On Gap Functions for Equilibrium Problems via Fenchel Duality

Lkhamsuren Altangerel[1]   Radu Ioan Boţ[2]   Gert Wanka[3]

Abstract: In this paper we deal with the construction of gap functions for equilibrium problems by using the Fenchel duality theory for convex optimization problems. For proving the properties which characterize a gap function weak and strong duality are used. Moreover, the proposed approach is applied to variational inequalities in a real Banach space.

Key words: equilibrium problem, Fenchel duality, weak regularity condition, gap functions, dual equilibrium problem, variational inequalities

AMS subject classification: 49N15, 58E35, 90C25

1 Introduction

Let X be a real topological vector space, $K \subseteq X$ be a nonempty closed and convex set. Assume that $f : X \times X \to \mathbb{R} \cup \{+\infty\}$ is a bifunction satisfying $f(x, x) = 0,\ \forall x \in K$. The equilibrium problem consists in finding $x \in K$ such that

$$(EP) \qquad f(x, y) \geq 0,\ \forall y \in K.$$

Since (EP) includes as special cases optimization problems, complementarity problems and variational inequalities (see [5]), some results for these problems have been extended to (EP) by several authors. In particular, the gap function approaches for solving variational inequalities (see for instance [2] and [13]) have been investigated for equilibrium problems in [4] and [10].
A function $\gamma : X \to \overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$ is said to be a gap function for (EP) [10, Definition 2.1] if it satisfies the properties

[1]Faculty of Mathematics, Chemnitz University of Technology, D-09107 Chemnitz, Germany, e-mail: lkal@mathematik.tu-chemnitz.de.

[2]Faculty of Mathematics, Chemnitz University of Technology, D-09107 Chemnitz, Germany, e-mail: radu.bot@mathematik.tu-chemnitz.de.

[3]Faculty of Mathematics, Chemnitz University of Technology, D-09107 Chemnitz, Germany, e-mail: gert.wanka@mathematik.tu-chemnitz.de.

(i) $\gamma(y) \geq 0, \ \forall y \in K$;

(ii) $\gamma(x) = 0$ and $x \in K$ if and only if x is a solution for (EP).

Recently, in [1] the construction of gap functions for finite-dimensional variational inequalities has been related to the conjugate duality of an optimization problem. On the other hand, in [6] very weak sufficient conditions for Fenchel duality regarding convex optimization problems have been established in infinite dimensional spaces. The combination of both results allows us to propose new gap functions for (EP) based on Fenchel duality.
This paper is organized as follows. In Section 2 we give some definitions and introduce the weak sufficient condition for the strong duality related to Fenchel duality. In the next section we propose some new functions by using Fenchel duality and we show that under certain assumptions they are gap functions for (EP). Finally, the proposed approach is applied to variational inequalities in a real Banach space.

2 Mathematical preliminaries

Let X be a real locally convex space and X^* be its topological dual, the set of all continuous linear functionals over X endowed with the weak* topology $w(X^*, X)$. By $\langle x^*, x\rangle$ we denote the value of $x^* \in X^*$ at $x \in X$. For the nonempty set $C \subseteq X$, the indicator function $\delta_C : X \to \mathbb{R} \cup \{+\infty\}$ is defined by

$$\delta_C(x) = \begin{cases} 0, & \text{if } x \in C, \\ +\infty, & \text{otherwise,} \end{cases}$$

while the support function is $\sigma_C(x^*) = \sup\limits_{x \in C} \langle x^*, x\rangle$. Considering now a function $f : X \to \mathbb{R} \cup \{+\infty\}$, we denote by $\text{dom } f = \Big\{x \in X | \ f(x) < +\infty\Big\}$ its effective domain and by

$$\text{epi } f = \Big\{(x, r) \in X \times \mathbb{R} | \ f(x) \leq r\Big\}$$

its epigraph. A function $f : X \to \mathbb{R} \cup \{+\infty\}$ is called proper if $\text{dom } f \neq \varnothing$. The (Fenchel-Moreau) conjugate function of f is $f^* : X^* \to \mathbb{R} \cup \{+\infty\}$ defined by

$$f^*(p) = \sup_{x \in X} [\langle p, x\rangle - f(x)].$$

Definition 2.1 Let the functions $f_i : X \to \mathbb{R} \cup \{+\infty\}, \ i = 1, ..., m$, be given.

