A general approach for studying duality in multiobjective optimization

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Abstract. A general duality framework in convex multiobjective optimization is established using the scalarization with K-strongly increasing functions and the conjugate duality for composed convex cone-constrained optimization problems. Other scalarizations used in the literature arise as particular cases and the general duality is specialized for some of them, namely linear scalarization, maximum(-linear) scalarization, set scalarization, (semi)norm scalarization and quadratic scalarization.

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1 Introduction

Multiobjective (vector, multicriteria) optimization is a modern and fruitful research field with many practical applications, concerning especially engineering, economy and finance but also location and transports, even medicine. From the large amount of relevant publications in vector optimization we mention just three books, namely [18, 21, 28], where most of the theoretical issues concerning multiobjective optimization are comprehensively treated. Moreover, almost all of the works cited in our article deal with multiobjective optimization and many of the references therein too. The rich literature on vector optimization mentions several types of solutions that can be attached to a multiobjective optimization

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problem. Let us enumerate here a few: efficient, properly efficient, strongly efficient, weakly efficient, strictly efficient, approximately efficient, critical efficient, ideal efficient, superefficient and epsilon-efficient solutions. In our paper we use efficient as well as properly and weakly efficient solutions.

Duality is an important tool in vector optimization. Dealing with a convex vector minimization problem via duality is realized mostly by attaching a scalar optimization problem to the initial one. Using the scalarized problem and its dual, it is tried to construct a multiobjective dual problem to the primal vector problem and some duality assertions are usually verified. Different scalarization methods were proposed in the literature, using linear functions, norms and other constructions, see for instance [3–9, 11–15, 17–25, 27, 29–36, 38–42]. The scalarization we consider within this paper has already been mentioned or used in the literature in various ways by Gerstewitz (cf. [11]), Gerstewitz and Iwanow (cf. [12]), Göpfert and Gerth (cf. [14]), Jahn (cf. [17, 18]) and Miglierina and Molho (cf. [24]), among others, and consists in attaching to the initial multiobjective optimization problem a scalar optimization problem whose objective function is the postcomposition of the objective vector function of the vector optimization problem with a K-strongly increasing function, called scalarization function. To this scalar optimization problem we attach a conjugate dual problem (cf. [1]), which is then used to formulate the multiobjective dual problem. The conjugate dual problem we use is a combination of the classical Fenchel and Lagrange dual problems, being introduced by Boţ and Wanka (see [1,2] for more) under the name Fenchel-Lagrange dual problem.

The underlying notion of solutions of the primal and dual multiobjective problem is the one of properly efficient solutions for the primal problem and efficient solutions for the dual problem. If the convex cone defining the partial ordering in the image space of the vectorial objective function has a non-empty interior we consider also strictly increasing scalarization functions. In this way we can consider subsets of weakly efficient solutions for the primal and dual multiobjective problems and state corresponding weak and strong duality assertions.

Some of the cited authors used also this kind of scalarization in order to introduce Lagrange-type multiobjective dual problems (see [11, 12, 14]), but without resorting to conjugate functions. As many of the other scalarizations used in the literature use strongly increasing functions, too, they can be rediscovered as special cases in the framework we describe here. This happens for the linear scalarization, maximum(-linear) scalarization, set scalarization, (semi)norm scalarization, quadratic scalarization and other scalarizations involving special K-strongly increasing functions which were introduced in the literature usually for computational reasons.

This paper is organized as follows. The second section contains some definitions of the notions needed later and the duality statements regarding the scalar convex composed optimization problem. Then we present the new approach for constructing a dual to a multiobjective convex optimization problem, giving also weak and strong duality assertions, as well as optimality conditions. The fourth part contains some special cases of our duality framework, namely the situations when the scalarization function has certain additional imposed properties. A short conclusive section closes the paper.

2 Preliminaries and duality for the scalar convex composed problem

2.1 Preliminaries

Let us state from the very beginning that all around this paper we work in finite dimensional real spaces. As usual, $\mathbb{R}^n$ denotes the n-dimensional real space for any positive integer n and throughout all the vectors are considered as column vectors. An upper index ^{T} transposes a column vector to a row one and viceversa. The *inner product* of two vectors $x = (x_1, \ldots, x_n)^T$ and $y = (y_1, \ldots, y_n)^T$ in the n-dimensional real space is denoted by $x^T y = \sum_{i=1}^n x_i y_i$. Given a set $X \subseteq \mathbb{R}^n$ we use the well-known *indicator function* $\delta_X : \mathbb{R}^n \to \overline{\mathbb{R}}$ defined by

$$\delta_X(x) = \begin{cases} 0, & \text{if } x \in X, \\ +\infty, & \text{if } x \notin X. \end{cases}$$

Another important function attached to the set X is the *support function* $\sigma_X : \mathbb{R}^n \to \overline{\mathbb{R}}$, $\sigma_X(\beta) = \sup_{x \in X} \beta^T x$. The interior of X is denoted by $\operatorname{int}(X)$, the relative interior by $\operatorname{ri}(X)$ and the closure by $\operatorname{cl}(X)$. The border of X is written $\operatorname{bd}(X)$ and $\operatorname{aff}(X)$ is the affine hull of X. Denote by "$\leqq$" the partial ordering introduced on $\mathbb{R}^k$ by the corresponding non-negative orthant. Having a non-empty cone $K \subseteq \mathbb{R}^k$, we denote by $K^* = \{\beta \in \mathbb{R}^k : \beta^T k \geq 0 \ \forall k \in K\}$ its *dual cone*.

For $X \subseteq \mathbb{R}^n$ and a function $f : X \to \mathbb{R}$ we recall the definition of the *conjugate function regarding the set* X

$$f_X^* : \mathbb{R}^n \to \overline{\mathbb{R}}, \ f_X^*(p) = \sup_{x \in X} \left\{ p^T x - f(x) \right\}.$$

When $X = \mathbb{R}^n$ the conjugate function regarding the set X is actually the classical (Legendre-Fenchel) conjugate function of f, denoted by f^*. It is easy to prove that $(\delta_X)^* = \sigma_X$. Concerning the conjugate functions we have the following inequality known as the Fenchel-Young inequality

$$f_X^*(p) + f(x) \geq p^T x \ \forall x \in X \ \forall p \in \mathbb{R}^n.$$

Given a convex cone $K \subseteq \mathbb{R}^k$ that contains the element 0, we define some properties involving this cone that play an important role throughout this paper. When $\operatorname{int}(K) \neq \emptyset$ denote $\hat{K} = \operatorname{int}(K) \cup \{0\}$. Take $X \subseteq \mathbb{R}^n$ and $D \subseteq \mathbb{R}^k$, both non-empty.

Definition 1. (see [17, 18]) A function $f : D \to \mathbb{R}$ is called *K-increasing* if for $x, y \in D$ such that $x - y \in K$, follows $f(x) \geq f(y)$. If, additionally, whenever $x \neq y$ there is $f(x) > f(y)$, the function f is called *K-strongly increasing.* If $\text{int}(K) \neq \emptyset$ and for $x, y \in D$ such that $x - y \in \text{int}(K)$, follows $f(x) > f(y)$ the function f is called *K-strictly increasing.*

Remark 1. Clearly, when $\text{int}(K) \neq \emptyset$ the K-strictly increasing functions coincide with the $\hat{K}$-strongly increasing functions. In the literature there are some other notions of increasing monotonicity for functions, some of them used in vector optimization, too. See for instance [21] where properly increasing functions are used or other works where pseudomonotone or polarly monotone functions are employed on vector optimization. We have to mention that in some works (see [14]) the strongly increasing functions are called strictly increasing. We have opted for the terminology in [17, 18].

Definition 2. A function $F : X \to \mathbb{R}^k$, where X is a convex set, is called *K-convex* if for any x and $y \in X$ and $\lambda \in [0, 1]$ one has

$$\lambda F(x) + (1 - \lambda)F(y) - F\big(\lambda x + (1 - \lambda)y\big) \in K.$$

Further definitions will be introduced in the sections dealing with multiobjective optimization problems. Due to the length of the paper we skipped some definitions and explanations borrowed from the literature, referring the reader to the sources we have used.

2.2 Duality for the scalar convex composed problem

Let K and C be convex cones in $\mathbb{R}^k$ and $\mathbb{R}^m$, respectively, each of them containing the zero element in the corresponding space. All around this paper the cones K and C will satisfy these properties. Take also D a non-empty convex subset of $\mathbb{R}^k$ and X a non-empty convex subset of $\mathbb{R}^n$. Consider moreover the K-increasing convex function $f : D \to \mathbb{R}$, the K-convex function $F : X \to \mathbb{R}^k$ with $F = (F_1, \ldots, F_k)^T$ and $g : X \to \mathbb{R}^m$ which is a C-convex function with $g = (g_1, \ldots, g_m)^T$. We impose also the feasibility condition $F(X) \subseteq D$.

The convex composed optimization problem we consider within this section, which is used later to attach a scalar problem to a vector minimization problem, is

$$(P_c) \qquad \inf_{\substack{x \in X, \\ g(x) \in -C}} f(F(x)).$$

There are several ways to attach a dual problem to (P_c), but the composition of functions $f \circ F$ remains in the objective function of the dual directly or through its conjugate. Wanting to have these functions separated within a new dual problem, we formulate the following optimization problem which is equivalent to (P_c) in the sense that their optimal objective values coincide,

$$(P'_c) \qquad \inf_{\substack{x\in X, y\in D,\\ g(x)\in -C,\\ F(x)-y\in -K}} f(y).$$

Proposition 1. *Denoting the optimal objective values of the problems (P_c) and (P'_c) by $v(P_c)$ and, respectively, $v(P'_c)$, there is $v(P_c) = v(P'_c)$.*

Proof. Let x be feasible to (P_c). Take $y = F(x)$. As $F(X) \subseteq D$, y belongs to D, too, while $F(x) - y = 0 \in -K$. Thus (x, y) is feasible to (P'_c) and $f(F(x)) = f(y) \geq v(P'_c)$. Since this is valid for any x feasible to (P_c) it is straightforward that $v(P_c) \geq v(P'_c)$.

On the other hand, for (x, y) feasible to (P'_c) we have $x \in X$ and $g(x) \in -C$, so x is feasible to (P_c). Since f is K-increasing we get $v(P_c) \leq f(F(x)) \leq f(y)$. Taking the infimum on the right-hand side over (x, y) feasible to (P'_c) we get $v(P_c) \leq v(P'_c)$. Therefore $v(P_c) = v(P'_c)$. □

The Fenchel-Lagrange dual problem to (P'_c) is (cf. [1,2])

$$(D_c) \qquad \sup_{\substack{\alpha\in C^*, \beta\in K^*,\\ u\in\mathbb{R}^n}} \Big\{ - f_D^*(\beta) - \big(\beta^T F\big)_X^*(u) - \big(\alpha^T g\big)_X^*(-u)\Big\},$$

where $\alpha^T g$ and $\beta^T F$ are real-valued functions defined on X defined by $\alpha^T g(x) = \sum_{j=1}^m \alpha_j g_j(x)$ and, respectively, $\beta^T F(x) = \sum_{t=1}^k \beta_t F_t(x)$ for all $x \in X$, with $\alpha = (\alpha_1, \ldots, \alpha_m)^T \in C^*$ and $\beta = (\beta_1, \ldots, \beta_k)^T \in K^*$. Thanks to Proposition 1 (D_c) is the Fenchel-Lagrange dual problem to (P_c), too. By $v(D_c)$ we denote the optimal objective value of the problem (D_c). Weak duality between (P_c) and (D_c), namely $v(P_c) \geq v(D_c)$, is always valid (see [1]). In order to achieve strong duality between (P'_c) and (D_c) we introduce the following constraint qualification (cf. [1])

$$(CQ_c) \qquad \exists x' \in \mathrm{ri}(X) : \begin{cases} g(x') \in -\,\mathrm{ri}(C), \\ F(x') \in \mathrm{ri}(D) - \mathrm{ri}(K). \end{cases}$$

Before giving the strong duality statement, we need the following result (cf. [1]).

