AN ANALYSIS OF SOME DUAL PROBLEMS IN MULTIOBJECTIVE OPTIMIZATION (I)

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In this work we study the duality for a general multiobjective optimization problem. Considering, first, a scalar problem, different duals using the conjugacy approach are presented. Starting from these scalar duals, we introduce six different multiobjective dual problems to the primal one, one depending on certain vector parameters. The existence of weak and, under certain conditions, of strong duality between the primal and the dual problems is shown.

Afterwards, some inclusion results for the image sets of the multiobjective dual problems (D_1), (D_α) and (D_{FL}) are derived. Moreover, we verify that the efficiency sets within the image sets of these problems coincide, but the image sets themselves do not.

Keywords: Conjugate duality; Multiobjective convex optimization; Pareto - efficiency; Weak and strong duality

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1 Introduction

This paper represents the first part of a study concerning duality for general multiobjective optimization problems with cone inequality constraints. Our intention is to construct, by means of scalarization, several multiobjective dual problems to a primal one and to relate these new duality concepts to each other and, more than that, to some well-known duality concepts from the literature (cf. [1], [2], [3], [4], [5], [6], [7]).

In the past, Isermann made in [8] a similar analysis, but for the duality in linear multiobjective optimization. He related the duality concept introduced by

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himself in [9] to the concepts introduced by Gale, Kuhn and Tucker in [10] and by Kornbluth in [11].

In this first part we associate to a primal multiobjective optimization problem a scalar one. Then we introduce three scalar dual problems to it, constructed by means of the conjugacy approach (cf. [12]). Starting from them, we formulate six different multiobjective duals and prove the existence of weak and, under certain conditions, of strong duality. Between these six duals one can recognize the dual presented by Jahn in [1], here in the finite-dimensional case, and a generalization of the dual introduced by Wanka and Boţ in [13].

Finally, we derive for the problems (D_1), (D_α) and (D_{FL}) some relations between the image sets and between their maximal elements sets, respectively.

A complete analysis of all the duals introduced here, which also includes a comparison with the duals of Nakayama, Wolfe and Weir-Mond (cf. [3], [6], [7]), is made in the second part of this study.

The primal optimization problem with cone inequality constraints which we consider is the following one

$$(P) \quad \underset{x\in\mathcal{A}}{\text{v-min}}\, f(x),$$

$$\mathcal{A} = \left\{ x \in \mathbb{R}^n : g(x) = (g_1(x), \ldots, g_k(x))^T \underset{K}{\leqq} 0 \right\},$$

where $f(x) = (f_1(x), \ldots, f_m(x))^T$ and $f_i : \mathbb{R}^n \to \overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$, $i = 1, ..., m$, are proper functions, $g_j : \mathbb{R}^n \to \mathbb{R}$, $j = 1, ..., k$, and $K \subseteq \mathbb{R}^k$ is assumed to be a convex closed cone with $intK \neq \emptyset$, defining a partial ordering according to $x_2 \underset{K}{\leqq} x_1$ if and only if $x_1 - x_2 \in K$.

The "v-min" term means that we ask for Pareto-efficient solutions to the problem (P).

DEFINITION 1.1 An element $\bar{x} \in \mathcal{A}$ is said to be Pareto-efficient with respect to (P) if from $f(x) \underset{\mathbb{R}^m_+}{\leqq} f(\bar{x})$ for $x \in \mathcal{A}$ follows $f(\bar{x}) = f(x)$.

Here the cone $\mathbb{R}^m_+$ is defined by $\mathbb{R}^m_+ = \{y = (y_1, ..., y_m)^T : y_i \geq 0, i = 1, ..., m\}$.

Another type of solutions which appear in the paper are the properly efficient solutions (cf. [14]).

DEFINITION 1.2 An element $\bar{x} \in \mathcal{A}$ is said to be properly efficient with respect to (P) if it is Pareto-efficient and if there exists a number $M > 0$ such that for each i and $x \in \mathcal{A}$ satisfying $f_i(x) < f_i(\bar{x})$, there exists at least one j such that $f_j(\bar{x}) < f_j(x)$ and

$$\frac{f_i(\bar{x}) - f_i(x)}{f_j(x) - f_j(\bar{x})} \leq M.$$

Let us now introduce three quite general assumptions which will play an important role in this study

$$(A_f)\ \left|\ \text{the functions } f_i,\ i=1,...,m, \text{ are convex and } \bigcap_{i=1}^{m} ri(dom f_i) \neq \emptyset,\right.$$

$$(A_g)\ \left|\ \begin{array}{l}\text{the function } g \text{ is convex relative to the cone } K, \text{ i.e. } \forall x_1, x_2 \in \\ \mathbb{R}^n,\ \forall \lambda \in [0,1], \lambda g(x_1) + (1-\lambda) g(x_2) - g(\lambda x_1 + (1-\lambda) x_2) \in K,\end{array}\right.$$

$$(A_{CQ})\ \left|\ \text{there exists } x' \in \bigcap_{i=1}^{m} ri(dom f_i) \text{ such that } g(x') \in -intK.\right.$$

Within this work we will mention if we are in the general case or if (A_f), (A_g) and/or (A_{CQ}) are assumed to be fulfilled.

2 The scalar optimization problem and its duals

Let be $\lambda = (\lambda_1, \ldots, \lambda_m)^T$ a fixed vector in $int\mathbb{R}^m_+$ and consider the scalar problem

$$(P_\lambda)\ \ \inf_{x \in \mathcal{A}} \sum_{i=1}^{m} \lambda_i f_i(x).$$

In order to study the duality for the multiobjective problem (P) we introduce, first, the duality for (P_λ), applying the so-called conjugacy approach (cf. [12]). This approach allows us to construct different dual problems to (P_λ).

The scalar dual problems we consider here are obtained by using the same method as in [13], [15] and [16]

$$(D_L^\lambda)\ \sup_{\substack{q \underset{K^*}{\geqq} 0}} \inf_{x \in \mathbb{R}^n} \left[\sum_{i=1}^{m} \lambda_i f_i(x) + q^T g(x) \right],$$

$$(D_F^\lambda)\ \sup_{p_i \in \mathbb{R}^n, i=1,\ldots,m} \left\{ -\sum_{i=1}^{m} \lambda_i f_i^*(p_i) - \chi_{\mathcal{A}}^* \left(-\sum_{i=1}^{m} \lambda_i p_i \right) \right\},$$

and

$$(D_{FL}^\lambda)\ \sup_{\substack{p_i \in \mathbb{R}^n, i=1,\ldots,m, \\ q \underset{K^*}{\geqq} 0}} \left\{ -\sum_{i=1}^{m} \lambda_i f_i^*(p_i) - (q^T g)^* \left(-\sum_{i=1}^{m} \lambda_i p_i \right) \right\}.$$

Here

$$\chi_{\mathcal{A}}(x) = \begin{cases} 0, & \text{if } x \in \mathcal{A}, \\ +\infty, & \text{if } x \notin \mathcal{A}, \end{cases}$$

denotes the indicator function of the set $\mathcal{A}$. On the other hand, for a function $f : \mathbb{R}^n \to \overline{\mathbb{R}}$, we denote by $f^* : \mathbb{R}^n \to \overline{\mathbb{R}}$, $f^*(p) = \sup\limits_{x\in\mathbb{R}^n}\{p^T x - f(x)\}$ its conjugate function. $K^* = \{q \in \mathbb{R}^k : q^T x \geq 0, \forall x \in K\}$ is the dual cone of K.

THEOREM 2.1 *Let us assume that the infimal value of (P_λ), $\inf(P_\lambda)$, is finite and that the assumptions (A_f), (A_g) and (A_{CQ}) are fulfilled. Then the dual problems (D_L^λ), (D_F^λ) and (D_{FL}^λ) have solutions and strong duality holds*

$$\inf(P_\lambda) = \max(D_L^\lambda) = \max(D_F^\lambda) = \max(D_{FL}^\lambda).$$

A proof of Theorem 2.1 has been given in [15] and, even under some weaker assumptions, in [16]. Let us observe that (D_L^λ) is the well-known Lagrange dual problem to (P_λ).

For later investigations we also need the optimality conditions regarding to the scalar problem (P_λ) and its dual (D_{FL}^λ). The following theorem gives us these conditions (cf. [13]).

THEOREM 2.2

(a) Let (A_f), (A_g) and (A_{CQ}) be fulfilled and let $\bar{x}$ be a solution to (P_λ). Then there exists $(\bar{p}, \bar{q}), \bar{p} = (\bar{p}_1, \ldots, \bar{p}_m) \in \mathbb{R}^n \times \ldots \times \mathbb{R}^n, \bar{q} \underset{K^*}{\geqq} 0$, that is solution to (D_{FL}^λ) such that the following optimality conditions are satisfied

$$\begin{aligned}
&(i) \quad f_i^*(\bar{p}_i) + f_i(\bar{x}) = \bar{p}_i^T \bar{x},\ i = 1, ..., m,\\
&(ii) \quad \bar{q}^T g(\bar{x}) = 0,\\
&(iii) \quad (\bar{q}^T g)^* \left(-\sum_{i=1}^m \lambda_i \bar{p}_i\right) = -\left(\sum_{i=1}^m \lambda_i \bar{p}_i\right)^T \bar{x}.
\end{aligned} \tag{2. 1}$$

(b) Let $\bar{x}$ be admissible to (P_λ) and $(\bar{p}, \bar{q})$ be admissible to (D_{FL}^λ), satisfying (i),(ii) and (iii).

