A new constraint qualification for the formula of the subdifferential of composed convex functions in infinite dimensional spaces

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In this paper we work in separated locally convex spaces where we give equivalent statements for the formulae of the conjugate function of the sum of a convex lower-semicontinuous function and the precomposition of another convex lower-semicontinuous function which is also K-increasing with a K-convex K-epi-closed function, where K is a non-empty closed convex cone. These statements prove to be the weakest constraint qualifications given so far under which the formulae for the subdifferential of the mentioned sum of functions are valid. Then we deliver constraint qualifications inspired from them that guarantee some conjugate duality assertions. Two interesting special cases taken from the literature conclude the paper.

1 Introduction

Many convex optimization problems may be considered as special cases of the so-called composed convex optimization problem which consists in minimizing the sum of a convex function and the precomposition of another convex function which is also K-increasing with a K-convex function, where K is a non-empty closed convex cone. Because of this important property, the problem of minimizing the mentioned sum of functions has been studied quite intensively under various conditions. We cite here the works of Combari, Laghdir and Thibault ([5], [6], [7]) and the book of Zălinescu ([15]), where various prerequisites and conditions that ensure the formulae of the conjugate and of the subdifferential of the mentioned sum of functions are given. Older results regarding these matters due to Gol'shtein, Levin, Lemaire, Kutateladze, Lescarret and Rubinov are mentioned, some of them being generalized or extended within these papers.

Other papers deal with a special case of the problem presenting results concerning only the mentioned composition of functions, renouncing the first term of the sum of functions, among which let us mention Lemaire's [10] and our article [2], where we say more about previous works containing optimization problems in which such composed functions appeared. Different to [2], we work here in separated locally convex spaces, considering a generalized notion of lower-semicontinuity for functions having their ranges in infinite dimensional spaces. Precisely, we assume that these functions are K -epi-closed (cf. [11]). This property is guaranteed in case these functions are K -lower-semicontinuous, a notion introduced by Penot and Théra ([12]) and further used also in [1], [7] and [11]. On the other hand, there exist functions which are K -epi-closed, but not K -lower-semicontinuous (cf. [12]).

The main section of the paper follows after the necessary preliminaries. After some auxiliary results we introduce a first constraint qualification and prove the main statement of the paper, which says that the known formula of the conjugate of the mentioned sum is equivalent to the fulfillment of this constraint qualification, which implies moreover the formula of the subdifferential of the same sum of functions. Digging further, we give a second constraint qualification, equivalent to a refined formula of the conjugate of our sum of functions. This also guarantees a further formula for the mentioned subdifferential. The next section is dedicated to conjugate

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duality. We give here some weak constraint qualifications that guarantee the formulae of the conjugate at 0 of the sum of a convex function and the precomposition of another convex function which is also K-increasing with a K-convex function, which are equivalent to the so-called strong duality between the problem of minimizing the mentioned sum of functions and its conjugate dual problem.

Finally, we have a section where we treat some important special cases, the already mentioned one when the first term of the sum vanishes and the situation when the post-composed function is a linear and continuous mapping. In [3] (see also [4]) Boţ and Wanka deal with this latter problem giving also solid and quite complete references to the literature, so we will not mention them here once more. We specialize the theorems given in the previous sections for the announced choices of functions and we rediscover some results due to two of the authors from [3], including the weakest constraint qualification known to us that guarantees the classical Fenchel duality statement.

2 Preliminaries

Some notions and previously known results are necessary in order to make this paper self-contained. Consider two separated locally convex vector spaces X and Y and their continuous dual spaces X^* and Y^*, endowed with the weak* topologies $w(X^*, X)$ and, respectively, $w(Y^*, Y)$.

Take also the non-empty closed convex cone $K \subseteq Y$ and its dual cone $K^* = \{y^* \in Y^* : \langle y^*, y\rangle \geq 0 \ \forall y \in Y\}$, where we denote by $\langle y^*, y\rangle = y^*(y)$ the value at y of the continuous linear functional y^*. We say that $K \subseteq Y$ is a cone if $\lambda y \in K \ \forall \lambda \geq 0$ and $y \in K$. Consider on Y the partial order induced by K, "$\leq_K$", defined by $x \leq_K y \Leftrightarrow y - x \in K \ \forall x, y \in Y$, whereas $x \nleq_K y$ means that $x \leq_K y$ is not satisfied. Moreover, we attach to Y a greatest element with respect to "$\leq_K$" denoted ∞ which does not belong to Y and let $Y^\bullet = Y \cup \{\infty\}$. Then for any $y \in Y^\bullet$ one has $y \leq_K \infty$ and we consider the following operations on $Y^\bullet$, $y + \infty = \infty + y = \infty$ and $t\infty = \infty \ \forall t \geq 0$.

For a subset C of X we have the *indicator* function $\delta_C : X \to \overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$, defined by

$$\delta_C(x) = \begin{cases} 0, & \text{if } x \in C, \\ +\infty, & \text{otherwise,} \end{cases}$$

and we denote by $\operatorname{cl}(C)$ its *closure* in the corresponding topology, while its *core* is $\operatorname{core}(C) = \{c \in C : \forall x \in X \ \exists \varepsilon > 0 : c + [-\varepsilon, \varepsilon]x \subseteq C\}$. Consider also the *identity* function on X defined as follows, $\mathrm{id}_X : X \to X$, $\mathrm{id}_X(x) = x \ \forall x \in X$ and the notation $\mathbb{R}_+ = [0, +\infty)$.

Having a function $f : X \to \overline{\mathbb{R}}$ we denote its *domain* by $\operatorname{dom}(f) = \{x \in X : f(x) < +\infty\}$ and its *epigraph* by $\operatorname{epi}(f) = \{(x, r) \in X \times \mathbb{R} : f(x) \leq r\}$. For $x \in X$ such that $f(x) \in \mathbb{R}$ we define the *subdifferential* of f at x by $\partial f(x) = \{x^* \in X^* : f(y) - f(x) \geq \langle x^*, y - x\rangle\}$. We call f *proper* if $f(x) > -\infty \ \forall x \in X$ and $\operatorname{dom}(f) \neq \emptyset$. The *conjugate* of the function f is $f^* : X^* \to \overline{\mathbb{R}}$ introduced by

$$f^*(y) = \sup\{\langle y, x\rangle - f(x) : x \in X\}.$$

Between a function and its conjugate there is the relation known as *Young-Fenchel inequality*

$$f^*(y) + f(x) \geq \langle y, x\rangle \ \forall x \in X \ \forall y \in X^*.$$

Given two proper functions $f, g : X \to \overline{\mathbb{R}}$, we have the *infimal convolution* of f and g defined by

$$f \Box g : X \to \overline{\mathbb{R}}, \ \big(f \Box g\big)(a) = \inf\{f(x) + g(a - x) : x \in X\},$$

which is called *exact* at some $a \in X$ when there is an $x \in X$ such that $\big(f \Box g\big)(a) = f(x) + g(a - x)$. When $f : X \to U$ and $g : Y \to V$, for U and V arbitrary linear spaces, we define also the function $f \times g : X \times Y \to U \times V$ by $f \times g(x, y) = (f(x), g(y))$, $(x, y) \in X \times Y$. Given a linear continuous mapping $A : X \to Y$, we have its *adjoint* $A^* : Y^* \to X^*$ given by $\langle A^* y^*, x\rangle = \langle y^*, Ax\rangle$ for any $(x, y^*) \in X \times Y^*$. For the proper function $f : X \to \overline{\mathbb{R}}$ we define also the *infimal function of f through A* as $Af : Y \to \overline{\mathbb{R}}$,

$Af(y) = \inf\big\{f(x) : x \in X, Ax = y\big\}, y \in Y.$

Some of the notions that exist for functions with extended real values may be given for functions having their ranges in infinite dimensional spaces, too. We call *domain* of the function $h : X \to Y^\bullet$ the set $\mathrm{dom}(h) = \big\{x \in X : h(x) \in Y\big\}$ and we say that h is *proper* when $\mathrm{dom}(h) \neq \emptyset$. For any subset $W \subseteq Y$ we denote $h^{-1}(W) = \big\{x \in X : \exists y \in W \text{ s.t. } h(x) = y\big\}$. According to [1], [7], [11] and [12] we have also the following extensions of the notion of lower-semicontinuity.

Definition 2.1 ([11]) We call the K-*epigraph* of the function $h : X \to Y^\bullet$ the set

$$\mathrm{epi}_K(h) = \big\{(x, y) \in X \times Y : y \in h(x) + K\big\}.$$

If h has a closed K-epigraph it is called K-*epi-closed*.

Definition 2.2 ([1], [7]) A function $h : X \to Y^\bullet$ is said to be K-*lower-semicontinuous* at $x \in X$ if for any neighborhood V of zero and for any $b \in Y$ satisfying $b \leq_K h(x)$, there exists a neighborhood U of x in X such that $h(U) \subseteq b + V + K \cup \{\infty\}$.

Remark 2.3 ([7]) If, for some $x \in X$, $h(x) \in Y$ the definition of K-lower-semicontinuity of h at x amounts to asking for any neighborhood $V \subseteq Y$ of zero (in Y) the existence of a neighborhood U of x such that $h(U) \subseteq h(x) + V + K \cup \{\infty\}$.

