Duality for convex partially separable optimization problems

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Abstract

This paper aims to extend duality investigations for the convex partially separable optimization problems. By using the results in [15] we formulate three dual problems for the optimization problem with convex inequality and affine equality constraints, which includes the convex partially separable one. For these duals we give a constraint qualification which guarantees the existence of strong duality. Optimality conditions for the convex partially separable optimization problem and some particular cases are also obtained.
Key words: Convex partially separable optimization problems, Lagrange and conjugate duality, strong duality, optimality conditions.

1. Introduction

Convexity and monotonicity conditions arising in spline approximation problems usually lead to the problem of finding $u_0, u_1, ..., u_n \in \mathbb{R}^s$ such that

$$(u_{i-1}, u_i) \in W_i \subseteq \mathbb{R}^{2s},\ i = \overline{1, n}, \tag{1.1}$$

where W_i are given closed and convex sets. If (1.1) is solvable, the number of the solutions of (1.1) may be infinite in general. In order to find a preferable spline we have to introduce a choice function, e.g. the Holliday function. Therefore we consider the optimization problem

$$\inf_{\substack{(u_{i-1},u_i)\in W_i \\ i=\overline{1,n}}} \sum_{i=1}^{n} F_i(u_{i-1}, u_i), \tag{1.2}$$

which is called a tridiagonally separable one, because the Hessian of the objective function has a block tridiagonal structure (see [2], [8], [10]).

Assume that $F_i : \mathbb{R}^{l_i} \to \mathbb{R}$ and $G_i : \mathbb{R}^{l_i} \to \mathbb{R}^m,\ i = \overline{1,n}$, are convex functions and $W_i \subseteq \mathbb{R}^{l_i},\ \ i = \overline{1,n}$, are convex sets. Let $A_i \in \mathbb{R}^{l_i \times (n+1)},\ l_i \in \{1, ..., n+1\}$ be given matrices.
Let us introduce the following optimization problem

$$(P^{cps}) \ \inf_{u \in W} \sum_{i=1}^{n} F_i(A_i u),$$

where

$$W = \left\{ u = (u_0, ..., u_n)^T \in \mathbb{R}^{n+1} |\ \sum_{i=1}^{n} G_i(A_i u) \underset{\mathbb{R}^m_+}{\leq} 0,\ A_i u \in W_i,\ i = \overline{1,n} \right\}.$$

(P^{cps}) is called the convex partially separable optimization problem and generalizes the problem (1.2). For $x, y \in \mathbb{R}^m,\ x \underset{\mathbb{R}^m_+}{\leq} y$ means $y - x \in \mathbb{R}^m_+ = \{z = (z_1, ..., z_m)^T \in \mathbb{R}^m |\ z_i \geq 0,\ i = \overline{1,m}\}$.
Introducing the auxiliary variables $v_i = A_i u \in \mathbb{R}^{l_i},\ i = \overline{1,n}$, (P^{cps}) can be rewritten as

$$(P^{cps}) \ \inf_{v \in V} \sum_{i=1}^{n} F_i(v_i),$$

where

$$V = \left\{ v \in \mathbb{R}^k |\ \sum_{i=1}^{n} G_i(v_i) \underset{\mathbb{R}^m_+}{\leq} 0,\ v_i - A_i u = 0,\ v_i \in W_i,\ i = \overline{1,n} \right\},$$

with $v = (u, v_1, ..., v_n) \in \mathbb{R}^k$ and $k = n + 1 + l_1 + ... + l_n$.
The following problems considered in [8] and in references therein, are also special cases of (P^{cps}).

(i) The convex partially separable optimization problem with affine constraints

$$\inf \sum_{i=1}^{n} F_i(A_i u),\ \text{s.t.} \sum_{i=1}^{n} B_i A_i u = b,\ A_i u \in W_i,\ i = \overline{1,n},$$

where the matrices $B_i \in \mathbb{R}^{m \times l_i},\ i = \overline{1,n}$ and the vector $b \in \mathbb{R}^m$ are given.

(ii) The convex separable optimization problem

$$\inf \sum_{i=1}^{n} F_i(u_i),\ \text{s.t.}\ \sum_{i=1}^{n} B_i u_i = b,\ u_i \in W_i,\ i = \overline{1,n}.$$

(iii) The tridiagonally separable optimization problem
$\inf \sum_{i=1}^{n} F_i(u_{i-1}, u_i),$ s.t. $\sum_{i=1}^{n}(B_i u_{i-1} + C_i u_i) = b,\ (u_{i-1}, u_i) \in W_i \subseteq \mathbb{R}^{2s}$, $i = \overline{1, n}$. By taking in the last one $B_i = C_i = 0,\ i = \overline{1, n}$, and $b = 0$ we get (1.2).

The Lagrange dual problems for the above particular cases were established and strong duality assertions were derived (see [8] and references therein). In most of these cases, the Lagrange dual problems are unconstrained and if solutions of them are known, then the solutions of the primal problems can be explicitly computed by the so-called return-formula. This is the idea which has been applied by solving tridiagonally separable optimization problems and then by different convex and monotone spline approximations problems. For details, we refer to [2], [6], [7], [8], [9] and [10].
A comprehensive introduction of the separable optimization including Lagrange duality is presented in [13]. Further investigations of the partial separability can be found in [4] and [5].
Let us also mention that another duality which has been used by different spline approximation problems including the convex and monotone interpolation with C^1 splines (cf. [2], [6]) is the so-called Fenchel duality.
The purpose of this paper is to obtain different dual problems for the convex partially separable optimization problem. By using the strong duality result for the optimization problem with convex inequality and affine equality constraints, we derive optimality conditions for the convex partially separable optimization problem and its particular cases.