Proposition 2. *Take a non-empty convex set $X \subseteq \mathbb{R}^n$, a convex cone $C \subseteq \mathbb{R}^m$ that contains the zero element and a C-convex function $g : X \to \mathbb{R}^m$. Then $0 \in \mathrm{ri}\,\big(g(X) + C\big)$ if and only if $0 \in g\big(\mathrm{ri}(X)\big) + \mathrm{ri}(C)$.*

Now we are ready to formulate the strong duality statement for (P_c) and (D_c), followed by the necessary and sufficient optimality conditions.

Theorem 1. (strong duality) *If the constraint qualification (CQ_c) is fulfilled and $v(P_c) > -\infty$ there is strong duality between the problem (P_c) and its dual (D_c), i.e. $v(P_c) = v(D_c)$ and the latter has an optimal solution.*

Proof. We show actually that there is strong duality between (P_c') and (D_c) and by Proposition 1 we obtain that the same property is valid for (P_c) and (D_c). First consider the Lagrange dual problem to (P_c')

$$(D_c^L) \qquad \sup_{\substack{\alpha\in C^*,\\ \beta\in K^*}} \inf_{\substack{x\in X,\\ y\in D}} \big[f(y) + \alpha^T g(x) + \beta^T(F(x) - y)\big].$$

According to [10] (see also [1]), the constraint qualification that assures strong duality between (P_c') and (D_c^L) is $0 \in \mathrm{ri}\big(G(X\times D)+C\times K\big)$, where $G : X\times D \to \mathbb{R}^m\times\mathbb{R}^k$ is defined by $G(x,y) = \big(g_1(x),\dots,g_m(x),F_1(x)-y_1,\dots,F_k(x)-y_k\big)^T$ for all $x\in X$, $y = (y_1,\dots,y_k)^T \in D$. By Proposition 2 this condition is equivalent to

$$0 \in G(\mathrm{ri}(X\times D)) + \mathrm{ri}(C\times K). \tag{1}$$

This means that there must exist some pair $(x', y') \in \mathrm{ri}(X\times D)$ such that $G(x',y') \in -\,\mathrm{ri}(C\times K)$. It is known that $\mathrm{ri}(X\times D) = \mathrm{ri}(X)\times\mathrm{ri}(D)$ and $\mathrm{ri}(C\times K) = \mathrm{ri}(C)\times\mathrm{ri}(K)$. Using also the definition of G, condition (1) becomes

$$\exists (x',y') \in \mathrm{ri}(X)\times\mathrm{ri}(D) : \begin{cases} g(x') \in -\,\mathrm{ri}(C), \\ F(x') - y' \in -\,\mathrm{ri}(K). \end{cases} \tag{2}$$

It is easy to notice that (2) is equivalent to (CQ_c), which is assumed to be true in the hypothesis. Hence $v(P_c') = v(D_c^L)$ and the latter has an optimal solution, say $(\bar\alpha,\bar\beta)\in C^*\times K^*$, since $v(P_c') = v(P_c) > -\infty$.

Now take the inner infimum in (D_c^L) for $\alpha = \bar\alpha$ and $\beta = \bar\beta$. It can be separated into a sum of two infima,

$$\inf_{\substack{x\in X,\\ y\in D}} \big[f(y)+\bar\alpha^T g(x)+\bar\beta^T(F(x)-y)\big] = \inf_{x\in X}\big[\bar\alpha^T g(x)+\bar\beta^T F(x)\big] + \inf_{y\in D}\big[f(y)-\bar\beta^T y\big].$$

Turning the infima into suprema and using the definition of the conjugate, the right-hand side of the equality above becomes $-\big(\bar\beta^T F + \bar\alpha^T g\big)_X^*(0) - f_D^*(\bar\beta)$. As $\bar\beta^T F$ and $\bar\alpha^T g$ are real-valued convex functions defined on X, we also have (see [26])

$$\big(\bar\beta^T F + \bar\alpha^T g\big)_X^*(0) = \inf_{u\in\mathbb{R}^n}\Big[\big(\bar\beta^T F\big)_X^*(u) + \big(\bar\alpha^T g\big)_X^T(-u)\Big],$$

the latter infimum being attained at some $\bar u \in \mathbb{R}^n$. Whence

$$v(P_c) = v(P_c') = v(D_c^L) = -f_D^*(\bar\beta) - \big(\bar\beta^T F\big)_X^*(\bar u) - \big(\bar\alpha^T g\big)_X^*(-\bar u) = v(D_c)$$

and (D_c) has the optimal solution $(\bar\alpha,\bar\beta,\bar u)$. $\square$

Theorem 2. (optimality conditions) (a) *If the constraint qualification* (CQ_c) *is fulfilled and the primal problem* (P_c) *has an optimal solution* $\bar{x}$*, then the dual problem* (D_c) *has an optimal solution* $(\bar{\alpha}, \bar{\beta}, \bar{u})$ *and the following optimality conditions are satisfied*

(i) $f_D^*(\bar{\beta}) + f(F(\bar{x})) = \bar{\beta}^T F(\bar{x})$,

(ii) $\left(\bar{\beta}^T F\right)_X^*(\bar{u}) + \bar{\beta}^T F(\bar{x}) = \bar{u}^T \bar{x}$,

(iii) $\left(\bar{\alpha}^T g\right)_X^*(-\bar{u}) + \bar{\alpha}^T g(\bar{x}) = -\bar{u}^T \bar{x}$,

(iv) $\bar{\alpha}^T g(\bar{x}) = 0$.

(b) *If* $\bar{x}$ *is a feasible point to the primal problem* (P_c) *and* $(\bar{\alpha}, \bar{\beta}, \bar{u})$ *is feasible to the dual problem* (D_c) *fulfilling the optimality conditions (i)-(iv), then there is strong duality between* (P_c) *and* (D_c) *and the mentioned feasible points turn out to be optimal solutions of the corresponding problems.*

Proof. The previous theorem yields the existence of an optimal solution $(\bar{\alpha}, \bar{\beta}, \bar{u})$ to the dual problem and that $v(P_c) = v(D_c)$, which means

$$f(F(\bar{x})) + f_D^*(\bar{\beta}) + \left(\bar{\beta}^T F\right)_X^*(\bar{u}) + \left(\bar{\alpha}^T g\right)_X^*(-\bar{u}) = 0. \tag{3}$$

The Fenchel-Young inequality asserts for the functions involved in (3)

$$f(F(\bar{x})) + f_D^*(\bar{\beta}) \geq \bar{\beta}^T F(\bar{x}), \tag{4}$$

$$\bar{\beta}^T F(\bar{x}) + \left(\bar{\beta}^T F\right)_X^*(\bar{u}) \geq \bar{u}^T \bar{x} \tag{5}$$

and

$$\bar{\alpha}^T g(\bar{x}) + \left(\bar{\alpha}^T g\right)_X^*(-\bar{u}) \geq -\bar{u}^T \bar{x}. \tag{6}$$

The last four relations lead to

$$0 \geq \bar{\beta}^T F(\bar{x}) + \bar{u}^T \bar{x} - \bar{\beta}^T F(\bar{x}) - \bar{u}^T \bar{x} - \bar{\alpha}^T g(\bar{x}) = -\bar{\alpha}^T g(\bar{x}) \geq 0,$$

as $\bar{\alpha} \in C^*$ and $g(\bar{x}) \in -C$. Therefore the inequalities above must be fulfilled as equalities. The last one implies the optimality condition (iv), while (i) arises from (4), (ii) from (5) and (iii) from (6).

The reverse assertion in (b) follows immediately, even without the fulfilment of (CQ_c) and of any convexity assumption we made concerning the involved functions and sets, because summing the equalities in $(i) - (iv)$ yields (3), that is equivalent to $v(P_c) = v(D_c)$, $\bar{x}$ solves (P_c) and $(\bar{\alpha}, \bar{\beta}, \bar{u})$ solves (D_c). □

We close this section with a result which simplifies the constraint qualification (CQ_c) in case D or K has a non-empty interior.

Proposition 3. *For any convex sets* $A, B \subseteq \mathbb{R}^k$ *such that* $\text{int}(B) \neq \emptyset$ *one has* $\text{ri}(A) + \text{int}(B) = A + \text{int}(B)$.

Proof. As $A + \text{int}(B)$ is a non-empty open set one has

$$A + \text{int}(B) = \text{int}(A + \text{int}(B)) = \text{ri}(A + \text{int}(B)).$$

Since A and $\text{int}(B)$ are convex sets, we get

$$\text{ri}(A + \text{int}(B)) = \text{ri}(A) + \text{ri}(\text{int}(B)) = \text{ri}(A) + \text{int}(B),$$

thus the conclusion follows. □

3 Duality for the multiobjective problem

Consider the convex multiobjective optimization problem

$$(P_v) \qquad \underset{\substack{x \in X, \\ g(x) \in -C}}{\text{v-min}} \; F(x),$$

where $K \neq \{0\}$, with $K \cap (-K) = \{0\}$, and C are convex cones in $\mathbb{R}^k$ and $\mathbb{R}^m$, respectively, that contain the zero element in the corresponding spaces, $F = (F_1, \ldots, F_k)^T : X \to \mathbb{R}^k$ is a K-convex function and $g = (g_1, \ldots, g_m)^T : X \to \mathbb{R}^m$ is a C-convex function. For simplicity let $\mathcal{A} = \{x \in X : g(x) \in -C\}$ be the *feasible set* of the convex vector minimization problem (P_v). By a solution to (P_v) one can understand different notions, we rely in this part of the paper to the following ones.

Definition 3. (see also [18, 28]) An element $\bar{x} \in \mathcal{A}$ is called a *(Pareto) efficient* solution to (P_v) if from $F(x) - F(\bar{x}) \in -K$ for $x \in \mathcal{A}$ follows $F(x) = F(\bar{x})$.

Let the convex set $D \subseteq \mathbb{R}^k$ be such that $F(X) \subseteq D$. Take an arbitrary set of K-strongly increasing convex functions $s : D \to \mathbb{R}$ denoted by $\mathcal{S}$.

Definition 4. (see also [11, 12, 14]) An element $\bar{x} \in \mathcal{A}$ is said to be an $\mathcal{S}$-*properly efficient* solution to (P_v) if there is some $s \in \mathcal{S}$ fulfilling $s(F(\bar{x})) \leq s(F(x))\ \forall x \in \mathcal{A}$.

Remark 2. It is easy to see that any $\mathcal{S}$-properly efficient solution to (P_v) is also an efficient one.

If $\text{int}(K) \neq \emptyset$ one can find in the literature also the so-called weakly efficient solutions to (P_v).