The function $f_1\Box\cdots\Box f_m : X \to \mathbb{R}\cup\{\pm\infty\}$ defined by

$$f_1\Box\cdots\Box f_m(x) = \inf\Big\{\sum_{i=1}^m f_i(x_i) \Big| \sum_{i=1}^m x_i = x\Big\}$$

is called the infimal convolution function of $f_1, ..., f_m$. The infimal convolution $f_1\Box\cdots\Box f_m$ is called to be exact at $x \in X$ if there exist some $x_i \in X,\ i = 1, ..., m$, such that $\sum\limits_{i=1}^m x_i = x$ and

$$f_1\Box\cdots\Box f_m(x) = f_1(x_1) + ... + f_m(x_m).$$

Furthermore, we say that $f_1\Box\cdots\Box f_m$ is exact if it is exact at every $x \in X$.

Let $f : X \to \mathbb{R}\cup\{+\infty\}$ and $g : X \to \mathbb{R}\cup\{+\infty\}$ be proper, convex and lower semicontinuous functions such that $\mathrm{dom}\ f \cap \mathrm{dom}\ g \neq \varnothing$. We consider the following optimization problem

$$(P) \qquad \inf_{x\in X}\Big\{f(x) + g(x)\Big\}.$$

The Fenchel dual problem to (P) is

$$(D) \qquad \sup_{p\in X^*}\Big\{-f^*(-p) - g^*(p)\Big\}.$$

In [6] a new weaker regularity condition has been introduced in a more general case in order to guarantee the existence of strong duality between a convex optimization problem and its Fenchel dual, namely that the optimal objective values of the primal and the dual are equal and the dual has an optimal solution. This regularity condition for (P) can be written as

(FRC) $\quad f^*\Box g^*$ is lower semicontinuous and

$$\text{epi }(f^*\Box g^*)\cap\Big(\{0\}\times\mathbb{R}\Big) = \Big(\text{epi}(f^*) + \text{epi}(g^*)\Big)\cap\Big(\{0\}\times\mathbb{R}\Big),$$

or, equivalently,

(FRC) $\quad f^*\Box g^*$ is a lower semicontinuous function and exact at 0.

Let us denote by $v(P)$ the optimal objective value of the optimization problem (P). The following theorem states (cf. [6]) the existence of strong duality between (P) and (D).

Proposition 2.1 *Let (FRC) be fulfilled. Then $v(P) = v(D)$ and (D) has an optimal solution.*

Remark that considering the perturbation function $\Phi : X \times X \to \mathbb{R} \cup \{+\infty\}$ defined by $\Phi(x, z) = f(x) + g(x + z)$, one can obtain the Fenchel dual (D). Indeed, the function Φ fulfills $\Phi(x, 0) = f(x) + g(x),\ \forall x \in X$ and choosing (D) as being

$$(D) \qquad \sup_{p \in X^*} \Big\{ -\Phi^*(0, p) \Big\}$$

(cf. [7]), this problem becomes actually the well-known Fenchel dual problem.

3 Gap functions based on Fenchel duality

In this section we consider the construction of gap functions for (EP) by using a similar approach like the one considered for finite-dimensional variational inequalities in [1]. Here, the Fenchel duality will play an important role. We assume that X is a real locally convex space and $K \subseteq X$ is a nonempty closed and convex set. Further, let $f : X \times X \to \mathbb{R} \cup \{+\infty\}$ be a given function such that $K \times K \subseteq \operatorname{dom} f$ and $f(x, x) = 0,\ \forall x \in K$.

Let $x \in X$ be given. Then (EP) can be reduced to the optimization problem

$$(P^{EP}; x) \qquad \inf_{y \in K} f(x, y).$$

We mention that $x^* \in K$ is a solution of (EP) if and only if it is a solution of $(P^{EP}; x^*)$.