Proposition 2.4 *([1], [11]) Any K-lower-semicontinuous function $h : X \to Y^\bullet$ is also K-epi-closed, but the reverse assertion is not always true.*

Remark 2.5 It is known that when $Y = \mathbb{R}$ and $K = \mathbb{R}_+$ the notions of K-lower-semicontinuity and K-epi-closedness coincide, both of them becoming the classical lower-semicontinuity. The reader is referred to [12] for an example of a function which is K-epi-closed, but not K-lower-semicontinuous.

There are some other notions extending the lower-semicontinuity to topological vector spaces. Alongside the two we have just presented, let us mention the so-called *level-closed* functions (cf. [11]) and the K-*sequentially lower-semicontinuous* functions (cf. [1], [7]). When X and Y are metrizable the latter notion coincides to the one given in Definition 2.2, while level-closedness is implied by K-epi-closedness and hence also by K-lower-semicontinuity, these notions coinciding provided that some conditions are fulfilled. For more on lower-semicontinuity on topological spaces we refer the reader to [1], [7], [11], [12] and [14].

Other definitions generalizing some notions by using the cone K follow.

Definition 2.6 A function $g : Y \to \overline{\mathbb{R}}$ is called K-*increasing*, if for $x, y \in Y$ such that $y \leq_K x$, follows $g(y) \leq g(x)$.

Definition 2.7 A function $h : X \to Y^\bullet$ is called K-*convex*, if for any x and $y \in X$ and $t \in [0, 1]$ one has

$$h\big(tx + (1-t)y\big) \leq_K th(x) + (1-t)h(y).$$

Let us introduce for any $\lambda \in K^*$ and $h : X \to Y^\bullet$ the function (λh) defined on X as follows

$$(\lambda h)(x) = \begin{cases} \langle \lambda, h(x)\rangle, & \text{for } x \in \mathrm{dom}(h), \\ +\infty, & \text{otherwise.} \end{cases}$$

Lemma 2.8 *Given a function $h : X \to Y^\bullet$ we have for any $x \in X$*

$$\delta_{\{x \in X : h(x) \leq_K 0\}}(x) = \sup_{\lambda \in K^*} (\lambda h)(x).$$

Proof. Let $x \in X$. We distinguish two cases. First, if $h(x) \leq_K 0$, we have $\delta_{\{x \in X : h(x) \leq_K 0\}}(x) = 0$ and $h(x) \in -K$. Further, for $\lambda \in K^*$ one has $(\lambda h)(x) = \langle \lambda, h(x)\rangle \leq 0$, value attained for $\lambda = 0$, so $\sup_{\lambda \in K^*}(\lambda h)(x) = 0 = \delta_{\{x \in X : h(x) \leq_K 0\}}(x)$.

When $h(x) \not\leq_K 0$ we have $\delta_{\{x \in X : h(x) \leq_K 0\}}(x) = +\infty$. It follows that $h(x) \notin -K = -K^{**}$, thus there is some $\bar{\lambda} \in K^*$ such that $(\bar{\lambda} h)(x) > 0$ and, as $\sup_{\lambda \in K^*}(\lambda h)(x) = +\infty$, the desired equality is valid. □

We give now two other important results concerning epigraphs of conjugate functions.

Lemma 2.9 *([3]) Let $f,\, g : X \to \mathbb{R}\cup\{+\infty\}$ be proper, convex and lower-semicontinuous functions such that* $\operatorname{dom}(f)\cap\operatorname{dom}(g)\neq\emptyset$. *Then*

$$\operatorname{epi}\big((f+g)^*\big)=\operatorname{cl}\big(\operatorname{epi}\big(f^*\Box g^*\big)\big)=\operatorname{cl}\big(\operatorname{epi}\big(f^*\big)+\operatorname{epi}\big(g^*\big)\big).$$

Lemma 2.10 *([3]) Let $f,\, g : X \to \mathbb{R}\cup\{+\infty\}$ be proper functions such that* $\operatorname{dom}(f)\cap\operatorname{dom}(g)\neq\emptyset$. *Then the following statements are equivalent*

(i) $\operatorname{epi}\big((f+g)^*\big)=\operatorname{epi}\big(f^*\big)+\operatorname{epi}\big(g^*\big)$,

(ii) $(f+g)^*=f^*\Box g^*$ *and* $f^*\Box g^*$ *is exact at every* $p\in X^*$.

We recall also a well - known characterization of the subdifferential which proves later to be useful.

Lemma 2.11 *Given any proper function $f : X \to \overline{\mathbb{R}}$, for some $x\in\operatorname{dom}(f)$ and $y\in X^*$ one has $y\in\partial f(x)$ if and only if $f^*(y)+f(x)=\langle y,x\rangle$.*

We conclude the preliminary section by introducing two new notions in order to present the main results of the paper in an easier way.

Definition 2.12 A set $M\subseteq X$ is said to be *closed regarding the subspace* $Z\subseteq X$ if $M\cap Z=\operatorname{cl}(M)\cap Z$.

Definition 2.13 A function $f : X\to\overline{\mathbb{R}}$ is said to be *lower-semicontinuous regarding the subspace* $Z\subseteq X$ if $\operatorname{epi}(f)\cap(Z\times\mathbb{R})=\operatorname{cl}(\operatorname{epi}(f))\cap(Z\times\mathbb{R})$, i.e. $\operatorname{epi}(f)$ is closed regarding the subspace $Z\times\mathbb{R}$.

3 Conjugate and subdifferential of composed functions

Within this section we give a constraint qualification that is equivalent to the formula of the conjugate of the sum between a proper convex lower-semicontinuous function $f : X\to\overline{\mathbb{R}}$ and the precomposition of another proper convex lower-semicontinuous function $g : Y\to\overline{\mathbb{R}}$ which is also K-increasing with a proper K-convex K -epi-closed function $h : X\to Y^\bullet$, provided that $\big(h(\operatorname{dom}(f))+K\big)\cap\operatorname{dom}(g)\neq\emptyset$. We show that the formula of the subdifferential of the function $f+g\circ h$ (cf. [7]) holds under this new constraint qualification. Without altering the properties of the function g we define, because h may take the value ∞, also $g(\infty)=+\infty$. In the following we write $\min$ $(\max)$ instead of $\inf$ $(\sup)$ when the infimum (supremum) is attained.

Proposition 3.1 *For any $p\in X^*$ we have*

$$\big(f+g\circ h\big)^*(p)\le\inf_{\lambda\in K^*}\big\{g^*(\lambda)+\big(f+(\lambda h)\big)^*(p)\big\}.$$

Proof. Let $p\in X^*$. We have

$$\begin{aligned}(f+g\circ h)^*(p) &= \sup_{x\in X}\big\{\langle p,x\rangle-(f+g\circ h)(x)\big\}=-\inf_{x\in X}\big\{(f+g\circ h)(x)-\langle p,x\rangle\big\}\\ &= -\inf_{\substack{x\in X,y\in Y,\\ h(x)-y\in -K}}\big[f(x)+g(y)-\langle p,x\rangle\big]\\ &= -\inf_{\substack{x\in X,\\ y\in Y}}\big[f(x)+g(y)-\langle p,x\rangle+\delta_{\{(x,y)\in X\times Y:h(x)-y\in -K\}}(x,y)\big].\end{aligned}$$

From Lemma 2.8 we know that for any $\lambda\in K^*$ one has $\delta_{\{(x,y)\in X\times Y:h(x)-y\in -K\}}(x,y)\ge(\lambda h)(x)-\langle\lambda,y\rangle$, so we obtain for all $\lambda\in K^*$

$$\begin{aligned}(f+g\circ h)^*(p) &\le -\inf_{\substack{x\in X,\\ y\in Y}}\big[f(x)+g(y)-\langle p,x\rangle-(\lambda h)(x)+\langle\lambda,y\rangle\big]\\ &= -\inf_{x\in X}\big[f(x)+(\lambda h)(x)-\langle p,x\rangle\big]-\inf_{y\in Y}\big[g(y)-\langle\lambda,y\rangle\big]\\ &= (f+(\lambda h))^*(p)+g^*(\lambda).\end{aligned}$$

Thus, for any $p \in X^*$ and $\lambda \in K^*$ it stands

$$(f + g \circ h)^*(p) \leq (f + (\lambda h))^*(p) + g^*(\lambda).$$

The desired inequality arises when taking the infimum over $\lambda \in K^*$ in the right-hand side. □

Proposition 3.2 *Let the functions $F, G : X \times Y \to \overline{\mathbb{R}}$ be defined by $F(x, y) = g(y)$ and $G(x, y) = f(x) + \delta_{\{(x,y)\in X\times Y: h(x)-y\in -K\}}(x, y)$, for $(x, y) \in X \times Y$.*

(a) *F and G are proper, convex and lower-semicontinuous functions and* $\mathrm{dom}\,(F) \cap \mathrm{dom}(G) \neq \emptyset$.

(b) *For $(p, r) \in X^* \times \mathbb{R}$: $(p, r) \in \mathrm{epi}\,\big(f + g \circ h\big)^* \Leftrightarrow (p, 0, r) \in \mathrm{epi}\,\big(F + G\big)^*$,*

(c) $\mathrm{epi}(F^*) = \{0\} \times \mathrm{epi}(g^*)$ *and* $\mathrm{epi}(G^*) = \underset{\lambda\in K^*}{\cup} \big\{(a, -\lambda, r) : (f + (\lambda h))^*(a) \leq r\big\}$.