Definition 5. (see also [17, 18]) An element $\bar{x} \in \mathcal{A}$ is said to be a *weakly efficient* solution to (P_v) if there is no $x \in \mathcal{A}$ such that $F(x) - F(\bar{x}) \in -\operatorname{int}(K)$.

One can easily notice that $\bar{x} \in \mathcal{A}$ is a weakly efficient solution to (P_v) if from $F(x) - F(\bar{x}) \in -\hat{K}$, where $\hat{K} = \operatorname{int}(K) \cup \{0\}$, for $x \in \mathcal{A}$ follows $F(x) = F(\bar{x})$, i.e. $\bar{x} \in \mathcal{A}$ is weakly efficient to (P_v) if and only if it is efficient when working with the cone $\hat{K}$. Something similar happens for the properly efficient solutions, too. By Remark 1 we know that when $\operatorname{int}(K) \neq \emptyset$ the K-strictly increasing functions are actually $\hat{K}$-strongly increasing and vice versa. Extending $\mathcal{S}$ to a set of K-strictly increasing convex functions (note that the class of K-strictly increasing functions contains the class of K-strongly increasing ones) denoted by $\mathcal{T}$, the relation in Definition 4 characterizes a new class of points as follows.

Definition 6. An element $\bar{x} \in \mathcal{A}$ is a said to be a $\mathcal{T}$*-weakly properly efficient* solution to (P_v) if there is some $s \in \mathcal{T}$ fulfilling $s(F(\bar{x})) \leq s(F(x))\ \forall x \in \mathcal{A}$.

Clearly, any $\mathcal{T}$-weakly properly efficient solution to (P_v) is also a weakly efficient solution to (P_v). Let us stress that when $\operatorname{int}(K) \neq \emptyset$ and $K = \hat{K}$ the $\mathcal{S}$-properly efficient solutions to (P_v) coincide with the $\mathcal{S}$-weakly properly efficient ones (see Remark 1) and, obviously, the efficient solutions with the weakly efficient ones.

In order to deal with (P_v) via duality we introduce, basing on Definitions 4 and 6, the following family of scalarized problems

$$(P_s) \qquad\qquad \inf_{x \in \mathcal{A}} s(F(x)), \text{ for } s \in \mathcal{S}.$$

Any function $s \in \mathcal{S}$ is called *scalarization function.* This type of scalarized problems has been used in the literature, but without having in mind conjugate duality for the primal multiobjective optimization problem. Gerstewitz (cf. [11]), Gerstewitz and Iwanow (cf. [12]) and Göpfert and Gerth (cf. [14]) gave Lagrange-type duality for non-convex vector maximization problems, where the scalarization functions are taken moreover continuous, while Jahn (cf. [17, 18]) and Miglierina and Molho (cf. [24]) mentioned this kind of scalarization in the context of characterizing solutions of vector minimization problems but without resorting to duality.

For any $s \in \mathcal{S}$, from the previous section (see (D_c)) we know that the Fenchel-Lagrange dual problem to (P_s) is

$$(D'_s) \qquad \sup_{\substack{\alpha \in C^*, \beta \in K^*, \\ u \in \mathbb{R}^n}} \Big\{ - s^*_D(\beta) - (\beta^T F)^*_X(u) - (\alpha^T g)^*_X(-u) \Big\}.$$

Using this, we introduce the following multiobjective dual problem to (P_v)

inspired by some dual problems given in [3, 4],

$$(D_v) \qquad \underset{(z,s,\alpha,\beta,u)\in\mathcal{B}}{\text{v-max}} z,$$

where

$$\begin{aligned}\mathcal{B} &= \Big\{(z,s,\alpha,\beta,u) \in D \times \mathcal{S} \times C^* \times K^* \times \mathbb{R}^n : \\ &\quad s(z) \leq -s_D^*(\beta) - \big(\beta^T F\big)_X^*(u) - \big(\alpha^T g\big)_X^*(-u)\Big\}.\end{aligned}$$

For vector maximization problems there are also several types of solutions in the literature. We use the following notion, similar to the one given earlier for vector minimization problems.

Definition 7. An element $(\bar{z}, \bar{s}, \bar{\alpha}, \bar{\beta}, \bar{u}) \in \mathcal{B}$ is said to be an *efficient* solution to (D_v) if from $z - \bar{z} \in K$ for $(z, s, \alpha, \beta, u) \in \mathcal{B}$ follows $z = \bar{z}$.

The weak and strong duality statements concerning (P_v) and (D_v) follow.

Theorem 3. (weak duality) *There is no $x \in \mathcal{A}$ and no $(z, s, \alpha, \beta, u) \in \mathcal{B}$ such that $z - F(x) \in K$ and $F(x) \neq z$.*

Proof. Assume that there are some $x \in \mathcal{A}$ and $(z, s, \alpha, \beta, u) \in \mathcal{B}$ contradicting the assumption. As s is K-strongly increasing it follows

$$s(F(x)) < s(z).$$

On the other hand,

$$s(z) \leq -s_D^*(\beta) - (\beta^T F)_X^*(u) - (\alpha^T g)_X^*(-u).$$

So we get

$$s(F(x)) < -s_D^*(\beta) - (\beta^T F)_X^*(u) - (\alpha^T g)_X^*(-u).$$

This last relation contradicts the weak duality that exists between (P_s) and (D_s'), therefore the supposition we made is false and weak duality holds. □

Theorem 4. (strong duality) *Assume (CQ_c) fulfilled and let $\bar{x} \in \mathcal{A}$ be an $\mathcal{S}$-properly efficient solution to (P_v). Then the dual problem (D_v) has an efficient solution $(\bar{z}, \bar{s}, \bar{\alpha}, \bar{\beta}, \bar{u})$ such that $F(\bar{x}) = \bar{z}$.*

Proof. According to Definition 4 there is an $\bar{s} \in \mathcal{S}$ such that $\bar{s}(F(\bar{x})) \leq \bar{s}(F(x))\ \forall x \in \mathcal{A}$. It is obvious that $\bar{x}$ is also an optimal solution to the scalarized problem $(P_{\bar{s}})$, therefore $v(P_{\bar{s}}) > -\infty$. As (CQ_c) is assumed to be valid there is strong duality between $(P_{\bar{s}})$ and $(D_{\bar{s}})$ because of Theorem 1. Therefore $(D_{\bar{s}})$ has an optimal solution, say $(\bar{\alpha}, \bar{\beta}, \bar{u}) \in C^* \times K^* \times \mathbb{R}^n$ and

$$\bar{s}(F(\bar{x})) = -\bar{s}_D^*(\bar{\beta}) - (\bar{\beta}^T F)_X^*(\bar{u}) - \left(\bar{\alpha}^T g\right)_X^*(-\bar{u}).$$

Denote $\bar{z} = F(\bar{x})$. It is obvious that $(\bar{z}, \bar{s}, \bar{\alpha}, \bar{\beta}, \bar{u}) \in \mathcal{B}$ and so we have found a feasible point to the dual problem. It remains to prove that $(\bar{z}, \bar{s}, \bar{\alpha}, \bar{\beta}, \bar{u})$ is an efficient solution to (D_v). Supposing that there is some $(z', s', \alpha', \beta', u') \in \mathcal{B}$ such that $z' - \bar{z} \in K$ and $\bar{z} \neq z'$, it follows that $z' - F(\bar{x}) \in K$ and $F(\bar{x}) \neq z'$, which contradicts Theorem 3. $\square$

The necessary and sufficient optimality conditions regarding (P_v) and (D_v) follow immediately from the ones concerning the problems (P_s) and (D_s').

Theorem 5. (optimality conditions) (a) *If the constraint qualification* (CQ_c) *is fulfilled and the primal problem* (P_v) *has an* $\mathcal{S}$*-properly efficient solution* $\bar{x}$*, then the dual problem* (D_v) *has an efficient solution* $(\bar{z}, \bar{s}, \bar{\alpha}, \bar{\beta}, \bar{u})$ *such that the following optimality conditions are satisfied*

(i) $F(\bar{x}) = \bar{z}$,

(ii) $\bar{s}_D^*(\bar{\beta}) + \bar{s}(F(\bar{x})) = \bar{\beta}^T F(\bar{x})$,

(iii) $\left(\bar{\beta}^T F\right)_X^*(\bar{u}) + \bar{\beta}^T F(\bar{x}) = \bar{u}^T \bar{x}$,

(iv) $\left(\bar{\alpha}^T g\right)_X^*(-\bar{u}) + \bar{\alpha}^T g(\bar{x}) = -\bar{u}^T \bar{x}$,

(v) $\bar{\alpha}^T g(\bar{x}) = 0$.

(b) *If* $\bar{x}$ *is a feasible point to the primal problem* (P_v) *and* $(\bar{z}, \bar{s}, \bar{\alpha}, \bar{\beta}, \bar{u})$ *is feasible to the dual problem* (D_v) *fulfilling the optimality conditions* $(i) - (v)$*, then* $\bar{x}$ *is an* $\mathcal{S}$*-properly efficient solution to* (P_v) *and* $(\bar{z}, \bar{s}, \bar{\alpha}, \bar{\beta}, \bar{u})$ *is efficient to the dual problem* (D_v).

Remark 3. Let $(\alpha, \beta, u) \in C^* \times K^* \times \mathbb{R}^n$. If $K = \mathbb{R}_+^k$ we have (see Theorem 16.4 in [26])

$$(\beta^T F)_X^*(u) = \min\left\{ \sum_{t=1}^{k} (\beta_t F_t)_X^*(p_t) : \sum_{t=1}^{k} p_t = u \right\},$$

while when $C = \mathbb{R}_+^m$ one gets

$$(\alpha^T g)_X^*(-u) = \min\left\{ \sum_{j=1}^{m} (\alpha_j g_j)_X^*(q_j) : \sum_{j=1}^{m} q_j = -u \right\}.$$

In both these special cases the dual problem as well as the optimality conditions can be modified correspondingly.

Remark 4. As one can notice further, the scalarizations used in the literature usually ask the cone K to have a non-empty interior. This additional assumption is not necessary when using our approach.

Remark 5. If $\text{int}(K) \neq \emptyset$ every K-strictly increasing real-valued function defined on D is actually $\hat{K}$-strongly increasing. Taking $\hat{K}$ instead of K and $\mathcal{S}$ a set of K-strictly increasing functions $s : D \to \mathbb{R}$, the aforementioned duality results turn into the following ones.

Theorem 6. (weak duality) *There is no $x \in \mathcal{A}$ and no $(z, s, \alpha, \beta, u) \in \mathcal{B}$ such that $z - F(x) \in \text{int}(K)$.*

Theorem 7. (strong duality) *Assume (CQ_c) fulfilled and let $\bar{x} \in \mathcal{A}$ be an $\mathcal{S}$-weakly properly efficient solution to (P_v). Then the dual problem (D_v) has a weakly efficient solution $(\bar{z}, \bar{s}, \bar{\alpha}, \bar{\beta}, \bar{u})$ such that $F(\bar{x}) = \bar{z}$.*

Theorem 8. (optimality conditions) (*a*) *If the constraint qualification (CQ_c) is fulfilled and the primal problem (P_v) has an $\mathcal{S}$-weakly properly efficient solution $\bar{x}$, then the dual problem (D_v) has a weakly efficient solution $(\bar{z}, \bar{s}, \bar{\alpha}, \bar{\beta}, \bar{u})$ such that the following optimality conditions are satisfied*

(i) $F(\bar{x}) = \bar{z}$,

(ii) $\bar{s}_D^*(\bar{\beta}) + \bar{s}(F(\bar{x})) = \bar{\beta}^T F(\bar{x})$,

(iii) $\left(\bar{\beta}^T F\right)_X^*(\bar{u}) + \bar{\beta}^T F(\bar{x}) = \bar{u}^T \bar{x}$,

(iv) $\left(\bar{\alpha}^T g\right)_X^*(-\bar{u}) + \bar{\alpha}^T g(\bar{x}) = -\bar{u}^T \bar{x}$,

(v) $\bar{\alpha}^T g(\bar{x}) = 0$.

