The results from the previous sections will help us prove the weak duality and, especially, the strong duality between the primal problem (P) and its dual (D).

A dual multiobjective optimization problem (D) is introduced by

$$(D) \qquad \underset{(u,v,\lambda,\delta,q^s,q^t) \in \mathcal{B}}{\text{v-max}} h(u, v, \lambda, \delta, q^s, q^t),$$

$$h(u,v,\lambda,\delta,q^s,q^t)=\begin{pmatrix} h_1(u,v,\lambda,\delta,q^s,q^t)\\ \vdots \\ h_m(u,v,\lambda,\delta,q^s,q^t)\end{pmatrix},$$

with

$$h_j(u,v,\lambda,\delta,q^s,q^t)=-q_j^s f_j^*\left(\frac{1}{q_j^s}u_j\right)-q_j^t(-g_j)^*\left(\frac{1}{q_j^t}v_j\right)-<\delta_j,b>,$$

for $j=1,\dots,m$. The dual variables are

$$u=(u_1,\dots,u_m),v=(v_1,\dots,v_m),\lambda=(\lambda_1,\dots,\lambda_m)^T,$$

$$\delta=(\delta_1,\dots,\delta_m),q^s=(q_1^s,\dots,q_m^s)^T,q^t=(q_1^t,\dots,q_m^t)^T,$$

$$u_i\in\mathbb{R}^n,v_i\in\mathbb{R}^n,\lambda_i\in\mathbb{R},\delta_i\in\mathbb{R}^l,q_i^s\in\mathbb{R},q_i^t\in\mathbb{R},i=1,\dots,m.$$

The set of constraints is defined by

$$\begin{aligned}\mathcal{B} \;=\; \Bigg\{ & (u,v,\lambda,\delta,q^s,q^t):\lambda\in int\mathbb{R}_+^m,\quad q^s,q^t \underset{R_+^m}{\geqq} 0,\quad \sum_{i=1}^m \lambda_i\delta_i \underset{R_+^l}{\geqq} 0,\\ & \sum_{i=1}^m \lambda_i(u_i+v_i+C^T\delta_i)=0,\quad (q_i^s)^2\le 4q_i^t,\quad i=1,\dots,m\Bigg\}. \end{aligned} \tag{7}$$

DEFINITION 5.1. An element $(\bar u,\bar v,\bar\lambda,\bar\delta,\bar q^s,\bar q^t)\in\mathcal{B}$ is said to be efficient (or maximal or Pareto-maximal) for (D) if from

$$h(u,v,\lambda,\delta,q^s,q^t)\underset{R_+^m}{\geqq} h(\bar u,\bar v,\bar\lambda,\bar\delta,\bar q^s,\bar q^t),\quad \text{for}\quad (u,v,\lambda,\delta,q^s,q^t)\in\mathcal{B},$$

follows $h(u,v,\lambda,\delta,q^s,q^t)=h(\bar u,\bar v,\bar\lambda,\bar\delta,\bar q^s,\bar q^t)$.

The following theorem states the weak duality assertion between the multiobjective problem (P) and its dual (D).

THEOREM 5.1. *There is no $x\in\mathcal{A}$ and no $(u,v,\lambda,\delta,q^s,q^t)\in\mathcal{B}$ such that $\frac{f_i^2(x)}{g_i(x)}\le h_i(u,v,\lambda,\delta,q^s,q^t)$ for $i=1,\dots,m$ and $\frac{f_j^2(x)}{g_j(x)}<h_j(u,v,\lambda,\delta,q^s,q^t)$ for at least one $j\in\{1,\dots,m\}$.*

Proof. Let us assume the contrary. This means that there exist $x \in \mathcal{A}$ and $(u, v, \lambda, \delta, q^s, q^t) \in \mathcal{B}$ such that

$$\sum_{i=1}^{m} \lambda_i \frac{f_i^2(x)}{g_i(x)} < \sum_{i=1}^{m} \lambda_i h_i(u, v, \lambda, \delta, q^s, q^t). \tag{8}$$