Now let us reformulate $(P^{EP}; x)$ using the indicator function $\delta_K(y)$ as

$$(P^{EP}; x) \qquad \inf_{y \in X} \Big\{ f(x, y) + \delta_K(y) \Big\}.$$

Then we can write the Fenchel dual to $(P^{EP}; x)$ as being

$$\begin{aligned}(D^{EP}; x) \qquad & \sup_{p \in X^*} \Big\{ -\sup_{y \in X}[\langle p, y\rangle - f(x, y)] - \delta_K^*(-p) \Big\} \\ = \ & \sup_{p \in X^*} \Big\{ -f_y^*(x, p) - \delta_K^*(-p) \Big\},\end{aligned}$$

where $f_y^*(x, p) := \sup\limits_{y \in X}[\langle p, y\rangle - f(x, y)]$ is the conjugate of $y \mapsto f(x, y)$ for a given $x \in X$. Let us introduce the following function for any $x \in X$

$$\begin{aligned}\gamma_F^{EP}(x) := -v(D^{EP}; x) &= -\sup_{p \in X^*} \Big\{ -f_y^*(x,p) - \delta_K^*(-p) \Big\} \\ &= \inf_{p \in X^*} \Big\{ f_y^*(x,p) + \sigma_K(-p) \Big\}.\end{aligned}$$

For $(P^{EP}; x)$, the regularity condition (FRC) can be written as follows

$(FRC^{EP}; x)$ $\quad f_y^*(x, \cdot)\Box\sigma_K$ is a lower semicontinuous function and exact at 0.

Theorem 3.1 *Assume that $\forall x \in K$ the regularity condition $(FRC^{EP}; x)$ is fulfilled. Let for each $x \in K$, $y \mapsto f(x,y)$ be convex and lower semicontinuous. Then γ_F^{EP} is a gap function for (EP).*

Proof:

(i) By weak duality it holds

$$v(D^{EP}; x) \leq v(P^{EP}; x) \leq 0, \ \forall x \in K.$$

Therefore one has $\gamma_F^{EP}(x) = -v(D^{EP}; x) \geq 0, \ \forall x \in K$.

(ii) If $\bar{x} \in K$ is a solution of (EP), then $v(P^{EP}; \bar{x}) = 0$. On the other hand, by Proposition 2.1 the strong duality between $(P^{EP}; \bar{x})$ and $(D^{EP}; \bar{x})$ holds. In other words

$$v(D^{EP}; \bar{x}) = v(P^{EP}; \bar{x}) = 0.$$

That means $\gamma_F^{EP}(\bar{x}) = 0$. Conversely, let $\gamma_F^{EP}(\bar{x}) = 0$ for $\bar{x} \in K$. Then

$$0 = v(D^{EP}; \bar{x}) \leq v(P^{EP}; \bar{x}) \leq 0.$$

Therefore $\bar{x}$ is a solution of (EP). $\square$

Remark 3.1 As it follows by Theorem 3.1, the gap function introduced above coincides under the assumption $(FRC^{EP}; x)$, $\forall x \in K$, with the well-known gap function

$$\sup_{y \in K}[-f(x,y)].$$

The advantage of considering γ_F^{EP} may come when computing it. In order to do this one has to minimize the sum of the conjugate of a given function, for whose calculation the well-developed apparatus existent in this field of convex analysis can be helpful, with the support function of a nonempty closed convex set. On the other hand, the formula above consists in solving a constraint maximization problem which can be a harder work. This aspect is underlined in Example 3.1.

Even if the assumption that $(FRC^{EP};x)$ must be fulfilled for all $x \in K$ seems complicate let us notice that it is valid under the natural assumption $\mathrm{int}(K) \neq \emptyset$. For a comprehensive study on regularity conditions for Fenchel duality we refer to [6].

Example 3.1 Take $X = \mathbb{R}^2$, $K = \{(x_1,x_2)^T \in \mathbb{R}^2 : x_1^2 + x_2^2 \leq 1\}$ and $f : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$, defined by $f(x_1,x_2,y_1,y_2) = y_1^2 - x_1^2 - y_2 + x_2$. Consider the equilibrium problem of finding $(x_1,x_2)^T \in K$ such that

$$y_1^2 - y_2 \geq x_1^2 - x_2, \forall (y_1,y_2)^T \in K.$$

Instead of using the formula given in Remark 3.1 we determine a gap function for this equilibrium problem by using our approach, as the calculations are easier.

