Farkas-type results for inequality systems with composed convex functions via conjugate duality

Radu Ioan Boţ *   Ioan Bogdan Hodrea †   Gert Wanka ‡

Abstract. We present some Farkas-type results for inequality systems involving finitely many functions. Therefore we use a conjugate duality approach applied to an optimization problem with a composed convex objective function and convex inequality constraints. Some recently obtained results are rediscovered as special cases of our main result.

Key Words. Farkas-type results, composed convex functions, conjugate functions, conjugate duality

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1 Introduction

During the last decades the optimization techniques have been intensively used in various fields of applications. Since the problems generated by the practical needs turn out to be more and more complex, the attention of many mathematicians has been focused on finding some methods and conditions which guarantee the existence of optimal solutions.

*Faculty of Mathematics, Chemnitz University of Technology, D-09107 Chemnitz, Germany, e-mail: radu.bot@mathematik.tu-chemnitz.de

†Faculty of Mathematics, Chemnitz University of Technology, D-09107 Chemnitz, Germany, e-mail: bogdan.hodrea@mathematik.tu-chemnitz.de.

‡Faculty of Mathematics, Chemnitz University of Technology, D-09107 Chemnitz, Germany, e-mail: gert.wanka@mathematik.tu-chemnitz.de

The problem treated within this paper consists in minimizing the composition of some convex functions when finitely many real-valued constraint functions are non-positive. From the large number of works dealing with composed optimization problems let us mention [3], [6], [8], [10], [11], [12], [14], [15], [18], [19]. In order to provide duality assertions for the problem treated in [18], the authors have worked with a duality concept based on conjugacy and perturbations.

However, similar results can be obtained using another approach (see [1]), namely considering an equivalent problem to the primal one, but whose dual can be easier established. The equivalent problem is introduced considering an auxiliary variable. For the new problem we consider the Lagrange dual problem. To the inner infimum of the Lagrange dual we attach the Fenchel dual problem and it can be easily seen that the final dual we obtain is actually a so-called Fenchel-Lagrange-type dual of the primal problem. The construction of the dual is described here in detail and a constraint qualification assuring strong duality between the primal problem and its dual is introduced. Regarding the Fenchel-Lagrange dual problem, let us mention that this type of dual problem has been introduced, named and studied by Boţ and Wanka (see, for example, [3], [4], [5], [17]).

Recently Boţ and Wanka have presented in [4] and [5] some Farkas-type results for inequality systems involving finitely many convex functions using an approach based on the theory of conjugate duality for convex problems. The aim of the present paper is to extend these results to convex optimization problems involving composed convex functions. The weak and strong duality assertions are used in order to deliver a Farkas-type statement for inequality systems involving composed convex functions. Moreover, it is shown that some results in the literature arise as special cases of the problem we treat.

The paper is organized as follows. In section 2 we present some definitions and results needed later within the paper. We give a dual for the optimization problem with composed convex functions and establish the weak and strong duality assertions in the third section. Section 4 contains the main result of the paper. Using the duality acquired in section 3 we give a Farkas-type theorem. In the last section Farkas-type results for some particular instances of the initial one are presented, rediscovering some recent results.

2 Notations and preliminaries

We use some well-known concepts briefly recalled here. The notations we use throughout the paper and some preliminary results are presented in the following, too. We consider all vectors as column vectors. Any column vector can be transposed to a row vector by an upper index ^{T}. By $x^T y = \sum_{i=1}^{n} x_i y_i$ is denoted the usual inner product of two vectors $x = (x_1, ..., x_n)^T$ and $y = (y_1, ..., y_n)^T$ in the real space $\mathbb{R}^n$. By "$\leqq$" we denote the partial order introduced by the non-negative orthant $\mathbb{R}^n_+$, defined by

$$x \leqq y \Leftrightarrow x_i \leq y_i, \forall i = 1, ..., n.$$

Let us consider an arbitrary set $X \subseteq \mathbb{R}^n$. The *relative interior*, the *convex hull* and the *closure* of the set X are denoted by $\mathrm{ri}(X)$, $\mathrm{co}(X)$ and $\mathrm{cl}(X)$, respectively. Furthermore, the *cone* and the *convex cone* generated by the set X are denoted by $\mathrm{cone}(X) = \bigcup_{\lambda \geq 0} \lambda X$ and, respectively, $\mathrm{coneco}(X) = \bigcup_{\lambda \geq 0} \lambda \,\mathrm{co}(X)$. By $v(P)$ we denote the optimal objective value of an optimization problem (P).

If $X \subseteq \mathbb{R}^n$ is given, we consider the following two functions, the *indicator function*

$$\delta_X : \mathbb{R}^n \to \overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}, \quad \delta_X(x) = \begin{cases} 0, & x \in X, \\ +\infty, & \text{otherwise,} \end{cases}$$

and the *support function*

$$\sigma_X : \mathbb{R}^n \to \overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}, \quad \sigma_X(u) = \sup_{x \in X} u^T x,$$

respectively.