Then $\bar{x}$ is a solution to (P_λ), $(\bar{p}, \bar{q})$ is a solution to (D_{FL}^λ) and strong duality holds

$$\sum_{i=1}^m \lambda_i f_i(\bar{x}) = -\sum_{i=1}^m \lambda_i f_i^*(\bar{p}_i) - (\bar{q}^T g)^* \left(-\sum_{i=1}^m \lambda_i \bar{p}_i\right).$$

3 The multiobjective dual (D_1) and the family of multiobjective duals (D_α), $\alpha \in \mathcal{F}$

The first multiobjective dual problem to (P) which we introduce here is

$$(D_1) \quad \underset{(p,q,\lambda,t)\in\mathcal{B}_1}{\text{v-max}} h^1(p,q,\lambda,t),$$

$$h^1(p,q,\lambda,t) = \begin{pmatrix} h_1^1(p,q,\lambda,t) \\ \vdots \\ h_m^1(p,q,\lambda,t) \end{pmatrix},$$

with

$$h_j^1(p,q,\lambda,t) = -f_j^*(p_j) - (q^Tg)^*\left(-\frac{1}{\sum\limits_{i=1}^m \lambda_i}\sum_{i=1}^m \lambda_i p_i\right) + t_j, j = 1,...,m,$$

the dual variables

$$p = (p_1,...,p_m) \in \mathbb{R}^n \times ... \times \mathbb{R}^n, q \in \mathbb{R}^k,$$

$$\lambda = (\lambda_1,...,\lambda_m)^T \in \mathbb{R}^m, t = (t_1,...,t_m)^T \in \mathbb{R}^m,$$

and the set of constraints

$$\mathcal{B}_1 = \left\{(p,q,\lambda,t) : \lambda \in int\mathbb{R}_+^m, \quad q \underset{K^*}{\geqq} 0, \quad \sum_{i=1}^m \lambda_i t_i = 0\right\}.$$

The dual (D_1) is a vector maximum problem for which we consider also Pareto - efficient solutions, but in the sense of maximum.

Next, we present the weak and strong duality theorems for the multiobjective problems (P) and (D_1).

THEOREM 3.1 (weak duality for (D_1)) *There is no $x \in \mathcal{A}$ and no $(p,q,\lambda,t) \in \mathcal{B}_1$ fulfilling $h^1(p,q,\lambda,t) \underset{\mathbb{R}_+^m}{\geqq} f(x)$ and $h^1(p,q,\lambda,t) \neq f(x)$.*

Proof We assume that there exist $x \in \mathcal{A}$ and $(p,q,\lambda,t) \in \mathcal{B}_1$ such that $f_i(x) \leq h_i^1(p,q,\lambda,t), \forall i \in \{1,\ldots,m\}$ and $f_j(x) < h_j^1(p,q,\lambda,t)$, for at least one $j \in \{1,\ldots,m\}$. This means that we have

$$\sum_{i=1}^m \lambda_i f_i(x) < \sum_{i=1}^m \lambda_i h_i^1(p,q,\lambda,t). \tag{3. 1}$$

On the other hand, by using the inequalities of Young (cf. [12]),

$$-f_i^*(p_i) \leq f_i(x) - p_i^T x, \ i = 1,...,m,$$

and

$$-(q^Tg)^*\left(-\frac{1}{\sum\limits_{j=1}^{m}\lambda_j}\sum_{j=1}^{m}\lambda_j p_j\right)\leq q^Tg(x)+\frac{1}{\sum\limits_{j=1}^{m}\lambda_j}\left(\sum_{j=1}^{m}\lambda_j p_j\right)^T x,$$

we have that

$$\begin{aligned}\sum_{i=1}^{m}\lambda_i h_i^1(p,q,\lambda,t) &= -\sum_{i=1}^{m}\lambda_i f_i^*(p_i)-\sum_{i=1}^{m}\lambda_i(q^Tg)^*\left(-\frac{1}{\sum\limits_{i=1}^{m}\lambda_i}\sum_{j=1}^{m}\lambda_j p_j\right)\\ &+ \sum_{i=1}^{m}\lambda_i t_i\leq\sum_{i=1}^{m}\lambda_i f_i(x)+\left(\sum_{i=1}^{m}\lambda_i q\right)^T g(x)\\ &- \left(\sum_{i=1}^{m}\lambda_i p_i\right)^T x+\left(\sum_{i=1}^{m}\lambda_i p_i\right)^T x\leq\sum_{i=1}^{m}\lambda_i f_i(x).\end{aligned}$$

The inequality obtained above, $\sum\limits_{i=1}^{m}\lambda_i h_i^1(p,q,\lambda,t)\leq\sum\limits_{i=1}^{m}\lambda_i f_i(x)$, contradicts relation (3. 1). $\square$

THEOREM 3.2 (strong duality for (D_1)) *Assume that (A_f), (A_g) and (A_{CQ}) are fulfilled. If $\bar{x}$ is a properly efficient solution to (P), then there exists an efficient solution $(\bar{p},\bar{q},\bar{\lambda},\bar{t})\in\mathcal{B}_1$ to the dual (D_1) and strong duality $f(\bar{x})=h^1(\bar{p},\bar{q},\bar{\lambda},\bar{t})$ holds.*

Proof If $\bar{x}$ is properly efficient to the problem (P), then there exists a vector $\bar{\lambda}=(\bar{\lambda}_1,\ldots,\bar{\lambda}_m)^T\in int\mathbb{R}_+^m$ (cf. [14]) such that $\bar{x}$ solves the scalar problem

$$(P_{\bar{\lambda}})\quad\inf_{x\in\mathcal{A}}\sum_{i=1}^{m}\bar{\lambda}_i f_i(x).$$

But, Theorem 2.2 assures the existence of a solution $(\tilde{p},\tilde{q})$ to the scalar dual $(D_{FL}^{\bar{\lambda}})$ such that the optimality conditions (i), (ii) and (iii) are satisfied.

Considering

$$\bar{p}:=\tilde{p},\bar{q}:=\frac{1}{\sum\limits_{i=1}^{m}\bar{\lambda}_i}\tilde{q}\underset{K^*}{\geqq}0$$

and

$$\bar{t}_i:=\bar{p}_i^T\bar{x}+(\bar{q}^Tg)^*\left(-\frac{1}{\sum\limits_{j=1}^{m}\bar{\lambda}_j}\sum_{j=1}^{m}\bar{\lambda}_j\bar{p}_j\right)\in\mathbb{R},i=1,...,m,$$

it holds $\sum\limits_{i=1}^{m} \bar{\lambda}_i \bar{t}_i = 0$ (cf. (2. 1)), so $(\bar{p}, \bar{q}, \bar{\lambda}, \bar{t})$ is feasible to (D_1), i.e. $(\bar{p}, \bar{q}, \bar{\lambda}, \bar{t}) \in \mathcal{B}_1$. Moreover, from Theorem 2.2 (i), it follows that $f_i(\bar{x}) = h_i^1(\bar{p}, \bar{q}, \bar{\lambda}, \bar{t})$, for $i = 1, ..., m$. The maximality of $(\bar{p}, \bar{q}, \bar{\lambda}, \bar{t})$ is given by Theorem 3.1. $\square$

In the second part of the section we introduce a family of dual multiobjective problems to (P). Therefore, let us consider the following family of functions,

$$\mathcal{F} = \left\{ \alpha : int\mathbb{R}_+^m \to \mathbb{R}_+^m \ : \begin{array}{c} \alpha(\lambda) = (\alpha_1(\lambda), ..., \alpha_m(\lambda))^T, \quad \text{such that} \\ \sum\limits_{i=1}^{m} \lambda_i \alpha_i(\lambda) = 1, \ \forall \lambda = (\lambda_1, ..., \lambda_m)^T \in int\mathbb{R}_+^m \end{array} \right\}.$$

For each $\alpha \in \mathcal{F}$ we consider the following dual problem

$$(D_\alpha) \quad \underset{(p,\tilde{q},\lambda,t) \in \mathcal{B}_\alpha}{\text{v-max}} \, h^\alpha(p, \tilde{q}, \lambda, t),$$

$$h^\alpha(p, \tilde{q}, \lambda, t) = \begin{pmatrix} h_1^\alpha(p, \tilde{q}, \lambda, t) \\ \vdots \\ h_m^\alpha(p, \tilde{q}, \lambda, t) \end{pmatrix},$$

with

$$h_j^\alpha(p, \tilde{q}, \lambda, t) = -f_j^*(p_j) - (q_j^T g)^* \left(-\alpha_j(\lambda) \sum_{i=1}^{m} \lambda_i p_i \right) + t_j, j = 1, \ldots, m,$$

the dual variables

$$p = (p_1, \ldots, p_m) \in \mathbb{R}^n \times ... \times \mathbb{R}^n, \tilde{q} = (q_1, ..., q_m) \in \mathbb{R}^k \times ... \times \mathbb{R}^k,$$

$$\lambda = (\lambda_1, \ldots, \lambda_m)^T \in \mathbb{R}^m, t = (t_1, \ldots, t_m)^T \in \mathbb{R}^m,$$

and the set of constraints

$$\mathcal{B}_\alpha = \left\{ (p, \tilde{q}, \lambda, t) : \lambda \in int\mathbb{R}_+^m, \quad \sum_{i=1}^{m} \lambda_i q_i \underset{K^*}{\geqq} 0, \quad \sum_{i=1}^{m} \lambda_i t_i = 0 \right\}.$$

Let us show now the existence of weak and strong duality between the primal problem and the problems just introduced.