(*b*) *If $\bar{x}$ is a feasible point to the primal problem (P_v) and $(\bar{z}, \bar{s}, \bar{\alpha}, \bar{\beta}, \bar{u})$ is feasible to the dual problem (D_v) fulfilling the optimality conditions $(i) - (v)$, then $\bar{x}$ is an $\mathcal{S}$-weakly properly efficient solution to (P_v) and $(\bar{z}, \bar{s}, \bar{\alpha}, \bar{\beta}, \bar{u})$ is weakly efficient to the dual problem (D_v).*

Remark 6. Let us mention that (*b*) in Theorems 5 and 8 is valid without supposing (CQ_c) fulfilled as well as any convexity assumptions as stated before.

4 Special cases: duals induced by some scalarizations in the literature

Next we show how the duality statements given in the previous section can be applied when the scalarization functions are taken in order to fulfill some additionally given conditions. Some scalarizations used in the literature on multiobjective optimization use different particular strongly increasing scalarization functions and they are actually special cases of the scalarization considered by us. In each situation we adapt the definition of the properly efficient elements to the particular formulation of the scalarization functions. When $\operatorname{int}(K) \neq \emptyset$ we work with weakly efficient and $\mathcal{S}$-weakly properly efficient solutions. In order to avoid possible confusions let us mention that some of these notions are known in the literature under different names, which we have not reminded here, pointing out just the works where we found the scalarizations that lead to them.

It is worth mentioning that in many papers on vector optimization the authors consider the functions involved also lower-semicontinuous, even continuous. In some cases these additional assumptions are necessary, but for our duality statements they would be redundant. That is why we have omitted them.

We have chosen five classes of scalarizations found in the literature to be included here, namely the linear scalarization, the maximum(-linear) scalarization, the set-scalarization, the (semi)norm scalarization and the quadratic scalarization. Some of these classes include more than one type of scalarization. Although in some papers the cone K is taken to be $\mathbb{R}^k_+$ or $\operatorname{int}(\mathbb{R}^k_+) \cup \{0\}$, we give our results in the most general case possible, keeping in mind the computational aspect, though. When the interior of the cone K is non-empty and the scalarization functions found in the literature are only K-strictly increasing instead of K-strongly increasing one could believe that our duality statements are not applicable. Fortunately this is not the case and in this situation we use Theorems 6-8, i.e. we deal with weakly efficient, respectively weakly properly efficient, solutions instead of efficient, respectively properly efficient, ones. Let us also mention that because of the length of the paper we do not give the necessary and sufficient optimality conditions regarding the duality statements in each special case, as they arise immediately from Theorem 5 or Theorem 8. Depending on the choice of $\mathcal{S}$ and K the optimality conditions (ii) and (iii) in Theorem 5 (8) turn into more specific formulations in each special case, while (i), (iv) and (v) remain unchanged. There are other types of scalarizations in the literature which do not belong to the classes we treat. We mention here those in [7, 9, 22, 33, 40, 42].

4.1 Linear scalarization

The most famous and used scalarization in vector optimization is the one with strongly increasing linear functionals, called linear (weighted) scalarization. From the large amount of papers dealing with this kind of scalarization we mention here [3,4,34–36], as Fenchel-Lagrange duality is involved there, too.

The cones K and C are taken like in the previous section. Denote the *quasi-interior* of the dual cone K^* by $K^{*o} = \{\lambda \in K^* : \lambda^T y > 0 \ \forall y \in K\backslash\{0\}\}$. For any fixed $\lambda = (\lambda_1, \ldots, \lambda_k)^T \in K^{*o}$, the scalarized primal problem is

$$(P_\lambda) \qquad \inf_{x \in \mathcal{A}} \left[\sum_{j=1}^{k} \lambda_j F_j(x) \right].$$

The linear scalarization is a special case of the general framework we presented as the objective function in (P_λ) can be written as $s_\lambda(F(x))$, for $s_\lambda(y) = \lambda^T y$ and it is clear that s_λ is K-strongly increasing and convex for any $\lambda \in K^{*o}$. In this case let $\mathcal{S} = \mathcal{S}_l$, the latter being defined as follows

$$\mathcal{S}_l = \left\{ s_\lambda : D \to \mathbb{R} : \ s_\lambda(y) = \lambda^T y, \lambda \in K^{*o} \right\}.$$

Thus an element $\bar{x} \in \mathcal{A}$ is called $\mathcal{S}_l$*-properly efficient* with respect to (P_v) when there is some $\lambda \in K^{*o}$ fulfilling $\sum_{j=1}^k \lambda_j F_j(\bar{x}) \leq \sum_{j=1}^k \lambda_j F(x) \ \forall x \in \mathcal{A}$.

Let us write now the dual problem to (P_v) that arises by using the scalarization function $s \in \mathcal{S}_l$. One can easily notice that the dual variable $s_\lambda \in \mathcal{S}_l$ that fulfills $s_\lambda(y) = \lambda^T y \ \forall y \in D$, where $\lambda \in K^{*o}$, can be represented and replaced by the variable $\lambda \in K^{*o}$. Moreover, $(s_\lambda)_D^*(\beta) = \sigma_D(\beta - \lambda) \ \forall \beta \in \mathbb{R}^k$. Knowing these, the dual problem to (P_v) obtained via the linear scalarization is

$$(D_l) \qquad \underset{(z,\lambda,\alpha,\beta,u) \in \mathcal{B}_l}{\text{v-max}} z,$$

where

$$\begin{aligned} \mathcal{B}_l \ = \ & \Bigg\{ (z, \lambda, \alpha, \beta, u) \in D \times K^{*o} \times C^* \times K^* \times \mathbb{R}^n : z = (z_1, \ldots, z_k)^T, \\ & \lambda = (\lambda_1, \ldots, \lambda_k)^T, \sum_{j=1}^{k} \lambda_j z_j \leq -\sigma_D(\beta - \lambda) - (\beta^T F)_X^*(u) - (\alpha^T g)_X^*(-u) \Bigg\}. \end{aligned}$$

Theorem 9. (weak duality) *There is no $x \in \mathcal{A}$ and no $(z, \lambda, \alpha, \beta, u) \in \mathcal{B}_l$ such that $z - F(x) \in K$ and $F(x) \neq z$.*

Theorem 10. (strong duality) *Assume (CQ_c) fulfilled and let $\bar{x} \in \mathcal{A}$ be an $\mathcal{S}_l$-properly efficient solution to (P_v). Then the dual problem (D_l) has an efficient solution $(\bar{z}, \bar{\lambda}, \bar{\alpha}, \bar{\beta}, \bar{u})$ such that $F(\bar{x}) = \bar{z}$.*

If $D = \mathbb{R}^k$ we get $\sigma_D(\beta - \lambda) = 0$ if $\beta = \lambda$ and $\sigma_D(\beta - \lambda) = +\infty$ otherwise, thus the variable $\beta \in K^*$ from (D_l) is no longer necessary since the inequality in the feasible set of the dual problem is not fulfilled unless $\beta = \lambda$. Therefore the dual problem obtained in this case to (P_v) is

$$(D'_l) \qquad \underset{(z,\lambda,\alpha,u)\in\mathcal{B}'_l}{\text{v-max}}\, z,$$

where

$$\begin{aligned}\mathcal{B}'_l \;=\; & \bigg\{(z,\lambda,\alpha,u) \in \mathbb{R}^k \times K^{*o} \times C^* \times \mathbb{R}^n : z = (z_1,\dots,z_k)^T, \\ & \lambda = (\lambda_1,\dots,\lambda_k)^T, \sum_{j=1}^k \lambda_j z_j \leq -(\lambda^T F)^*_X(u) - (\alpha^T g)^*_X(-u)\bigg\}.\end{aligned}$$

When $K = \mathbb{R}^k_+$ it is clear that $K^{*o} = \text{int}(\mathbb{R}^k_+)$ and by Theorem 16.4 in [26] (see also Remark 3) we have for λ and u taken like in $\mathcal{B}'_l$

$$(\lambda^T F)^*_X(u) = \min\left\{\sum_{j=1}^k (\lambda_j F_j)^*_X(p_j) : p_j \in \mathbb{R}^n, j = 1,\dots,k,\ \sum_{j=1}^k p_j = u\right\}$$

and, as $\lambda_j > 0$, $j = 1,\dots,k$, this turns into

$$(\lambda^T F)^*_X(u) = \min\left\{\sum_{j=1}^k \lambda_j F^*_{jX}\Big(\frac{1}{\lambda_j} p_j\Big) : p_j \in \mathbb{R}^n, j = 1,\dots,k,\ \sum_{j=1}^k p_j = u\right\}.$$

Denoting $y_j = (1/\lambda_j)p_j$ for $j = 1,\dots,k$, and $y = (y_1,\dots,y_k)$, the latter dual problem turns into

$$(D''_l) \qquad \underset{(z,\lambda,\alpha,y)\in\mathcal{B}'_l}{\text{v-max}}\, z,$$

with

$$\begin{aligned}\mathcal{B}''_l = & \bigg\{(z,\lambda,\alpha,y) \in \mathbb{R}^k \times \text{int}(\mathbb{R}^k_+) \times C^* \times (\mathbb{R}^n \times \dots \mathbb{R}^n) : y = (y_1,\dots,y_k), \\ & z = (z_1,\dots,z_k)^T, \sum_{j=1}^k \lambda_j z_j \leq -\sum_{j=1}^k \lambda_j F^*_{jX}(y_j) - (\alpha^T g)^*_X\bigg(-\sum_{j=1}^k \lambda_j y_j\bigg)\bigg\},\end{aligned}$$

which is exactly the dual problem obtained by Boţ and Wanka in [3,4].

Let us notice that the constraint qualification needed in this particular case for strong duality becomes, as $\text{ri}(D) = \mathbb{R}^k$,

$$(CQ_v) \qquad \exists x' \in \mathrm{ri}(X) : g(x') \in -\mathrm{ri}(C),$$

and it is weaker than the one considered in [3, 4] for strong duality between (P_v) and (D_l'').