For a given function $f : \mathbb{R}^n \to \overline{\mathbb{R}}$, we denote by $\mathrm{dom}(f) = \big\{x \in \mathbb{R}^n : f(x) < +\infty\big\}$ its *effective domain*, by $\mathrm{epi}(f) = \big\{(x, r) : x \in \mathbb{R}^n, r \in \mathbb{R}, f(x) \leq r\big\}$ its *epigraph* and by $\mathrm{cl}(f)$ its *closure*, i.e. the function whose epigraph is the closure of $\mathrm{epi}(f)$, respectively. The function f is called *proper* if its effective domain is a nonempty set and $f(x) > -\infty$ for all $x \in \mathbb{R}^n$.

We consider also the linear operator

$$\mathcal{T} : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}^n, \quad \mathcal{T}\big(x, r\big) = (r, x).$$

When X is a nonempty subset of $\mathbb{R}^n$ we define for the function f the *conjugate relative to the set* X by

$$f_X^*: \mathbb{R}^n \to \overline{\mathbb{R}}, \quad f_X^*(p) = \sup_{x \in X} \left\{ p^T x - f(x) \right\}.$$

It is easy to remark that for $X = \mathbb{R}^n$ the conjugate relative to the set X is actually the *(Fenchel-Moreau) conjugate function* of f denoted by f^*. Even more, it can be easily proved that

$$f_X^* = (f + \delta_X)^* \text{ and } \delta_X^* = \sigma_X.$$

Definition 2.1 Let the function $f: \mathbb{R}^k \to \overline{\mathbb{R}}$ be given. The function is called $\mathbb{R}_+^k$*-increasing* if for all $x = (x_1, ..., x_k)^T$ and $y = (y_1, ..., y_k)^T$ in $\mathbb{R}^k$ such that $x_i \leq y_i$, $i = 1, ..., k$, it holds $f(x) \leq f(y)$.

Definition 2.2 Given the proper functions $f_1, ..., f_m : \mathbb{R}^n \to \overline{\mathbb{R}}$, we call their *infimal convolution* the function

$$f_1 \Box ... \Box f_m : \mathbb{R}^n \to \overline{\mathbb{R}}, \quad (f_1 \Box ... \Box f_m)(x) = \inf \left\{ \sum_{i=1}^m f_i(x_i) : x = \sum_{i=1}^m x_i \right\}.$$

The following statements close this preliminary section.

Theorem 2.1 (cf. [16]) Let $f_1, ..., f_m : \mathbb{R}^n \to \overline{\mathbb{R}}$ be proper convex functions. If the set $\bigcap_{i=1}^m \operatorname{ri}(\operatorname{dom}(f_i))$ is nonempty, then

$$\left(\sum_{i=1}^m f_i \right)^* (p) = (f_1^* \Box ... \Box f_m^*)(p) = \inf \left\{ \sum_{i=1}^m f_i^*(p_i) : p = \sum_{i=1}^m p_i \right\},$$

and for each $p \in \mathbb{R}^n$ the infimum is attained.

Corollary 2.2 Let $f_1, ..., f_m : \mathbb{R}^n \to \overline{\mathbb{R}}$ be proper convex functions. If the set $\bigcap_{i=1}^m \operatorname{ri}(\operatorname{dom}(f_i))$ is nonempty, then

$$\operatorname{epi} \left(\left(\sum_{i=1}^m f_i \right)^* \right) = \sum_{i=1}^m \operatorname{epi}(f_i^*).$$

Proof. "$\subseteq$" Let us consider an arbitrary $(p, r) \in \operatorname{epi} \left(\left(\sum\limits_{i=1}^m f_i \right)^* \right)$. Since the hypotheses of Theorem 2.1 are fulfilled, there exist $p_1, ..., p_m \in \mathbb{R}^n$ such that

$$p = \sum_{i=1}^{m} p_i$$

and

$$r \geq \left(\sum_{i=1}^{m} f_i \right)^* (p) = \sum_{i=1}^{m} f_i^*(p_i).$$

The last relation implies $f_1^*(p_1) \leq r - \sum\limits_{i=2}^{m} f_i^*(p_i)$ and it occurs

$$\left(p_1, r - \sum_{i=2}^{m} f_i^*(p_i) \right) \in \text{epi}\left(f_1^*\right).$$

Thus

$$(p, r) = \left(p_1, r - \sum_{i=2}^{m} f_i^*(p_i) \right) + \sum_{i=2}^{m} \left(p_i, f_i^*(p_i)\right) \in \sum_{i=1}^{m} \text{epi}(f_i^*).$$

"$\supseteq$" For the reverse inclusion, consider $(p_i, r_i) \in \text{epi}\left(f_i^*\right)$, $i = 1, ..., m$. Since

$$\sum_{i=1}^{m} r_i \geq \sum_{i=1}^{m} f_i^*(p_i) \geq \left(\sum_{i=1}^{m} f_i \right)^* \left(\sum_{i=1}^{m} p_i \right),$$

we can state that

$$\sum_{i=1}^{m} (p_i, r_i) = \left(\sum_{i=1}^{m} p_i, \sum_{i=1}^{m} r_i \right) \in \text{epi}\left(\left(\sum_{i=1}^{m} f_i \right)^* \right).$$