THEOREM 3.3 (weak duality for $(D_\alpha), \alpha \in \mathcal{F}$) *For each $\alpha \in \mathcal{F}$ there is no $x \in \mathcal{A}$ and no $(p, \tilde{q}, \lambda, t) \in \mathcal{B}_\alpha$ fulfilling $h^\alpha(p, \tilde{q}, \lambda, t) \underset{\mathbb{R}_+^m}{\geqq} f(x)$ and $h^\alpha(p, \tilde{q}, \lambda, t) \neq f(x)$.*

Proof Let be $\alpha \in \mathcal{F}$, fixed. We assume that there exist $x \in \mathcal{A}$ and $(p, \tilde{q}, \lambda, t) \in \mathcal{B}_\alpha$ such that $f_i(x) \leq h_i^\alpha(p, \tilde{q}, \lambda, t), \forall i \in \{1, \ldots, m\}$ and $f_j(x) < h_j^\alpha(p, \tilde{q}, \lambda, t)$, for at least one $j \in \{1, \ldots, m\}$. This means that

$$\sum_{i=1}^{m} \lambda_i f_i(x) < \sum_{i=1}^{m} \lambda_i h_i^\alpha(p, \tilde{q}, \lambda, t). \tag{3. 2}$$

Applying again the Young's inequalities, it results

$$\begin{aligned}
\sum_{i=1}^{m} \lambda_i h_i^\alpha(p, \tilde{q}, \lambda, t) &= -\sum_{i=1}^{m} \lambda_i f_i^*(p_i) - \sum_{i=1}^{m} \lambda_i (q_i^T g)^* \left(-\alpha_i(\lambda) \sum_{j=1}^{m} \lambda_j p_j \right) \\
&+ \sum_{i=1}^{m} \lambda_i t_i \leq \sum_{i=1}^{m} \lambda_i f_i(x) - \left(\sum_{i=1}^{m} \lambda_i p_i \right)^T x \\
&+ \left(\sum_{i=1}^{m} \lambda_i q_i \right)^T g(x) + \left(\sum_{i=1}^{m} \lambda_i \alpha_i(\lambda) \right) \left(\sum_{i=1}^{m} \lambda_i p_i \right)^T x \\
&\leq \sum_{i=1}^{m} \lambda_i f_i(x).
\end{aligned}$$

But the inequality $\sum\limits_{i=1}^{m} \lambda_i h_i^\alpha(p, \tilde{q}, \lambda, t) \leq \sum\limits_{i=1}^{m} \lambda_i f_i(x)$ contradicts relation (3. 2), so theorem's assertion holds. □

THEOREM 3.4 (strong duality for $(D_\alpha), \alpha \in \mathcal{F}$) *Let be $\alpha \in \mathcal{F}$ and assume that (A_f), (A_g) and (A_{CQ}) are fulfilled. If $\bar{x}$ is a properly efficient solution to (P), then there exists an efficient solution $(\bar{p}, \bar{\tilde{q}}, \bar{\lambda}, \bar{t}) \in \mathcal{B}_\alpha$ to the dual (D_α) and strong duality $f(\bar{x}) = h^\alpha(\bar{p}, \bar{\tilde{q}}, \bar{\lambda}, \bar{t})$ holds.*

Proof If $\bar{x}$ is properly efficient to the problem (P), then there exists a vector $\bar{\lambda} = (\bar{\lambda}_1, \ldots, \bar{\lambda}_m)^T \in int\mathbb{R}_+^m$ (cf. [14]) such that $\bar{x}$ solves the scalar problem

$$(P_{\bar{\lambda}}) \ \inf_{x \in \mathcal{A}} \sum_{i=1}^{m} \bar{\lambda}_i f_i(x).$$

Like in the proof of Theorem 3.2, we notice that Theorem 2.2 assures the existence of a solution $(\tilde{p}, \tilde{q})$ to $(D_{FL}^{\bar{\lambda}})$ such that the optimality conditions (i), (ii) and (iii) are satisfied.

Considering

$$\bar{p} := \tilde{p}, \bar{q}_i := \alpha_i(\bar{\lambda})\tilde{q} \in \mathbb{R}^k, i = 1, ..., m,$$

and

$$\bar{t}_i := \bar{p}_i^T \bar{x} + (\bar{q}^T g)^* \left(-\alpha_i(\bar{\lambda}) \sum_{j=1}^{m} \bar{\lambda}_j \bar{p}_j \right) \in \mathbb{R}, i = 1, ..., m,$$

it holds $\sum\limits_{i=1}^{m} \bar{\lambda}_i \bar{q}_i = \tilde{q} \underset{K^*}{\geqq} 0$ and $\sum\limits_{i=1}^{m} \bar{\lambda}_i \bar{t}_i = 0$ (cf. (2. 1)). This means that $(\bar{p}, \bar{\tilde{q}}, \bar{\lambda}, \bar{t})$, $\bar{\tilde{q}} = (\bar{q}_1, ..., \bar{q}_m)$, is feasible to (D_α). Moreover, from Theorem 2.2 (i), it follows that $f_i(\bar{x}) = h_i^\alpha(\bar{p}, \bar{\tilde{q}}, \bar{\lambda}, \bar{t})$, for $i = 1, ..., m$. The maximality of $(\bar{p}, \bar{\tilde{q}}, \bar{\lambda}, \bar{t})$ is given by Theorem 3.3. □

Remark 3.1

(a) The set $\mathcal{B}_\alpha$ does not depend on the function $\alpha \in \mathcal{F}$.

(b) For $\alpha : int\mathbb{R}^m_+ \to \mathbb{R}^m_+$, $\alpha(\lambda) = \left(\frac{1}{m\lambda_1}, ..., \frac{1}{m\lambda_m}\right)^T$, $\lambda \in int\mathbb{R}^m_+$, it holds $\sum\limits_{i=1}^{m} \lambda_i$ $\alpha_i(\lambda) = 1$, which implies that $\alpha \in \mathcal{F}$. The dual problem (D_α) obtained for this choice of the function α is actually the multiobjective dual problem introduced by Wanka and Boţ in [13].

4 The multiobjective dual problems (D_{FL}), (D_F), (D_L) and (D_P)

In this section we continue to introduce other multiobjective dual problems to the primal (P). Therefore we use the expressions which appear in the formulation of the scalar dual problems presented in section 2. For all the multiobjective duals we prove the existence of weak and strong duality between them and the primal problem. Let us begin with the following problem

$$(D_{FL}) \quad \underset{(p,q,\lambda,y)\in\mathcal{B}_{FL}}{\text{v-max}} h^{FL}(p, q, \lambda, y),$$

$$h^{FL}(p, q, \lambda, y) = \begin{pmatrix} h_1^{FL}(p, q, \lambda, y) \\ \vdots \\ h_m^{FL}(p, q, \lambda, y) \end{pmatrix} = \begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix},$$

with

$$h_j^{FL}(p, q, \lambda, y) = y_j, j = 1, ..., m,$$

the dual variables

$$p = (p_1, \ldots, p_m) \in \mathbb{R}^n \times ... \times \mathbb{R}^n, q \in \mathbb{R}^k,$$

$$\lambda = (\lambda_1, \ldots, \lambda_m)^T \in \mathbb{R}^m, y = (y_1, \ldots, y_m)^T \in \mathbb{R}^m,$$

and the set of constraints

$$\mathcal{B}_{FL} = \Big\{(p, q, \lambda, y) : \quad \lambda = (\lambda_1, \ldots, \lambda_m)^T \in int\mathbb{R}^m_+,\ q \underset{K^*}{\geqq} 0,$$
$$\sum_{i=1}^{m} \lambda_i y_i \leq -\sum_{i=1}^{m} \lambda_i f_i^*(p_i) - (q^T g)^* \left(-\sum_{i=1}^{m} \lambda_i p_i\right)\Big\}.$$