By definition, for $(x_1,x_2)^T \in \mathbb{R}^2$, one has

$$\gamma_F^{EP}(x_1,x_2) =$$

$$\inf_{(p_1,p_2)^T \in \mathbb{R}^2} \left[\sup_{(y_1,y_2)^T \in \mathbb{R}^2} (p_1 y_1 + p_2 y_2 - y_1^2 + x_1^2 + y_2 - x_2) + \delta_K^*(-p_1,-p_2) \right].$$

As

$$\sup_{(y_1,y_2)^T \in \mathbb{R}^2} (p_1 y_1 + p_2 y_2 - y_1^2 + y_2) = \begin{cases} \frac{p_1^2}{4}, & \text{if } p_2 = -1, \\ +\infty, & \text{otherwise,} \end{cases}$$

and

$$\delta_K^*(-p_1,-p_2) = \sqrt{p_1^2 + p_2^2},$$

we have

$$\gamma_F^{EP}(x_1,x_2) = x_1^2 - x_2 + \inf_{p \in \mathbb{R}} \left\{ \frac{p^2}{4} + \sqrt{p^2+1} \right\} = x_1^2 - x_2 + 1.$$

Since for $(x_1,x_2)^T \in K$ one has $\gamma_F^{EP}(x_1,x_2) \geq 1 - x_2 \geq 0$, property (i) in the definition of the gap function is fulfilled. On the other hand, if for an $(x_1,x_2)^T \in K$, $\gamma_F^{EP}(x_1,x_2) = 0$, then x_2 must be equal to 1 and x_1 must be equal to 0. As $(0,1)^T$ is the only solution of the equilibrium problem considered within this example, γ_F^{EP} is a gap function.

An alternative proof of the fact that γ_F^{EP} is a gap function comes from verifying the fulfillment of the hypotheses of Theorem 3.1, which are surely fulfilled. As $\operatorname{int}(K)$ is nonempty, the regularity condition $(FRC^{EP};x)$ is obviously valid for all $x \in K$.

Example 3.2 Let $X = \mathbb{R}^2$, $K = \{0\}\times\mathbb{R}_+$ and $f : \mathbb{R}^2\times\mathbb{R}^2 \to \mathbb{R}\cup\{+\infty\}$, $f = \delta_{\mathbb{R}^2_+\times\mathbb{R}^2_+}$. One can see that $K \times K \subseteq \operatorname{dom} f$, $f(x,x) = 0, \forall x \in K$, and that for all $x \in K$ the mapping $y \mapsto f(x,y)$ is convex and lower semicontinuous. We show that although $\operatorname{int}(K) \neq \emptyset$ fails, the regularity condition $(FRC^{EP};x)$ is fulfilled for all $x \in K$.

Let $x \in K$ be fixed. For all $p \in \mathbb{R}^2$ we have

$$f^*_y(x,p) = \sup_{y\in\mathbb{R}^2_+}[p^T y] = \delta_{-\mathbb{R}^2_+}(p)$$

and

$$\sigma_K(p) = \sup_{y\in\{0\}\times\mathbb{R}_+}[p^T y] = \delta_{\mathbb{R}\times(-\mathbb{R}_+)}(p).$$

As $f^*_y(x,\cdot)\Box\sigma_K = \delta_{\mathbb{R}\times(-\mathbb{R}_+)}$, it is obvious that this function is lower semicontinuous and exact at 0. The regularity condition $(FRC^{EP};x)$ is fulfilled for all $x \in K$ and one can apply Theorem 3.1.

Remark 3.2 In the following we stress the connections between the gap function we have just introduced and convex optimization. Therefore let $K \subseteq X$ be a convex and closed set and $u : X \to \mathbb{R}\cup\{+\infty\}$ be a convex and lower semicontinuous function with $K \subseteq \operatorname{dom} u$. We consider the following optimization problem with geometrical constraints

$$(P_u) \qquad\qquad \inf_{x\in K} u(x).$$

Take $f : X \times X \to \mathbb{R}\cup\{+\infty\}$, $f(x,y) = u(y) - u(x)$ and assume, by convention, that $(+\infty) - (+\infty) = +\infty$. For all $x \in X$ the gap function γ_F^{EP} becomes $\gamma_F^{EP}(x) = \inf\limits_{p\in X^*}\left\{u^*(p) + \sigma_K(-p)\right\} + u(x)$. Assuming that $u^*\Box\sigma_K$ is lower semicontinuous and exact at 0, the hypotheses of Theorem 3.1 are fulfilled and, so, γ_F^{EP} turns out to be a gap function for the equilibrium problem which consists in finding $x \in K$ such that

$$f(x,y) = u(y) - u(x) \geq 0, \forall y \in K \Leftrightarrow u(y) \geq u(x), \forall y \in K.$$