Theorem 11. (weak duality) *There is no $x \in \mathcal{A}$ and no $(z, \lambda, \alpha, y) \in \mathcal{B}_l''$ such that $F(x) \leqq z$ and $F(x) \neq z$.*

Theorem 12. (strong duality) *Assume (CQ_v) fulfilled and let $\bar{x} \in \mathcal{A}$ be an $\mathcal{S}_l$-properly efficient solution to (P_v). Then the dual problem (D_l'') has an efficient solution $(\bar{z}, \bar{\lambda}, \bar{\alpha}, \bar{y})$ such that $F(\bar{x}) = \bar{z}$.*

Getting back to the general case of the linear scalarization, an interesting situation occurs when $\mathrm{int}(K) \neq \emptyset$. Consider the set

$$\mathcal{S}_{lw} = \Big\{ s_\lambda : D \to \mathbb{R} : \ s_\lambda(y) = \lambda^T y, \lambda \in K^* \backslash \{0\} \Big\}.$$

It is known (see [18], for instance) that $\mathcal{S}_{lw}$ is a set of K-strictly increasing functions, i.e. it contains only $\hat{K}$-strongly increasing functions. One could define $\bar{x} \in \mathcal{A}$ to be $\mathcal{S}_{lw}$*-weakly properly efficient* with respect to (P_v) if there exists some $\lambda \in K^* \backslash \{0\}$ such that $\sum_{j=1}^k \lambda_j F_j(\bar{x}) \leq \sum_{j=1}^k \lambda_j F(x)\ \forall x \in \mathcal{A}$. Using Theorem 5.4 in [18] it is not difficult to show that any weakly efficient element with respect to (P_v) is actually a $\mathcal{S}_{lw}$-weakly properly efficient solution to (P_v). On the other hand the $\mathcal{S}_{lw}$-weakly properly efficient solutions to (P_v) are also weakly efficient with respect to (P_v), thus in this special case the two notions coincide. Now we can give a dual problem (D_{lw}) to (P_v) in an analogous manner as done with (D_l), by replacing K^{*o} with $K^* \backslash \{0\}$ within the definition of $\mathcal{B}_l$, which becomes $\mathcal{B}_{lw}$. Weak and strong duality, as well as necessary and sufficient optimality conditions follow by Theorems 6, 7 and 8 for weakly efficient solutions to (P_v) and weakly efficient solutions to (D_{lw}). Like everywhere within this section we give here only the duality statements.

Theorem 13. (weak duality) *There is no $x \in \mathcal{A}$ and no $(z, \lambda, \alpha, \beta, u) \in \mathcal{B}_{lw}$ such that $z - F(x) \in \mathrm{int}(K)$.*

Theorem 14. (strong duality) *Assume (CQ_c) fulfilled and let $\bar{x} \in \mathcal{A}$ be a weakly efficient solution to (P_v). Then the dual problem (D_{lw}) has a weakly efficient solution $(\bar{z}, \bar{\lambda}, \bar{\alpha}, \bar{\beta}, \bar{u})$ such that $F(\bar{x}) = \bar{z}$.*

These duality statements can be further specialized for the special cases $D = \mathbb{R}^k$ and $K = \mathbb{R}^k_+$ as done above, by replacing K^{*o} with $K^* \backslash \{0\}$ in $\mathcal{B}_l'$ and, respectively $\mathrm{int}(\mathbb{R}^k_+)$ with $\mathbb{R}^k_+ \backslash \{0\}$ in $\mathcal{B}_l''$.

4.2 Maximum(-linear) scalarization

Another scalarization met especially in the applications of vector optimization is the so-called Tchebyshev scalarization or maximum scalarization, where the objective function of the scalarized problem consists in the maximal entry of the vector function at each point. Among the papers dealing with this kind of scalarization we cite here Mbunga's [23], mentioning also [9]. The weighted Tchebyshev scalarization (see [18, 33]) is slightly more general than it and we found an even more general scalarization based on the weighted maximum function combined with a linear function, namely the one in [25]. There this scalarization is applied in diet planning. Take $K = \mathbb{R}^k_+$, $D = \mathbb{R}^k$ and $\eta \geq 0$. Clearly, $K^* = \mathbb{R}^k_+$. The family of scalarized primal problems is

$$(P_{w,a}) \qquad \inf_{x\in\mathcal{A}} \left[\max_{j=1,\ldots,k} \left\{ w_j(F_j(x) - a_j) \right\} + \eta \sum_{j=1}^{k} w_j F_j(x) \right],$$

where $w = (w_1, \ldots, w_k)^T \in \operatorname{int}(\mathbb{R}^k_+)$ and $a = (a_1, \ldots, a_k)^T \in \mathbb{R}^k$. The scalarization functions are $s_{w,a} : \mathbb{R}^k \to \mathbb{R}$, $s_{w,a}(y) = \max\limits_{j=1,\ldots,k} \left\{ w_j(y_j - a_j) \right\} + \eta \sum_{j=1}^k w_j y_j$ $\forall y = (y_1, \ldots, y_k)^T \in \mathbb{R}^k$ and it is easy to notice that they are convex and $\mathbb{R}^k_+$-strictly increasing for all $w = (w_1, \ldots, w_k)^T \in \operatorname{int}(\mathbb{R}^k_+)$ and $a = (a_1, \ldots, a_k)^T \in \mathbb{R}^k$. Since the scalarization functions are $\mathbb{R}^k_+$-strictly increasing we can apply the theory given in the general case by characterizing the weakly efficient solutions of the dual instead of the efficient ones, as done in the final part of the previous section. Take $\mathcal{S} = \mathcal{S}_{ml}$, where

$$\mathcal{S}_{ml} = \Bigg\{ s_{w,a} : \mathbb{R}^k \to \mathbb{R} : \ s_{w,a}(y) = \max_{j=1,\ldots,k} \left\{ w_j(y_j - a_j) \right\} + \eta \sum_{j=1}^{k} w_j y_j \ \forall y \in \mathbb{R}^k,$$

$$y = (y_1, \ldots, y_k)^T, w = (w_1, \ldots, w_k)^T \in \operatorname{int}(\mathbb{R}^k_+), a = (a_1, \ldots, a_k)^T \in \mathbb{R}^k \Bigg\}.$$

We call an element $\bar{x} \in \mathcal{A}$ $\mathcal{S}_{ml}$*-weakly properly efficient* with respect to (P_v) when there are some $w \in \operatorname{int}(\mathbb{R}^k)$ and $a \in \mathbb{R}^k$ such that $\max\limits_{j=1,\ldots,k} \left\{ w_j \left(F_j(\bar{x}) - a_j \right) \right\} + \eta \sum_{j=1}^k w_j F_j(\bar{x}) \leq \max\limits_{j=1,\ldots,k} \left\{ w_j(F_j(x) - a_j) \right\} + \eta \sum_{j=1}^k w_j F_j(x)$ $\forall x \in \mathcal{A}$.

Let $w = (w_1, \ldots, w_k)^T \in \operatorname{int}(\mathbb{R}^k_+)$ and $a = (a_1, \ldots, a_k)^T \in \mathbb{R}^k$. Regarding the conjugate of the function $s_{w,a} \in \mathcal{S}_{ml}$, we have, for $\beta = (\beta_1, \ldots, \beta_k)^T \in \mathbb{R}^k$,

$$\begin{aligned} s^*_{w,a}(\beta) &= \sup_{y\in\mathbb{R}^k} \left\{ \beta^T y - \max_{j=1,\ldots,k} \left\{ w_j(y_j - a_j) \right\} - \eta \sum_{j=1}^{k} w_j y_j \right\} \\ &= \sup_{y\in\mathbb{R}^k} \left\{ (\beta - \eta w)^T y - \max_{j=1,\ldots,k} \left\{ w_j(y_j - a_j) \right\} \right\}. \end{aligned}$$

Denoting $u = y - a$ and using the formula of the conjugate of the weighted maximum, the conjugate above becomes

$$\begin{aligned} s^*_{w,a}(\beta) &= \sup_{u\in\mathbb{R}^k}\left\{(\beta-\eta w)^T(u+a) - \max_{j=1,\dots,k}\{w_j u_j\}\right\} \\ &= (\beta-\eta w)^T a + \begin{cases} 0, & \text{if } \eta w \leqq \beta \text{ and } \sum\limits_{j=1}^k \frac{\beta_j}{w_j} = k\eta+1, \\ +\infty, & \text{otherwise.} \end{cases} \end{aligned}$$

Let us write now the dual problem to (P_v) when the scalarization function s belongs to $\mathcal{S}_{ml}$. The variable $s \in \mathcal{S}_{ml}$ can be identified with a pair $w = (w_1,\dots,w_k)^T \in \operatorname{int}(\mathbb{R}^k_+)$ and $a = (a_1,\dots,a_k)^T \in \mathbb{R}^k$. The dual problem obtained in this case to (P_v) is

$$(D_{ml}) \qquad \underset{(z,w,a,\alpha,\beta,u)\in\mathcal{B}_{ml}}{\text{v-max}} z,$$

where

$$\begin{aligned} \mathcal{B}_{ml} = \Bigg\{ &(z,w,a,\alpha,\beta,u) \in \mathbb{R}^k \times \operatorname{int}(\mathbb{R}^k_+) \times \mathbb{R}^k \times C^* \times \mathbb{R}^k_+ \times \mathbb{R}^n : \eta w \leqq \beta, \\ &z = (z_1,\dots,z_k)^T, \sum_{j=1}^k \frac{\beta_j}{w_j} = k\eta+1,\ \max_{j=1,\dots,k}\{w_j(z_j-a_j)\} \\ &+\eta\sum_{j=1}^k w_j z_j \leq (\beta-\eta w)^T a - (\beta^T F)^*_X(u) - (\alpha^T g)^*_X(-u) \Bigg\}. \end{aligned}$$

Theorem 15. (weak duality) *There is no $x \in \mathcal{A}$ and no $(z,w,a,\alpha,\beta,u) \in \mathcal{B}_{ml}$ such that $z - F(x) \in \operatorname{int}(\mathbb{R}^k_+)$.*

Theorem 16. (strong duality) *Assume (CQ_v) fulfilled and let $\bar{x} \in \mathcal{A}$ be an $\mathcal{S}_{ml}$-weakly properly efficient to (P_v). Then the dual problem (D_{ml}) has a weakly efficient solution $(\bar{z},\bar{w},\bar{a},\bar{\alpha},\bar{\beta},\bar{u})$ such that $F(\bar{x}) = \bar{z}$.*

4.2.1 Maximum scalarization

When $\eta = 0$ the maximum-linear scalarization becomes the weighted Tchebyshev scalarization. If the scalarization function is actually the maximum function, i.e. $a_j = 0$ and $w_j = 1$ for all $j = 1,\dots,k$, the scalarized problem attached to (P_v) is

$$(P_{\max}) \qquad \inf_{x\in\mathcal{A}} \max_{j=1,\dots,k} F_j(x).$$

One can easily notice that $(P_{\max})$ is actually a min-max convex optimization problem. The maximum scalarization is a special case of the general framework we presented as the objective function in $(P_{\max})$ is $\mathbb{R}^k_+$-strictly increasing and convex. The set $\mathcal{S}$ is in this case

$$\mathcal{S}_m = \Big\{ s : \mathbb{R}^k \to \mathbb{R},\ s(y) = \max_{j=1,\ldots,k} y_j,\ y = (y_1, \ldots, y_k)^T \in \mathbb{R}^k \Big\}.$$

We call an element $\bar{x} \in \mathcal{A}$ $\mathcal{S}_m$*-weakly properly efficient* with respect to (P_v) when $\max\limits_{j=1,\ldots,k} F_j(\bar{x}) \leq \max\limits_{j=1,\ldots,k} F_j(x)\ \forall x \in \mathcal{A}$.