2. Duality for the optimization problem with convex inequality and affine equality constraints

Let us consider the convex optimization problem

$$(P) \quad \inf_{x \in G} f(x), \qquad G = \{x \in X |\ g(x) \underset{\mathbb{R}^t_+}{\leq} 0,\ h(x) = 0\},$$

where $X \subseteq \mathbb{R}^l$ is a convex set, $f : \mathbb{R}^l \to \mathbb{R},\ g = (g_1, ..., g_t)^T : \mathbb{R}^l \to \mathbb{R}^t,\ h = (h_1, ..., h_w)^T : \mathbb{R}^l \to \mathbb{R}^w$ are given such that $f,\ g_i,\ i = \overline{1, t}$ are convex functions and $h_j,\ j = \overline{1, w}$ are affine functions.
Recently, in [15] different dual problems for (P) have been derived. A general perturbation approach and the theory of conjugate functions have been used

there. This leads to the following three dual problems for (P)

$$(D_L) \quad \sup_{\substack{q_1 \underset{\mathbb{R}^t_+}{\geq} 0 \\ q_2 \in \mathbb{R}^w}} \inf_{x \in X} \{f(x) + \langle q_1, g(x)\rangle + \langle q_2, h(x)\rangle\},$$

$$(D_F) \quad \sup_{p \in \mathbb{R}^l} \left\{ -f^*(p) + \inf_{x \in G} \langle p, x\rangle \right\}$$

and

$$(D_{FL}) \quad \sup_{\substack{p \in \mathbb{R}^l, q_1 \underset{\mathbb{R}^t_+}{\geq} 0 \\ q_2 \in \mathbb{R}^w}} \left\{ -f^*(p) + \inf_{x \in X} [\langle p, x\rangle + \langle q_1, g(x)\rangle + \langle q_2, h(x)\rangle] \right\}.$$

Here $f^* : \mathbb{R}^l \to \overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\}$ is defined by $f^*(\xi) = \sup_{x \in \mathbb{R}^l} \{\langle \xi, x\rangle - f(x)\}$ and is called the conjugate function of f. With $\langle \cdot, \cdot \rangle$ we denote the Euclidean scalar product for the corresponding space $R^t,\ \mathbb{R}^w,\ \mathbb{R}^l$ etc.

The problems (D_L) and (D_F) are the classical Lagrange and Fenchel dual problems, respectively. The dual problem (D_{FL}) is called the Fenchel-Lagrange dual and it is a "combination" of the Fenchel and Lagrange dual problems. By construction weak duality always holds, i.e., the optimal objective values of the mentioned dual problems are less than or equal to the optimal objective value of (P). In order to formulate the strong duality for (P) we need a constraint qualification. Because, in this case,

$$\text{rint} X \cap \text{rint}(\text{dom} f) = \text{rint} X \cap \mathbb{R}^l = \text{rint} X,$$

where $\text{rint}X$ and $\text{dom}f$ denotes the relative interior of X and the effective domain of f, respectively, the constraint qualification looks like (cf. [15])

$$(CQ) \qquad \exists x' \in \text{rint} X : \left[\begin{array}{l} g_i(x') \leq 0,\ i \in L, \\ g_i(x') < 0,\ i \in N, \\ h_j(x') = 0,\ j = \overline{1, w}. \end{array} \right.$$

Here

$$L = \{i \in \{1, ..., t\} \mid g_i \text{ is an affine function}\}$$

and

$$N = \{i \in \{1, ..., t\} \mid g_i \text{ is not an affine function}\}.$$

Denoting by $v(P)$ the optimal objective value of (P) and by $v(D_L),\ v(D_F),\ v(D_{FL})$ the optimal objective values of $(D_L),\ (D_F)$ and (D_{FL}), respectively, we have the following assertion (cf. [15]).

Proposition 2.1 (**Strong duality**)
Assume that the constraint qualification (CQ) is fulfilled. If $v(P)$ is finite then (D_L), (D_F), (D_{FL}) have solutions and it holds

$$v(P) = v(D_L) = v(D_F) = v(D_{FL}).$$

By using the same scheme, in the second part of this section we will formulate three other dual problems to (P). Therefore we reformulate (P) in the following equivalent form

$$(\widetilde{P}) \quad \inf_{x \in D} (f + \delta_X)(x), \quad D = \{x \in \mathbb{R}^l |\ g(x) \underset{\mathbb{R}^t_+}{\leq} 0,\ h(x) = 0\},$$

where $\delta_X(x) = \begin{cases} 0, & \text{if } x \in X, \\ +\infty, & \text{if } x \notin X \end{cases}$ is the indicator function of X. Obviously, the optimal objective values of (P) and $(\widetilde{P})$ coincide. The three dual problems look like

$$(\widetilde{D}_L) \qquad \sup_{\substack{q_1 \underset{\mathbb{R}^t_+}{\geq} 0 \\ q_2 \in \mathbb{R}^w}} \inf_{x \in X} \Big\{ f(x) + \langle q_1, g(x) \rangle + \langle q_2, h(x) \rangle \Big\},$$

$$(\widetilde{D}_F) \qquad \sup_{p \in \mathbb{R}^l} \Big\{ -f^*_X(p) + \inf_{x \in D} \langle p, x \rangle \Big\},$$

$$(\widetilde{D}_{FL}) \qquad \sup_{\substack{q_1 \underset{\mathbb{R}^t_+}{\geq} 0, p \in \mathbb{R}^l \\ q_2 \in \mathbb{R}^w}} \Big\{ -f^*_X(p) + \inf_{x \in \mathbb{R}^l} [\langle p, x \rangle + \langle q_1, g(x) \rangle + \langle q_2, h(x) \rangle] \Big\},$$

where $f^*_X : \mathbb{R}^l \to \overline{\mathbb{R}}$ is defined by $f^*_X(p) = (f + \delta_X)^*(p) = \sup_{x \in X} \{\langle p, x \rangle - f(x)\}$ and is called the conjugate of f relative to the set X. Let us observe that $(\widetilde{D}_L)$ and (D_L) have similar formulations.
Since

$$\text{rint}(\mathbb{R}^l) \cap \text{rint}(\text{dom}(f + \delta_X)) = \mathbb{R}^l \cap \text{rint}X = \text{rint}X,$$

we can take the same constraint qualification as for (P) also for the strong duality assertion for $(\widetilde{P})$.

Proposition 2.2 (**Strong duality**)
Assume that the constraint qualification (CQ) is fulfilled. If $v(\widetilde{P})$ is finite then $(\widetilde{D}_L)$, $(\widetilde{D}_F)$, $(\widetilde{D}_{FL})$ have solutions and it holds

$$v(\widetilde{P}) = v(\widetilde{D}_L) = v(\widetilde{D}_F) = v(\widetilde{D}_{FL}).$$

Proposition 2.1 and Proposition 2.2 state that, under the assumptions we made in this section, the optimal objective values of all dual problems, which introduced above, are equal.