Taking into account that

$$(D_u) \qquad\qquad \sup_{p\in X^*}\{-u^*(p) - \sigma_K(-p)\}$$

is the Fenchel dual problem of (P_u), we observe that the property (i) in the definition of the gap function is nothing else than weak duality between these problems. The second requirement claims that $x \in K$ to be a solution for (P_u) if and only if $\gamma_F^{EP}(x) = 0$, which is nothing else than $u(x) = \sup_{p \in X^*} \{-u^*(p) - \sigma_K(-p)\}$.

Thus we rediscover a well-known statement in convex optimization as a particular instance of our main result.

In the second part of the section we assume that dom $f = X \times X$ and under this assumption we deal with the so-called dual equilibrium problem (cf. [8]) which is closely related to (EP) and consists in finding $x \in K$ such that

$$(DEP) \qquad f(y,x) \leq 0, \ \forall y \in K,$$

or, equivalently,

$$(DEP) \qquad - f(y,x) \geq 0, \ \forall y \in K.$$

By K^{EP} and K^{DEP} we denote the solution sets of problems (EP) and (DEP), respectively. In order to suggest another gap function for (EP) we need some definitions and results.

Definition 3.1
The bifunction $f : X \times X \to \mathbb{R}$ is said to be

(i) monotone if, for each pair of points $x, y \in X$, we have

$$f(x,y) + f(y,x) \leq 0;$$

(ii) pseudomonotone if, for each pair of points $x, y \in X$, we have

$$f(x,y) \geq 0 \text{ implies } f(y,x) \leq 0.$$

Definition 3.2
Let $K \subseteq X$ and $\varphi : X \to \mathbb{R}$. The function φ is said to be

(i) quasiconvex on K if, for each pair of points $x, y \in K$ and for all $\alpha \in [0,1]$, we have

$$\varphi(\alpha x + (1-\alpha)y) \leq \max\Big\{\varphi(x), \varphi(y)\Big\};$$

(ii) explicitly quasiconvex on K if it is quasiconvex on K and for each pair of points $x, y \in K$ such that $\varphi(x) \neq \varphi(y)$ and for all $\alpha \in (0,1)$, we have

$$\varphi(\alpha x + (1-\alpha)y) < \max\Big\{\varphi(x), \varphi(y)\Big\}.$$

(iii) (explicitly) quasiconcave on K if $-\varphi$ is (explicitly) quasiconvex on K.

Definition 3.3
Let $K \subseteq X$ and $\varphi : X \to \mathbb{R}$. The function φ is said to be u-hemicontinuous on K if, for all $x, y \in K$ and $\alpha \in [0,1]$, the function $\tau(\alpha) = \varphi(\alpha x + (1-\alpha)y)$ is upper semicontinuous at 0.

Proposition 3.1 (cf. [8, Proposition 2.1])

(i) *If f is pseudomonotone, then $K^{EP} \subseteq K^{DEP}$.*

(ii) *If $f(\cdot, y)$ is u-hemicontinuous on K for all $y \in K$ and $f(x, \cdot)$ is explicitly quasiconvex on K for all $x \in K$ then $K^{DEP} \subseteq K^{EP}$.*