□

Proposition 2.3 Let $f : \mathbb{R}^k \to \overline{\mathbb{R}}$ be a proper function and $\alpha > 0$ a real number. One has

$$\text{epi}\left((\alpha f)^*\right) = \alpha \, \text{epi}\left(f^*\right).$$

Proof. The following equivalences are fulfilled:

$$\begin{aligned}(p, r) \in \text{epi}\left((\alpha f)^*\right) \quad &\Leftrightarrow \quad (\alpha f)^*(p) \leq r \Leftrightarrow \alpha f^* \left(\frac{1}{\alpha} p \right) \leq r \\ &\Leftrightarrow \quad \left(\frac{1}{\alpha} p, \frac{1}{\alpha} r \right) \in \text{epi}\left(f^*\right) \Leftrightarrow (p, r) \in \alpha \, \text{epi}\left(f^*\right).\end{aligned}$$

□

3 Duality for the composed programming problem

Let X be a nonempty convex set in $\mathbb{R}^n$. Consider the functions $f : \mathbb{R}^k \to \overline{\mathbb{R}}$, $F : \mathbb{R}^n \to \mathbb{R}^k$, $F = \big(F_1, ..., F_k\big)^T$ and $G : \mathbb{R}^n \to \mathbb{R}^m$, $G = \big(G_1, ..., G_m\big)^T$ such that f is proper, $\mathbb{R}^k_+$-increasing and convex, while $F_1, ..., F_k$ and $G_1, ..., G_m$ are convex. Moreover assume that

$$F^{-1}(\mathrm{dom}(f)) \cap X \neq \emptyset,$$

where $F^{-1}\big(\mathrm{dom}(f)\big) = \{x \in \mathbb{R}^n : F(x) \in \mathrm{dom}(f)\}$. The optimization problem we treat within this paper is

$$(P) \qquad \inf_{\substack{x \in X,\\ G(x) \leqq 0}} f\big(F(x)\big).$$

It is not hard to prove that the function $f \circ F$ is actually a convex function (see [18]) and thus the problem (P) is nothing but a convex optimization problem, with convex objective function and finitely many convex constraints.

We associate to the problem (P) the following convex optimization problem

$$(P') \qquad \inf_{\substack{x \in X,\, y \in \mathbb{R}^k,\\ G_i(x) \leq 0,\, i=1,...,m\\ F_j(x) - y_j \leq 0,\, j=1,...,k}} f(y).$$

Regarding the optimal values of the problems (P) and (P'), the following result can be established.

Theorem 3.1 $v(P) = v(P')$.

Proof. For an arbitrary x feasible to (P) take $y = F(x)$, i.e. $F_j(x) - y_j = 0$, $j = 1, ..., k$, and since the feasibility of x involves $G_i(x) \leq 0$, $i = 1, ..., m$, we can conclude that (x, y) is feasible to (P'). As x is arbitrarily chosen, it follows that for all $x \in X$ such that $G_i(x) \leq 0$, $i = 1, ..., m$, we can find an element (x, y) feasible to (P') such that $f(y) = f\big(F(x)\big)$. Thus $f\big(F(x)\big) \geq v(P')$ for all x feasible to (P). That implies $v(P) \geq v(P')$.

In order to prove the opposite inequality, let us consider (x, y) feasible to (P'). Since $G_i(x) \leq 0$, $i = 1, ..., m$, it follows immediately that x is feasible to (P). But $F_j(x) - y_j \leq 0$, $j = 1, ..., k$, implies $F(x) \leqq y$ and, since f is an $\mathbb{R}^k_+$-increasing function, we have also $v(P) \leq f\big(F(x)\big) \leq f(y)$. Taking the infimum on the right-side regarding (x, y) feasible to (P') we obtain $v(P) \leq v(P')$. □

Our next step is to construct a dual problem to (P') and to give sufficient conditions in order to achieve strong duality, i.e. the situation when the optimal objective value of the primal coincides with the optimal objective value of the dual and the dual has an optimal solution.

We consider first the Lagrange dual problem to (P') with $\alpha = (\alpha_1, ..., \alpha_m)^T \in \mathbb{R}^m_+$ and $\beta = (\beta_1, ..., \beta_k) \in \mathbb{R}^k_+$ as dual variables

$$(D) \qquad \sup_{\substack{\alpha \geqq 0, \\ \beta \geqq 0}} \inf_{\substack{x \in X, \\ y \in \mathbb{R}^k}} \left\{ f(y) + \alpha^T G(x) + \beta^T \big(F(x) - y\big) \right\}.$$