3. The convex partially separable optimization problem

For the convex partially separable optimization problem (P^{cps}) we obtain the following dual problems, which follows from (D_L), (D_F), (D_{FL}), respectively:

$$(D_L^{cps}) \qquad \sup_{\substack{q_i \in \mathbb{R}^{l_i}, i=\overline{1,n} \\ \sum_{i=1}^{n} A_i^T q_i = 0 \\ q_{n+1} \underset{\mathbb{R}^m_+}{\geq} 0}} \left\{ \sum_{i=1}^{n} \inf_{v_i \in W_i} [F_i(v_i) + \langle q_{n+1}, G_i(v_i) \rangle + \langle q_i, v_i \rangle] \right\},$$

$$(D_F^{cps}) \qquad \sup_{p_i \in \mathbb{R}^{l_i}, i=\overline{1,n}} \left\{ -\sum_{i=1}^{n} F_i^*(p_i) + \inf_{v \in V} \sum_{i=1}^{n} \langle p_i, v_i \rangle \right\}$$

and

$$(D_{FL}^{cps}) \sup_{\substack{q_i, p_i \in \mathbb{R}^{l_i}, i=\overline{1,n} \\ \sum_{i=1}^{n} A_i^T q_i = 0 \\ q_{n+1} \underset{\mathbb{R}^m_+}{\geq} 0}} \left\{ -\sum_{i=1}^{n} F_i^*(p_i) + \sum_{i=1}^{n} \inf_{v_i \in W_i} [\langle p_i + q_i, v_i \rangle + \langle q_{n+1}, G_i(v_i) \rangle] \right\}.$$

The functions F_i^* are the conjugates of F_i, $i = \overline{1,n}$.
Indeed, let us observe that the convex partially separable optimization problem (P^{cps}) is a particular case of (P), namely taking

$$\begin{cases} X = \mathbb{R}^{n+1} \times W_1 \times \cdots \times W_n \quad G = V, \\ f : \mathbb{R}^k \to \mathbb{R}, \quad f(v) = \sum_{i=1}^{n} F_i(v_i), \\ g : \mathbb{R}^k \to \mathbb{R}^m, \quad g(v) = \sum_{i=1}^{n} G_i(v_i), \\ h : \mathbb{R}^k \to \mathbb{R}^{l_1} \times \cdots \times \mathbb{R}^{l_n} \\ h(v) = (v_1 - A_1 u, \cdots, v_n - A_n u)^T, \\ v = (u, v_1, ..., v_n) \in \mathbb{R}^{n+1} \times \mathbb{R}^{l_1} \times \cdots \times \mathbb{R}^{l_n}. \end{cases} \tag{3.1}$$

1. Lagrange duality

Substituting (3.1) in (D_L), we have

$$\sup_{\substack{q_1 \underset{\mathbb{R}^m_+}{\geq} 0 \\ q_2 \in \mathbb{R}^{l_1}\times\cdots\times\mathbb{R}^{l_n} \\ q_2=(q_{21},...,q_{2n})}} \inf_{v\in X} \left\{ \sum_{i=1}^n F_i(v_i) + \sum_{i=1}^n \langle q_1, G_i(v_i)\rangle + \sum_{i=1}^n \langle q_{2i}, v_i - A_i u\rangle \right\}$$

$$= \sup_{\substack{q_1 \underset{\mathbb{R}^m_+}{\geq} 0 \\ q_2 \in \mathbb{R}^{l_1}\times\cdots\times\mathbb{R}^{l_n}}} \inf_{\substack{u\in\mathbb{R}^{n+1} \\ v_i\in W_i, i=\overline{1,n}}} \Big\{ \sum_{i=1}^n F_i(v_i) + \sum_{i=1}^n \langle q_1, G_i(v_i)\rangle + \sum_{i=1}^n \langle q_{2i}, v_i\rangle - \sum_{i=1}^n \langle q_{2i}, A_i u\rangle \Big\}$$

$$= \sup_{\substack{q_1 \underset{\mathbb{R}^m_+}{\geq} 0 \\ q_2 \in \mathbb{R}^{l_1}\times\cdots\times\mathbb{R}^{l_n}}} \Big\{ \inf_{u\in\mathbb{R}^{n+1}} \Big\langle -\sum_{i=1}^n A_i^T q_{2i}, u \Big\rangle + \sum_{i=1}^n \inf_{v_i\in W_i} [F_i(v_i) + \langle q_1, G_i(v_i)\rangle + \langle q_{2i}, v_i\rangle] \Big\}.$$

Because of
$$\inf_{u\in\mathbb{R}^{n+1}} \Big\langle -\sum_{i=1}^n A_i^T q_{2i}, u \Big\rangle = \begin{cases} 0, & \text{if } \sum_{i=1}^n A_i^T q_{2i} = 0, \\ -\infty, & \text{otherwise}, \end{cases} \tag{3.2}$$

we get (D_L^{cps}), where we take $q_i := q_{2i},\ i = \overline{1,n}$ and $q_{n+1} := q_1$.

2. Fenchel duality

For $p = (p_u, p_{v_1}, ..., p_{v_n})$, we calculate $f^*(p)$ that appears in the formulation of (D_F). By definition, it holds

$$\begin{aligned} f^*(p) &= \sup_{v\in\mathbb{R}^k} \{\langle p, v\rangle - f(v)\} = \sup_{v\in\mathbb{R}^k} \left\{ \langle p, v\rangle - \sum_{i=1}^n F_i(v_i) \right\} \\ &= \sup_{\substack{u\in\mathbb{R}^{n+1} \\ v_i\in\mathbb{R}^{l_i}, i=\overline{1,n}}} \left\{ \langle p_u, u\rangle + \sum_{i=1}^n \langle p_{v_i}, v_i\rangle - \sum_{i=1}^n F_i(v_i) \right\} \\ &= \sup_{u\in\mathbb{R}^{n+1}} \langle p_u, u\rangle + \sum_{i=1}^n \sup_{v_i\in\mathbb{R}^{l_i}} \{\langle p_{v_i}, v_i\rangle - F_i(v_i)\}. \end{aligned}$$

Thus, in view of

$$\sup_{u\in\mathbb{R}^{n+1}} \langle p_u, u\rangle = \begin{cases} 0, & \text{if } p_u = 0, \\ +\infty, & \text{otherwise}, \end{cases} \tag{3.3}$$

and taking into account that $\inf\limits_{v\in V}\langle p,v\rangle = \inf\limits_{v\in V}\sum\limits_{i=1}^{n}\langle p_{v_i},v_i\rangle$, (D_F^{cps}) is immediately obtained, where $p_i := p_{v_i},\ i=\overline{1,n}$.