THEOREM 4.1 (weak duality for (D_{FL})) *There is no $x \in \mathcal{A}$ and no $(p, q, \lambda, y) \in \mathcal{B}_{FL}$ fulfilling $h^{FL}(p, q, \lambda, y) \underset{\mathbb{R}^m_+}{\geqq} f(x)$ and $h^{FL}(p, q, \lambda, y) \neq f(x)$.*

Proof We assume that there exist $x \in \mathcal{A}$ and $(p, q, \lambda, y) \in \mathcal{B}_{FL}$ such that $f_i(x) \leq y_i, \forall i \in \{1, \ldots, m\}$ and $f_j(x) < y_j$, for at least one $j \in \{1, \ldots, m\}$. This means that

$$\sum_{i=1}^{m} \lambda_i f_i(x) < \sum_{i=1}^{m} \lambda_i y_i. \tag{4. 1}$$

On the other hand, the inequality of Young for $f_i, i = 1, ..., m$, gives us

$$\begin{aligned}\sum_{i=1}^{m} \lambda_i y_i &\leq -\sum_{i=1}^{m} \lambda_i f_i^*(p_i) - (q^T g)^* \left(-\sum_{i=1}^{m} \lambda_i p_i\right) \\ &\leq \sum_{i=1}^{m} \lambda_i f_i(x) - \left(\sum_{i=1}^{m} \lambda_i p_i\right)^T x + q^T g(x) + \left(\sum_{i=1}^{m} \lambda_i p_i\right)^T x \\ &\leq \sum_{i=1}^{m} \lambda_i f_i(x).\end{aligned}$$

But the inequality $\sum\limits_{i=1}^{m} \lambda_i y_i \leq \sum\limits_{i=1}^{m} \lambda_i f_i(x)$ contradicts relation (4. 1). □

THEOREM 4.2 (strong duality for (D_{FL})) *Assume that (A_f), (A_g) and (A_{CQ}) are fulfilled. If $\bar{x}$ is a properly efficient solution to (P), then there exists an efficient solution $(\bar{p}, \bar{q}, \bar{\lambda}, \bar{y}) \in \mathcal{B}_{FL}$ to the dual (D_{FL}) and strong duality $f(\bar{x}) = h^{FL}(\bar{p}, \bar{q}, \bar{\lambda}, \bar{y}) = \bar{y}$ holds.*

Proof If $\bar{x}$ is properly efficient to the problem (P), then there exists a vector $\bar{\lambda} = (\bar{\lambda}_1, \ldots, \bar{\lambda}_m)^T \in int\mathbb{R}^m_+$ (cf. [14]) such that $\bar{x}$ solves the scalar problem

$$(P_{\bar{\lambda}}) \quad \inf_{x \in \mathcal{A}} \sum_{i=1}^{m} \bar{\lambda}_i f_i(x).$$

By the strong duality Theorem 2.1 it results the existence of a solution $(\bar{p}, \bar{q})$ to $(D^{\bar{\lambda}}_{FL})$ such that

$$\sum_{i=1}^{m} \bar{\lambda}_i f_i(\bar{x}) = \inf(P_{\bar{\lambda}}) = \max(D^{\bar{\lambda}}_{FL}) = -\sum_{i=1}^{m} \bar{\lambda}_i f_i^*(\bar{p}_i) - (\bar{q}^T g)^* \left(-\sum_{i=1}^{m} \bar{\lambda}_i \bar{p}_i\right).$$

Taking $\bar{y}_i := f_i(\bar{x})$, for $i = 1, ..., m$, we have that $(\bar{p}, \bar{q}, \bar{\lambda}, \bar{y}) \in \mathcal{B}_{FL}$ and $f(\bar{x}) = h^{FL}(\bar{p}, \bar{q}, \bar{\lambda}, \bar{y}) = \bar{y}$. The maximality of $(\bar{p}, \bar{q}, \bar{\lambda}, \bar{y})$ comes from Theorem 4.1. □

Following the same scheme, using the form of the objective functions of the scalar duals (D_F^λ) and (D_L^λ), we can formulate two other dual multiobjective duals to (P),

$$(D_F) \quad \underset{(p,\lambda,y)\in\mathcal{B}_F}{\text{v-max}}\ h^F(p,\lambda,y),$$

$$h^F(p,\lambda,y) = \begin{pmatrix} h_1^F(p,\lambda,y) \\ \vdots \\ h_m^F(p,\lambda,y) \end{pmatrix} = \begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix},$$

with

$$h_j^F(p,\lambda,y) = y_j, j = 1,...,m,$$

the dual variables

$$p = (p_1,\ldots,p_m) \in \mathbb{R}^n \times ... \times \mathbb{R}^n, \lambda = (\lambda_1,\ldots,\lambda_m)^T \in \mathbb{R}^m, y = (y_1,\ldots,y_m)^T \in \mathbb{R}^m,$$

and the set of constraints

$$\mathcal{B}_F = \{(p,\lambda,y) \quad : p = (p_1,...,p_m), \quad \lambda = (\lambda_1,\ldots,\lambda_m)^T \in int\mathbb{R}^m_+, \\ \sum_{i=1}^m \lambda_i y_i \leq -\sum_{i=1}^m \lambda_i f_i^*(p_i) - \chi_{\mathcal{A}}^*\left(-\sum_{i=1}^m \lambda_i p_i\right)\},$$

and

$$(D_L) \quad \underset{(q,\lambda,y)\in\mathcal{B}_L}{\text{v-max}}\ h^L(q,\lambda,y),$$

$$h^L(q,\lambda,y) = \begin{pmatrix} h_1^L(q,\lambda,y) \\ \vdots \\ h_m^L(q,\lambda,y) \end{pmatrix} = \begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix},$$

with

$$h_j^L(q,\lambda,y) = y_j, j = 1,...,m,$$

the dual variables

$$q \in \mathbb{R}^k, \lambda = (\lambda_1,\ldots,\lambda_m)^T \in \mathbb{R}^m, y = (y_1,\ldots,y_m)^T \in \mathbb{R}^m,$$

and the set of constraints

$$\mathcal{B}_L = \{(q,\lambda,y) \quad : \lambda = (\lambda_1,\ldots,\lambda_m)^T \in int\mathbb{R}^m_+, \quad q \underset{K^*}{\geqq} 0, \\ \sum_{i=1}^m \lambda_i y_i \leq \inf_{x\in\mathbb{R}^n}\left[\sum_{i=1}^m \lambda_i f_i(x) + q^T g(x)\right]\}.$$

Next, we show that also for these two dual problems weak and strong duality hold.

THEOREM 4.3 (weak duality for (D_F)) *There is no* $x \in \mathcal{A}$ *and no* $(p, \lambda, y) \in \mathcal{B}_F$ *fulfilling* $h^F(p, \lambda, y) \underset{\mathbb{R}^m_+}{\geqq} f(x)$ *and* $h^F(p, \lambda, y) \neq f(x)$.

Proof We assume that there exist $x \in \mathcal{A}$ and $(p, \lambda, y) \in \mathcal{B}_F$ such that $f_i(x) \leq y_i, \forall i \in \{1, \dots, m\}$ and $f_j(x) < y_j$, for at least one $j \in \{1, \dots, m\}$. Using all these inequalities yields

$$\sum_{i=1}^{m} \lambda_i f_i(x) < \sum_{i=1}^{m} \lambda_i y_i. \tag{4. 2}$$

But

$$\begin{aligned}
\sum_{i=1}^{m} \lambda_i y_i &\leq -\sum_{i=1}^{m} \lambda_i f_i^*(p_i) - \chi_{\mathcal{A}}^*\left(-\sum_{i=1}^{m} \lambda_i p_i\right) \\
&\leq \sum_{i=1}^{m} \lambda_i f_i(x) - \left(\sum_{i=1}^{m} \lambda_i p_i\right)^T x + \chi_{\mathcal{A}}(x) + \left(\sum_{i=1}^{m} \lambda_i p_i\right)^T x \\
&= \sum_{i=1}^{m} \lambda_i f_i(x),
\end{aligned}$$

and the inequality $\sum\limits_{i=1}^{m} \lambda_i y_i \leq \sum\limits_{i=1}^{m} \lambda_i f_i(x)$ contradicts relation (4. 2). $\square$

THEOREM 4.4 (weak duality for (D_L)) *There is no* $x \in \mathcal{A}$ *and no* $(q, \lambda, y) \in \mathcal{B}_L$ *fulfilling* $h^L(q, \lambda, y) \underset{\mathbb{R}^m_+}{\geqq} f(x)$ *and* $h^L(q, \lambda, y) \neq f(x)$.