Proof. (a) As g is proper we know that it takes nowhere the value $-\infty$, so $F(x, y) > -\infty\ \forall (x, y) \in X \times Y$. Moreover, $\mathrm{dom}(g) \neq \emptyset$, thus $\mathrm{dom}(F) \neq \emptyset$ and F turns out to be proper. Because $\mathrm{epi}(F) = \big\{(x, y, r) \in X \times Y \times \mathbb{R} : g(y) \leq r\big\} = X \times \mathrm{epi}(g)$, which is convex and closed, F is convex and lower-semicontinuous.

As f is proper and the indicator function takes nowhere the value $-\infty$ it is clear that there is no pair $(x, y) \in X \times Y$ such that $G(x, y) = -\infty$. Moreover, from the initial assumption on the domains of the functions involved follows the existence of some $y \in \big(h(\mathrm{dom}(f)) + K\big) \cap \mathrm{dom}(g)$, so there is an $x \in X$ such that $f(x) < +\infty$ and $h(x) \leq_K y$. Thus $\delta_{\{(x,y)\in X\times Y: h(x)-y\in -K\}}(x, y) = 0$ and G is proper, too. The epigraph of G

$$\begin{aligned}\mathrm{epi}(G) &= \big\{(x, y, r) \in X \times Y \times \mathbb{R} : h(x) - y \in -K, f(x) \leq r\big\} \\ &= \big\{(x, y, r) \in X \times Y \times \mathbb{R} : y \in h(x) + K, f(x) \leq r\big\},\end{aligned}$$

is a closed convex set, thus G is convex and lower-semicontinuous.

Now let us prove the non-emptiness of the intersection of the domains of F and G. We have $\mathrm{dom}(F) = X \times \mathrm{dom}(g)$ and $\mathrm{dom}(G) = \big\{(x, y) \in X \times Y : x \in \mathrm{dom}(f), h(x) - y \in -K\big\}$. We know that there is a pair $(x, y) \in X \times Y$ such that $y \in \mathrm{dom}(g)$, $h(x) - y \in -K$ and $f(x) < +\infty$. It is clear that $(x, y) \in \mathrm{dom}(F)$ and $(x, y) \in \mathrm{dom}(G)$, too.

(b) First we prove that for any $p \in X^*$ it holds

$$\inf_{x\in X} \big[(f + g \circ h)(x) - \langle p, x\rangle\big] = \inf_{\substack{x\in X, y\in Y,\\ h(x)-y\in -K}} \big[f(x) + g(y) - \langle p, x\rangle\big]. \tag{3.1}$$

"$\geq$" Take first $x \notin \mathrm{dom}(f) \cap h^{-1}(\mathrm{dom}(g) - K)$. If $x \notin \mathrm{dom}(f)$ it follows $f(x) = +\infty$, so $(f + g \circ h)(x) = +\infty$, which is greater than or equal to any value which the term in the right-hand side of (3.1) may take. If this is not the case, then we get that $h(x) \notin \mathrm{dom}(g)$, so $g(h(x)) = +\infty$, thus $(f + g \circ h)(x) = +\infty$ and this is greater than or equal to the infimum in the right-hand side of (3.1).

Now let us take an $x \in \mathrm{dom}(f) \cap h^{-1}(\mathrm{dom}(g) - K)$. We have $f(x) \in \mathbb{R}$, as f is proper, and $h(x) \in \mathrm{dom}(g) - K$. Assuming $h(x) = \infty$ leads to the existence of some $y \in \mathrm{dom}(g)$ and $k \in K$ such that $\infty = y - k$, thus $y = \infty + k = \infty$. But $g(\infty) = +\infty$, so $y \notin \mathrm{dom}(g)$ and we reach a contradiction. Whence $h(x) \in Y$. As the value $(f + g \circ h)(x) - \langle p, x\rangle$ is taken by the function to be minimized in the right-hand side of this inequality for $y = h(x)$ it follows

$$(f + g \circ h)(x) - \langle p, x\rangle \geq \inf_{\substack{x\in X, y\in Y,\\ h(x)-y\in -K}} \big[f(x) + g(y) - \langle p, x\rangle\big].$$

This being fulfilled for all $x \in X$ it follows

$$\inf_{x\in X} \big[(f + g \circ h)(x) - \langle p, x\rangle\big] \geq \inf_{\substack{x\in X, y\in Y,\\ h(x)-y\in -K}} \big[f(x) + g(y) - \langle p, x\rangle\big].$$

"$\leq$" Let $x \in X$ and $y \in Y$ such that $h(x) - y \leq_K 0$, i.e. $h(x) \leq_K y$. Hence, $g(h(x)) \leq g(y)$, so $(f + g \circ h)(x) - \langle p, x\rangle \leq f(x) + g(y) - \langle p, x\rangle$. Further, taking the infimum over $x \in X$ in the left-hand side of (3.1) we get $\inf_{x\in X} \big[(f + g \circ h)(x) - \langle p, x\rangle\big] \leq f(x) + g(y) - \langle p, x\rangle$, followed by

$$\inf_{x\in X} \big[(f + g \circ h)(x) - \langle p, x\rangle\big] \leq \inf_{\substack{x\in X, y\in Y,\\ h(x)-y\in -K}} \big[f(x) + g(y) - \langle p, x\rangle\big].$$

As the converse inequality holds, too, the validity of (3.1) is guaranteed. Using F and G, relation (3.1) may be equivalently written for any $p \in X^*$

$$\begin{aligned}\inf_{x\in X} \big[(f + g \circ h)(x) - \langle p, x\rangle\big] &= \inf_{\substack{x\in X, y\in Y,\\ h(x)-y\in -K}} \big[f(x) + g(y) - \langle p, x\rangle\big]\\ &= \inf_{\substack{x\in X,\\ y\in Y}} \big[f(x) + g(y) - \langle p, x\rangle + \delta_{\{(x,y)\in X\times Y: h(x)-y\in -K\}}\big]\\ &= \inf_{\substack{x\in X,\\ y\in Y}} \big[F(x, y) + G(x, y) - \langle p, x\rangle\big] = -(F + G)^*(p, 0).\end{aligned}$$

For $(p, r) \in \operatorname{epi}(f + g \circ h)^*$ we have $(f + g \circ h)^*(p) = \sup_{x\in X} \big\{\langle p, x\rangle - (f + g \circ h)(x)\big\} \leq r$, equivalent to $-r \leq \inf_{x\in X} \big[(f + g \circ h)(x) - \langle p, x\rangle\big]$, so $-r \leq -(F + G)^*(p, 0)$, which means $(p, 0, r) \in \operatorname{epi}(F + G)^*$.

(c) Let us determine the conjugate functions of F and G. We have, for some $(a, b) \in X^* \times Y^*$,

$$F^*(a, b) = \sup_{x\in X, y\in Y} \big\{\langle a, x\rangle + \langle b, y\rangle - g(y)\big\} = \sup_{x\in X} \langle a, x\rangle + \sup_{y\in Y} \big\{\langle b, y\rangle - g(y)\big\}$$

and

$$\begin{aligned}G^*(a, b) &= \sup_{\substack{x\in X,\\ y\in Y}} \big\{\langle a, x\rangle + \langle b, y\rangle - f(x) - \delta_{\{(x,y)\in X\times Y: h(x)-y\in -K\}}(x, y)\big\}\\ &= \sup_{\substack{x\in X, y\in Y,\\ h(x)-y\in -K}} \big\{\langle a, x\rangle + \langle b, y\rangle - f(x)\big\}.\end{aligned}$$

Denoting $z = h(x) - y$ it follows

$$\begin{aligned}G^*(a, b) &= \sup_{x\in X, z\in -K} \big\{\langle a, x\rangle + \langle b, h(x) - z\rangle - f(x)\big\}\\ &= \sup_{x\in X} \big\{\langle a, x\rangle - \langle -b, h(x)\rangle - f(x)\big\} + \sup_{z\in -K} \langle -b, z\rangle\end{aligned}$$

and we get

$$F^*(a, b) = \begin{cases} g^*(b), & \text{if } a = 0,\\ +\infty, & \text{otherwise,}\end{cases} \quad \text{and } G^*(a, b) = \begin{cases} \big(f + (-bh)\big)^*(a), & \text{if } b \in -K^*,\\ +\infty, & \text{otherwise.}\end{cases}$$

Now we determine the epigraphs of these conjugate functions. For F^*, $(a, b, r) \in \operatorname{epi}\big(F^*\big)$ means $F^*(a, b) \leq r$, which is equivalent to $a = 0$ and $g^*(b) \leq r$, i.e. $(a, b, r) \in \{0\} \times \operatorname{epi}\big(g^*\big)$. Thus $\operatorname{epi}\big(F^*\big) = \{0\} \times \operatorname{epi}(g^*)$. For G^*, $(a, b, r) \in \operatorname{epi}(G^*)$ means $-b \in K^*$ and $\big(f + (-bh)\big)^*(a) \leq r$, so, denoting $\lambda = -b$ we have $\operatorname{epi}(G^*) = \cup_{\lambda\in K^*} \big\{(a, -\lambda, r) : (f + (\lambda h))^*(a) \leq r\big\}$. □

In order to prove the main result we introduce now the constraint qualification

(CQ) $\quad \{0\} \times \operatorname{epi}(g^*) + \underset{\lambda\in K^*}{\cup} \big\{(a, -\lambda, r) : (a, r) \in \operatorname{epi}\big(f + (\lambda h)\big)^*\big\}$ is closed regarding the subspace $X^* \times \{0\} \times \mathbb{R}$.