3. Fenchel-Lagrange duality
As we have seen

$$f^*(p) = \sup_{u\in\mathbb{R}^{n+1}}\langle p_u,u\rangle + \sum_{i=1}^{n}F_i^*(p_{v_i}).$$

By (3.3), we can omit p_u in the second term of (D_{FL}). Thus, this looks like

$$\begin{aligned}
&\inf_{v\in X}\left\{\sum_{i=1}^{n}\langle p_{v_i},v_i\rangle + \sum_{i=1}^{n}\langle q_1,G_i(v_i)\rangle + \sum_{i=1}^{n}\langle q_{2i},v_i-A_iu\rangle\right\}\\
= &\inf_{\substack{u\in\mathbb{R}^{n+1}\\ v_i\in W_i,i=\overline{1,n}}}\left\{\sum_{i=1}^{n}\langle p_{v_i},v_i\rangle + \sum_{i=1}^{n}\langle q_1,G_i(v_i)\rangle + \sum_{i=1}^{n}\langle q_{2i},v_i\rangle - \sum_{i=1}^{n}\langle A_i^Tq_{2i},u\rangle\right\}\\
= &\inf_{u\in\mathbb{R}^{n+1}}\left\langle -\sum_{i=1}^{n}A_i^Tq_{2i},u\right\rangle + \sum_{i=1}^{n}\inf_{v_i\in W_i}[\langle p_{v_i}+q_{2i},v_i\rangle + \langle q_1,G_i(v_i)\rangle].
\end{aligned}$$

In view of (3.2) and replacing $p_{v_i},\ q_{2i},\ i=\overline{1,n}$, and q_1 by $p_i,\ q_i,\ i=\overline{1,n}$, and q_{n+1}, respectively, we get (D_{FL}^{cps}).

As in Section 2, for $D = \left\{v\in\mathbb{R}^k \mid \sum\limits_{i=1}^{n}G_i(v_i) \underset{\mathbb{R}^m_+}{\leq} 0,\ v_i - A_iu = 0,\ i=\overline{1,n}\right\}$ and $\sum\limits_{i=1}^{n}\Big(F_i(v_i)+\delta_{W_i}(v_i)\Big)$ as objective function, we can formulate further dual problems for (P).

$$(\widetilde{D}_L^{cps})\qquad \sup_{\substack{q_i\in\mathbb{R}^{l_i},i=\overline{1,n}\\ \sum_{i=1}^{n}A_i^Tq_i=0\\ q_{n+1}\underset{\mathbb{R}^m_+}{\geq}0}}\left\{\sum_{i=1}^{n}\inf_{v_i\in W_i}[F_i(v_i)+\langle q_{n+1},G_i(v_i)\rangle + \langle q_i,v_i\rangle]\right\},$$

$$(\widetilde{D}_F^{cps})\qquad \sup_{p_i\in\mathbb{R}^{l_i},i=\overline{1,n}}\left\{-\sum_{i=1}^{n}(F_i)^*_{W_i}(p_i) + \inf_{v\in D}\sum_{i=1}^{n}\langle p_i,v_i\rangle\right\},$$

$$(\widetilde{D}_{FL}^{cps})\qquad \sup_{\substack{p_i,q_i\in\mathbb{R}^{l_i},i=\overline{1,n}\\ \sum_{i=1}^{n}A_i^Tq_i=0\\ q_{n+1}\underset{\mathbb{R}^m_+}{\geq}0}}\left\{-\sum_{i=1}^{n}(F_i)^*_{W_i}(p_i) + \sum_{i=1}^{n}\inf_{v_i\in\mathbb{R}^{l_i}}[\langle p_i+q_i,v_i\rangle\right.$$

$$+\langle q_{n+1}, G_i(v_i)\rangle]\Big\}.$$

Further, we concentrate to the dual problems (D_L^{cps}), (D_F^{cps}), (D_{FL}^{cps}). By using Proposition 2.1 we derive for (P) and its duals necessary and sufficient optimality conditions . Similar results can be obtained for the other three duals.

Theorem 3.1 (**Optimality conditions for** (P^{cps}) **and** (D_L^{cps}))
1. *Assume that the constraint qualification* (CQ) *is fulfilled (with the denotations given in* (3.1)*). Let* $\bar{u} \in \mathbb{R}^{n+1}$ *be a solution for* (P^{cps})*. Then there exists an element* $\bar{q} = (\bar{q}_1, ..., \bar{q}_n, \bar{q}_{n+1}) \in \mathbb{R}^{l_1} \times \cdots \times \mathbb{R}^{l_n} \times \mathbb{R}^m$, $\bar{q}_{n+1} \underset{\mathbb{R}^m_+}{\geq} 0$, $\sum\limits_{i=1}^{n} A_i^T \bar{q}_i = 0$ *such that the following optimality conditions are satisfied:*

(i) $F_i(\bar{v}_i) + \langle \bar{q}_{n+1}, G_i(\bar{v}_i)\rangle + \langle \bar{q}_i, \bar{v}_i\rangle$
$= \inf\limits_{v_i \in W_i} \{F_i(v_i) + \langle \bar{q}_{n+1}, G_i(v_i)\rangle + \langle \bar{q}_i, v_i\rangle\}$, $i = \overline{1, n}$,

(ii) $\Big\langle \bar{q}_{n+1}, \sum\limits_{i=1}^{n} G_i(\bar{v}_i)\Big\rangle = 0$,

(iii) $\bar{v}_i = A_i \bar{u}$, $i = \overline{1, n}$.

2. *Let* $\bar{u} \in W$ *and* $\bar{q} = (\bar{q}_1, ..., \bar{q}_n, \bar{q}_{n+1}) \in \mathbb{R}^{l_1} \times \cdots \times \mathbb{R}^{l_n} \times \mathbb{R}^m$ *be feasible to* (D_L^{cps})*, satisfying* $(i)-(iii)$*. Then* $\bar{u}$ *and* $\bar{q}$ *are solutions for* (P^{cps}) *and* (D_L^{cps})*, respectively, and strong duality holds.*

Proof: Let $\bar{u}$ be a solution for (P^{cps}). Then $v(P^{cps}) = \sum\limits_{i=1}^{n} F_i(\bar{v}_i) \in \mathbb{R}$, where $\bar{v}_i = A_i \bar{u}$, $i = \overline{1, n}$. Therefore, by Proposition 2.1, there exists $\bar{q} = (\bar{q}_1, ..., \bar{q}_n, \bar{q}_{n+1}) \in \mathbb{R}^{l_1} \times \cdots \times \mathbb{R}^{l_n} \times \mathbb{R}^m$, solution for (D_L^{cps}) such that $\bar{q}_{n+1} \underset{\mathbb{R}^m_+}{\geq} 0$, $\sum\limits_{i=1}^{n} A_i^T \bar{q}_i = 0$, and strong duality holds

$$\sum_{i=1}^{n} F_i(\bar{v}_i) = \sum_{i=1}^{n} \inf_{v_i \in W_i} \{F_i(v_i) + \langle \bar{q}_{n+1}, G_i(v_i)\rangle + \langle \bar{q}_i, v_i\rangle\}.$$