Regarding the inner infimum concerning $(x, y) \in X \times \mathbb{R}^k$, by using the definition of the conjugate relative to a set, we have

$$\begin{aligned}
& \inf_{\substack{x \in X, \\ y \in \mathbb{R}^k}} \left\{ f(y) + \alpha^T G(x) + \beta^T \big(F(x) - y)\big) \right\} \\
= \ & \inf_{x \in X} \left\{ \alpha^T G(x) + \beta^T F(x) \right\} + \inf_{y \in \mathbb{R}^k} \left\{ f(y) - \beta^T y \right\} \\
= \ & - \sup_{x \in X} \left\{ -\alpha^T G(x) - \beta^T F(x) \right\} - \sup_{y \in \mathbb{R}^k} \left\{ \beta^T y - f(y) \right\} \\
= \ & - \left(\alpha^T G + \beta^T F \right)^*_X (0) - f^*(\beta).
\end{aligned}$$

Since X is a nonempty convex set we have $\operatorname{ri}(X) \neq \emptyset$ and thus, by Theorem 2.1,

$$\begin{aligned}
\left(\alpha^T G + \beta^T F \right)^*_X (0) &= \left(\alpha^T G + \beta^T F + \delta_X \right)^* (0) \\
&= \inf_{p \in \mathbb{R}^n} \left\{ \big(\beta^T F\big)^*(p) + \big(\alpha^T G + \delta_X(x)\big)^*(-p) \right\} \\
&= \inf_{p \in \mathbb{R}^n} \left\{ \big(\beta^T F\big)^*(p) + \big(\alpha^T G\big)^*_X(-p) \right\},
\end{aligned}$$

and the last infimum is attained for some $p \in \mathbb{R}^n$.

Considering the latter relations we obtain the following formulation for the dual problem

$$(D) \qquad \sup_{\substack{p \in \mathbb{R}^n, \\ \alpha \geqq 0, \beta \geqq 0}} \Big\{ - f^*(\beta) - \big(\beta^T F\big)^*(p) - \big(\alpha^T G\big)^*_X(-p) \Big\}.$$

The optimal objective value of the problem (P') is always greater than or equal to the optimal objective value of its Lagrange dual, i.e. $v(P') \geq v(D)$. Because of Theorem 3.1 (D) is also a dual problem to (P) and thus the following assertion arises easily.

Theorem 3.2 Between the primal problem (P) and the dual (D) weak duality is always satisfied, i.e. $v(P) \geq v(D)$.

It can be easily shown that in the general case strong duality between the primal problem and its dual can fail (see [17]). In order to avoid this situation the following constraint qualification is considered (see [1])

$$(CQ) \qquad \exists x' \in \mathrm{ri}(X) \text{ such that } \begin{cases} F(x') \in \mathrm{ri}\big(\mathrm{dom}(f)\big) - \mathrm{int}\big(\mathbb{R}^k_+\big), \\ G_i(x') \leq 0, i \in L, \\ G_i(x') < 0, i \in N, \end{cases}$$

where

$$L := \big\{ i \in \{1, ..., m\} : G_i \text{ is an affine function} \big\}$$

and $N := \{1, ..., m\} \setminus L$.

As the authors proved in [1], (CQ) is weaker than the constraint qualifications concerning composed convex optimization problems given until now in the literature (see [9], [13]).

The following assertion states strong duality between the problems (P) and (D).

Theorem 3.3 Assume that $v(P)$ is finite. If (CQ) is fulfilled, then between (P) and (D) strong duality holds, i.e. $v(P) = v(D)$ and the dual problem has an optimal solution.

Proof. In order to prove that the problems (P) and (D) have the same optimal objective value, we will actually prove that this happens between the problems (P') and (D) and, using again Theorem 3.1, the desired result will arise as a direct consequence. Therefore to the problem

$$(P') \qquad \inf_{\substack{x\in X,\, y\in\mathbb{R}^k,\\ G_i(x)\leq 0,\, i=1,\ldots,m,\\ F_j(x)-y_j\leq 0,\, j=1,\ldots,k}} f(y)$$

we associate first its Lagrange dual

$$(D') \qquad \sup_{\substack{\alpha\geqq 0,\\ \beta\geqq 0}} \inf_{\substack{x\in X,\\ y\in\mathbb{R}^k}} \left\{ f(y) + \alpha^T G(x) + \beta^T \big(F(x) - y\big) \right\}.$$

It is not hard to see that we can always find a $y \in \operatorname{ri}\big(\operatorname{dom}(f)\big)$ such that $F_j(x') < y_j$, $j = 1, \ldots, k$, and, since the condition (CQ) is fulfilled and the functions involved are convex, it is well-known from the literature (see Theorem 28.2 in [16]) that between between (P') and its Lagrange dual strong duality holds. This means that the optimal objective values of (P') and (D') are equal and, moreover, there exist some $\overline{\alpha} \geqq 0$ and $\overline{\beta} \geqq 0$ such that

$$\begin{aligned}
v(P') &= \sup_{\substack{\alpha\geqq 0,\\ \beta\geqq 0}} \inf_{x\in X, y\in\mathbb{R}^k} \left\{ f(y) + \alpha^T G(x) + \beta^T \big(F(x) - y\big) \right\} \\
&= \inf_{x\in X, y\in\mathbb{R}^k} \left\{ f(y) + \overline{\alpha}^T G(x) + \overline{\beta}^T \big(F(x) - y\big) \right\} \\
&= -\sup_{x\in X, y\in\mathbb{R}^k} \left\{ \big(\overline{\beta}^T y - f(y)\big) - \big(\overline{\alpha}^T G(x) + \overline{\beta}^T F(x)\big) \right\} \\
&= -\sup_{y\in\mathbb{R}^k} \left\{ \overline{\beta}^T y - f(y) \right\} - \sup_{x\in\mathbb{R}^n} \left\{ -\overline{\alpha}^T G(x) - \overline{\beta}^T F(x) - \delta_X(x) \right\} \\
&= -f^*(\overline{\beta}) - \Big(\big(\overline{\beta}^T F\big) + \big(\overline{\alpha}^T G + \delta_X\big) \Big)^*(0).
\end{aligned}$$