On the other hand, applying the Young inequalities (5) and (6), we have that

$$\sum_{i=1}^{m} \lambda_i \frac{f_i^2(x)}{g_i(x)} - \sum_{i=1}^{m} \lambda_i h_i(u, v, \lambda, \delta, q^s, q^t) = \sum_{i=1}^{m} \lambda_i \frac{f_i^2(x)}{g_i(x)} + < \sum_{i=1}^{m} \lambda_i \delta_i, b >$$

$$+ \sum_{i=1}^{m} \lambda_i \left[q_i^s f_i^* \left(\frac{1}{q_i^s} u_i \right) + q_i^t (-g_i)^* \left(\frac{1}{q_i^t} v_i \right) \right] \geq \sum_{i=1}^{m} \lambda_i \frac{f_i^2(x)}{g_i(x)}$$

$$+ < \sum_{i=1}^{m} \lambda_i \delta_i, b > + \sum_{i=1}^{m} \lambda_i \left[-q_i^s f_i(x) + q_i^t g_i(x) + < u_i + v_i, x > \right]$$

$$= < \sum_{i=1}^{m} \lambda_i \delta_i, b - Cx > + \sum_{i=1}^{m} \lambda_i g_i(x) \left[\frac{f_i^2(x)}{g_i^2(x)} - q_i^s \frac{f_i(x)}{g_i(x)} + q_i^t \right]$$

$$\geq < \sum_{i=1}^{m} \lambda_i \delta_i, b - Cx > + \sum_{i=1}^{m} \lambda_i g_i(x) \left[\frac{f_i^2(x)}{g_i^2(x)} - q_i^s \frac{f_i(x)}{g_i(x)} + \frac{(q_i^s)^2}{4} \right]$$

$$= < \sum_{i=1}^{m} \lambda_i \delta_i, b - Cx > + \sum_{i=1}^{m} \lambda_i g_i(x) \left[\frac{f_i(x)}{g_i(x)} - \frac{q_i^s}{2} \right]^2 \geq 0.$$

This contradicts the strict inequality (8). ∎

The following theorem expresses the so-called strong duality between the two problems (P) and (D).

THEOREM 5.2. *Assume that $b \neq (0, \ldots, 0)^T$. If $\bar{x} \in \mathcal{A}$ is a properly efficient solution to (P), then there exists an efficient solution $(\bar{u}, \bar{v}, \bar{\lambda}, \bar{\delta}, \bar{q}^s, \bar{q}^t)$ $\in \mathcal{B}$ to the dual (D), such that strong duality $\frac{f_i^2(\bar{x})}{g_i(\bar{x})} = h_i(\bar{u}, \bar{v}, \bar{\lambda}, \bar{\delta}, \bar{q}^s, \bar{q}^t), i = 1, \ldots, m$, holds.*

Proof. ¿From the proper efficiency of $\bar{x}$, by Definition 2.2, we get a corresponding vector $\bar{\lambda} = (\bar{\lambda}_1, \ldots, \bar{\lambda}_m)^T \in int\mathbb{R}_+^m$ with the property that $\bar{x}$

solves the scalar optimization problem

$$(P_{\bar\lambda}) \quad \inf_{x\in\mathcal{A}} \sum_{i=1}^{m} \bar\lambda_i \frac{f_i^2(x)}{g_i(x)}.$$

Theorem 4.3 assures the existence of a solution $(\hat u, \hat v, \hat q^x, \hat q^s, \hat q^t)$ for the dual of $(P_{\bar\lambda})$ such that the optimality conditions (i)-(vi) are satisfied.