After some transformations we get

$$\begin{aligned} 0 &= \sum_{i=1}^{n} \{F_i(\bar{v}_i) + \langle \bar{q}_{n+1}, G_i(\bar{v}_i)\rangle + \langle \bar{q}_i, \bar{v}_i\rangle \\ &- \inf_{v_i \in W_i} [F_i(v_i) + \langle \bar{q}_{n+1}, G_i(v_i)\rangle + \langle \bar{q}_i, v_i\rangle]\} \\ &+ \Big\langle \bar{q}_{n+1}, -\sum_{i=1}^{n} G_i(\bar{v}_i)\Big\rangle + \Big\langle -\sum_{i=1}^{n} A_i^T \bar{q}_i, \bar{u}\Big\rangle. \end{aligned}$$

Taking into account that $\bar{u}, \bar{q}$ are feasible to (P^{cps}) and (D_L^{cps}), respectively, and since the inequality

$$F_i(\bar{v}_i)+\langle \bar{q}_{n+1}, G_i(\bar{v}_i)\rangle+\langle \bar{q}_i, \bar{v}_i\rangle \geq \inf_{v_i \in W_i}[F_i(v_i)+\langle \bar{q}_{n+1}, G_i(v_i)\rangle+\langle \bar{q}_i, v_i\rangle], i=\overline{1,n},$$

is true, $(i)-(iii)$ follows.
The same calculations can be done in the opposite direction. Therefore we obtain assertion 2. □

Theorem 3.2 (**Optimality conditions for** (P^{cps}) **and** (D_F^{cps}))
1. *Assume that the constraint qualification* (CQ) *is fulfilled. Let* $\bar{u} \in \mathbb{R}^{n+1}$ *be a solution for* (P^{cps}). *Then there exists an element* $\bar{p} = (\bar{p}_1, ..., \bar{p}_n) \in \mathbb{R}^{l_1} \times \cdots \times \mathbb{R}^{l_n}$ *such that the following optimality conditions are satisfied:*

(i) $F_i(\bar{v}_i) + F_i^*(\bar{p}_i) = \langle \bar{p}_i, \bar{v}_i\rangle, \ i=\overline{1,n},$

(ii) $\sum_{i=1}^n \langle \bar{p}_i, \bar{v}_i\rangle = \inf_{v \in V} \sum_{i=1}^n \langle \bar{p}_i, v_i\rangle,$

(iii) $\bar{v}_i = A_i \bar{u}, \ i=\overline{1,n}$

2. *Let* $\bar{u} \in W$ *and* $\bar{p} \in \mathbb{R}^{l_1} \times \cdots \times \mathbb{R}^{l_n}$ *be such that* $(i)-(iii)$ *are satisfied. Then* $\bar{u}$ *and* $\bar{p}$ *are solutions for* (P^{cps}) *and* (D_F^{cps}), *respectively, and strong duality holds.*

Proof: Let $\bar{u}$ be a solution for (P^{cps}). Then $v(P^{cps}) = \sum_{i=1}^n F_i(\bar{v}_i) \in \mathbb{R}$, where $\bar{v}_i = A_i\bar{u}, \ i=\overline{1,n}$. Therefore, by Proposition 2.1, there exists $\bar{p} = (\bar{p}_1, ..., \bar{p}_n) \in \mathbb{R}^{l_1} \times \cdots \times \mathbb{R}^{l_n}$, solution for (D_F^{cps}), and it holds

$$\sum_{i=1}^n F_i(\bar{v}_i) = -\sum_{i=1}^n F_i^*(\bar{p}_i) + \inf_{v \in V} \sum_{i=1}^n \langle \bar{p}_i, v_i\rangle.$$

The last relation can be rewritten as

$$\begin{aligned} 0 &= \sum_{i=1}^n \{F_i(\bar{v}_i) + F_i^*(\bar{p}_i) - \langle \bar{p}_i, \bar{v}_i\rangle\} \\ &+ \sum_{i=1}^n \langle \bar{p}_i, \bar{v}_i\rangle - \inf_{v \in V} \sum_{i=1}^n \langle \bar{p}_i, v_i\rangle. \end{aligned} \quad (3.4)$$

Since the inequalities

$$\begin{aligned} F_i(\bar{v}_i) + F_i^*(\bar{p}_i) &\geq \langle \bar{p}_i, \bar{v}_i\rangle, \ i=\overline{1,n} \text{ (Young inequality)}, \\ \sum_{i=1}^n \langle \bar{p}_i, \bar{v}_i\rangle &- \inf_{v \in V} \sum_{i=1}^n \langle \bar{p}_i, v_i\rangle \geq 0 \end{aligned}$$

are always true, all terms in (3.4) must be equal to zero. Therefore $(i)-(iii)$ follows.
In order to get the second part of the theorem one has to make the same calculations, but in the opposite direction. $\square$

Theorem 3.3 (**Optimality conditions for** (P^{cps}) **and** (D^{cps}_{FL}))
1. *Assume that the constraint qualification* (CQ) *is fulfilled. Let* $\bar{u} \in \mathbb{R}^{n+1}$ *be a solution for* (P^{cps}). *Then there exists an element* $(\bar{p}, \bar{q})$, $\bar{p} = (\bar{p}_1, ..., \bar{p}_n) \in \mathbb{R}^{l_1} \times \cdots \times \mathbb{R}^{l_n}$, $\bar{q} = (\bar{q}_1, ..., \bar{q}_n, \bar{q}_{n+1}) \in \mathbb{R}^{l_1} \times \cdots \times \mathbb{R}^{l_n} \times \mathbb{R}^m$, $\bar{q}_{n+1} \underset{\mathbb{R}^m_+}{\geq} 0$, $\sum\limits_{i=1}^{n} A_i^T \bar{q}_i = 0$ *such that the following optimality conditions are satisfied:*

(i) $F_i(\bar{v}_i) + F_i^*(\bar{p}_i) = \langle \bar{p}_i, \bar{v}_i \rangle, \ i = \overline{1, n}$,

(ii) $\langle \bar{p}_i + \bar{q}_i, \bar{v}_i \rangle + \langle \bar{q}_{n+1}, G_i(\bar{v}_i) \rangle$
$= \inf\limits_{v_i \in W_i} \{ \langle \bar{p}_i + \bar{q}_i, v_i \rangle + \langle \bar{q}_{n+1}, G_i(v_i) \rangle \}, \ i = \overline{1, n}$,

(iii) $\left\langle \bar{q}_{n+1}, \sum\limits_{i=1}^{n} G_i(\bar{v}_i) \right\rangle = 0$,

(iv) $\bar{v}_i = A_i \bar{u}, \ i = \overline{1, n}$.