Since

$$\operatorname{ri}\big(\operatorname{dom}(\beta^T F)\big) \cap \operatorname{ri}\big(\operatorname{dom}(\alpha^T G + \delta_X)\big) = \operatorname{ri}(X) \neq \emptyset,$$

by Theorem 2.1 we get further

$$v(P') = -f^*(\overline{\beta}) - \inf_{p \in \mathbb{R}^n} \left\{ \left(\overline{\beta}^T F\right)^*(p) + \left(\overline{\alpha}^T G\right)_X^*(-p) \right\},$$

completed with the existence of some $\overline{p} \in \mathbb{R}^n$ where the infimum in the equality above is attained. Therefore

$$v(P') = -f^*(\overline{\beta}) - \left(\overline{\beta}^T F\right)^*(\overline{p}) - \left(\overline{\alpha}^T G\right)_X^*(-\overline{p}),$$

and from here follows immediately that $v(P) = v(P') = v(D)$ and $(\overline{p}, \overline{\alpha}, \overline{\beta})$ is an optimal solution for (D). $\square$

4 Farkas-type results via conjugate duality

The results presented in the previous section are the backbone in the proof of the following Farkas-type result.

Theorem 4.1 Suppose that (CQ) holds. Then the following assertions are equivalent:

(i) $x \in X$, $G(x) \leqq 0 \Rightarrow f\big(F(x)\big) \geq 0$;

(ii) there exist $p \in \mathbb{R}^n$, $\alpha \geqq 0$ and $\beta \geqq 0$ such that

$$f^*(\beta) + \left(\beta^T F\right)^*(p) + \left(\alpha^T G\right)_X^*(-p) \leq 0. \tag{1}$$

Proof. "$(i) \Rightarrow (ii)$" The statement (i) implies $v(P) \geq 0$ and, since the assumptions of Theorem 3.3 are fulfilled, strong duality holds, i.e. $v(D) = v(P) \geq 0$ and the dual (D) has an optimal solution. Thus there exist $p \in \mathbb{R}^n$, $\alpha \geqq 0$ and $\beta \geqq 0$ fulfilling (1).

"$(ii) \Rightarrow (i)$" As we can find some $p \in \mathbb{R}^n$, $\alpha \geqq 0$ and $\beta \geqq 0$ fulfilling (1), it follows right away that

$$v(D) \geq -f^*(\beta) - \left(\beta^T F\right)^*(p) - \left(\alpha^T G\right)_X^*(-p) \geq 0.$$

Weak duality between (P) and (D) always holds and thus we obtain $v(P) \geq 0$, i.e. (i) is true. $\square$

The previous statement can be reformulated as a theorem of the alternative.

Corollary 4.2 Assume the hypothesis of Theorem 4.1 being fulfilled. Then either the inequality system

$$(I) \quad x \in X, G(x) \leqq 0, f\big(F(x)\big) < 0$$

has a solution or the system

$$\begin{aligned}(II) \quad & f^*(\beta) + \big(\beta^T F\big)^*(p) + \big(\alpha^T G\big)_X^*(-p) \leq 0, \\ & p \in \mathbb{R}^n, \alpha \geqq 0, \beta \geqq 0\end{aligned}$$

has a solution, but never both.

As in [4] and [10], we give an equivalent formulation for the statement (ii) in Theorem 4.1 using the epigraphs of the involved functions.

Theorem 4.3 The statement (ii) in Theorem 4.1 is equivalent to

$$\begin{aligned}(0,0,0) \in \quad & \{0\} \times \mathcal{T}\big(\operatorname{epi}(f^*)\big) + \bigcup_{\beta \geqq 0} \left(\operatorname{epi}\big((\beta^T F)^*\big) \times \{-\beta\} \right) \\ & + \operatorname{coneco}\left(\bigcup_{i=1}^{m} \operatorname{epi}(G_i^*) \right) \times \{0\} + \operatorname{epi}(\sigma_X) \times \{0\}.\end{aligned}$$

Proof. "$\Rightarrow$" Since the statement (ii) holds, there exist $p \in \mathbb{R}^n$, $\alpha \geqq 0$ and $\beta \geqq 0$ such that

$$f^*(\beta) + \big(\beta^T F\big)^*(p) + \big(\alpha^T G\big)_X^*(-p) \leq 0.$$

As $f^*(\beta)$ and $\big(\beta^T F\big)^*(p)$ have both finite real values, it is clear that

$$\big(\beta, f^*(\beta)\big) \in \operatorname{epi}\big(f^*\big)$$

and

$$\big(p, \big(\beta^T F\big)^*(p)\big) \in \operatorname{epi}\big((\beta^T F)^*\big).$$