Proof We assume that there exist $x \in \mathcal{A}$ and $(q, \lambda, y) \in \mathcal{B}_L$ such that $f_i(x) \leq y_i, \forall i \in \{1, \dots, m\}$ and $f_j(x) < y_j$, for at least one $j \in \{1, \dots, m\}$. This gives us that

$$\sum_{i=1}^{m} \lambda_i f_i(x) < \sum_{i=1}^{m} \lambda_i y_i. \tag{4. 3}$$

Again,

$$\begin{aligned}
\sum_{i=1}^{m} \lambda_i y_i &\leq \inf_{x \in \mathbb{R}^n} \left[\sum_{i=1}^{m} \lambda_i f_i(x) + q^T g(x)\right] \\
&\leq \sum_{i=1}^{m} \lambda_i f_i(x) + q^T g(x) \\
&\leq \sum_{i=1}^{m} \lambda_i f_i(x),
\end{aligned}$$

which contradicts the inequality from (4. 3). $\square$

THEOREM 4.5 (strong duality for (D_F)) *Assume that (A_f), (A_g) and (A_{CQ}) are fulfilled. If $\bar{x}$ is a properly efficient solution to (P), then there exists an efficient solution $(\bar{p}, \bar{\lambda}, \bar{y}) \in \mathcal{B}_F$ to the dual (D_F) and strong duality $f(\bar{x}) = h^F(\bar{p}, \bar{\lambda}, \bar{y}) = \bar{y}$ holds.*

Proof If $\bar{x}$ is properly efficient to the problem (P), then there exists a vector $\bar{\lambda} = (\bar{\lambda}_1, \dots, \bar{\lambda}_m)^T \in int\mathbb{R}^m_+$ (cf. [14]) such that $\bar{x}$ solves the scalar problem

$$(P_{\bar{\lambda}}) \ \inf_{x \in \mathcal{A}} \sum_{i=1}^{m} \bar{\lambda}_i f_i(x).$$

By the strong duality theorem 2.1, it results the existence of a solution $\bar{p} = (\bar{p}_1, ..., \bar{p}_m)$ to $(D_F^{\bar{\lambda}})$ such that

$$\sum_{i=1}^{m} \bar{\lambda}_i f_i(\bar{x}) = \inf(P_{\bar{\lambda}}) = \max(D_F^{\bar{\lambda}}) = -\sum_{i=1}^{m} \bar{\lambda}_i f_i^*(\bar{p}_i) - \chi_{\mathcal{A}}^* \left(-\sum_{i=1}^{m} \bar{\lambda}_i \bar{p}_i \right).$$

By taking $\bar{y}_i := f_i(\bar{x})$, for $i = 1, ..., m$, we have that $(\bar{p}, \bar{\lambda}, \bar{y}) \in \mathcal{B}_F$ and $f(\bar{x}) = h^F(\bar{p}, \bar{\lambda}, \bar{y}) = \bar{y}$. By Theorem 4.3 it follows that the element $(\bar{p}, \bar{\lambda}, \bar{y})$ is maximal for (D_F). $\square$

THEOREM 4.6 (strong duality for (D_L)) *Assume that (A_f), (A_g) and (A_{CQ}) are fulfilled. If $\bar{x}$ is a properly efficient solution to (P), then there exists an efficient solution $(\bar{q}, \bar{\lambda}, \bar{y}) \in \mathcal{B}_L$ to the dual (D_L) and strong duality $f(\bar{x}) = h^L(\bar{q}, \bar{\lambda}, \bar{y}) = \bar{y}$ holds.*

Proof If $\bar{x}$ is properly efficient to the problem (P), then there exists a vector $\bar{\lambda} = (\bar{\lambda}_1, \dots, \bar{\lambda}_m)^T \in int\mathbb{R}^m_+$ (cf. [14]) such that $\bar{x}$ solves the scalar problem

$$(P_{\bar{\lambda}}) \ \inf_{x \in \mathcal{A}} \sum_{i=1}^{m} \bar{\lambda}_i f_i(x).$$

By the strong duality theorem 2.1, it results the existence of a solution $\bar{q} \underset{K^*}{\geqq} 0$ to the dual $(D_L^{\bar{\lambda}})$ such that

$$\sum_{i=1}^{m} \bar{\lambda}_i f_i(\bar{x}) = \inf(P_{\bar{\lambda}}) = \max(D_L^{\bar{\lambda}}) = \inf_{x \in \mathbb{R}^n} \left[\sum_{i=1}^{m} \bar{\lambda}_i f_i(x) + \bar{q}^T g(x) \right].$$

By taking $\bar{y}_i := f_i(\bar{x})$, for $i = 1, ..., m$, we have that $(\bar{q}, \bar{\lambda}, \bar{y}) \in \mathcal{B}_L$ and $f(\bar{x}) = h^L(\bar{q}, \bar{\lambda}, \bar{y}) = \bar{y}$. By Theorem 4.4 it follows that the element $(\bar{q}, \bar{\lambda}, \bar{y})$ is maximal in (D_L). $\square$

Remark 4.1 The dual problem (D_L) represents the transcription in finite-dimensional spaces of the multiobjective dual introduced by Jahn in [1] and [2].

For the last multiobjective problem which we present here we use exclusively the scalarized problem (P_λ). So, let this dual be

$$(D_P) \quad \underset{(\lambda,y)\in\mathcal{B}_P}{\text{v-max}}\, h^P(\lambda,y),$$

$$h^P(\lambda,y) = \begin{pmatrix} h_1^P(\lambda,y) \\ \vdots \\ h_m^P(\lambda,y) \end{pmatrix} = \begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix},$$

with

$$h_j^P(\lambda,y) = y_j, j = 1, ..., m,$$

the dual variables

$$\lambda = (\lambda_1, \ldots, \lambda_m)^T \in \mathbb{R}^m, y = (y_1, \ldots, y_m)^T \in \mathbb{R}^m,$$

and the set of constraints

$$\mathcal{B}_P = \{(\lambda, y) : \lambda = (\lambda_1, \ldots, \lambda_m)^T \in int\mathbb{R}^m_+, \sum_{i=1}^m \lambda_i y_i \leq \inf_{x\in\mathcal{A}} \sum_{i=1}^m \lambda_i f_i(x)\}.$$

Between the primal problem (P) and the dual (D_P) weak and strong duality hold. We omit the proofs of the following two theorems because of their simplicity.

THEOREM 4.7 (weak duality for (D_P)) *There is no $x \in \mathcal{A}$ and no $(\lambda, y) \in \mathcal{B}_P$ fulfilling $h^P(\lambda,y) \underset{\mathbb{R}^m_+}{\geqq} f(x)$ and $h^F(\lambda,y) \neq f(x)$.*

THEOREM 4.8 (strong duality for (D_P)) *Assume that (A_f) and (A_g) are fulfilled. If $\bar{x}$ is a properly efficient solution to (P), then there exists an efficient solution $(\bar{\lambda}, \bar{y}) \in \mathcal{B}_P$ to the dual (D_P) and strong duality $f(\bar{x}) = h^P(\bar{\lambda}, \bar{y}) = \bar{y}$ holds.*

Remark 4.2 Let us notice that, in order to have strong duality between (P) and (D_P), we do not need the assumption (A_{CQ}) to be fulfilled. Only (A_f) and (A_g) are here necessary, assuring the convexity of the problem (P). This permits us to characterize the properly efficient solutions of (P) via scalarization (cf. [14]).

5 The relations between the duals (D_1), (D_α), $\alpha \in \mathcal{F}$, and (D_{FL})

Now, we investigate the existence of some relations of inclusion between the three dual problems (D_1), (D_α), $\alpha \in \mathcal{F}$, and (D_{FL}).

For the beginning, let us notice that to find the Pareto-efficient solutions of a multiobjective dual problem means actually to determine the maximal elements of the image set of its objective function over the set of constraints. This is the reason why, in order to compare the duals (D_1), (D_α), $\alpha \in \mathcal{F}$, and (D_{FL}), we analyze the relations between the corresponding image sets. Therefore, let be $D_1 := h^1(\mathcal{B}_1)$, $D_\alpha := h^\alpha(\mathcal{B}_\alpha), \alpha \in \mathcal{F}$, and $D_{FL} := h^{FL}(\mathcal{B}_{FL})$. It is obvious that $D_1 \subseteq \underbrace{\overline{\mathbb{R}} \times ... \times \overline{\mathbb{R}}}_{m}$, $D_\alpha \subseteq \underbrace{\overline{\mathbb{R}} \times ... \times \overline{\mathbb{R}}}_{m}, \alpha \in \mathcal{F}$, and $D_{FL} \subseteq \mathbb{R}^m$.