2. *Let* $\bar{u} \in W$ *and* $(\bar{p}, \bar{q})$, $\bar{p} = (\bar{p}_1, ..., \bar{p}_n) \in \mathbb{R}^{l_1} \times ... \times \mathbb{R}^{l_n}$, $\bar{q} = (\bar{q}_1, ..., \bar{q}_n, \bar{q}_{n+1}) \in \mathbb{R}^{l_1} \times ... \times \mathbb{R}^{l_n} \times \mathbb{R}^m$ *be feasible to* (D^{cps}_{FL}), *satisfying* $(i)-(iv)$. *Then* $\bar{u}$ *and* $(\bar{p}, \bar{q})$ *are solutions for* (P^{cps}) *and* (D^{cps}_{FL}), *respectively, and strong duality holds.*

Proof: Let $\bar{u}$ be a solution for (P^{cps}). Then $v(P^{cps}) = \sum\limits_{i=1}^{n} F_i(\bar{v}_i) \in \mathbb{R}$, where $\bar{v}_i = A_i \bar{u}, \ i = \overline{1, n}$. Therefore by Proposition 2.1, there exists $(\bar{p}, \bar{q})$, $\bar{p} = (\bar{p}_1, ..., \bar{p}_n) \in \mathbb{R}^{l_1} \times \cdots \times \mathbb{R}^{l_n}$, $\bar{q} = (\bar{q}_1, ..., \bar{q}_n, \bar{q}_{n+1}) \in \mathbb{R}^{l_1} \times \cdots \times \mathbb{R}^{l_n} \times \mathbb{R}^m$, solution for (P^{cps}_{FL}) such that $\bar{q}_{n+1} \underset{\mathbb{R}^m_+}{\geq} 0$, $\sum\limits_{i=1}^{n} A_i^T \bar{q}_i = 0$, and it holds

$$\sum_{i=1}^{n} F_i(\bar{v}_i) = -\sum_{i=1}^{n} F_i^*(\bar{p}_i) + \sum_{i=1}^{n} \inf_{v_i \in W_i} \{ \langle \bar{p}_i + \bar{q}_i, v_i \rangle + \langle \bar{q}_{n+1}, G(v_i) \rangle \}.$$

The last equality is rewritable as

$$\begin{aligned} 0 &= \sum_{i=1}^{n} \{ F_i(\bar{v}_i) + F_i^*(\bar{p}_i) - \langle \bar{p}_i, \bar{v}_i \rangle \} \\ &+ \sum_{i=1}^{n} \{ \langle \bar{p}_i + \bar{q}_i, \bar{v}_i \rangle + \langle \bar{q}_{n+1}, G_i(\bar{v}_i) \rangle \end{aligned}$$

$$- \inf_{v_i \in W_i} [\langle \bar{p}_i + \bar{q}_i, v_i \rangle + \langle \bar{q}_{n+1}, G_i(v_i) \rangle]\}$$
$$+ \Big\langle \bar{q}_{n+1}, -\sum_{i=1}^{n} G_i(\bar{v}_i) \Big\rangle + \Big\langle -\sum_{i=1}^{n} A_i^T \bar{q}_i, \bar{u} \Big\rangle.$$

Because $\bar{u}$ and $(\bar{p}, \bar{q})$ are feasible to (P^{cps}) and (D_{FL}^{cps}), respectively, and since the inequalities

$$\begin{aligned} F_i(\bar{v}_i) + F_i^*(\bar{p}_i) &\geq \langle \bar{p}_i, \bar{v}_i \rangle, \ i = \overline{1,n} \text{ (Young inequality)}, \\ \langle \bar{p}_i + \bar{q}_i, \bar{v}_i \rangle + \langle \bar{q}_{n+1}, G_i(\bar{v}_i) \rangle &\geq \inf_{v_i \in W_i} [\langle \bar{p}_i + \bar{q}_i, v_i \rangle + \langle \bar{q}_{n+1}, G_i(v_i) \rangle], \ i = \overline{1,n}, \end{aligned}$$

are true, we obtain $(i) - (iv)$.
The second part of the theorem follows by making the same calculations, but in the opposite direction. □

4. Special cases of the convex partially separable optimization problem

4.1 The convex partially separable optimization problem with affine constraints

Consider the problem

$$(P^{lps}) \quad \inf_{u \in W} \sum_{i=1}^{n} F_i(A_i u),$$

where

$$W = \left\{ u \in \mathbb{R}^{n+1} | \ \sum_{i=1}^{n} B_i A_i u = b, \ A_i u \in W_i, \ i = \overline{1,n} \right\}$$

and $B_i \in R^{m \times l_i}, \ i = \overline{1,n}, \ b \in \mathbb{R}^m$ are given.
It is obvious that (P^{lps}) is a special case of (P), whose feasible set containing only affine constraints. The dual problems of (P^{lps}) look like

$$(D_L^{lps}) \qquad \sup_{\substack{q_i \in \mathbb{R}^{l_i}, \ i=\overline{1,n} \\ \sum_{i=1}^{n} A_i^T q_i = 0 \\ q_{n+1} \in \mathbb{R}^m}} \left\{ \langle q_{n+1}, b \rangle - \sum_{i=1}^{n} (F_i)_{W_i}^* (B_i^T q_{n+1} + q_i) \right\},$$

$$(D_F^{lps}) \qquad \sup_{p_i \in \mathbb{R}^{l_i}, \ i=\overline{1,n}} \left\{ -\sum_{i=1}^{n} F_i^*(p_i) + \inf_{u \in W} \sum_{i=1}^{n} \langle p_i, A_i u \rangle \right\}$$

and

$$(D_{FL}^{lps}) \qquad \sup_{\substack{q_i,p_i\in\mathbb{R}^{l_i},\ i=\overline{1,n} \\ \sum\limits_{i=1}^{n} A_i^T q_i=0 \\ q_{n+1}\in\mathbb{R}^m}} \Big\{ \langle q_{n+1}, b\rangle - \sum_{i=1}^{n} F_i^*(p_i)$$

$$+ \sum_{i=1}^{n} \inf_{v_i\in W_i} \langle p_i + q_i + B_i^T q_{n+1}, v_i\rangle \Big\}.$$

As we have seen in Section 3, optimality conditions for all these three dual problems can be derived. But, further we restrict our work by treating only the Fenchel-Lagrange dual.