Let us write now the dual problem to (P_v) generated by the scalarization function $s \in \mathcal{S}_m$. It comes directly from (D_{ml}) for $\eta = 0$, by removing the variables a and w which are constant, namely $a_j = 0$ and $w_j = 1$ for all $j = 1, \ldots, k$, being

$$(D_m) \qquad \operatorname*{v-max}_{(z,\alpha,\beta,u)\in\mathcal{B}_m} z,$$

where

$$\begin{aligned}\mathcal{B}_m \ = \ & \Big\{ (z, \alpha, \beta, u) \in \mathbb{R}^k \times C^* \times \mathbb{R}^k_+ \times \mathbb{R}^n : z = (z_1, \ldots, z_k)^T, \\ & \sum_{j=1}^k \beta_j = 1,\ \max_{j=1,\ldots,k} \{z_j\} \leq -(\beta^T F)^*_X(u) - (\alpha^T g)^*_X(-u) \Big\}.\end{aligned}$$

Theorem 17. (weak duality) *There is no $x \in \mathcal{A}$ and no $(z, \alpha, \beta, u) \in \mathcal{B}_m$ such that $z - F(x) \in \operatorname{int}(\mathbb{R}^k_+)$.*

Theorem 18. (strong duality) *Assume (CQ_c) fulfilled and let $\bar{x} \in \mathcal{A}$ be an $\mathcal{S}_m$-weakly properly efficient solution to (P_v). Then the dual problem (D_m) has a weakly efficient solution $(\bar{z}, \bar{\alpha}, \bar{\beta}, \bar{u})$ such that $F(\bar{x}) = \bar{z}$.*

4.3 Set scalarization

Some quite recent scalarization methods are based on already given or constructed sets which have to satisfy some conditions. We gather here some of them under the name of set scalarization since the scalarization functions are defined with the help of some sets and they are K-strictly increasing when these sets fulfill some inclusions. The most general among the scalarizations we treat in this subsection is connected to the one due to Gerth and Weidner (cf. [13]), used also by Tammer and Göpfert (cf. [31]), Tammer and Winkler (cf. [32]) and Weidner (cf. [38]), for instance.

Take the convex cone K such that $\operatorname{int}(K) \neq \emptyset$. Let the non-empty convex set $E \subseteq \mathbb{R}^k$ fulfilling $\operatorname{cl}(E) + \operatorname{int}(K) \subseteq \operatorname{int}(E)$. Let moreover $D = \mathbb{R}^k$. The scalarization functions are

$$s_\mu : \mathbb{R}^k \to \mathbb{R},\ s_\mu(y) = \inf \big\{ t \in \mathbb{R} : y \in t\mu - \operatorname{cl}(E) \big\},\ \mu \in \operatorname{int}(K)$$

and we have in this case $\mathcal{S} = \mathcal{S}_s = \{s_\mu : \mu \in \text{int}(K)\}$. According to [13, 31] the functions s_μ are convex and K-strictly increasing. Because of this fact we work within this subsection with $\mathcal{S}_s$-weakly properly efficient and, respectively, weakly efficient solutions, obtaining the duality statements from Theorems 6-8. The family of scalarized primal problems is

$$(P_\mu) \qquad \inf_{x\in\mathcal{A}} \inf\big\{t \in \mathbb{R} : F(x) \in t\mu - \text{cl}(E)\big\},\ \mu \in \text{int}(K).$$

An element $\bar{x} \in \mathcal{A}$ is called *$\mathcal{S}_s$-weakly properly efficient* with respect to (P_v) when there is some $\mu \in \text{int}(K)$ such that $s_\mu(F(\bar{x})) \leq s_\mu(F(x))\ \forall x \in \mathcal{A}$.

In order to formulate the multiobjective dual problem to (P_v) that arises in this case we need the conjugate function of s_μ, when a $\mu \in \text{int}(K)$ is fixed. It is $s_\mu^* : \mathbb{R}^k \to \overline{\mathbb{R}}$,

$$\begin{aligned}
s_\mu^*(\beta) &= \sup_{y\in\mathbb{R}^k}\left\{\beta^T y - \inf_{\substack{t\in\mathbb{R},\\ y\in t\mu-\text{cl}(E)}} t\right\} = \sup_{y\in\mathbb{R}^k}\left\{\beta^T y + \sup_{\substack{t\in\mathbb{R},\\ y\in t\mu-\text{cl}(E)}} -t\right\}\\
&= \sup_{\substack{y\in t\mu-\text{cl}(E),\\ t\in\mathbb{R}}}\big\{\beta^T y - t\big\} = \sup_{t\in\mathbb{R}}\left\{-t + \sup_{u=y-t\mu\in-\text{cl}(E)} \beta^T(u+t\mu)\right\}\\
&= \sup_{t\in\mathbb{R}}\left\{t\beta^T\mu - t + \sup_{u\in-\text{cl}(E)} \beta^T u\right\} = \sup_{t\in\mathbb{R}}\left\{t(\beta^T\mu - 1) + \sigma_{-\text{cl}(E)}(\beta)\right\}\\
&= \begin{cases} \sigma_{-\text{cl}(E)}(\beta), & \text{if } \beta^T\mu = 1,\\ +\infty, & \text{otherwise.}\end{cases}
\end{aligned}$$

Now we are able to formulate the multiobjective dual problem attached to (P_v) via the set scalarization. It is

$$(D_s) \qquad \operatorname*{v-max}_{(z,\mu,\alpha,\beta,u)\in\mathcal{B}_s} z,$$

where

$$\begin{aligned}
\mathcal{B}_s = \Big\{&(z,\mu,\alpha,\beta,u) \in \mathbb{R}^k \times \text{int}(K) \times C^* \times K^* \times \mathbb{R}^n : \beta^T\mu = 1,\\
&s_\mu(z) \leq -\sigma_{-\text{cl}(E)}(\beta) - \big(\beta^T F\big)_X^*(u) - \big(\alpha^T g\big)_X^*(-u)\Big\}.
\end{aligned}$$

Theorem 19. (weak duality) *There is no $x \in \mathcal{A}$ and no $(z,\mu,\alpha,\beta,u) \in \mathcal{B}_s$ such that $z - F(x) \in \text{int}(K)$.*

Theorem 20. (strong duality) *Assume (CQ_v) fulfilled and let $\bar{x} \in \mathcal{A}$ be an $\mathcal{S}_s$-weakly properly efficient solution to (P_v). Then the dual problem (D_s) has a weakly efficient solution $(\bar{z}, \bar{\mu}, \bar{\alpha}, \bar{\beta}, \bar{u})$ such that $F(\bar{x}) = \bar{z}$.*

In the remaining part of this subsection we treat some special cases of this scalarization which arise for particular choices of the set E. In this framework could be brought the scalarization in [39] which involves polyhedral sets, too. The reader is referred to [38] for a deeper analysis of the way some older scalarization functions are embedded into the set scalarization.

4.3.1 Set scalarization with conical sets

Keeping the notations above, take $E = K$. The condition $\mathrm{cl}(E)+\mathrm{int}(K) \subseteq \mathrm{int}(E)$ is automatically satisfied since K is a convex cone. For each $\nu \in \mathrm{int}(K)$ the scalarization function is in this case

$$s_\nu : \mathbb{R}^k \to \mathbb{R},\ s_\nu(y) = \inf\big\{t \in \mathbb{R} : y \in t\nu - \mathrm{cl}(K)\big\}.$$

We have in this case $\mathcal{S} = \mathcal{S}_{sc} = \{s_\nu : \nu \in \mathrm{int}(K)\}$. Among the authors who have used this kind of scalarization function in the literature we cite here Kaliszewski (cf. [19]), Rubinov and Gasimov (cf. [27]) and Tammer (cf. [30]), where it is mentioned that it is convex and K-strictly increasing. For each $\nu \in \mathrm{int}(K)$, the scalarized primal problem is

$$(P_\nu) \qquad \inf_{x\in\mathcal{A}} \inf\big\{t \in \mathbb{R} : F(x) \in t\nu - \mathrm{cl}(K)\big\}.$$

Using this scalarization an element $\bar{x} \in \mathcal{A}$ is called $\mathcal{S}_{sc}$-*weakly properly efficient* with respect to (P_v) when there is a $\nu \in \mathrm{int}(K)$ such that $s_\nu(F(\bar{x})) \leq s_\nu(F(x))$ $\forall x \in \mathcal{A}$.

Taking $\nu \in \mathrm{int}(K)$, from the earlier calculations we known that

$$s_\nu^*(\beta) = \begin{cases} \sigma_{-\,\mathrm{cl}(K)}(\beta), & \text{if } \beta^T\nu = 1, \\ +\infty, & \text{otherwise.} \end{cases}$$

From [16] we know that $\sigma_{-\,\mathrm{cl}(K)} = \delta_{K^*}$, so the multiobjective dual problem attached to (P_v) via the scalarization using conical sets is

$$(D_{sc}) \qquad \operatorname*{v-max}_{(z,\nu,\alpha,\beta,u)\in\mathcal{B}_{sc}} z,$$

where

$$\begin{aligned} \mathcal{B}_{sc} \;=\; & \Big\{(z,\nu,\alpha,\beta,u) \in \mathbb{R}^k \times \mathrm{int}(K) \times C^* \times K^* \times \mathbb{R}^n : \beta^T\nu = 1, \\ & s_\nu(z) \leq -\big(\beta^T F\big)_X^*(u) - \big(\alpha^T g\big)_X^*(-u)\Big\}. \end{aligned}$$

Theorem 21. (weak duality) *There is no $x \in \mathcal{A}$ and no $(z,\nu,\alpha,\beta,u) \in \mathcal{B}_{sc}$ such that $z - F(x) \in \mathrm{int}(K)$.*

Theorem 22. (strong duality) *Assume* (CQ_v) *fulfilled and let* $\bar{x} \in \mathcal{A}$ *be an* $\mathcal{S}_{sc}$*-weakly properly efficient solution to* (P_v)*. Then the dual problem* (D_{sc}) *has a weakly efficient solution* $(\bar{z}, \bar{\nu}, \bar{\alpha}, \bar{\beta}, \bar{u})$ *such that* $F(\bar{x}) = \bar{z}$*.*

4.3.2 Set scalarization with sets generated by norms

The scalarization we deal with in the following is by construction a special case of the general set scalarization treated earlier, but on the other hand it is more general since the family of scalarization functions depends on three variables, not on a single one like there. In the following we attach to (P_v) a scalarized problem obtained with the scalarization function used by Tammer and Winkler in [32] and by the latter also in [41]. In order to proceed we need to introduce some special classes of norms, about which more is available in [29] and some references therein. Take the cone K with non-empty interior.

Definition 8. A subset $A \subseteq \mathbb{R}^k$ is called *polyhedral* if it can be expressed as the intersection of a finite collection of closed half-spaces.

Definition 9. A norm $\gamma : \mathbb{R}^k \to \mathbb{R}$ is called *block norm* if its unit ball B_γ is polyhedral.

Definition 10. A norm $\gamma : \mathbb{R}^k \to \mathbb{R}$ is called *absolute* if $\forall \bar{y} \in \mathbb{R}^k$ one has $\gamma(y) = \gamma(\bar{y})$ for all $y \in \big\{z = (z_1, \ldots, z_k)^T \in \mathbb{R}^k : |z_j| = |\bar{y}_j| \ \forall j = 1, \ldots, k\big\}$.