The main result in this paper follows.

Theorem 3.3 (a) *(CQ) is fulfilled if and only if for any $p \in X^*$ one has*

$$\big(f + g \circ h\big)^*(p) = \min_{\lambda \in K^*} \big\{g^*(\lambda) + \big(f + (\lambda h)\big)^*(p)\big\}.$$

(b) *If (CQ) is fulfilled then for any $x \in \operatorname{dom}(f) \cap h^{-1}(\operatorname{dom}(g))$, one has*

$$\partial\big(f + g \circ h\big)(x) = \bigcup_{\lambda \in \partial g(h(x))} \partial\big(f + (\lambda h)\big)(x).$$

Proof. (a) "$\Leftarrow$" Take $(p, 0, r) \in \operatorname{cl}\big(\{0\} \times \operatorname{epi}(g^*) + \cup_{\lambda \in K^*} \big\{(a, -\lambda, r) : (a, r) \in \operatorname{epi}\big(f + \lambda h)^*\big\}\big) \cap (X^* \times \{0\} \times \mathbb{R})$. From the formulae of the epigraphs of F^* and G^* it follows instantly $(p, 0, r) \in \operatorname{cl}(\operatorname{epi}(F^*) + \operatorname{epi}(G^*)) \cap (X^* \times \{0\} \times \mathbb{R})$ and by Lemma 2.9 we get $(p, 0, r) \in \operatorname{epi}\big(F + G\big)^* \cap (X^* \times \{0\} \times \mathbb{R})$. This actually means by Proposition 3.2, $(p, r) \in \operatorname{epi}\big(\big(f + g \circ h\big)^*\big)$, i.e. $\big(f + g \circ h\big)^*(p) \le r$. By the formula in the hypothesis it follows the existence of some $\bar{\lambda} \in K^*$ fulfilling

$$\big(f + g \circ h\big)^*(p) = \min_{\lambda \in K^*} \big\{g^*(\lambda) + \big(f + (\lambda h)\big)^*(p)\big\} = g^*(\bar{\lambda}) + \big(f + (\bar{\lambda} h)\big)^*(p).$$

We have then

$$g^*(\bar{\lambda}) + \big(f + (\bar{\lambda} h)\big)^*(p) \le r,$$

so

$$\big(f + (\bar{\lambda} h)\big)^*(p) \le r - g^*(\bar{\lambda}).$$

It is not difficult to notice that $(p, 0, r) = (0, \bar{\lambda}, g^*(\bar{\lambda})) + \big(p, -\bar{\lambda}, r - g^*(\bar{\lambda})\big)$, where the first term in the right-hand side sum belongs to $\{0\} \times \operatorname{epi}(g^*)$ and the second to $\cup_{\lambda \in K^*} \big\{(a, -\lambda, r) : (a, r) \in \operatorname{epi}\big(f + \lambda h)^*\big\}$, therefore $\operatorname{cl}\big(\{0\} \times \operatorname{epi}(g^*) + \cup_{\lambda \in K^*} \big\{(a, -\lambda, r) : (a, r) \in \operatorname{epi}\big(f + \lambda h)^*\big\}\big) \cap (X^* \times \{0\} \times \mathbb{R}) \subseteq \big(\{0\} \times \operatorname{epi}(g^*) + \cup_{\lambda \in K^*} \big\{(a, -\lambda, r) : (a, r) \in \operatorname{epi}\big(f + \lambda h)^*\big\}\big) \cap (X^* \times \{0\} \times \mathbb{R})$, i.e. (CQ) stands.

"$\Rightarrow$" Let $p \in X^*$. If $\big(f + g \circ h\big)^*(p) = +\infty$ the conclusion arises trivially by Proposition 3.1, so let us consider further $\big(f + g \circ h\big)^*(p) \in \mathbb{R}$. It is clear that $\big(p, \big(f + g \circ h\big)^*(p)\big) \in \operatorname{epi}\big(\big(f + g \circ h\big)^*\big)$, which gives by Proposition 3.2(b)

$$\big(p, 0, \big(f + g \circ h\big)^*(p)\big) \in \operatorname{epi}\big(F + G\big)^* \cap \big(X^* \times \{0\} \times \mathbb{R}\big).$$

By Lemma 2.9 and Proposition 3.2 it follows $(p, 0, \big(f + g \circ h\big)^*(p)) \in \operatorname{cl}(\operatorname{epi}(F^*) + \operatorname{epi}(G^*)) \cap (X^* \times \{0\} \times \mathbb{R}) = \operatorname{cl}\big(\{0\} \times \operatorname{epi}(g^*) + \cup_{\lambda \in K^*} \big\{(a, -\lambda, r) : (a, r) \in \operatorname{epi}\big(f + \lambda h)^*\big\}\big) \cap (X^* \times \{0\} \times \mathbb{R})$, so (CQ) yields $(p, 0, \big(f + g \circ h\big)^*(p)) \in \big(\{0\} \times \operatorname{epi}(g^*) + \cup_{\lambda \in K^*} \big\{(a, -\lambda, r) : (a, r) \in \operatorname{epi}\big(f + \lambda h)^*\big\}$. Therefore there is some $\bar{\lambda} \in K^*$ such that

$$\big(p, 0, (f + g \circ h)^*(p)\big) = \big(0, \bar{\lambda}, g^*(\bar{\lambda})\big) + \big(p, -\bar{\lambda}, \big(f + g \circ h\big)^*(p) - g^*(\bar{\lambda})\big),$$

where $\big(\bar{\lambda}, g^*(\bar{\lambda})\big) \in \operatorname{epi}\big(g^*\big)$ and $\big(f + (\bar{\lambda} h)\big)^*(p) \le \big(f + g \circ h\big)^*(p) - g^*(\bar{\lambda})$, the latter actually stating that

$$\exists \bar{\lambda} : \ \big(f + g \circ h\big)^*(p) \ge g^*(\bar{\lambda}) + \big(f + (\bar{\lambda} h)\big)^*(p).$$

As the reverse inequality holds for any $\lambda \in K^*$ (cf. Proposition 3.1), it follows that we have obtained a $\bar{\lambda} \in K^*$ such that

$$\big(f + g \circ h\big)^*(p) = g^*(\bar{\lambda}) + \big(f + (\bar{\lambda} h)\big)^*(p) = \min_{\lambda \in K^*} \big\{g^*(\lambda) + \big(f + (\lambda h)\big)^*(p)\big\}. \tag{3.2}$$

(b) Let $x \in \operatorname{dom}(f) \cap h^{-1}(\operatorname{dom}(g))$. To characterize the subdifferentials we use the definition as well as Lemma 2.11.

"$\supseteq$" Take $\lambda \in \partial g\big(h(x)\big)$, i.e. $\forall s \in Y$ one has $\langle \lambda, s - h(x) \rangle \le g(s) - g(h(x))$ and $z \in \partial\big(f + (\lambda h)\big)(x)$, which means that for any $t \in X$ we have $\langle z, t - x \rangle \le \big(f + (\lambda h)\big)(t) - \big(f + (\lambda h)\big)(x)$. If for some $t \in X$ we have

$h(t) = \infty$, then $g(h(t)) = +\infty$, thus $(f + g \circ h)(t) = +\infty$. Hence $\langle z, t - x\rangle \leq (f + g \circ h)(t) - (f + g \circ h)(x)$. If $h(t) \in Y$, by rewriting the term in the right-hand side and applying the first inequality for $s = h(t)$ we have

$$\begin{aligned}\langle z, t - x\rangle &\leq f(t) - f(x) + \langle \lambda, h(t) - h(x)\rangle \leq f(t) - f(x) + g(h(t)) - g(h(x)) \\ &= \big(f + g \circ h\big)(t) - \big(f + g \circ h\big)(x),\end{aligned}$$

so $\langle z, t - x\rangle \leq \big(f + g \circ h\big)(t) - \big(f + g \circ h\big)(x)\ \forall t \in X$, i.e. $z \in \partial\big(f + g \circ h\big)(x)$. Let us remark that the inclusion proven here holds even without assuming (CQ) fulfilled.