Let us now construct by means of $\bar x$ and $(\hat u, \hat v, \hat q^x, \hat q^s, \hat q^t)$ a solution for (D). Therefore, for $i = 1, \ldots, m$, let be $\bar u_i = \frac{1}{\bar\lambda_i}\hat u_i$, $\bar v_i = \frac{1}{\bar\lambda_i}\hat v_i$, $\bar q_i^s = \frac{\hat q_i^s}{\bar\lambda_i}$, $\bar q_i^t = \frac{\hat q_i^t}{\bar\lambda_i}$ and

$$\bar\delta_i = \begin{cases} -\frac{1}{\bar\lambda_i}\frac{<\hat u_i+\hat v_i,\bar x>}{<\hat q^x, b>}\hat q^x, & \text{if } <\hat q^x, b> \neq 0,\\[2ex] \frac{1}{m\bar\lambda_i}\hat q^x - \frac{<\hat u_i+\hat v_i,\bar x>}{\bar\lambda_i}\hat q, & \text{if } <\hat q^x, b> = 0, \text{ with } \hat q \in \mathbb{R}^l : <\hat q, b> = 1.\end{cases}$$

By (iii) and (iv) (cf. Theorem 4.3), for the element $(\bar u, \bar v, \bar\lambda, \bar\delta, \bar q^s, \bar q^t)$ with $\bar\delta = (\bar\delta_1, \ldots, \bar\delta_m)$, it holds $\bar\lambda \in int\mathbb{R}^m_+$, $\bar q^s, \bar q^t \geqq 0$, $\sum\limits_{i=1}^{m} \bar\lambda_i\bar\delta_i = \hat q^x \geqq 0$ and $\sum\limits_{i=1}^{m} \bar\lambda_i(\bar u_i + \bar v_i + C^T\bar\delta_i) = 0$.

Additionally, by (v) and (vi), we have, for $i = 1, \ldots, m$,

$$(\bar q_i^s)^2 = \left(\frac{\hat q_i^s}{\bar\lambda_i}\right)^2 = 4\frac{f_i^2(\bar x)}{g_i^2(\bar x)} = 4\frac{\hat q_i^t}{\bar\lambda_i} = 4\bar q_i^t,$$

and this means that $(\bar u, \bar v, \bar\lambda, \bar\delta, \bar q^s, \bar q^t) \in \mathcal{B}$, i.e. it is feasible for (D).

Moreover, by (i)-(ii) and (v)-(vi), for $i = 1, \ldots, m$, it holds

$$h_i(\bar u, \bar v, \bar\lambda, \bar\delta, \bar q^s, \bar q^t) = -\bar q_i^s f_i^*\left(\frac{1}{\bar q_i^s}\bar u_i\right) - \bar q_i^t(-g_i)^*\left(\frac{1}{\bar q_i^t}\bar v_i\right) - <\bar\delta_i, b> =$$

$$-\frac{\hat q_i^s}{\bar\lambda_i} f_i^*\left(\frac{1}{\hat q_i^s}\hat u_i\right) - \frac{\hat q_i^t}{\bar\lambda_i}(-g_i)^*\left(\frac{1}{\hat q_i^t}\hat v_i\right) + \frac{1}{\bar\lambda_i} <\hat u_i + \hat v_i, \bar x> = \frac{\hat q_i^s}{\bar\lambda_i} f_i(\bar x)$$

$$-\frac{1}{\bar\lambda_i} <\hat u_i, \bar x> - \frac{\hat q_i^t}{\bar\lambda_i} g_i(\bar x) - \frac{1}{\bar\lambda_i} <\hat v_i, \bar x> + \frac{1}{\bar\lambda_i} <\hat u_i + \hat v_i, \bar x> =$$

$$2\frac{f_i^2(\bar x)}{g_i(\bar x)} - \frac{f_i^2(\bar x)}{g_i(\bar x)} = \frac{f_i^2(\bar x)}{g_i(\bar x)}.$$

The maximality of $(\bar u, \bar v, \bar\lambda, \bar\delta, \bar q^s, \bar q^t)$ follows immediately by Theorem 5.1. ∎

6. A SPECIAL CASE

In the last section of this paper we will consider the multiobjective optimization problem for one of the two special cases presented in [11] and we will find out how its dual looks like.