Proposition 4.1 (**Optimality conditions for** (P^{lps}) **and** (D_{FL}^{lps}))
1. *Assume that the constraint qualification* (CQ) *is fulfilled. Let* $\bar{u} \in \mathbb{R}^{n+1}$ *be a solution for* (P^{lps})*. Then there exists an element* $(\bar{p}, \bar{q}),\ \bar{p} = (\bar{p}_1, ..., \bar{p}_n) \in \mathbb{R}^{l_1} \times \cdots \times \mathbb{R}^{l_n},\ \bar{q} = (\bar{q}_1, ..., \bar{q}_n, \bar{q}_{n+1}) \in \mathbb{R}^{l_1} \times \cdots \times \mathbb{R}^{l_n} \times \mathbb{R}^m,\ \sum\limits_{i=1}^{n} A_i^T \bar{q}_i = 0$ *such that the following optimality conditions are satisfied:*

(i) $F_i(\bar{v}_i) + F_i^*(\bar{p}_i) = \langle \bar{p}_i, \bar{v}_i\rangle,\ i = \overline{1,n},$

(ii) $\langle \bar{p}_i + \bar{q}_i + B_i^T \bar{q}_{n+1}, \bar{v}_i\rangle = \inf\limits_{v_i\in W_i} \langle \bar{p}_i + \bar{q}_i + B_i^T \bar{q}_{n+1}, v_i\rangle,\ i = \overline{1,n},$

(iii) $\bar{v}_i = A_i \bar{u},\ i = \overline{1,n}$

2. *Let* $\bar{u} \in W$ *and* $(\bar{p}, \bar{q}),\ \bar{p} = (\bar{p}_1, ..., \bar{p}_n) \in \mathbb{R}^{l_1} \times \cdots \times \mathbb{R}^{l_n},\ \bar{q} = (\bar{q}_1, ..., \bar{q}_n, \bar{q}_{n+1}) \in \mathbb{R}^{l_1} \times \cdots \times \mathbb{R}^{l_n} \times \mathbb{R}^m$ *be feasible to* (D_{FL}^{lps})*, satisfying* $(i)-(iii)$*. Then* $\bar{u}$ *and* $(\bar{p}, \bar{q})$ *are solutions for* (P^{lps}) *and* (D_{FL}^{lps})*, respectively, and strong duality holds.*

4.2 Tridiagonally separable optimization problem

Let us now treat the problem

$$(P^{ts}) \quad \inf_{u\in W} \sum_{i=1}^{n} F_i(u_{i-1}, u_i),$$

where

$$W = \Big\{ u = (u_0, ..., u_n) \in \underbrace{\mathbb{R}^s \times \cdots \times \mathbb{R}^s}_{n+1} \ \Big|\ \sum_{i=1}^{n} (B_i u_{i-1} + C_i u_i) = b,\ (u_{i-1}, u_i) \in W_i \subseteq \mathbb{R}^{2s},\ i = \overline{1,n} \Big\}$$

and $B_i,\ C_i \in \mathbb{R}^{m\times s},\ i=\overline{1,n},\ b\in\mathbb{R}^m$ are given.
For (P^{ts}) we can use the dual schemes of (P^{lps}). The duals of (P^{lps}) become in this situation

$$(D_L^{ts}) \quad \sup_{\substack{q_i\in\mathbb{R}^s, i=\overline{0,n}\\ q_0=q_n=0\\ q_{n+1}\in\mathbb{R}^m}} \left\{ \langle q_{n+1}, b\rangle - \sum_{i=1}^{n} (F_i)^*_{W_i}(q_{i-1}+B_i^T q_{n+1}, -q_i + C_i^T q_{n+1}) \right\},$$

$$(D_F^{ts}) \quad \sup_{\substack{(p_{i1},p_{i2})\in\mathbb{R}^{2s}\\ i=\overline{1,n}}} \left\{ -\sum_{i=1}^{n} F_i^*(p_{i1},p_{i2}) + \inf_{u\in W} \sum_{i=1}^{n} [\langle p_{i1}, u_{i-1}\rangle + \langle p_{i2}, u_i\rangle] \right\},$$

$$(D_{FL}^{ts}) \quad \sup_{\substack{(p_{i1},p_{i2})\in\mathbb{R}^{2s}\\ q_i\in\mathbb{R}^s, i=\overline{0,n}\\ q_0=q_n=0\\ q_{n+1}\in\mathbb{R}^m}} \Bigg\{ \langle q_{n+1}, b\rangle - \sum_{i=1}^{n} F_i^*(p_{i1},p_{i2}) + \sum_{i=1}^{n} \inf_{(u_{i-1},u_i)\in W_i}$$
$$[\langle p_{i1} - q_{i-1} - B_i^T q_{n+1}, u_{i-1}\rangle + \langle p_{i2} + q_i - C_i^T q_{n+1}, u_i\rangle] \Bigg\}.$$

The next proposition provide optimality conditions for (P^{ts}) and (D_{FL}^{ts}).

Proposition 4.2 (**Optimality conditions for** (P^{ts}) **and** (D_{FL}^{ts}))
1. *Assume that the constraint qualification* (CQ) *is fulfilled. Let* $\bar{u}\in\mathbb{R}^{n+1}$ *be a solution for* (P^{ts}). *Then there exists an element* $(\bar{p},\bar{q}),\ \bar{p}=(\bar{p}_1,...,\bar{p}_n)\in \underbrace{\mathbb{R}^s\times\cdots\times\mathbb{R}^s}_{n},\ \bar{q}=(\bar{q}_0,\bar{q}_1,...,\bar{q}_n,\bar{q}_{n+1})\in \underbrace{\mathbb{R}^s\times\cdots\times\mathbb{R}^s}_{n+1}\times\mathbb{R}^m,\ \bar{q}_0=\bar{q}_n=0$ *such that the following optimality conditions are satisfied:*

(i) $F_i(\bar{u}_{i-1},\bar{u}_i) + F_i^*(\bar{p}_{i1},\bar{p}_{i2}) = \langle \bar{p}_{i1},\bar{u}_{i-1}\rangle + \langle \bar{p}_{i2},\bar{u}_i\rangle,\ i=\overline{1,n},$

(ii) $\langle \bar{p}_{i1} - \bar{q}_{i-1} - B_i{}^T\bar{q}_{n+1}, \bar{u}_{i-1}\rangle + \langle \bar{p}_{i2} + \bar{q}_i - C_i{}^T\bar{q}_{n+1}, \bar{u}_i\rangle$
$= \inf_{(u_{i-1},u_i)^T\in W_i} [\langle \bar{p}_{i1} - \bar{q}_{i-1} - B_i{}^T\bar{q}_{n+1}, u_{i-1}\rangle + \langle \bar{p}_{i2} + \bar{q}_i - C_i{}^T\bar{q}_{n+1}, u_i\rangle],$
$i=\overline{1,n}.$