"$\subseteq$" For some $z \in \partial\big(f + g \circ h\big)(x)$ we have $\big(f + g \circ h\big)^*(z) + \big(f + g \circ h\big)(x) = \langle z, x\rangle$. As (CQ) is fulfilled there is a $\bar{\lambda} \in K^*$ fulfilling (3.2). We have, by applying Young's inequality twice,

$$\begin{aligned}\langle z, x\rangle &= \big(f + g \circ h\big)(x) + g^*(\bar{\lambda}) + \big(f + (\bar{\lambda}h)\big)^*(z) = f(x) + \big(f + (\bar{\lambda}h)\big)^*(z) + g(h(x)) + g^*(\bar{\lambda}) \\ &\geq f(x) + \big(f + (\bar{\lambda}h)\big)^*(z) + \langle \bar{\lambda}, h(x)\rangle = \big(f + (\bar{\lambda}h)\big)(x) + \big(f + (\bar{\lambda}h)\big)^*(z) \geq \langle z, x\rangle.\end{aligned}$$

This means that all the inequalities above must hold as equalities, i.e. we have equality in both places where we have used Young's inequality, so

$$g(h(x)) + g^*(\bar{\lambda}) = \langle \bar{\lambda}, h(x)\rangle \text{ and } \big(f + (\bar{\lambda}h)\big)(x) + \big(f + (\bar{\lambda}h)\big)^*(z) = \langle z, x\rangle.$$

The latter relations mean $\bar{\lambda} \in \partial g\big(h(x)\big)$ and $z \in \partial\big(f + (\bar{\lambda}h)\big)(x)$, exactly what we needed. □

Remark 3.4 The constraint qualification we give is weaker than others considered in the literature. Proposition 4.11 in [7] states that the formulae for the conjugate and subdifferential of $(f + g \circ h)$ in Theorem 3.3 hold under one of the following constraint qualifications

(CQR) X and Y are Fréchet spaces, f and g are lower-semicontinuous, h is K-sequentially lower-semicontinuous and $0 \in \operatorname{core}[\operatorname{dom}(g) - h(\operatorname{dom}(f) \cap \operatorname{dom}(h))]$,

$(CQAB)$ X and Y are Fréchet spaces, f and g are lower-semicontinuous, h is K-sequentially lower-semicontinuous and $\mathbb{R}_+[\operatorname{dom}(g) - h(\operatorname{dom}(f) \cap \operatorname{dom}(h))]$ is a closed vector subspace of Y.

The first of them was inspired by Rockafellar's work [13], while the second belongs to the class of so-called Attouch-Brézis-type constraint qualifications. It is also known (cf. [7], for instance) that (CQR) implies $(CQAB)$. As Theorem 3.3(a) states the equivalence of the formula of $(f + g \circ h)^*$ with (CQ), it follows that (CQ) is satisfied, too, when $(CQAB)$ or (CQR) is fulfilled.

An example showing that (CQ) is indeed weaker than $(CQAB)$, i.e. it may hold even without assuming $(CQAB)$ true, follows.

Example 3.5 Let $X = Y = \mathbb{R}$, $K = \{0\}$, $f = \delta_{(-\infty,0]}$, $g = \delta_{[0,+\infty)}$ and $h = \mathrm{id}_{\mathbb{R}}$. We have $X^* = Y^* = K^* = \mathbb{R}$, $\operatorname{dom}(f) = (-\infty, 0]$, $\operatorname{dom}(g) = [0, +\infty)$ and $\operatorname{dom}(h) = \mathbb{R}$, so $\operatorname{dom}(g) - h(\operatorname{dom}(f) \cap \operatorname{dom}(h)) = \mathbb{R}_+ - h((-\infty, 0]) = \mathbb{R}_+ - (-\infty, 0] = \mathbb{R}_+$. Hence, $\mathbb{R}_+[\operatorname{dom}(g) - h(\operatorname{dom}(f) \cap \operatorname{dom}(h))] = \mathbb{R}_+$, which is not a subspace. Therefore $(CQAB)$ is not fulfilled, so neither is (CQR). On the other hand, for any $\lambda, p \in \mathbb{R}$,

$$g^*(p) = \begin{cases} 0, & \text{if } p \in (-\infty, 0], \\ +\infty, & \text{if } p \in (0, +\infty) \end{cases}, \quad (f + (\lambda h))^*(p) = \begin{cases} 0, & \text{if } p \geq \lambda, \\ +\infty, & \text{if } p < \lambda \end{cases},$$

so $\operatorname{epi}(g^*) = (-\infty, 0] \times \mathbb{R}_+$ and $\operatorname{epi}((f + (\lambda h))^*) = [\lambda, +\infty) \times \mathbb{R}_+$. We have $\{0\} \times \operatorname{epi}(g^*) + \bigcup_{\lambda \in K^*} \big\{(p, -\lambda, r) : (p, r) \in \operatorname{epi}\big(f + (\lambda h)\big)^*\big\} = \{0\} \times (-\infty, 0] \times \mathbb{R}_+ + \bigcup_{\lambda \in \mathbb{R}} [\lambda, +\infty) \times \{-\lambda\} \times \mathbb{R}_+ = \bigcup_{\lambda \in \mathbb{R}} \big[[\lambda, +\infty) \times (-\infty, -\lambda] \times \mathbb{R}_+\big] = \mathbb{R} \times \mathbb{R} \times \mathbb{R}_+$, which is closed, so (CQ) holds. □

Let us consider now another constraint qualification, namely that the set

$$\{0\} \times \operatorname{epi}(g^*) + \big\{(p, 0, r) : (p, r) \in \operatorname{epi}(f^*)\big\} + \bigcup_{\lambda \in K^*} \big\{(p, -\lambda, r) : (p, r) \in \operatorname{epi}\big(\lambda h\big)^*\big\}$$

is closed regarding the subspace $X^* \times \{0\} \times \mathbb{R}$. We call it $(\overline{CQ})$.

To compare the two constraint qualifications we have introduced within this paper we need the following result.

Proposition 3.6 *We have* $\operatorname{cl}\big(\{0\}\times\operatorname{epi}(g^*)+\cup_{\lambda\in K^*}\big\{(p,-\lambda,r):(p,r)\in\operatorname{epi}\big(f+(\lambda h)\big)^*\big\}\big)=\operatorname{cl}\big(\{0\}\times\operatorname{epi}(g^*)+\big\{(p,0,r):(p,r)\in\operatorname{epi}(f^*)\big\}+\cup_{\lambda\in K^*}\big\{(p,-\lambda,r):(p,r)\in\operatorname{epi}\big(\lambda h\big)^*\big\}\big)$.

Proof. We introduce the functions $G_1:X\times Y\to\overline{\mathbb{R}}$, $G_1(x,y)=f(x)$ and $G_2:X\times Y\to\overline{\mathbb{R}}$, $G_2(x,y)=\delta_{\{(x,y)\in X\times Y:h(x)-y\in-K\}}(x,y),\ (x,y)\in X\times Y$. The functions G_1 and G_2 are proper, convex and lower-semicontinuous and $\operatorname{dom}(G_1)\cap\operatorname{dom}(G_2)\neq\emptyset$. Moreover, it holds $G=G_1+G_2$. Thus, by Lemma 2.9, one has

$$\operatorname{epi}(G^*)=\operatorname{epi}((G_1+G_2)^*)=\operatorname{cl}(\operatorname{epi}(G_1^*)+\operatorname{epi}(G_2^*)).$$

Further, applying the same lemma, we get

$$\operatorname{epi}((F+G)^*)=\operatorname{cl}(\operatorname{epi}(F^*)+\operatorname{epi}(G^*))=$$

$$\operatorname{cl}(\operatorname{epi}(F^*)+\operatorname{cl}(\operatorname{epi}(G_1^*)+\operatorname{epi}(G_2^*)))=\operatorname{cl}(\operatorname{epi}(F^*)+\operatorname{epi}(G_1^*)+\operatorname{epi}(G_2^*)). \tag{3.3}$$

By Proposition 3.2 it holds

$$\operatorname{epi}(F^*)+\operatorname{epi}(G^*)=\{0\}\times\operatorname{epi}(g^*)+\underset{\lambda\in K^*}{\cup}\big\{(p,-\lambda,r):(p,r)\in\operatorname{epi}\big(f+(\lambda h)\big)^*\big\}.$$

Simple calculations give

$$\operatorname{epi}(G_1^*)=\big\{(p,0,r):(p,r)\in\operatorname{epi}(f^*)\big\},$$

$$\operatorname{epi}(G_2^*)=\underset{\lambda\in K^*}{\cup}\big\{(p,-\lambda,r):(p,r)\in\operatorname{epi}\big(\lambda h\big)^*\big\},$$

which implies

$$\operatorname{epi}(F^*)+\operatorname{epi}(G_1^*)+\operatorname{epi}(G_2^*)=$$

$$\{0\}\times\operatorname{epi}(g^*)+\big\{(p,0,r):(p,r)\in\operatorname{epi}(f^*)\big\}+\underset{\lambda\in K^*}{\cup}\big\{(p,-\lambda,r):(p,r)\in\operatorname{epi}\big(\lambda h\big)^*\big\}.$$

Relation (3.3) leads now to the desired conclusion. □

Remark 3.7 The closures proven to coincide in the previous statement are also equal to $\operatorname{epi}((F+G)^*)$. Since $\{0\}\times\operatorname{epi}(g^*)+\cup_{\lambda\in K^*}\big\{(p,-\lambda,r):(p,r)\in\operatorname{epi}\big(f+(\lambda h)\big)^*\big\}=\operatorname{epi}(F^*)+\operatorname{epi}(G^*)\supseteq\operatorname{epi}(F^*)+\operatorname{epi}(G_1^*)+\operatorname{epi}(G_2^*)=\{0\}\times\operatorname{epi}(g^*)+\big\{(p,0,r):(p,r)\in\operatorname{epi}(f^*)\big\}+\cup_{\lambda\in K^*}\big\{(p,-\lambda,r):(p,r)\in\operatorname{epi}\big(\lambda h\big)^*\big\}$, by Proposition 3.6 it follows that the fulfillment of $(\overline{CQ})$ implies the validity of (CQ).