Thus

$$\big(p, \big(\beta^T F\big)^*(p), -\beta\big) \in \operatorname{epi}\big((\beta^T F)^*\big) \times \{-\beta\}$$

and it follows

$$\big(p, \big(\beta^T F\big)^*(p), -\beta\big) \in \bigcup_{\beta \geqq 0} \Bigg(\operatorname{epi}\big((\beta^T F)^*\big) \times \{-\beta\} \Bigg). \tag{2}$$

Taking into consideration the definition of the operator $\mathcal{T}$ introduced in the first section of the paper, the relation

$$\big(0, f^*(\beta), \beta\big) \in \{0\} \times \mathcal{T}(\operatorname{epi}(f^*)) \tag{3}$$

follows at once.

On the other hand the inequality

$$\big(\alpha^T G\big)^*_X(-p) \leq -f^*(\beta) - \big(\beta^T F\big)^*(p)$$

is also fulfilled, and, as the value in the right-hand side is finite, it holds

$$\big(-p, -f^*(\beta) - \big(\beta^T F\big)^*(p)\big) \in \operatorname{epi}\big((\alpha^T G)^*_X\big). \tag{4}$$

Further we deal with two cases. First suppose that $\alpha = 0$. It is trivial to verify that in this case we have

$$\operatorname{epi}\big((\alpha^T G)^*_X\big) = \operatorname{epi}\big(\delta^*_X\big) = \operatorname{epi}(\sigma_X)$$

and therefore

$$\big(-p, -f^*(\beta) - \big(\beta^T F\big)^*(p)\big) \in \operatorname{epi}\big(\sigma_X\big).$$

Moreover, as

$$(0,0,0) \in \operatorname{coneco}\left(\bigcup_{i=1}^m \operatorname{epi}(G_i^*)\right) \times \{0\},$$

we have

$$\begin{aligned}
(0,0,0) &= \big(0,f^*(\beta),\beta\big) + \big(p,\big(\beta^T F\big)^*(p),-\beta\big) + \big(-p,-f^*(\beta) - \big(\beta^T F\big)^*(p),0\big) \\
&\in \{0\} \times \mathcal{T}\big(\operatorname{epi}(f^*)\big) + \bigcup_{\beta \geqq 0} \Bigg(\operatorname{epi}\big((\beta^T F)^*\big) \times \{-\beta\} \Bigg) + \operatorname{epi}(\sigma_X) \times \{0\} \\
&\subseteq \{0\} \times \mathcal{T}\big(\operatorname{epi}(f^*)\big) + \bigcup_{\beta \geqq 0} \Bigg(\operatorname{epi}\big((\beta^T F)^*\big) \times \{-\beta\} \Bigg) \\
&+ \operatorname{coneco}\left(\bigcup_{i=1}^m \operatorname{epi}(G_i^*)\right) \times \{0\} + \operatorname{epi}(\sigma_X) \times \{0\},
\end{aligned}$$

and the desired result is secured.

The second case occurs for $\alpha \neq 0$. The set $I_\alpha = \{i : \alpha_i \neq 0\}$ is obviously nonempty. Since the hypothesis of Corollary 2.2 are fulfilled, relation (4) becomes

$$\big(-p, -f^*(\beta) - \big(\beta^T F\big)^*(p)\big) \in \operatorname{epi}\left(\left(\sum_{i \in I_\alpha} \alpha_i G_i\right)^*\right) + \operatorname{epi}(\sigma_X).$$

But

$$\begin{aligned}
\operatorname{epi}\left(\left(\sum_{i \in I_\alpha} \alpha_i G_i\right)^*\right) &= \sum_{i \in I_\alpha} \operatorname{epi}\big((\alpha_i G_i)^*\big) = \sum_{i \in I_\alpha} \alpha_i \operatorname{epi}\big(G_i^*\big) \\
&= \left(\sum_{i \in I_\alpha} \alpha_i\right) \sum_{i \in I_\alpha} \frac{\alpha_i}{\sum\limits_{i \in I_\alpha} \alpha_i} \operatorname{epi}\big(G_i^*\big) \\
&\subseteq \operatorname{coneco}\left(\bigcup_{i=1}^m \operatorname{epi}(G_i^*)\right).
\end{aligned}$$

Combining relations (2) and (3) with the last two we get

$$\begin{aligned}
(0,0,0) &= \big(0, f^*(\beta), \beta\big) + \big(p, \big(\beta^T F\big)^*(p), -\beta\big) + \big(-p, -f^*(\beta) - \big(\beta^T F\big)^*(p), 0\big) \\
&\in \{0\} \times \mathcal{T}\big(\operatorname{epi}(f^*)\big) + \bigcup_{\beta \geqq 0} \Big(\operatorname{epi}\big((\beta^T F)^*\big) \times \{-\beta\}\Big) \\
&+ \operatorname{coneco}\left(\bigcup_{i=1}^m \operatorname{epi}(G_i^*)\right) \times \{0\} + \operatorname{epi}(\sigma_X) \times \{0\}.
\end{aligned}$$