PROPOSITION 5.1 *For each $\alpha \in \mathcal{F}$ it holds $D_1 \subseteq D_\alpha$.*

Proof Let be $\alpha \in \mathcal{F}$ fixed and $d = (d_1, ..., d_m)^T$ an element in D_1. Then there exists $(p, q, \lambda, t) \in \mathcal{B}_1$ such that

$$d_j = -f_j^*(p_j) - (q^T g)^* \left(-\frac{1}{\sum\limits_{i=1}^m \lambda_i} \sum_{i=1}^m \lambda_i p_i \right) + t_j, j = 1, ..., m.$$

Let us define now $\bar{p}_j := p_j$, $\bar{\lambda}_j := \lambda_j$, $\bar{q}_j := \alpha_j(\bar{\lambda}) \left(\sum\limits_{i=1}^m \bar{\lambda}_i \right) q$, for $j = 1, ..., m$, and $\bar{\bar{q}} := (\bar{q}_1, ..., \bar{q}_m)$. It holds

$$\sum_{i=1}^m \bar{\lambda}_i \bar{q}_i = \left(\sum_{j=1}^m \bar{\lambda}_j \alpha_j(\bar{\lambda}) \right) \left(\sum_{i=1}^m \bar{\lambda}_i \right) q = \left(\sum_{i=1}^m \bar{\lambda}_i \right) q \underset{K^*}{\geqq} 0,$$

and, for $j = 1, ..., m$,

$$(\bar{q}_j^T g)^* \left(-\alpha_j(\bar{\lambda}) \sum_{i=1}^m \bar{\lambda}_i \bar{p}_i \right) = \left(\alpha_j(\bar{\lambda}) \sum_{i=1}^m \bar{\lambda}_i \right) (q^T g)^* \left(-\frac{1}{\sum\limits_{i=1}^m \bar{\lambda}_i} \sum_{i=1}^m \bar{\lambda}_i \bar{p}_i \right). \quad (5.\ 1)$$

First, let us consider the case when $(q^T g)^* \left(-\frac{1}{\sum\limits_{i=1}^m \bar{\lambda}_i} \sum\limits_{i=1}^m \bar{\lambda}_i \bar{p}_i \right) = +\infty$. This means that $d_j = -\infty, \forall j = 1, ..., m$. By taking $\bar{t}_j := t_j$, for $j = 1, ..., m$, we have that $(\bar{p}, \bar{\bar{q}}, \bar{\lambda}, \bar{t}) \in \mathcal{B}_\alpha$ and $d = (-\infty, ..., -\infty)^T = h^\alpha(\bar{p}, \bar{\bar{q}}, \bar{\lambda}, \bar{t}) \in h^\alpha(\mathcal{B}_\alpha) = D_\alpha$.

In the other case, when $(q^T g)^* \left(-\frac{1}{\sum\limits_{i=1}^m \bar{\lambda}_i} \sum\limits_{i=1}^m \bar{\lambda}_i \bar{p}_i \right) \in \mathbb{R}$, let us take, for $j = 1, ..., m$,

$$\bar{t}_j := t_j + (\bar{q}_j^T g)^* \left(-\alpha_j(\bar{\lambda}) \sum_{i=1}^m \bar{\lambda}_i \bar{p}_i \right) - (q^T g)^* \left(-\frac{1}{\sum\limits_{i=1}^m \bar{\lambda}_i} \sum_{i=1}^m \bar{\lambda}_i \bar{p}_i \right) \in \mathbb{R}.$$

From (5. 1), we have $\sum\limits_{i=1}^m \bar{\lambda}_i \bar{t}_i = \sum\limits_{i=1}^m \lambda_i t_i = 0$, which implies that $(\bar{p}, \bar{\tilde{q}}, \bar{\lambda}, \bar{t}) \in \mathcal{B}_\alpha$. Moreover, for $j = 1, ..., m$,

$$d_j = -f_j^*(\bar{p}_j) - (\bar{q}_j^T g)^* \left(-\alpha_j(\bar{\lambda}) \sum_{i=1}^m \bar{\lambda}_i \bar{p}_i \right) + \bar{t}_j.$$

Therefore, $d = h^\alpha(\bar{p}, \bar{\tilde{q}}, \bar{\lambda}, \bar{t}) \in h^\alpha(\mathcal{B}_\alpha) = D_\alpha$ meaning $D_1 \subseteq D_\alpha$. $\square$

Example 5.1 Let be $\alpha \in \mathcal{F}$ fixed, $m = 2, n = 1, k = 1$ and $K = \mathbb{R}_+$. Considering $f_1, f_2 : \mathbb{R} \to \mathbb{R}$, $f_1(x) = f_2(x) = x^2$, $g : \mathbb{R} \to \mathbb{R}$, $g(x) = x^2 - 1$, $\lambda = (2, 1)^T$, $\tilde{q} = (q_1, q_2) = (1, -1)$, $t = (1, -2)^T$, we have that $\lambda_1 q_1 + \lambda_2 q_2 = 1$ and $\lambda_1 t_1 + \lambda_2 t_2 = 0$. For $p = (0, 0)$, it holds $f_1^*(p_1) = f_2^*(p_2) = 0$ and

$$d = (-(q_1 g)^*(0) + t_1, -(q_2 g)^*(0) + t_2)^T = (0, -\infty)^T = h^\alpha(p, \tilde{q}, \lambda, t) \in D_\alpha.$$

But, let us notice that $d \notin D_1$. It means that the inclusion $D_1 \subseteq D_\alpha$ may be strict. We denote this by $D_1 \subsetneq D_\alpha$, $\alpha \in \mathcal{F}$.

Example 5.2 Let be again $\alpha \in \mathcal{F}$ fixed, but $m = 2, n = 1, k = 2$ and $K = \mathbb{R}^2$. Considering now $f_1, f_2 : \mathbb{R} \to \mathbb{R}$, $f_1(x) = f_2(x) = 0$, $g_1, g_2 : \mathbb{R} \to \mathbb{R}$,

$$g_1(x) = \begin{cases} 1, & \text{if } x < 0, \\ e^{-x}, & \text{if } x \geq 0, \end{cases}$$

$$g_2(x) = \begin{cases} e^{x}, & \text{if } x < 0, \\ 1, & \text{if } x \geq 0, \end{cases}$$

$p = (0, 0)$, $q_1 = (1, -1), q_2 = (-1, 1)$, $\tilde{q} = (q_1, q_2)$, $\lambda = (1, 1)^T$ and $t = \left(\frac{1}{2}, -\frac{1}{2}\right)^T$, we have $\lambda_1 q_1 + \lambda_2 q_2 = (0, 0)^T \in K^*$, $\lambda_1 t_1 + \lambda_2 t_2 = 0$ and $f_1^*(0) = f_2^*(0) = 0$.

This means that

$$d = \left(-\frac{1}{2}, -\frac{3}{2} \right)^T = \left(-(q_1^T g)^*(0) + t_1, -(q_2^T g)^*(0) + t_2 \right)^T = h^\alpha(p, \tilde{q}, \lambda, t) \in D_\alpha \cap \mathbb{R}^2.$$

Let us show now that $d \notin D_1$. If this were not true, then there would exist a tuple $(\bar{p}, \bar{q}, \bar{\lambda}, \bar{t}) \in \mathcal{B}_1$ such that

$$\begin{pmatrix} -\frac{1}{2} \\ \\ -\frac{3}{2} \end{pmatrix} = \begin{pmatrix} -f_1^*(\bar{p}_1) - (\bar{q}^T g)^* \left(-\frac{\bar{\lambda}_1 \bar{p}_1 + \bar{\lambda}_2 \bar{p}_2}{\bar{\lambda}1 + \bar{\lambda}2}\right) + \bar{t}_1 \\ \\ -f_2^*(\bar{p}_2) - (\bar{q}^T g)^* \left(-\frac{\bar{\lambda}_1 \bar{p}_1 + \bar{\lambda}_2 \bar{p}_2}{\bar{\lambda}1 + \bar{\lambda}2}\right) + \bar{t}_2 \end{pmatrix}.$$

It follows that $f_1^*(\bar{p}_1), f_2^*(\bar{p}_2) \in \mathbb{R}$, but, in order to happen this, we must have $\bar{p}_1 = \bar{p}_2 = 0$, $f_1^*(\bar{p}_1) = f_2^*(\bar{p}_2) = 0$ and, because of $\bar{q} \in (\mathbb{R}^2)^* = \{0\}$, $(\bar{q}^T g)^*(0) = 0$. So,

$$\begin{pmatrix} -\frac{1}{2} \\ \\ -\frac{3}{2} \end{pmatrix} = \begin{pmatrix} \bar{t}_1 \\ \\ \bar{t}_2 \end{pmatrix},$$

and, from here,

$$\bar{\lambda}_1 \bar{t}_1 + \bar{\lambda}_2 \bar{t}_2 = -\frac{\bar{\lambda}_1 + 3\bar{\lambda}_2}{2} < 0. \tag{5. 2}$$

Obviously, relation (5. 2) contradicts $\bar{\lambda}_1 \bar{t}_1 + \bar{\lambda}_2 \bar{t}_2 = 0$ and this means that $d = (-\frac{1}{2}, -\frac{3}{2})^T \notin D_1 \cap \mathbb{R}^2$. In conclusion, $D_1 \cap \mathbb{R}^m \subsetneq D_\alpha \cap \mathbb{R}^m$, i.e. the inclusion may be strict.