Because $(\overline{CQ})$ yields (CQ) it is supposed to guarantee some other results somehow similar to the ones given in Theorem 3.3. Indeed, we show that it is equivalent to a deeper formula for the conjugate of $f+g\circ h$, where f^* and $(\lambda h)^*$ are separated, and implies another formula for the subdifferential of $f+g\circ h$, where ∂f and $\partial(\lambda h)$ are no more together.

Theorem 3.8 (a) *$(\overline{CQ})$ is fulfilled if and only if for any $p\in X^*$ it holds*

$$\big(f+g\circ h\big)^*(p)=\min_{\substack{\lambda\in K^*,\\ \beta\in X^*}}\big\{g^*(\lambda)+f^*(\beta)+\big(\lambda h\big)^*(p-\beta)\big\}.$$

(b) *If $(\overline{CQ})$ is fulfilled then for any $x\in\operatorname{dom}(f)\cap h^{-1}(\operatorname{dom}(g))$, one has*

$$\partial\big(f+g\circ h\big)(x)=\partial f(x)+\underset{\lambda\in\partial g(h(x))}{\cup}\partial(\lambda h)(x).$$

Proof. (a) "⇒" As for any $q\in X^*$ and $\lambda\in K^*$ we have $(f+(\lambda h))^*(p)\leq f^*(q)+(\lambda h)^*(p-q)$, by Proposition 3.1 we get

$$(f+g\circ h)^*(p)\leq(\lambda h)^*(p-q)+g^*(\lambda)+f^*(q)\ \forall p,q\in X^*\ \forall\lambda\in K^*. \tag{3.4}$$

Let $p\in X^*$. If $(f+g\circ h)^*(p)=+\infty$, the assertion follows by (3.4). Consider further $(f+g\circ h)^*(p)\in\mathbb{R}$. We have $(p,(f+g\circ h)^*(p))\in\operatorname{epi}((f+g\circ h)^*)$, so, by Proposition 3.2$(b)$ we have $(p,0,(f+g\circ h)^*(p))\in$

$\text{epi}((F+G)^*)$. From Remark 3.7 we know that $\text{epi}((F+G)^*) = \text{cl}\big(\{0\}\times\text{epi}(g^*) + \big\{(p,0,r) : (p,r)\in \text{epi}(f^*)\big\} + \cup_{\lambda\in K^*}\big\{(p,-\lambda,r) : (p,r)\in\text{epi}\,(\lambda h)^*\big\}\big)$, so the fulfillment of $(\overline{CQ})$ yields $(p,0,(f+g\circ h)^*(p)) \in \big(\{0\}\times\text{epi}(g^*) + \big\{(a,0,r) : (a,r)\in\text{epi}(f^*)\big\} + \cup_{\lambda\in K^*}\big\{(a,-\lambda,r) : (a,r)\in\text{epi}\,(\lambda h)^*\big\}\big)\cap(X^*\times\{0\}\times\mathbb{R})$. Thus there exist some $\bar\lambda\in K^*$ and $\bar\beta\in X^*$ such that $(p,0,(f+g\circ h)^*(p)) = (0,\bar\lambda,g^*(\bar\lambda)) + (\bar\beta,0,f^*(\bar\beta)) + (p-\bar\beta,-\bar\lambda,(f+g\circ h)^*(p)-g^*(\bar\lambda)-f^*(\bar\beta))$, the first term in the right-hand side being in $\{0\}\times\text{epi}(g^*)$, the second in $\big\{(a,0,r) : (a,r)\in\text{epi}(f^*)\big\}$, while the third belongs to $\big\{(a,-\bar\lambda,r) : (a,r)\in\text{epi}\,(\bar\lambda h)^*\big\}$. Hence, $(\bar\lambda h)^*(p-\bar\beta)\le (f+g\circ h)^*(p) - g^*(\bar\lambda) - f^*(\bar\beta)$, so

$$(\bar\lambda h)^*(p-\bar\beta) + g^*(\bar\lambda) + f^*(\bar\beta) \le (f+g\circ h)^*(p).$$

By (3.4) we get the desired formula.

"$\Leftarrow$" Let $(p,0,r)\in\text{cl}\big(\{0\}\times\text{epi}(g^*) + \big\{(p,0,r) : (p,r)\in\text{epi}(f^*)\big\} + \cup_{\lambda\in K^*}\big\{(p,-\lambda,r) : (p,r)\in\text{epi}\,(\lambda h)^*\big\}\big)$. By Remark 3.7 we get $(p,0,r)\in\text{epi}((F+G)^*)$, thus by Proposition 3.2(b) $(p,r)\in\text{epi}((f+g\circ h)^*)$, i.e. $(f+g\circ h)^*(p)\le r$. From hypothesis we known that there are some $\bar\lambda\in K^*$ and $\bar\beta\in X^*$ satisfying

$$(\bar\lambda h)^*(p-\bar\beta) + g^*(\bar\lambda) + f^*(\bar\beta) = (f+g\circ h)^*(p),$$

so we have $(\bar\lambda h)^*(p-\bar\beta) + g^*(\bar\lambda) + f^*(\bar\beta)\le r$. This yields $(\bar\lambda h)^*(p-\bar\beta)\le r - g^*(\bar\lambda) - f^*(\bar\beta)$. Writing

$$(p,0,r) = (0,\bar\lambda,g^*(\bar\lambda)) + (\bar\beta,0,f^*(\bar\beta)) + (p-\bar\beta,-\bar\lambda,r-g^*(\bar\lambda)-f^*(\bar\beta)),$$

it is not difficult to notice that the first term in the right-hand side belongs to $\{0\}\times\text{epi}(g^*)$, the second to $\big\{(q,0,r) : (q,r)\in\text{epi}(f^*)\big\}$ and the third is in $\big\{(q,-\bar\lambda,r) : (q,r)\in\text{epi}\,(\bar\lambda h)^*\big\}$, so $\big(\text{cl}\big(\{0\}\times\text{epi}(g^*) + \big\{(p,0,r) : (p,r)\in\text{epi}(f^*)\big\} + \cup_{\lambda\in K^*}\big\{(p,-\lambda,r) : (p,r)\in\text{epi}\,(\lambda h)^*\big\}\big)\big)\cap(X^*\times\{0\}\times\mathbb{R}) \subseteq \big(\{0\}\times\text{epi}(g^*) + \big\{(p,0,r) : (p,r)\in\text{epi}(f^*)\big\} + \cup_{\lambda\in K^*}\big\{(p,-\lambda,r) : (p,r)\in\text{epi}\,(\lambda h)^*\big\}\big)\cap(X^*\times\{0\}\times\mathbb{R})$ and, as the opposite inclusion is always true, $(\overline{CQ})$ is surely valid.

(b) We skip this part of the proof as it is similar to the one of Theorem 3.3(b). □

Remark 3.9 To the best of our knowledge this formula for the conjugate has not been given elsewhere so far, while in Corollary 4.12 in [7] the formula for the subdifferential given above is proven to hold provided that h is continuous at some point in $\text{dom}(f)$ and assuming that one of the conditions $(CQAB)$ and (CQR) is fulfilled. The validity of each of these two constraint qualifications ensures the satisfaction of (CQ), so the formula given in Theorem 3.3(a) for $(f+g\circ h)^*$ holds at any $p\in X^*$. Theorem 2.8.7(iii) in [15] yields that when h is continuous at some point in $\text{dom}(f)$ then for any $\lambda\in K^*$ and $p\in X^*$ one gets $(f+(\lambda h))^*(p) = \min_{\beta\in X^*}\{f^*(\beta)+(\lambda h)^*(p-\beta)\}$. Assembling the last two formulae we get that under the conditions imposed in Corollary 4.12 in [7] the formula in Theorem 3.8(a) stands, and, as it is equivalent to $(\overline{CQ})$, the latter is also satisfied.

We conclude the section proving that in general (CQ) can be really weaker than $(\overline{CQ})$.