"$\Leftarrow$" Since

$$\begin{aligned}
(0,0,0) \in \quad & \{0\} \times \mathcal{T}\big(\operatorname{epi}(f^*)\big) + \bigcup_{\beta \geqq 0} \operatorname{epi}\big((\beta^T F)^*\big) \times \{-\beta\} \\
& + \operatorname{coneco}\left(\bigcup_{i=1}^m \operatorname{epi}(G_i^*)\right) \times \{0\} + \operatorname{epi}(\sigma_X) \times \{0\},
\end{aligned}$$

we can find some $p \in \mathbb{R}^n$ and $r \in \mathbb{R}$ such that

$$(p, r, 0) \in \{0\} \times \mathcal{T}(\operatorname{epi}(f^*)) + \bigcup_{\beta \geqq 0} \operatorname{epi}\big((\beta^T F)^*\big) \times \{-\beta\} \tag{5}$$

and

$$(-p,-r,0)\in \operatorname{coneco}\left(\bigcup_{i=1}^{m}\operatorname{epi}(G_i^*)\right)\times\{0\}+\operatorname{epi}(\sigma_X)\times\{0\}. \tag{6}$$

The definition of the operator $\mathcal{T}$ and relation (5) imply that there exist $\beta \geqq 0$ and two real numbers r_1 and r_2 such that $r = r_1 + r_2$, while the pairs (β, r_1) and (p, r_2) are in $\operatorname{epi}(f^*)$ and $\operatorname{epi}\big((\beta^T F)^*\big)$, respectively. Thus

$$f^*(\beta)+(\beta^T F)^*(p)\le r_1+r_2=r. \tag{7}$$

Since relation (6) is also fulfilled, there exist $\lambda \ge 0$, and $\mu_i \ge 0$, $i = 1,...,m$, $\sum_{i=1}^m \mu_i = 1$, such that

$$(-p,-r)\in\lambda\sum_{i=1}^{m}\mu_i\operatorname{epi}\left(G_i^*\right)+\operatorname{epi}(\sigma_X). \tag{8}$$

If $\lambda = 0$ we have $(-p,-r)\in\operatorname{epi}(\sigma_X)$. Considering $\alpha = (0,...,0)\in\mathbb{R}^m$ it follows

$$(-p,-r)\in\operatorname{epi}\left((\alpha^T G)_X^*\right)$$

and, taking into consideration also relation (7), we obtain

$$f^*(\beta)+\left(\beta^T F\right)^*(p)+\left(\alpha^T G\right)_X^*(-p)\le 0,$$

so the conclusion is straightforward in this case.

If $\lambda > 0$ let us consider the vector $\alpha = (\lambda\mu_1, ..., \lambda\mu_m)\in\mathbb{R}_+^m$. Since $\sum_{i=1}^m \mu_i = 1$, the set I_α is obviously nonempty and relation (8) becomes

$$(-p,-r)\in\sum_{i\in I_\alpha}\alpha_i\operatorname{epi}\left(G_i^*\right)+\operatorname{epi}(\sigma_X).$$

But $\alpha_i > 0$ for all $i\in I_\alpha$ and using Corollary 2.2 and Proposition 2.3 we get

$$\sum_{i\in I_\alpha}\alpha_i\operatorname{epi}\left(G_i^*\right)=\sum_{i\in I_\alpha}\operatorname{epi}\left((\alpha_i G_i)^*\right)=\operatorname{epi}\left(\left(\sum_{i\in I_\alpha}\alpha_i G_i\right)^*\right)=\operatorname{epi}\left((\alpha^T G)^*\right).$$

From the last two relations we have

$$(-p,-r)\in\operatorname{epi}\left((\alpha^T G)^*\right)+\operatorname{epi}(\sigma_X)=\operatorname{epi}\left((\alpha^T G)_X^*\right)$$

and, combining this last result with relation (7), the desired conclusion follows easily. $\square$

5 Special cases

In this section we give Farkas-type results for some problems which turn out to be special cases of the problem (P).

5.1 The max-function

This special case of our initial problem is similar to the one studied in [5]. It is well known (see [8]) that the conjugate of the function

$$f : \mathbb{R}^k \to \mathbb{R}, \quad f(x) = \max\{x_1, ..., x_k\}, \quad x = (x_1, ..., x_k)^T \in \mathbb{R}^k$$

is

$$f^* : \mathbb{R}^k \to \overline{\mathbb{R}}, \quad f^*(\beta) = \begin{cases} 0, & \beta \geqq 0, \sum\limits_{j=1}^{k} \beta_j = 1, \\ +\infty, & \text{otherwise.} \end{cases}$$

As $\mathrm{dom}(f) = \mathbb{R}^k$, one may notice that the constraint qualification becomes

$$(\widetilde{CQ}) \qquad \exists x' \in \mathrm{ri}(X) \text{ such that } \begin{cases} G_i(x') \leq 0, & i \in L, \\ G_i(x') < 0, & i \in N. \end{cases}$$

Using these remarks, the following result can be formulated as a special case of Theorem 4.1.