PROPOSITION 5.2 *For each $\alpha \in \mathcal{F}$ it holds $D_\alpha \cap \mathbb{R}^m \subseteq D_{FL}$.*

Proof Let be $\alpha \in \mathcal{F}$ fixed and $d = (d_1, ..., d_m)^T$ an element in $D_\alpha \cap \mathbb{R}^m$. Then there exists $(p, \tilde{q}, \lambda, t) \in \mathcal{B}_\alpha$ such that $d = h^\alpha(p, \tilde{q}, \lambda, t)$. From here, by using the inequalities of Young for $q_j^T g, j = 1, ..., m$, we have

$$\begin{aligned}
\sum_{j=1}^m \lambda_j d_j &= \sum_{i=j}^m \lambda_j h_j^\alpha(p, \tilde{q}, \lambda, t) = -\sum_{i=j}^m \lambda_j f_j^*(p_j) \\
&- \sum_{j=1}^m \lambda_j (q_j^T g)^* \left(-\alpha_j(\lambda) \sum_{i=1}^m \lambda_i p_i\right) + \sum_{j=1}^m \lambda_j t_j \\
&\leq -\sum_{j=1}^m \lambda_j f_j^*(p_j) + \left(\sum_{j=1}^m \lambda_j q_j\right)^T g(x) \\
&+ \left(\sum_{j=1}^m \lambda_j \alpha_j(\lambda)\right) \left(\sum_{i=1}^m \lambda_i p_i\right)^T x = -\sum_{j=1}^m \lambda_j f_j^*(p_j) \\
&+ \left(\sum_{j=1}^m \lambda_j p_j\right)^T x + \left(\sum_{j=1}^m \lambda_j q_j\right)^T g(x), \forall x \in \mathbb{R}^n.
\end{aligned}$$

We have then

$$\begin{aligned}\sum_{j=1}^{m}\lambda_j d_j &\leq -\sum_{j=1}^{m}\lambda_j f_j^*(p_j) + \inf_{x\in\mathbb{R}^n}\left[\left(\sum_{i=1}^{m}\lambda_i p_i\right)^T x + \left(\sum_{j=1}^{m}\lambda_j q_j\right)^T g(x)\right]\\ &= -\sum_{j=1}^{m}\lambda_j f_j^*(p_j) - \left(\left(\sum_{j=1}^{m}\lambda_j q_j\right)^T g\right)^* \left(-\sum_{i=1}^{m}\lambda_i p_i\right)\end{aligned}$$

and this means that $(p, \sum_{j=1}^{m}\lambda_j q_j, \lambda, d) \in \mathcal{B}_{FL}$. In conclusion,

$$d = h^{FL}(p, \sum_{j=1}^{m}\lambda_j q_j, \lambda, d) \in h^{FL}(\mathcal{B}_{FL}) = D_{FL}.$$

$\square$

Example 5.3 Let be $\alpha \in \mathcal{F}$ fixed, $m = 2, n = 1, k = 1$ and $K = \mathbb{R}$. Considering $f_1, f_2 : \mathbb{R} \to \mathbb{R}$, $f_1(x) = f_2(x) = 0$, $g : \mathbb{R} \to \mathbb{R}$, $g(x) = x^2$, $p = (0, 0)$, $q = 0 \in K^* = \{0\}$, $\lambda = (1, 1)^T$ and $d = (-1, -1)^T$, we have

$$\lambda_1 d_1 + \lambda_2 d_2 = -2 < 0 = -\lambda_1 f_1^*(p_1) - \lambda_2 f_2^*(p_2) - (qg)^*(-\lambda_1 p_1 - \lambda_2 p_2),$$

which implies that $(p, q, \lambda, d) \in \mathcal{B}_{FL}$ and $d = (-1, -1)^T \in D_{FL}$.

Let us show now that $d \notin D_\alpha$. If this were not true, then there would exist $(\bar{p}, \bar{\bar{q}}, \bar{\lambda}, \bar{t}) \in \mathcal{B}_\alpha$, $\bar{\bar{q}} = (\bar{q}_1, \bar{q}_2)$ such that

$$\begin{pmatrix} -1 \\ \\ -1 \end{pmatrix} = \begin{pmatrix} -f_1^*(\bar{p}_1) - (\bar{q}_1 g)^*\left(-\alpha_1(\bar{\lambda})(\bar{\lambda}_1\bar{p}_1 + \bar{\lambda}_2\bar{p}_2)\right) + \bar{t}_1 \\ \\ -f_2^*(\bar{p}_2) - (\bar{q}_2 g)^*\left(-\alpha_2(\bar{\lambda})(\bar{\lambda}_1\bar{p}_1 + \bar{\lambda}_2\bar{p}_2)\right) + \bar{t}_2 \end{pmatrix}.$$

Again, $f_2^*(\bar{p}_2), f_2^*(\bar{p}_2) \in \mathbb{R}$, but, in order to happen this, we must have $\bar{p}_1 = \bar{p}_2 = 0$ and $f_2^*(\bar{p}_2) = f_2^*(\bar{p}_2) = 0$. In this case, we obtain

$$\begin{pmatrix} -1 \\ \\ -1 \end{pmatrix} = \begin{pmatrix} -(\bar{q}_1 g)^*(0) + \bar{t}_1 \\ \\ -(\bar{q}_2 g)^*(0) + \bar{t}_2 \end{pmatrix}.$$

From the last relation, it follows $-(\bar{q}_1 g)^*(0) = \inf_{x\in\mathbb{R}}(\bar{q}_1 x^2) \in \mathbb{R}$ and $-(\bar{q}_2 g)^*(0) = \inf_{x\in\mathbb{R}}(\bar{q}_2 x^2) \in \mathbb{R}$, which hold just if $\bar{q}_1 \geq 0$ and $\bar{q}_2 \geq 0$. On the other hand, we have that $\bar{\lambda}_1\bar{q}_1 + \bar{\lambda}_2\bar{q}_2 \in K^* = \{0\}$, whence $\bar{q}_1 = \bar{q}_2 = 0$.

So, $\bar{t}_1 = \bar{t}_2 = -1$ and $\bar{\lambda}_1\bar{t}_1 + \bar{\lambda}_2\bar{t}_2 = -\bar{\lambda}_1 - \bar{\lambda}_2 < 0$, which is a contradiction to $(\bar{p}, \bar{\bar{q}}, \bar{\lambda}, \bar{t}) \in \mathcal{B}_\alpha$. Our assumption that $d \in D_\alpha$ proves to be false. In conclusion,

$D_\alpha \cap \mathbb{R}^m \subsetneq D_{FL}$, i.e. the inclusion may be strict.

By the propositions 5.1, 5.2 and the examples 5.1-5.3, we have, for each $\alpha \in \mathcal{F}$,

$$D_1 \cap \mathbb{R}^m \subsetneq D_\alpha \cap \mathbb{R}^m \subsetneq D_{FL}. \tag{5. 3}$$

In the last part of this section, let us prove that, even the sets D_1, $D_\alpha, \alpha \in \mathcal{F}$, and D_{FL} may be different (cf. (5. 3)), they have the same maximal elements. In order to do this, we consider their corresponding sets of maximal elements $vmaxD_1$, $vmaxD_\alpha$, $\alpha \in \mathcal{F}$, and $vmaxD_{FL}$, respectively. All these sets are subsets of $\mathbb{R}^m$.

We are now able to prove the main results of the first part of this study.

THEOREM 5.3 *It holds $vmaxD_1 = vmaxD_{FL}$.*

Proof

"$vmaxD_1 \subseteq vmaxD_{FL}$". Let be $d \in vmaxD_1$. It means that $d \in D_1 \cap \mathbb{R}^m$ and, from (5. 3), we have $d \in D_{FL}$. Then there exists an element $(p, q, \lambda, y) \in \mathcal{B}_{FL}$ such that $y = h^{FL}(p, q, \lambda, y) = d$.