By using (DEP), in the same way as before, we introduce a new gap function for (EP). Let $x \in K$ be a solution of (DEP). This is equivalent to that x is a solution to the optimization problem

$$(P^{DEP}; x) \qquad \inf_{y \in K} [-f(y, x)].$$

Now we consider $(P^{DEP}; x)$ for all $x \in X$. The corresponding Fenchel dual problem to $(P^{DEP}; x)$ is

$$(D^{DEP}; x) \qquad \sup_{p \in X^*} \Big\{ -\sup_{y \in X} [\langle p, y\rangle + f(y, x)] - \delta_K^*(-p) \Big\},$$

if we rewrite $(P^{DEP}; x)$ again using δ_K similarly as done for $(P^{EP}; x)$. Let us define the function

$$\begin{aligned} \gamma_F^{DEP}(x) : &= -v(D^{DEP}; x) \\ &= -\sup_{p \in X^*} \Big\{ -\sup_{y \in X} [\langle p, y\rangle + f(y, x)] - \delta_K^*(-p) \Big\} \\ &= \inf_{p \in X^*} \Big\{ \sup_{y \in X} [\langle p, y\rangle + f(y, x)] + \sigma_K(-p) \Big\}. \end{aligned}$$

Assuming that for all $x \in K$ the function $y \mapsto -f(y, x)$ is convex and lower-semicontinuous one can give, in analogy to Theorem 3.1, some weak regularity conditions such that γ_F^{DEP} becomes a gap function for (DEP).

Next result shows under which conditions γ_F^{DEP} becomes a gap function for the equilibrium problem (EP).

Proposition 3.2 *Assume that f is a monotone bifunction. Then it holds*

$$\gamma_F^{DEP}(x) \leq \gamma_F^{EP}(x),\ \forall x \in X.$$

Proof: By the monotonicity of f, we have

$$f(x,y) + f(y,x) \leq 0,\ \forall x, y \in X,$$

or, equivalently, $f(y,x) \leq -f(x,y),\ \forall x, y \in X$. Let $p \in X^*$ be fixed. Adding $\langle p, y\rangle$ and taking the supremum in both sides over all $y \in X$ yields

$$\sup_{y\in X}[\langle p, y\rangle + f(y,x)] \leq \sup_{y\in X}[\langle p, y\rangle - f(x,y)].$$

After adding $\sigma_K(-p)$ and taking the infimum in both sides over $p \in X^*$, we conclude that $\gamma_F^{DEP}(x) \leq \gamma_F^{EP}(x),\ \forall x \in X$. □

Theorem 3.2 *Let the assumptions of Theorem* 3.1, *Proposition* 3.1(ii) *and Proposition 3.2 be fulfilled. Then γ_F^{DEP} is a gap function for* (EP).

Proof:

(i) By weak duality it holds

$$\gamma_F^{DEP}(x) = -v(D^{DEP}; x) \geq -v(P^{DEP}; x) \geq 0,\ \forall x \in K.$$

(ii) Let $\bar{x}$ be a solution of $(EP.)$ By Theorem 3.1, $\bar{x}$ is solution of (EP) if and only if $\gamma_F^{EP}(\bar{x}) = 0$. In view of (i) and Proposition 3.2, we get

$$0 \leq \gamma_F^{DEP}(\bar{x}) \leq \gamma_F^{EP}(\bar{x}) = 0.$$

Whence $\gamma_F^{DEP}(\bar{x}) = 0$. Let now $\gamma_F^{DEP}(\bar{x}) = 0$. By weak duality it holds

$$0 = v(D^{DEP}; \bar{x}) \leq v(P^{DEP}; \bar{x}) \leq 0.$$

Consequently $v(P^{DEP}; \bar{x}) = 0$. That means $\bar{x} \in K^{DEP}$. Hence, according to Proposition 3.1(ii), $\bar{x}$ is a solution of (EP). □

4 Applications to variational inequalities

In this section we apply the approach proposed in Section 3 to variational inequalities in a real Banach space. Let us notice that the approach based on the conjugate duality including Fenchel one, has been first considered for finite-dimensional variational inequalities (cf. [1]). We assume that X is a Banach space. The variational inequality problem consists in finding $x \in K$ such that

$$(VI) \qquad \langle F(x), y - x\rangle \geq 0, \ \forall y \in K,$$

where $F : K \to X^*$ is a given mapping and $K \subseteq X$ is a closed and convex set. Considering $f : X \times X \to \mathbb{R} \cup \{+\infty\}$,

$$f(x,y) = \begin{cases} \langle F(x), y - x\rangle, & \text{if } (x,y) \in K \times X, \\ +\infty, & \text{otherwise,} \end{cases}$$

the problem (VI) can be seen as a particular case of the equilibrium problem (EP).