Example 3.10 Let $X=\mathbb{R}^2$, $Y=\mathbb{R}$ and $K=\mathbb{R}_+$. Therefore $X^*=\mathbb{R}^2$, $Y^*=\mathbb{R}$ and $K^*=\mathbb{R}_+$. Consider the sets $C=\big\{(x_1,x_2)^T\in\mathbb{R}^2 : x_1\ge 0\big\}$ and $D=\big\{(x_1,x_2)^T\in\mathbb{R}^2 : 2x_1+x_2^2\le 0\big\}$. Take $f=\delta_C$, $g=\text{id}_{\mathbb{R}}$ and $h=\delta_D$. As $\text{dom}(g)=\mathbb{R}$ it follows $\mathbb{R}_+[\text{dom}(g)-h(\text{dom}(f)\cap\text{dom}(h))]=\mathbb{R}_+\cdot\mathbb{R}=\mathbb{R}$, which is clearly a closed subspace of itself. Thus $(CQAB)$ stands, so (CQ) is valid, too. Let us see whether is $(\overline{CQ})$ satisfied or not in this situation. We have, for $(y_1,y_2)\in\mathbb{R}^2$ and $\lambda\in\mathbb{R}$,

$$f^*(y_1,y_2)=\begin{cases}0, & \text{if } y_1\le 0, y_2=0,\\ +\infty, & \text{otherwise,}\end{cases}\quad g^*(\lambda)=\begin{cases}0, & \text{if } \lambda=1,\\ +\infty, & \text{if } \lambda\ne 1,\end{cases}$$

and

$$h^*(y_1,y_2)=\begin{cases}\frac{y_2^2}{y_1}, & \text{if } y_1>0,\\ 0, & \text{if } y_1=y_2=0,\\ +\infty, & \text{otherwise.}\end{cases}$$

For $(p_1, p_2) \in \mathbb{R}^2$ it holds

$$\begin{aligned}(f + g \circ h)^*(p_1, p_2) &= \sup_{x_1, x_2 \in \mathbb{R}} \big\{p_1 x_1 + p_2 x_2 - f(x_1, x_2) - g(h(x_1, x_2))\big\} \\ &= \sup_{x_1, x_2 \in \mathbb{R}} \big\{p_1 x_1 + p_2 x_2 - \delta_C(x_1, x_2) - \delta_D(x_1, x_2)\big\} \\ &= \sup_{(x_1, x_2) \in C \cap D = \{(0,0)\}} \big\{p_1 x_1 + p_2 x_2\big\} = 0,\end{aligned}$$

while at $(p_1, p_2) = (1, 1)$ we have
$$\inf_{\substack{\lambda \in \mathbb{R}_+, \\ (\beta_1, \beta_2) \in \mathbb{R}^2}} \big\{g^*(\lambda) + f^*((\beta_1, \beta_2)) + \big(\lambda h\big)^*((1,1) - (\beta_1, \beta_2))\big\} = \inf_{\substack{\lambda = 1, \\ \beta_1 \leq 0, \\ \beta_2 = 0}} \big\{h^*((1,1) - (\beta_1, \beta_2))\big\} = \inf_{\beta_1 \leq 0} \frac{1}{1 - \beta_1} = 0,$$
but there is no $\beta_1 \leq 0$ where this value is attained. Thus even if in this case $(f + g \circ h)^*(1, 1) = \inf \big\{g^*(\lambda) + f^*((\beta_1, \beta_2)) + \big(\lambda h\big)^*((1,1) - (\beta_1, \beta_2)) : \lambda \in \mathbb{R}_+,\ (\beta_1, \beta_2) \in \mathbb{R}^2\big\}$, the infimum in the right-hand side is not attained, hence the formula in Theorem 3.8(a) is not satisfied for our choice of the functions and sets, and, in conclusion, $(\overline{CQ})$ is violated. □

4 Conjugate duality

Within this part of the paper we give weak constraint qualifications for conjugate duality inspired by the constraint qualifications given in the previous section. We have proven that (CQ) and, respectively, $(\overline{CQ})$ are equivalent to some formulae for $(f + g \circ h)^*(p)$ that are valid for any $p \in X^*$. It is known that $\inf_{x \in X}(f + g \circ h)(x) = -(f + g \circ h)^*(0)$, so we introduce some constraint qualifications derived from (CQ) and, respectively, $(\overline{CQ})$ that ensure the validity of the mentioned formulae at 0, which are actually the strong duality assertions between the mentioned minimization problem and two of its dual problems. We use also the functions F and G defined before.

Lemma 4.1 *The constraint qualification* (CQ) *is equivalent to*

$$\mathrm{epi}((F + G)^*) \cap (X^* \times \{0\} \times \mathbb{R}) = \big(\mathrm{epi}(F^*) + \mathrm{epi}(G^*)\big) \cap (X^* \times \{0\} \times \mathbb{R}).$$

Proof. By Proposition 3.2 one can see that (CQ) may be equivalently written

$$\mathrm{cl}\big(\mathrm{epi}(F^*) + \mathrm{epi}(G^*)\big) \cap (X^* \times \{0\} \times \mathbb{R}) = \big(\mathrm{epi}(F^*) + \mathrm{epi}(G^*)\big) \cap (X^* \times \{0\} \times \mathbb{R}).$$

The conclusion arises by Lemma 2.9. □

Remark 4.2 It may be proven also that (CQ) is equivalent to the fact that $F^* \Box G^*$ is lower-semicontinuous regarding the subspace $X^* \times \{0\}$ and is exact at any $(p, 0) \in X^* \times \{0\}$. By Lemma 2.9 we have

$$\mathrm{epi}((F + G)^*) = \mathrm{cl}(\mathrm{epi}(F^* \Box G^*)) \supseteq \mathrm{epi}(F^* \Box G^*) \supseteq \mathrm{epi}(F^*) + \mathrm{epi}(G^*). \tag{4.1}$$

Lemma 4.1 says that (CQ) states that both inclusions in (4.1) must be fulfilled as equalities when intersecting them in both sides with the subspace $X^* \times \{0\} \times \mathbb{R}$. The first of them, $\mathrm{cl}(\mathrm{epi}(F^* \Box G^*)) \cap (X^* \times \{0\} \times \mathbb{R}) = (\mathrm{epi}(F^* \Box G^*)) \cap (X^* \times \{0\} \times \mathbb{R})$ means actually that $F^* \Box G^*$ is lower-semicontinuous regarding the subspace $X^* \times \{0\}$, while the other one, namely $(\mathrm{epi}\ (F^* \Box G^*)) \cap (X^* \times \{0\} \times \mathbb{R}) = (\mathrm{epi}(F^*) + \mathrm{epi}(G^*)) \cap (X^* \times \{0\} \times \mathbb{R})$ is nothing but the fact that $F^* \Box G^*$ is exact at any point $(p, 0) \in X^* \times \{0\}$.

The formula given in Theorem 3.3(a) for $(f + g \circ h)^*$ is valid for any $p \in X^*$ if and only if (CQ) holds, i.e. if and only if $F^* \Box G^*$ is lower-semicontinuous regarding the subspace $X^* \times \{0\}$ and it is exact at any $(p, 0) \in X^* \times \{0\}$. Being interested to find a sufficient condition for the mentioned formula only at 0, we introduce another constraint qualification. Let us calculate first $F^* \Box G^*$. Taking some pair $(p, q) \in X^* \times Y^*$ we have by the definition

$$\begin{aligned} F^*\Box G^*(p,q) &= \inf_{\substack{\alpha\in X^*,\\ \beta\in Y^*}} \big\{F^*(\alpha,\beta)+G^*(p-\alpha,q-\beta)\big\} \\ &= \inf_{\substack{\alpha=0,\\ q-\beta\in -K^*}} \big\{g^*(\beta)+(f+(-(q-\beta)h))^*(p-\alpha)\big\} \\ &= \inf_{\beta\in K^*+q} \big\{g^*(\beta)+(f+((\beta-q)h))^*(p)\big\}. \end{aligned}$$

Now let us introduce the constraint qualification

(CQD) The function $(p,q)\mapsto \inf\limits_{\beta\in K^*+q} \big\{g^*(\beta)+(f+((\beta-q)h))^*(p)\big\}$ is lower-semicontinuous regarding the subspace $X^*\times\{0\}$ and the infimum is attained at $(0,0)$.

Using Remark 4.2 it is noticeable that (CQD) is implied by (CQ). Both these constraint qualifications ask $F^*\Box G^*$ to be lower-semicontinuous regarding the subspace $X^*\times\{0\}$, but (CQ) means moreover that this infimal convolution is exact at any $(p,0)\in X^*\times\{0\}$, whence also at $(0,0)$ as (CQD) wants.

The next statement proves that (CQD) is sufficient to ensure the formula given in Theorem 3.3(a) for $(f+g\circ h)^*$ at 0.

Theorem 4.3 *Assume (CQD) valid. Then*

$$\inf_{x\in X}\big[f(x)+g\circ h(x)\big] = \max_{\lambda\in K^*}\big\{-g^*(\lambda)-(f+(\lambda h))^*(0)\big\}.$$

Proof. If $(f+g\circ h)^*(0)=+\infty$, which means $\inf_{x\in X}\big[f(x)+g\circ h(x)\big]=-\infty$, Proposition 3.1 yields $\inf_{\lambda\in K^*}\big\{g^*(\lambda)+(f+\lambda h)^*(0)\big\}=+\infty$, thus $\sup_{\lambda\in K^*}\big\{-g^*(\lambda)-(f+\lambda h)^*(0)\big\}=-\infty$. Thus in this case the required equality holds. Assume further $(f+g\circ h)^*(0)<+\infty$. As $(0,(f+g\circ h)^*(0))\in \operatorname{epi}((f+g\circ h)^*)$, by Proposition 3.2 we have $(0,0,(f+g\circ h)^*(0))\in \operatorname{epi}(F+G)^*$. By (CQD) and Lemma 2.9 we have $\operatorname{epi}((F+G)^*)\cap(X^*\times\{0\}\times\mathbb{R}) = (\operatorname{epi}(F^*\Box G^*))\cap(X^*\times\{0\}\times\mathbb{R})$ and $F^*\Box G^*$ must be exact at $(0,0)$, i.e. there exists some $\bar\lambda\in K^*$ such that $(F^*\Box G^*)(0,0)=g^*(\bar\lambda)+(f+(\bar\lambda h))^*(0)$. This means $(0,0,(f+g\circ h)^*(0))\in \operatorname{epi}(F^*\Box G^*)$, i.e.

$$g^*(\bar\lambda)+(f+(\bar\lambda h))^*(0)=(F^*\Box G^*)(0,0)\le (f+g\circ h)^*(0).$$

By Proposition 3.1 it follows

$$(f+g\circ h)^*(0)=\min_{\lambda\in K^*}\big\{g^*(\lambda)+(f+(\lambda h))^*(0)\big\}$$

and by the definition of the conjugate one has

$$\inf_{x\in X}\big[f(x)+g\circ h(x)\big] = -(f+g\circ h)^*(0).$$

The assertion arises by combining the latter two relations. □

Remark 4.4 As $\sup_{\lambda\in K^*}\big\{-g^*(\lambda)-(f+(\lambda h))^*(0)\big\}$ is a *dual* problem to $\inf_{x\in X}\big[f(x)+g\circ h(x)\big]$, called *primal* problem, the latter statement may be seen also as a strong duality assertion, i.e. the case when the optimal objective values of the primal and dual coincide and the dual has an optimal solution (the weak duality arises from Proposition 3.1).