Let us assume now that $d \notin vmaxD_{FL}$. By the definition of the maximal elements, it follows that there exists $(\bar{p}, \bar{q}, \bar{\lambda}, \bar{d}) \in \mathcal{B}_{FL}$ such that $d \in \bar{d} - \big(\mathbb{R}^m_+ \setminus \{0\}\big)$ $(\bar{d} \gneqq d)$, i.e. $d_i \leq \bar{d}_i$, $\forall i = 1, ..., m$, and $d_j < \bar{d}_j$, for at least one $j \in \{1, ..., m\}$. Thus,

$$\sum_{i=1}^m \bar{\lambda}_i d_i < \sum_{i=1}^m \bar{\lambda}_i \bar{d}_i \leq -\sum_{i=1}^m \bar{\lambda}_i f_i^*(\bar{p}_i) - (\bar{q}^T g)^* \left(-\sum_{i=1}^m \bar{\lambda}_i \bar{p}_i \right). \tag{5. 4}$$

Let be now $\bar{\bar{d}} \in \mathbb{R}^m$ such that $\bar{\bar{d}} \in \bar{d} + \mathbb{R}^m_+$ $(\bar{\bar{d}} \geqq \bar{d})$ and

$$\sum_{i=1}^m \bar{\lambda}_i \bar{\bar{d}}_i = -\sum_{i=1}^m \bar{\lambda}_i f_i^*(\bar{p}_i) - (\bar{q}^T g)^* \left(-\sum_{i=1}^m \bar{\lambda}_i \bar{p}_i \right). \tag{5. 5}$$

Considering $\bar{\bar{q}} := \frac{1}{\sum\limits_{i=1}^m \bar{\lambda}_i} \bar{q} \underset{K^*}{\geqq} 0$ and, for $j = 1, ..., m$,

$$\begin{aligned} \bar{t}_j \quad &:= \quad f_j^*(\bar{p}_j) + (\bar{\bar{q}}^T g)^* \left(-\frac{1}{\sum\limits_{i=1}^m \bar{\lambda}_i} \sum_{i=1}^m \bar{\lambda}_i \bar{p}_i \right) + \bar{\bar{d}}_j \\ &= \quad f_j^*(\bar{p}_j) + \frac{1}{\sum\limits_{i=1}^m \bar{\lambda}_i} (\bar{q}^T g)^* \left(-\sum_{i=1}^m \bar{\lambda}_i \bar{p}_i \right) + \bar{\bar{d}}_j \in \mathbb{R}, \end{aligned}$$

we obtain an element $(\bar{p}, \bar{\bar{q}}, \bar{\lambda}, \bar{t})$ with the properties $\bar{\bar{q}} \underset{K^*}{\geqq} 0$, $\bar{\lambda} \in int\mathbb{R}^m_+$ and, by (5. 5),

$$\sum_{i=1}^m \bar{\lambda}_i \bar{t}_i = \sum_{i=1}^m \bar{\lambda}_i \bar{\bar{d}}_i + \sum_{i=1}^m \bar{\lambda}_i f_i^*(\bar{p}_i) + (\bar{\bar{q}}^T g)^* \left(- \sum_{i=1}^m \bar{\lambda}_i \bar{p}_i \right) = 0.$$

So, $(\bar{p}, \bar{\bar{q}}, \bar{\lambda}, \bar{t}) \in \mathcal{B}_1$, $\bar{\bar{d}} = h^1(\bar{p}, \bar{\bar{q}}, \bar{\lambda}, \bar{t}) \in h^1(\mathcal{B}_1) = D_1$ and

$$\bar{\bar{d}} \in \bar{d} + \mathbb{R}^m_+ \in d + \mathbb{R}^m_+ \setminus \{0\} + \mathbb{R}^m_+ = d + \mathbb{R}^m_+ \setminus \{0\} \ (\bar{\bar{d}} \gneqq d). \tag{5. 6}$$

This contradicts the maximality of d in D_1 and implies that d must be maximal in D_{FL}.

"vmax$D_{FL} \subseteq$ vmaxD_1". Let be now $d \in vmaxD_{FL}$. Then there exist $p_i \in \mathbb{R}^n, i = 1, ..., m$, $q \in \mathbb{R}^k$ and $\lambda \in int\mathbb{R}^m_+$ such that

$$\sum_{i=1}^m \lambda_i d_i \leq - \sum_{i=1}^m \lambda_i f_i^*(p_i) - (q^T g)^* \left(- \sum_{i=1}^m \lambda_i p_i \right).$$

Let be again $\bar{d} \in \mathbb{R}^m$ such that $\bar{d} \in d + \mathbb{R}^m_+$ $(\bar{d} \geqq d)$ and

$$\sum_{i=1}^m \bar{\lambda}_i \bar{d}_i = - \sum_{i=1}^m \lambda_i f_i^*(p_i) - (q^T g)^* \left(- \sum_{i=1}^m \lambda_i p_i \right). \tag{5. 7}$$

For $\bar{p} := p$, $\bar{\lambda} := \lambda$, $\bar{q} := \frac{1}{\sum_{i=1}^m \bar{\lambda}_i} q \underset{K^*}{\geqq} 0$ and

$$\bar{t}_j := f_j^*(\bar{p}_j) + (\bar{q}^T g)^* \left(- \frac{1}{\sum_{i=1}^m \bar{\lambda}_i} \sum_{i=1}^m \bar{\lambda}_i \bar{p}_i \right) + \bar{d}_j \in \mathbb{R}, j = 1, ..., m,$$

we have $\sum_{i=1}^m \bar{\lambda}_i \bar{t}_i = 0$ (cf. (5. 7)), whence $(\bar{p}, \bar{q}, \bar{\lambda}, \bar{t}) \in \mathcal{B}_1$.

Moreover, the value of the objective function on this element is $h^1(\bar{p}, \bar{q}, \bar{\lambda}, \bar{t}) = \bar{d} \in d + \mathbb{R}^m_+$ $(\bar{d} \geqq d)$. On the other hand, (5. 7) assures that $(\bar{p}, \bar{q}, \bar{\lambda}, \bar{d}) \in \mathcal{B}_{FL}$ and the maximality of d in D_{FL} implies that it is impossible to have $d \in \bar{d} - \left(\mathbb{R}^m_+ \setminus \{0\} \right)$ $(\bar{d} \gneqq d)$. Then we must have $h^1(\bar{p}, \bar{q}, \bar{\lambda}, \bar{t}) = \bar{d} = d$ and, so, $d \in h^1(\mathcal{B}_1) = D_1$.

It remains to show that, actually, $d \in vmaxD_1$. If this does not happen, then there exists an element $(\bar{\bar{p}}, \bar{\bar{q}}, \bar{\bar{\lambda}}, \bar{\bar{t}}) \in \mathcal{B}_1$ such that $\bar{\bar{d}} = h^1(\bar{\bar{p}}, \bar{\bar{q}}, \bar{\bar{\lambda}}, \bar{\bar{t}}) \in d + \left(\mathbb{R}^m_+ \setminus \{0\} \right)$ $(\bar{\bar{d}} \gneqq d)$. But, relation (5. 3) states that $D_1 \cap \mathbb{R}^m \subseteq D_{FL}$ and, from here, $\bar{\bar{d}} \in D_{FL}$. The fact that $\bar{\bar{d}} \in d + \left(\mathbb{R}^m_+ \setminus \{0\}\right)$ ($\bar{\bar{d}} \gneqq d$) contradicts the maximality of d in D_{FL}. We can conclude now that $d \in vmaxD_1$. $\square$

THEOREM 5.4 *For each $\alpha \in \mathcal{F}$ it holds $vmaxD_\alpha = vmaxD_{FL}$.*

Proof Let be $\alpha \in \mathcal{F}$ fixed.

"$vmaxD_\alpha \subseteq vmaxD_{FL}$". Let be $d \in vmaxD_\alpha$. Then it holds $d \in D_\alpha \cap \mathbb{R}^m$ and this implies that $d \in D_{FL}$ (cf. (5. 3)). Let us assume that $d \notin vmaxD_{FL}$. As in the proof of Theorem 5.3, there exists $\bar{\bar{d}} \in D_1$ such that (cf. (5. 6))

$$\bar{\bar{d}} \in d + \mathbb{R}^m_+ \setminus \{0\} \ (\bar{\bar{d}} \gneqq d).$$

But, by (5. 3) we have $\bar{\bar{d}} \in D_\alpha$ and this contradicts the maximality of d in D_α. In conclusion, $d \in vmaxD_{FL}$.

"$vmaxD_{FL} \subseteq vmaxD_\alpha$." Let be now $d \in vmaxD_{FL}$. By Theorem 5.3 it follows that $d \in vmaxD_1$ and, from here, $d \in D_1 \cap \mathbb{R}^m \subseteq D_\alpha \cap \mathbb{R}^m$.

Assuming that $d \notin vmaxD_\alpha$, there must exist an $\bar{\bar{d}} \in D_\alpha$ such that $\bar{\bar{d}} \in d + \mathbb{R}^m_+ \setminus \{0\}$ ($\bar{\bar{d}} \gneqq d$). On the other hand, because of $D_\alpha \cap \mathbb{R}^m \subseteq D_{FL}$, it holds $\bar{\bar{d}} \in D_{FL}$ and this leads us to a contradiction to $d \in vmaxD_{FL}$. So, d must belong to $vmaxD_\alpha$. $\square$

As a conclusion of this first part, we obtain from the last two theorems that, for each $\alpha \in \mathcal{F}$,

$$vmaxD_1 = vmaxD_\alpha = vmaxD_{FL}. \tag{5. 8}$$

Remark 5.1

(a) Let us notice that the inclusions in (5. 3) can be strict, even if the assumptions (A_f), (A_g) and (A_{CQ}) are fulfilled (cf. Example 5.2 and Example 5.3). On the other hand, the equality (5. 8) holds without asking the fulfillment of any of these three assumptions. The equality in (5. 8) holds in the most general case.

(b) Further comparisons between the image sets and the maximal elements sets of the image sets of these three duals and of other multiobjective duals will be discussed within the second part of this study.

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