For $x \in K$, (VI) can be rewritten as the optimization problem

$$(P^{VI}; x) \qquad \inf_{y \in X} \Big\{ \langle F(x), y - x\rangle + \delta_K(y) \Big\},$$

in the sense that x is a solution of (VI) if and only if it is a solution of $(P^{VI}; x)$. In view of γ_F^{EP}, we introduce the function based on Fenchel duality for (VI) by

$$\begin{aligned} \gamma_F^{VI}(x) &= \inf_{p \in X^*} \Big\{ \sup_{y \in X} [\langle p, y\rangle - \langle F(x), y - x\rangle] + \sigma_K(-p) \Big\} \\ &= \inf_{p \in X^*} \Big\{ \sup_{y \in X} \langle p - F(x), y\rangle + \sigma_K(-p) \Big\} + \langle F(x), x\rangle, \forall x \in K. \end{aligned}$$

From

$$\sup_{y \in X} \langle p - F(x), y\rangle = \begin{cases} 0, & \text{if } p = F(x), \\ +\infty, & \text{otherwise,} \end{cases}$$

follows that

$$\gamma_F^{VI}(x) = \inf_{p = F(x)} \sup_{y \in K} \langle -p, y\rangle + \langle F(x), x\rangle = \sup_{y \in K} \langle F(x), x - y\rangle, \forall x \in K.$$

In accordance to the definition of γ_F^{EP} in the previous, we have that for $x \notin K$, $\gamma_F^{VI}(x) = -\infty$.

Let us notice that for all $x \in K$, $y \mapsto f(x,y)$ is an affine function, thus continuous. On the other hand, the set $\operatorname{epi}(f_y^*(x,\cdot)) + \operatorname{epi}(\sigma_K) =$

$\{F(x)\} \times [\langle F(x), x\rangle, +\infty) + \mathrm{epi}(\sigma_K)$ is closed for all $x \in K$. This means that for all $x \in K$, $f_y^*(x, \cdot)\Box\sigma_K$ is lower semicontinuous and exact everywhere in X^* (cf. [6]). Thus the hypotheses of Theorem 3.1 are verified and γ_F^{VI} turns out to be a gap function for the problem (VI). γ_F^{VI} is actually the so-called Auslender's gap function (see [1] and [3]).

The problem (VI) can be associated with the following variational inequality introduced by Minty, which consists in finding $x \in K$ such that

$$(MVI) \qquad \langle F(y), y - x\rangle \geq 0, \ \forall y \in K.$$

As in Section 3, before introducing another gap function for (VI), let us consider some definitions and assertions.

Definition 4.1 A mapping $F : K \to X^*$ is said to be

(i) monotone if, for each pair of points $x, y \in K$, we have

$$\langle F(y) - F(x), y - x\rangle \geq 0;$$

(ii) pseudo-monotone if, for each pair of points $x, y \in K$, we have

$$\langle F(x), y - x\rangle \geq 0 \text{ implies } \langle F(y), y - x\rangle \geq 0;$$

(iii) continuous on finite-dimensional subspaces if for any finite-dimensional subspace M of X with $K \cap M \neq \varnothing$ the restricted mapping $F : K \cap M \to X^*$ is continuous from the norm topology of $K \cap M$ to the weak* topology of X^*.

Proposition 4.1 [12, Lemma 3.1] *Let $F : K \to X^*$ be a pseudo-monotone mapping which is continuous on finite-dimensional subspaces. Then $x \in K$ is a solution of (VI) if and only if it is a solution of (MVI).*

Minty variational inequality (MVI) is equivalent to the equilibrium problem which consists in finding $x \in K$ such that

$$-f(y, x) \geq 0, \forall y \in K.$$

As

$$-f(y, x) = \begin{cases} \langle F(y), y - x\rangle, & \text{if } (x, y) \in X \times K, \\ -\infty, & \text{otherwise,} \end{cases}$$

using the formula of γ_F^{DEP}, we get

$$\gamma_F^{MVI}(x) := \inf_{p \in X^*} \Big\{ \sup_{y \in K} [\langle p, y\rangle - \langle F(y), y - x\rangle] + \sigma_K(-p) \Big\} = \sup_{y \in K} \langle F(y), y - x\rangle.$$

One can notice that γ_F^{MVI} is nothing else than Auslender's gap function for Minty variational inequality (see, for instance, [9] and [11]). The following two results close the last section of the paper.