2. *Let* $\bar{u}\in W$ *and* $(\bar{p},\bar{q}),\ \bar{p}=(\bar{p}_1,...,\bar{p}_n)\in\underbrace{\mathbb{R}^s\times\cdots\times\mathbb{R}^s}_{n},$
$\bar{q}=(\bar{q}_0,\bar{q}_1,...,\bar{q}_n,\bar{q}_{n+1})\in\underbrace{\mathbb{R}^s\times\cdots\times\mathbb{R}^s}_{n+1}\times\mathbb{R}^m$ *be feasible to* (D_{FL}^{ts}), *satisfying* $(i)-(ii)$. *Then* $\bar{u}$ *and* $(\bar{p},\bar{q})$ *are solutions of* (P^{ts}) *and* (D_{FL}^{ts}), *respectively, and strong duality holds.*

4.3 Convex interpolation with cubic C^1 splines

The aim of this last subsection is to show how it is possible to reformulate the convex interpolation problem with C^1 splines as a tridiagonally separable optimization problem. The role of the duality by solving this problem will also be discussed.
Let $(x_i, y_i) \in \mathbb{R}^2,\ i = \overline{0,n}$ be given data points defined on the grid

$$\Delta_n:\ x_0 < x_1 < ... < x_n.$$

A cubic spline S on Δ_n can be given for $[x_{i-1}, x_i]$ by the formula

$$\begin{aligned} S(x) &= y_{i-1} + m_{i-1}(x - x_{i-1}) \\ &+ (3\tau_i - 2m_{i-1} - m_i)\frac{(x - x_{i-1})^2}{h_i} + (m_{i-1} + m_i - 2\tau_i)\frac{(x - x_{i-1})^3}{h_i^2} \end{aligned}$$

with $h_i = x_i - x_{i-1},\ \tau_i = \frac{y_i - y_{i-1}}{h_i},\ i = \overline{1,n}$. It holds $S \in C^1[x_0, x_n]$ and $S(x_i) = y_i,\ S'(x_i) = m_i,\ i = \overline{0,n}$.
The points $(x_0, y_0), ..., (x_n, y_n)$ associated with Δ_n are said to be in convex position if

$$\tau_1 \le \tau_2 \le \cdots \le \tau_n. \tag{4.1}$$

By (4.1), the necessary and sufficient convexity condition for S on $[0, 1]$ leads to the following problem

$$(m_{i-1}, m_i)^T \in W_i \tag{4.2}$$

where

$$W_i = \{(m_{i-1}, m_i)^T \in \mathbb{R}^2 |\ 2m_{i-1} + m_i \le 3\tau_i \le m_{i-1} + 2m_i\},\ i = \overline{1,n}. \tag{4.3}$$

If the inequality $a_i \le b_i,\ i = \overline{1,n}$ where $a_0 = -\infty,\ b_0 = +\infty$ and $a_i = \max\{\tau_i, \frac{1}{2}(3\tau_i - b_{i-1})\},\ b_i = 3\tau_i - 2a_{i-1},\ i = \overline{1,n}$, is fulfilled, then the problem (4.2) is solvable, but not uniquely in general. In order to select an unique convex interpolant one can minimize the mean curvature of S. It is easy to verify that

$$\begin{aligned} \int_{x_0}^{x_n} S''(x)^2 dx &= \sum_{i=1}^{n} \frac{4}{h_i^2}\{m_i^2 + m_i m_{i-1} + m_{i-1}^2 - 3\tau_i(m_i + m_{i-1}) + 3\tau_i^2\} \\ &= \sum_{i=1}^{n} F_i(m_{i-1}, m_i), \end{aligned}$$

and therefore we get the following optimization problem

$$(P^{sca}) \qquad \min_{\substack{(m_{i-1},m_i)^T \in W_i \\ i=\overline{1,n}}} \sum_{i=1}^{n} F_i(m_{i-1}, m_i),$$

where $W_i,\ i = \overline{1,n}$ is given by (4.3). Obviously, (P^{csa}) is a particular case of (1.2). As we have seen, the Lagrange dual problem for (P^{csa}) is

$$(D_L^{csa}) \qquad \sup_{\substack{q \in \mathbb{R}^{n+1} \\ q=(q_0,q_1,...,q_n)^T \\ q_0=q_n=0}} -\left\{ \sum_{i=1}^{n} (F_i)^*_{W_i}(q_{i-1}, -q_i) \right\},$$

where (see [2], [8]),

$$(F_i)^*_{W_i}(\xi,\eta) = \begin{cases} \tau_i(\xi+\eta) + \frac{h_i}{12}(\xi^2 - \xi\eta + \eta^2), & \text{if } \ \xi \leq 0, \eta \geq 0, \\ \tau_i(\xi+\eta) + \frac{h_i}{12}(\frac{\xi}{2} - \eta)^2 & \text{if } \ 0 \leq \xi \leq 2\eta, \\ \tau_i(\xi+\eta) + \frac{h_i}{12}(\xi - \frac{\eta}{2})^2, & \text{if } \ 2\xi \leq \eta \leq 0, \\ \tau_i(\xi+\eta), & \text{if } \ \xi \geq 2\eta, 2\xi \geq \eta. \end{cases}$$

So (P^{csa}) was solved in the literature by means of the so-called return- formula (see for example, [8])

$$(u_{i-1}, u_i)^T = \operatorname{grad}[(F_i)^*_{W_i}(\bar{q}_{i-1}, -\bar{q}_i)],$$

where $(\bar{q}_0, \bar{q}_1, ..., \bar{q}_n)^T \in \mathbb{R}^{n+1}$ is a solution of (D_L^{csa}).
As one can see, the return-formula requires the differentiability of the conjugate function of $F_i + \delta_{W_i},\ i = \overline{1,n}$. If this is not fulfilled, one can try to derive an algorithm for solving (P^{csa}) by using its Fenchel-Lagrange dual and the optimality conditions presented in Proposition 4.2. These conditions have the advantage that the conjugate functions are easy to calculate. For (P^{csa}) they look like

$$F_i^*(\xi,\eta) = \sup_{x,y \in \mathbb{R}} \{\langle x, \xi\rangle + \langle y, \eta\rangle - F_i(x,y)\} = \tau_i(\xi+\eta) + \frac{h_i}{12}(\xi^2+\eta^2-\xi\eta),\ i = \overline{1,n}.$$

How the optimality conditions derived by using the Fenchel-Lagrange dual can be used to construct a dual algorithm for solving the convex interpolation problems with C^1 splines is the subject of future research.

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