Definition 11. A block norm $\gamma : \mathbb{R}^k \to \mathbb{R}$ is called *oblique* if it is absolute and satisfies $\big(y - \mathbb{R}^k_+\big) \cap \mathbb{R}^k_+ \cap \operatorname{bd}(B_\gamma) = \{y\}$ for all $y \in \mathbb{R}^k_+ \cap \operatorname{bd}(B_\gamma)$.

Example 1. The Euclidean norm $\|\cdot\|_2$ in $\mathbb{R}^k$ is absolute, but not block, thus not oblique. We refer to [29, 32] for more on such norms.

According to [29] and [32] (see Definition 9), for a block norm γ there are some $r \in \mathbb{N}$, $a_i \in \mathbb{R}^k$ and $\eta_i \in \mathbb{R}$, $i = 1, \ldots, r$, such that the unit ball generated by γ is

$$B_\gamma = \Big\{y \in \mathbb{R}^k : a_i^T y \leq \eta_i, i = 1, \ldots, r\Big\}.$$

We need also the following sets

$$I_\gamma = \Big\{i \in \{1, \ldots, r\} : \Big\{y \in \mathbb{R}^k : a_i^T y = \eta_i\Big\} \cap B_\gamma \cap \operatorname{int}(\mathbb{R}^k_+) \neq \emptyset\Big\}$$

and

$$E_\gamma = \Big\{y \in \mathbb{R}^k : a_i^T y \leq \eta_i \ \forall i \in I_\gamma\Big\}.$$

Theorem 23. (cf. [32]) *The function* $\zeta_{\gamma,l,v} : \mathbb{R}^k \to \mathbb{R}$*, defined by*

$$\zeta_{\gamma,l,v}(y) = \inf\big\{t \in \mathbb{R} : y \in tl + E_\gamma + v\big\},$$

where γ is an absolute norm on $\mathbb{R}^k$, $l \in \mathrm{int}(\mathbb{R}^k_+)$ and $v \in \mathbb{R}^k$, is convex and K-strictly increasing when $\mathrm{bd}(E_\gamma) - (K\backslash\{0\}) \subseteq \mathrm{int}(E_\gamma)$.

Remark 7. If γ is an absolute norm on $\mathbb{R}^k$, $l \in \mathrm{int}(\mathbb{R}^k_+)$, $v \in \mathbb{R}^k$ and $\mathrm{bd}(E_\gamma) - (K\backslash\{0\}) \subseteq \mathrm{int}(E_\gamma)$, the function $\zeta_{\gamma,l,v}$ defined above is $\hat{K}$-strongly increasing.

Corollary 1. (cf. [32]) *When γ is an absolute block norm, $\zeta_{\gamma,l,v}$ is $\mathbb{R}^k_+$-strictly increasing for any $l \in \mathrm{int}(\mathbb{R}^k_+)$ and $v \in \mathbb{R}^k$.*

Corollary 2. (cf. [32]) *When γ is an oblique norm, $\zeta_{\gamma,l,v}$ is $\mathbb{R}^k_+$-strongly increasing for any $l \in \mathrm{int}(\mathbb{R}^k_+)$ and $v \in \mathbb{R}^k$.*

Denote by $\mathcal{O}$ the set of the absolute norms $\gamma : \mathbb{R}^k \to \mathbb{R}$ for which $\mathrm{bd}(E_\gamma) - \mathrm{int}(K) \subseteq \mathrm{int}(E_\gamma)$ and consider the following set

$$\begin{aligned}\mathcal{S}_{sn} \quad = \quad & \Big\{\zeta_{\gamma,l,v} : \mathbb{R}^k \to \mathbb{R} : \ \gamma \in \mathcal{O}, l \in \mathrm{int}(\mathbb{R}^k_+), v \in \mathbb{R}^k, \\ & \quad \zeta_{\gamma,l,v}(y) = \inf\big\{t \in \mathbb{R} : y \in tl + E_\gamma + v\big\} \ \forall y \in \mathbb{R}^k\Big\}.\end{aligned}$$

The family of scalarized problems attached to (P_v) in this case is

$$(P_{\gamma,l,v}) \qquad\qquad \inf_{x\in\mathcal{A}} \zeta_{\gamma,l,v}(F(x)),$$

where $(\gamma, l, v) \in \mathcal{O} \times \mathrm{int}(\mathbb{R}^k_+) \times \mathbb{R}^k$. According to the definitions above and Theorem 23 this fits into our framework, too, by taking $\mathcal{S} = \mathcal{S}_{sn}$. In this case an element $\bar{x} \in \mathcal{A}$ is called $\mathcal{S}_{sn}$*-weakly properly efficient* with respect to (P_v) when there is an absolute norm $\gamma \in \mathcal{O}$, some $l \in \mathrm{int}(\mathbb{R}^k_+)$ and a $v \in \mathbb{R}^k$ such that $\zeta_{\gamma,l,v}(F(\bar{x})) \le \zeta_{\gamma,l,v}(F(x)) \ \forall x \in \mathcal{A}$.

Remark 8. Restricting moreover the set $\mathcal{S}_{sn}$ to contain only the functions that satisfy the hypotheses in the corollaries above we get other scalarizations which could be treated separately, too.

To obtain the dual problem to (P_v) that arises by using the scalarization just presented, let us calculate the conjugate of the scalarization functions $\zeta_{\gamma,l,v}$, for some fixed $(\gamma, l, v) \in \mathcal{O} \times \mathrm{int}(\mathbb{R}^k_+) \times \mathbb{R}^k$. We have

$$\begin{aligned}\zeta^*_{\gamma,l,v}(\beta) \quad &= \quad \sup_{y\in\mathbb{R}^k}\Big\{\beta^T y - \inf\big[t \in \mathbb{R} : y \in tl + E_\gamma + v\big]\Big\} \\ &= \quad \sup_{y\in\mathbb{R}^k}\Big\{\beta^T y + \sup\big\{-t \in \mathbb{R} : y \in tl + E_\gamma + v\big\}\Big\}.\end{aligned}$$

Denoting $w = y - tl - v$, one gets

$$\begin{aligned}
\zeta^*_{\gamma,l,v}(\beta) &= \sup_{t\in\mathbb{R}} \left\{ -t + \sup_{w\in E_\gamma} \left\{ \beta^T(w + tl + v) \right\} \right\} \\
&= \sup_{t\in\mathbb{R}} \left\{ -t + t\beta^T l + \sup_{w\in E_\gamma} \beta^T w \right\} + \beta^T v \\
&= \sup_{t\in\mathbb{R}} \left\{ t\Big(\beta^T l - 1\Big) \right\} + \sigma_{E_\gamma}(\beta) + \beta^T v \\
&= \begin{cases} \sigma_{E_\gamma}(\beta) + \beta^T v, & \text{if } \beta^T l = 1, \\ +\infty, & \text{otherwise.} \end{cases}
\end{aligned}$$

The dual problem to (P_v) obtained in this case is

$$(D_{sn}) \qquad \underset{(z,\gamma,l,v,\alpha,\beta,u)\in\mathcal{B}_{sn}}{\text{v-max}} z,$$

where

$$\begin{aligned}
\mathcal{B}_{sn} = \Big\{ &(z,\gamma,l,v,\alpha,\beta,u) \in \mathbb{R}^k \times \mathcal{O} \times \operatorname{int}(\mathbb{R}^k_+) \times \mathbb{R}^k \times C^* \times K^* \times \mathbb{R}^n : \\
&\beta^T l = 1, \zeta_{\gamma,l,v}(z) \leq -\sigma_{E_\gamma}(\beta) - \beta^T v - (\beta^T F)^*_X(u) - (\alpha^T g)^*_X(-u) \Big\}.
\end{aligned}$$

Theorem 24. (weak duality) *There is no $x \in \mathcal{A}$ and no $(z,\gamma,l,v,\alpha,\beta,u) \in \mathcal{B}_{sn}$ such that $z - F(x) \in \operatorname{int}(K)$.*

Theorem 25. (strong duality) *Assume (CQ_v) fulfilled and let $\bar{x} \in \mathcal{A}$ be an $\mathcal{S}_{sn}$-weakly properly efficient solution to (P_v). Then the dual problem (D_{sn}) has a weakly efficient solution $(\bar{z}, \overline{\gamma}, \bar{l}, \bar{v}, \bar{\alpha}, \bar{\beta}, \bar{u})$ such that $F(\bar{x}) = \bar{z}$.*

The case when γ is an oblique norm, not treated here separately because of the limited space, could bring some interesting results since, according to Corollary 2, the functions $\zeta_{\gamma,l,v}$, $(\gamma,l,v) \in \mathcal{O} \times \operatorname{int}(\mathbb{R}^k_+) \times \mathbb{R}^k$, are $\mathbb{R}^k_+$-strongly increasing. Given these, when γ is an oblique norm one can give the strong duality statement not for $\mathcal{S}_{sn}$-weakly properly efficient and weakly efficient solutions as done within this subsection, but for properly efficient and efficient ones like in Section 3.

4.4 (Semi)Norm scalarization

Now K is again a convex cone such that $K \cap (-K) = \{0\}$. In some circumstances the (semi)norms turn out to be K-strongly increasing functions and this fact could not remain unnoticed by many authors working in the vast field of multiobjective programming. We cite here only a few of them, namely Jahn (cf. [18]), Khánh (cf. [20]), Schandl, Klamroth and Wiecek (cf. [29]) and Wierzbicki (cf. [40]). Some of the references mentioned in the works cited above contain also other types of scalarizations involving norms and seminorms (see moreover [6, 17, 24]). In the following we use scalarization functions based on K-strongly increasing gauges, which are seminorms. Such scalarizations are successfully used in location problems (see [37]) and goal programming (cf. [6]). Then we consider the case when the scalarization functions are based on a norm, where we mention also two special cases. First we have to introduce two notions.

Definition 12. (cf. [16]) Let $E \subseteq \mathbb{R}^k$ a closed convex set containing the origin. The function

$$\gamma_E : \mathbb{R}^k \to \mathbb{R},\ \gamma_E(y) = \inf\{t > 0 : y \in tE\}$$

is called the *gauge (Minkowski functional)* of E. When there is no $t > 0$ such that $y \in tE$ one sets $\gamma_E(y) = +\infty$. The set E is the *unit ball* associated to γ_E.

Let us assume that there exists some $b \in \mathbb{R}^k$ such that $F(X) \subseteq b + K$. Take $D = b + K$, thus the feasibility condition is fulfilled, and consider the closed convex set $E \subseteq \mathbb{R}^k$ such that $0 \in \operatorname{int}(E)$ and its gauge γ_E is K-strongly increasing on K. One can notice then that $\gamma_E(y) \in \mathbb{R}$ for all $y \in \mathbb{R}^k$.

Remark 9. Assuming that $E \subseteq \mathbb{R}^k$ is the Euclidean unit ball, the assumption on γ_E is fulfilled if and only if $K \subseteq K^*$ (see [18]). This is the case if, for instance, K is the non-negative orthant in $\mathbb{R}^k$.