Proposition 4.2 *Let $F : K \to X^*$ be a monotone mapping. Then it holds*

$$\gamma_F^{MVI}(x) \leq \gamma_F^{VI}(x), \ \forall x \in K.$$

Theorem 4.1 *Let $F : K \to X^*$ be a monotone mapping which is continuous on finite-dimensional subspaces. Then γ_F^{MVI} is a gap function for (VI).*

Proof:

(i) $\gamma_F^{DEP}(x) \geq 0$ implies that $\gamma_F^{MVI}(x) \geq 0, \ \forall x \in K$, as this is a special case.

(ii) By the definition of the gap function, $\bar{x} \in K$ is a solution of (VI) if and only if $\gamma_F^{VI}(\bar{x}) = 0$. Taking into account (i) and Proposition 4.2, one has
$$0 \leq \gamma_F^{MVI}(\bar{x}) \leq \gamma_F^{VI}(\bar{x}) = 0.$$
In other words, $\gamma_F^{MVI}(\bar{x}) = 0$. Let now $\gamma_F^{MVI}(\bar{x}) = 0$. We can easily see that $\bar{x} \in K$ is a solution of (MVI). This follows using an analogous argumentation as in the proof of Theorem 3.2. Whence, according to Proposition 4.1, $\bar{x}$ solves (VI). □

5 Concluding remarks

In this paper we deal with the construction of gap functions for equilibrium problems by using the Fenchel duality theory for convex optimization problems. The gap functions we introduce here are defined by means of the optimal objective value of the duals of some primal optimization problems associated to the equilibrium problem, but also to the so-called dual equilibrium problem, respectively. In the particular case of the variational inequality problem we rediscover Auslender's gap functions for Stampacchia and Minty variational inequalities.

The present research opens the door for using the well-developed theory of duality when working with gap functions for equilibrium problems. As future research one can consider alongside the conjugate duality also other types of duality for convex as well as non-convex problems.

Acknowledgements. The authors are grateful to anonymous reviewers for remarks and suggestions that helped us to improve the paper.

References

[1] L. Altangerel, R.I. Boţ and G. Wanka, On the construction of gap functions for variational inequalities via conjugate duality, *Asia-Pac. J. Oper. Res.* (to appear).

[2] G. Auchmuty, Variational principles for variational inequalities, *Numer. Funct. Anal. Opt.* 10 (1989) 863-874.

[3] A. Auslender, *Optimisation: Méthods Numériques*, Masson, Paris, 1976.

[4] E. Blum and W. Oettli, Variational principles for equilibrium problems, in *Parametric Optimization and Related Topics.III.*, J. Guddat, H. Jongen, B. Kummer and F. Nožička (eds.), Peter Lang Verlag, Frankfurt am Main, 1993, pp. 79-88.

[5] E. Blum and W. Oettli, From optimization and variational inequalities to equilibrium problems, *Math. Student* 63 (1994) 123-145.

[6] R.I. Boţ and G. Wanka, A weaker regularity condition for subdifferential calculus and Fenchel duality in infinite dimensional spaces, *Nonlinear Anal.* (to appear).

[7] I. Ekeland, and R. Temam, *Convex Analysis and Variational Problems,* North-Holland Publishing Company, Amsterdam, 1976.

[8] I.V. Konnov and S. Schaible, Duality for equilibrium problems under generalized monotonicity, *J. Optim. Theory Appl.* 104 (2000) 395-408.

[9] P. Marcotte and D.L. Zhu, Weak sharp solutions of variational inequalities, *SIAM J. Optim.* 9 (1999) 179-189.

[10] G. Mastroeni, Gap functions for equilibrium problems, *J. Global Optim.* 27 (2003) 411-426.

[11] X.Q. Yang, On the gap functions of prevariational inequalities, *J. Optim. Theory Appl.* 116 (2003) 437-452.

[12] J.C. Yao, Variational inequalities with generalized monotone operators, *Math. Oper. Res.* 19 (1994) 691-705.

[13] D.L. Zhu and P. Marcotte, An extended descent framework for variational inequalities, *J. Optim. Theory Appl.* 80 (1994) 349-366.