Theorem 5.1 Suppose that $(\widetilde{CQ})$ holds. Then the following assertions are equivalent:

(i) $x \in X$, $G(x) \leqq 0 \Rightarrow \max\left\{F_1(x), ..., F_k(x)\right\} \geq 0$;

(ii) there exist $p \in \mathbb{R}^n$, $\alpha \geqq 0$ and $\beta \geqq 0$, $\sum\limits_{j=1}^{k} \beta_j = 1$ such that

$$\left(\beta^T F\right)^*(p) + \left(\alpha^T G\right)_X^*(-p) \leq 0.$$

Theorem 5.2 The statement (ii) in Theorem 5.1 is equivalent to

$$(0,0) \in \mathrm{co}\left(\bigcup_{j=1}^{k} \mathrm{epi}\left(F_j^*\right)\right) + \mathrm{coneco}\left(\bigcup_{i=1}^{m} \mathrm{epi}(G_i^*)\right) + \mathrm{epi}(\sigma_X). \qquad (9)$$

Proof. Theorem 4.3 ensures that the statement (ii) in Theorem 5.1 is actually equivalent to

$$(0,0,0) \in \{0\} \times \mathcal{T}\big(\operatorname{epi}(f^*)\big) + \bigcup_{\beta \geqq 0} \Big(\operatorname{epi}\big((\beta^T F)^*\big) \times \{-\beta\} \Big)$$
$$+ \operatorname{coneco}\left(\bigcup_{i=1}^{m} \operatorname{epi}(G_i^*) \right) \times \{0\} + \operatorname{epi}(\sigma_X) \times \{0\}.$$

This holds if and only if there exists $\beta \geqq 0$ such that $\sum_{j=1}^{k} \beta_j = 1$ and

$$(0,0) \in \{0\} \times [0,+\infty) + \operatorname{epi}\big((\beta^T F)^*\big) + \operatorname{coneco}\left(\bigcup_{i=1}^{m} \operatorname{epi}(G_i^*) \right) + \operatorname{epi}(\sigma_X).$$

Using the definition of the epigraph of a function it can be easily shown that $\operatorname{epi}(\sigma_X) = \operatorname{epi}(\sigma_X) + \{0\} \times [0,+\infty)$. Moreover, since $\beta \neq 0$, we have

$$\operatorname{epi}\big((\beta^T F)^*\big) = \operatorname{epi}\left(\left(\sum_{\beta_j \neq 0} \beta_j F_j \right)^* \right) = \sum_{\beta_j \neq 0} \beta_j \operatorname{epi}(F_j^*) = \sum_{j=1}^{k} \beta_j \operatorname{epi}(F_j^*).$$

Thus the initial relation is equivalent to the existence of a vector $\beta \geqq 0$, $\sum_{j=1}^{k} \beta_j = 1$, such that

$$(0,0) \in \sum_{j=1}^{k} \beta_j \operatorname{epi}(F_j^*) + \operatorname{coneco}\left(\bigcup_{i=1}^{m} \operatorname{epi}(G_i^*) \right) + \operatorname{epi}(\sigma_X)$$

and it is not hard to observe that this condition is fulfilled if and only if the desired conclusion holds. $\square$

5.2 The ordinary convex optimization problem

The next two theorems are direct consequences of Theorem 5.1 and 5.2, respectively. If we take $k = 1$, as the constraint qualification $(\widetilde{CQ})$ remains unchanged and the set $\operatorname{epi}(F^*)$ is convex, the proofs are obvious.

Theorem 5.3 Assume $(\widetilde{CQ})$ fulfilled. Then the following assertions are equivalent:

(i) $x \in X$, $G(x) \leqq 0 \Rightarrow F(x) \geq 0$;

(ii) there exist $p \in \mathbb{R}^n$ and $\alpha \geqq 0$ such that

$$F^*(p) + \left(\alpha^T G\right)^*_X(-p) \leq 0.$$

Theorem 5.4 The statement (ii) in Theorem 5.3 is equivalent to

$$(0,0) \in \operatorname{epi}(F^*) + \operatorname{coneco}\left(\bigcup_{i=1}^{m} \operatorname{epi}(G_i^*)\right) + \operatorname{epi}(\sigma_X).$$

This result had been also obtained by Boţ and Wanka in [4] and by Jeyakumar in [10].

6 Conclusions

In this paper we present a Farkas-type result for inequality systems with finitely many convex functions. The approach we use is based on conjugate duality for an optimization problem consisting in minimizing the composition of an $\mathbb{R}^k_+$-increasing and convex function with a convex vector function, subject to finitely many convex inequality constraints. The result we present generalizes some Farkas-type results presented by Boţ and Wanka in [4] and [5]. The connections between the Farkas-type results and the theory of the alternative and, respectively, the theory of duality are exposed once more.

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