The scalarization functions we use here are

$$s_a : (b + K) \to \mathbb{R},\ s_a(y) = \gamma_E(y - a) = \inf\{t > 0 : y \in a + tE\},\ a \in b - K.$$

Let us remark that whenever $a \in b - K$ one gets $F(X) \subseteq a + K$. It is straightforward to see that s_a is K-strongly increasing on $D = b + K$ for every $a \in b - K$. We have in this case $\mathcal{S} = \mathcal{S}_g = \{s_a : a \in b - K\}$ and the family of scalarized primal problems is

$$(P_a) \qquad \inf_{x \in \mathcal{A}} \inf \big\{t > 0 : F(x) \in a + tE\big\},\ a \in b - K,$$

i.e.

$$(P_a) \qquad \inf_{x \in \mathcal{A}} \gamma_E(F(x) - a),\ a \in b - K.$$

For this scalarizations $\bar{x} \in \mathcal{A}$ is called $\mathcal{S}_g$*-properly efficient* with respect to (P_v) when there is some $a \in b - K$ such that $s_a(F(\bar{x})) \leq s_a(F(x))\ \forall x \in \mathcal{A}$.

In order to formulate the multiobjective dual problem to (P_v) that arises in this case we need the conjugate functions regarding $b + K$ of s_a, $a \in b - K$. Let

$a \in b - K$ and $\beta \in \mathbb{R}^k$. We have $(s_a)^*_{b+K} : \mathbb{R}^k \to \overline{\mathbb{R}}$,

$$(s_a)^*_{b+K}(\beta) = (\gamma_E(\cdot - a) + \delta_{b+K})^*(\beta) = \min_{\varsigma \in \mathbb{R}^k} \big[(\gamma_E(\cdot - a))^*(\beta - \varsigma) + (\delta_{b+K})^*(\varsigma)\big],$$

where we have applied Theorem 16.4 in [26]. Further,

$$(\gamma_E(\cdot - a))^*(\beta - \varsigma) = \sup_{y \in \mathbb{R}^k} \big\{(\beta - \varsigma)^T y - \gamma_E(y - a)\big\}.$$

Denoting $u = y - a$ we get

$$(\gamma_E(\cdot - a))^*(\beta - \varsigma) = \sup_{u \in \mathbb{R}^k} \big\{(\beta - \varsigma)^T (u + a) - \gamma_E(u)\big\} = (\beta - \varsigma)^T a + (\gamma_E)^*(\beta - \varsigma).$$

For the conjugate of the gauge one gets at some $\tau \in \mathbb{R}^k$

$$\begin{aligned}
(\gamma_E)^*(\tau) &= \sup_{y \in \mathbb{R}^k} \big\{\tau^T y - \inf\{t > 0 : y \in tE\}\big\} = \sup_{y \in \mathbb{R}^k} \left\{ \tau^T y + \sup_{\substack{t>0,\\ y \in tE}} -t \right\} \\
&= \sup_{t>0} \left\{ -t + \sup_{y \in tE} \tau^T y \right\} = \sup_{t>0} \left\{ -t + \sup_{\substack{w = \frac{1}{t} y,\\ w \in E}} \tau^T (tw) \right\} \\
&= \sup_{t>0} \left\{ t \left(\sup_{w \in E} \tau^T w - 1 \right) \right\} = \begin{cases} 0, & \text{if } \sigma_E(\tau) \leq 1, \\ +\infty, & \text{otherwise.} \end{cases}
\end{aligned}$$

It is also known that $(\delta_{b+K})^*(\varsigma) = \varsigma^T b + (\delta_K)^*(\varsigma)\ \forall \varsigma \in \mathbb{R}^k$, consequently $(\delta_{b+K})^*(\varsigma) = \varsigma^T b$ if $\varsigma \in -K^*$ and $(\delta_{b+K})^*(\varsigma) = +\infty$ otherwise. From all these partial results we get that the conjugates of our scalarization functions are

$$(s_a)^*_{b+K}(\beta) = \min_{\substack{\varsigma \in -K^*,\\ \sigma_E(\beta - \varsigma) \leq 1}} \big[(\beta - \varsigma)^T a + \varsigma^T b\big] = \beta^T a + \min_{\substack{\varsigma \in -K^*,\\ \sigma_E(\beta - \varsigma) \leq 1}} \varsigma^T (b - a),\ a \in b - K.$$

Now we are able to formulate the multiobjective dual problem attached to (P_v) via the gauge scalarization. It is

$$(D_g) \qquad \operatorname*{v-max}_{(z, a, \alpha, \beta, \varsigma, u) \in \mathcal{B}_g} z,$$

where

$$\mathcal{B}_g = \Big\{(z, a, \alpha, \beta, \varsigma, u) \in (b+K) \times (b-K) \times C^* \times K^* \times (-K^*) \times \mathbb{R}^n : \sigma_E(\beta - \varsigma) \leq 1,$$

$$\gamma_E(z - a) \leq \varsigma^T (a - b) - \beta^T a - \big(\beta^T F\big)^*_X(u) - \big(\alpha^T g\big)^*_X(-u)\Big\}.$$

Remark 10. We emphasize that σ_E defines the so-called dual gauge to the gauge γ_E and if γ_E is a norm it turns out to be indeed the dual norm.

Theorem 26. (weak duality) *There is no $x \in \mathcal{A}$ and no $(z, a, \alpha, \beta, \varsigma, u) \in \mathcal{B}_g$ such that $z - F(x) \in K$ and $F(x) \neq z$.*

Theorem 27. (strong duality) *Assume (CQ_c) fulfilled and let $\bar{x} \in \mathcal{A}$ be a $\mathcal{S}_g$-properly efficient solution to (P_v). Then the dual problem (D_g) has an efficient solution $(\bar{z}, \bar{a}, \bar{\alpha}, \bar{\beta}, \bar{\varsigma}, \bar{u})$ such that $F(\bar{x}) = \bar{z}$.*

4.4.1 Norm scalarization

When the scalarization function is based on a norm we can use the previous results since it is known that a gauge γ_E satisfying $\gamma_E(y) = \gamma_E(-y)\ \forall y \in \mathbb{R}^k$ is a norm with the unit ball E. When E is polyhedral γ_E is a block norm. We cite [17, 18, 29, 32] and the references therein for more on the way norms are described as gauges of their unit balls and applications of the norm scalarization in various fields. For instance, the scalarization with the l^1 norm is used in goal programming (cf. [18]). A family of scalarization functions similar to the one used in [40] (see also [19]) is

$$s_a : D \to \mathbb{R},\ s_a(y) = \|y - a\|,$$

where the non-empty convex set $D \subseteq \mathbb{R}^k$ and $a \in \mathbb{R}^k$ are conveniently chosen, K is a convex cone fulfilling $K \cap (-K) = \{0\}$ and $\|\cdot\|$ is a norm which is K-strongly increasing on D. Let us notice that conditions under which a norm is K-strongly increasing on D are given in [18, 40].

Remark 10. Along the Euclidean norm which is $\mathbb{R}^k_+$-strongly increasing on $\mathbb{R}^k_+$, the oblique norms are $\mathbb{R}^k_+$-strongly increasing on the non-negative orthant, too. One can provide duality statements similar to the ones given in the general case by using some scalarization functions based on such norms.

4.5 Quadratic scalarization

Some authors have noticed that in some circumstances also the quadratic functions are strongly increasing on certain sets. More precisely let Q be a symmetric positive semidefinite $k \times k$ matrix, K a non-empty closed convex cone in $\mathbb{R}^k$ and $D \subseteq \mathbb{R}^k$ a relatively open set, i.e. $D = \mathrm{ri}(D)$. Denote by L the subspace parallel to $\mathrm{aff}(D)$. If $\mathrm{int}(K^* + L^\perp) \neq \emptyset$ and $QD \subseteq K^* + L^\perp$, where $L^\perp$ is the orthogonal subspace to L, then (cf. [8]) the function

$$s_q : D \to \mathbb{R},\ s_q(y) = y^T Q y$$

is K-strongly increasing on D. We have $\mathcal{S} = \mathcal{S}_q = \{s_q\}$. The scalarized primal problem is

$$(P_{quad}) \qquad \inf_{x\in\mathcal{A}} \left[F(x)^T QF(x)\right].$$

An element $\bar{x} \in \mathcal{A}$ is called *$\mathcal{S}_q$-properly efficient* with respect to (P_v) when $F(\bar{x})^T QF(\bar{x}) \le F(x)^T QF(x)\ \forall x \in \mathcal{A}$.

In order to formulate the multiobjective dual problem to (P_v) that arises in this case we need the conjugate function of s_q regarding D. As in the literature this conjugate is computed when D is a subspace, we assume further this, too. According to [16] the conjugate of the scalarization function is $(s_q)^*_D : \mathbb{R}^k \to \overline{\mathbb{R}}$,

$$(s_q)^*_D(\beta) = \begin{cases} \frac{1}{4}\beta^T(P_D \circ Q \circ P_D)^\dagger\beta, & \text{if } \beta \in \mathrm{Im}(Q) + D^\perp,\\ +\infty, & \text{otherwise,}\end{cases}$$

where $\mathrm{Im}(Q)$ is the image of Q seen as a symmetric positive semidefinite operator on $\mathbb{R}^k$, P_D is the operator of orthogonal projection onto D and $Q^\dagger$ is the Moore-Penrose pseudo inverse of Q (cf. [16]).

Now we are able to formulate the multiobjective dual problem attached to (P_v) via the quadratic scalarization. As $\mathcal{S}_q$ contains only an element, namely s_q, the multiobjective dual problem has four variables. It is

$$(D_q) \qquad \underset{(z,\alpha,\beta,u)\in\mathcal{B}_q}{\text{v-max}}\ z,$$

where

$$\begin{aligned}\mathcal{B}_q \ = \ \Big\{&(z,\alpha,\beta,u) \in \mathbb{R}^k_+ \times C^* \times K^* \times \mathbb{R}^n : \beta \in \mathrm{Im}(Q) + D^\perp,\\ &z^TQz \le -\frac{1}{4}\beta^T(P_D \circ Q \circ P_D)^\dagger\beta - \left(\beta^T F\right)^*_X(u) - \left(\alpha^T g\right)^*_X(-u)\Big\}.\end{aligned}$$

Theorem 28. (weak duality) *There is no $x \in \mathcal{A}$ and no $(z,\alpha,\beta,u) \in \mathcal{B}_q$ such that $z - F(x) \in K$ and $F(x) \ne z$.*

Theorem 29. (strong duality) *Assume (CQ_c) fulfilled and let $\bar{x} \in \mathcal{A}$ be a $\mathcal{S}_q$-properly efficient solution to (P_v). Then the dual problem (D_q) has an efficient solution $(\bar{z},\bar{\alpha},\bar{\beta},\bar{u})$ such that $F(\bar{x}) = \bar{z}$.*

5 Conclusions

We introduce a general duality framework for convex multiobjective optimization problems based on conjugate duality. The multiobjective dual problem to a given convex vector minimization problem is constructed by using the scalarization with K-strongly increasing functions and the Fenchel-Lagrange duality for composed convex cone-constrained optimization problems (cf. [1]). When $\mathrm{int}(K) \ne \emptyset$ the duality statements are given also for the scalarization with K-strictly increasing functions. After presenting the general framework we show that some other scalarizations used in the literature on multiobjective optimization arise as particular cases and the general duality is specialized for each of them. This happens for the linear scalarization, maximum(-linear) scalarization, set scalarization, (semi)norm scalarization and quadratic scalarization.

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