As primal multiobjective problem we consider

$$(P_1) \quad \operatorname*{v-min}_{x \in \mathcal{A}} \left(\frac{x^T Q_1 x}{(d_1)^T x + e_1}, \ldots, \frac{x^T Q_m x}{(d_m)^T x + e_m} \right),$$

$$\mathcal{A} = \{x \in \mathbb{R}^n : Cx \leqq b\},$$

where Q_i is a symmetric positive definite $n \times n$ matrix, $f_i(x) = \sqrt{x^T Q_i x}$ and $g_i(x) = (d_i)^T x + e_i$ are convex functions, for each $i = 1, \ldots, m$.

Let be $d_i \in \mathbb{R}^n, e_i \in \mathbb{R},\ i = 1, \ldots, m$, and the polyhedral set $A = \{x \in R^n : Cx \leqq b\}$ selected so that $g_i(x) = (d_i)^T x + e_i > 0$, for all $x \in A$.

For the conjugate of f_i and g_i we have, for $i = 1, \ldots, m$,

$$f_i^* \left(\frac{1}{q_i^s} u_i \right) = \begin{cases} 0, & \text{if } \sqrt{u_i^T Q_i^{-1} u_i} \leq q_i^s, \\ +\infty, & \text{otherwise,} \end{cases}$$

and

$$(-g_i)^* \left(\frac{1}{q_i^t} v_i \right) = \begin{cases} e_i, & \text{if } \frac{1}{q_i^t} v_i = -d_i, \\ +\infty, & \text{otherwise.} \end{cases}$$

Owing to the general approach presented within section 5, the dual of (P_1) turns out to be

$$(D_1) \quad \text{v} - \max \begin{pmatrix} -q_1^t e_1 - < \delta_1, b > \\ \vdots \\ -q_m^t e_m - < \delta_m, b > \end{pmatrix},$$

$$\text{s.t.} \qquad (u, v, \lambda, \delta, q^t, q^s) \in \mathcal{B}$$

with

$$\mathcal{B} = \Bigg\{ (u, v, \lambda, \delta, q^t, q^s) : \lambda \in int R_+^m, q^s, q^t \underset{\mathbb{R}_+^m}{\geqq} 0, \sum_{i=1}^m \lambda_i (u_i + v_i + C^T \delta_i) = 0,$$

$$\sum_{i=1}^m \lambda_i \delta_i \underset{\mathbb{R}_+^l}{\geqq} 0,\ (q_i^s)^2 \leq 4 q_i^t,\ \sqrt{u_i^T Q_i^{-1} u_i} \leq q_i^s,\ v_i = -q_i^t d_i,\ i = 1, \ldots, m \Bigg\},$$

or, equivalently,

$$(D_1) \quad \text{v}-\max \begin{pmatrix} -q_1^t e_1 - < \delta_1, b > \\ \vdots \\ -q_m^t e_m - < \delta_m, b > \end{pmatrix},$$

$$\text{s.t.} \qquad (u, \lambda, \delta, q^t) \in \mathcal{B}$$

with

$$\mathcal{B} = \{(u, \lambda, \delta, q^t) : \lambda \in intR_+^m, \ q^t \underset{\mathbb{R}_+^m}{\geqq} 0, \ \sum_{i=1}^m \lambda_i (u_i - d_i q_i^t + C^T \delta_i) = 0,$$
$$\sum_{i=1}^m \lambda_i \delta_i \underset{\mathbb{R}_+^l}{\geqq} 0, \ u_i^T Q_i^{-1} u_i \leq 4 q_i^t, \ i = 1, \ldots, m\}.$$

Remark 6. 1. The problem (P_1) also can be considered as a special case of a general multiobjective fractional optimization problem. For this class of optimization problems, in [15] and [7] different dual problems have been presented. But, calculating the multiobjective dual for (P_1) by the methods proposed there, one may find out that (D_1) is different from the duals introduced in the papers mentioned above.

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