Remark 4.5 Within the next section we shall prove that (CQD) does not always guarantee (CQ).

Similar results are determinable also from $(\overline{CQ})$ concerning the formula given in Theorem 3.8(a) at 0.

Lemma 4.6 *The constraint qualification $(\overline{CQ})$ is equivalent to* $\operatorname{epi}((F+G)^*)\cap(X^*\times\{0\}\times\mathbb{R}) = \big(\{0\}\times\operatorname{epi}(g^*)+\big\{(p,0,r):(p,r)\in\operatorname{epi}(f^*)\big\}+\cup_{\lambda\in K^*}\big\{(p,-\lambda,r):(p,r)\in\operatorname{epi}\big(\lambda h\big)^*\big\}\big)\cap(X^*\times\{0\}\times\mathbb{R})$.

Proof. The conclusion arises from the proof of Proposition 3.6. □

The new constraint qualification we introduce is

$(\overline{CQD})$ The function $(p,q) \mapsto \inf\limits_{\beta \in K^*+q} \big\{g^*(\beta) + (f + ((\beta - q)h))^*(p)\big\}$ is lower-semicontinuous regarding the subspace $X^* \times \{0\}$ and $\operatorname{epi}(F^* \Box G^*) \cap (\{0\} \times \{0\} \times \mathbb{R}) = \big(\{0\} \times \operatorname{epi}(g^*) + \big\{(p,0,r) : (p,r) \in \operatorname{epi}(f^*)\big\} + \bigcup\limits_{\lambda \in K^*} \big\{(p,-\lambda,r) : (p,r) \in \operatorname{epi}\big(\lambda h\big)^*\big\}\big) \cap (\{0\} \times \{0\} \times \mathbb{R})$.

Theorem 4.7 *Assume* $(\overline{CQD})$ *valid. Then*

$$\inf_{x \in X} \big[f(x) + g \circ h(x)\big] = \max_{\substack{\lambda \in K^*,\\ \beta \in X^*}} \big\{ - g^*(\lambda) - f^*(\beta) - (\lambda h)^*(-\beta)\big\}.$$

Proof. We have $\operatorname{epi}((F+G)^*) \supseteq \operatorname{epi}(F^* \Box G^*) \supseteq \operatorname{epi}(F^*) + \operatorname{epi}(G^*) \supseteq \{0\} \times \operatorname{epi}(g^*) + \big\{(p,0,r) : (p,r) \in \operatorname{epi}(f^*)\big\} + \cup_{\lambda \in K^*} \big\{(p,-\lambda,r) : (p,r) \in \operatorname{epi}\big(\lambda h\big)^*\big\}$. If $(f + g \circ h)^*(0) = +\infty$ the assertion follows by (3.4). Assume further $(f + g \circ h)^*(0) < +\infty$. As $(0,(f+g\circ h)^*(0)) \in \operatorname{epi}((f + g \circ h)^*)$, by Proposition 3.2 we have $(0,0,(f+g\circ h)^*(0)) \in \operatorname{epi}(F+G)^*$. By $(\overline{CQD})$ we get $(0,0,(f+g\circ h)^*(0)) \in \operatorname{epi}(F^* \Box G^*)$ and, moreover, $(0,0,(f+g\circ h)^*(0)) \in \{0\} \times \operatorname{epi}(g^*) + \big\{(p,0,r) : (p,r) \in \operatorname{epi}(f^*)\big\} + \cup_{\lambda \in K^*} \big\{(p,-\lambda,r) : (p,r) \in \operatorname{epi}\big(\lambda h\big)^*\big\}$. Thus there are some $\lambda \in K^*$ and $\beta \in X^*$ such that $(0,0,(f+g\circ h)^*(0)) = (0,\lambda,g^*(\lambda)) + (\beta,0,f^*(\beta)) + (-\beta,-\lambda,(f+g\circ h)^*(0) - g^*(\lambda) - f^*(\beta))$, the first term of the sum in the right-hand side belonging to $\{0\} \times \operatorname{epi}(g^*)$, the second to $\big\{(p,0,r) : (p,r) \in \operatorname{epi}(f^*)\big\}$ and the last one to $\cup_{\lambda \in K^*} \big\{(p,-\lambda,r) : (p,r) \in \operatorname{epi}\big(\lambda h\big)^*\big\}$. Thus $(\lambda h)^*(-\beta) \le (f + g \circ h)^*(0) - g^*(\lambda) - f^*(\beta)$, so

$$g^*(\lambda) + f^*(\beta) + (\lambda h)^*(-\beta) \le (f + g \circ h)^*(0).$$

By (3.4) we get

$$\begin{aligned} \inf_{x \in X} \big[f(x) + g \circ h(x)\big] &= -(f + g \circ h)^*(0) = - \min_{\substack{\lambda \in K^*,\\ \beta \in X^*}} \big\{g^*(\lambda) + f^*(\beta) + (\lambda h)^*(-\beta)\big\} \\ &= \max_{\substack{\lambda \in K^*,\\ \beta \in X^*}} \big\{ - g^*(\lambda) - f^*(\beta) - (\lambda h)^*(-\beta)\big\}. \end{aligned}$$

□

Remark 4.8 This statement may also be considered as a strong duality assertion for the primal problem $\inf_{x \in X} \big[f(x) + g \circ h(x)\big]$ and another dual of it, namely $\sup_{\lambda \in K^*, \beta \in X^*} \big\{ - g^*(\lambda) - f^*(\beta) - (\lambda h)^*(-\beta)\big\}$.

Remark 4.9 We have that $(\overline{CQD})$ implies (CQD). This comes quickly from the beginning of the proof of Theorem 4.7 and the way these constraint qualifications are formulated.

Remark 4.10 When $h = \operatorname{id}_X$ and $X = Y$ we obtain the classical Fenchel duality assertion as special case of the, then equivalent, Theorems 4.3 and 4.7. The corresponding statement will be given in Section 5.2.

5 Special cases

5.1 The case $f = 0$

When $f(x) = 0 \ \forall x \in X$ and $(h(X) + K) \cap \operatorname{dom}(g) \neq \emptyset$, then the constraint qualifications (CQ) and $(\overline{CQ})$ become both

(CQ_1) $\{0\} \times \operatorname{epi}(g^*) + \bigcup\limits_{\lambda \in K^*} \big\{(a,-\lambda,r) : (a,r) \in \operatorname{epi}((\lambda h)^*)\big\}$ is closed regarding the subspace $X^* \times \{0\} \times \mathbb{R}$.

We have the following assertion.

Theorem 5.1 (a) (CQ_1) *is fulfilled if and only if for any* $p \in X^*$ *one has*

$$\big(g \circ h\big)^*(p) = \min_{\lambda \in K^*} \big\{g^*(\lambda) + \big(\lambda h\big)^*(p)\big\}.$$

(b) *If* (CQ_1) *is fulfilled then for any* $x \in h^{-1}(\mathrm{dom}(g))$ *one has*

$$\partial\big(g \circ h\big)(x) = \bigcup_{\lambda \in \partial g(h(x))} \partial(\lambda h)(x).$$

Remark 5.2 The formula in Theorem 5.1(b) is given also in [10], but there g is required to be continuous.

We deliver also the strong duality assertion for the primal problem $\inf_{x \in X} \big[g \circ h(x)\big]$ and its dual problem $\sup_{\lambda \in K^*} \big\{ - g^*(\lambda) - (\lambda h)^*(0)\big\}$. The constraint qualifications (CQD) and $(\overline{CQD})$ turn both into

(CQD_1) The function $(p, q) \mapsto \inf\limits_{\beta \in K^* + q} \big\{g^*(\beta) + ((\beta - q)h)^*(p)\big\}$ is lower-semicontinuous regarding the subspace $X^* \times \{0\}$ and the infimum is attained at $(0, 0)$,

and we have the following statement.

Theorem 5.3 *Assume* (CQD_1) *valid. Then*

$$\inf_{x \in X} \big[g \circ h(x)\big] = \max_{\lambda \in K^*} \big\{ - g^*(\lambda) - (\lambda h)^*(0)\big\}.$$

5.2 The case h linear

Let $A : X \to Y$ be a linear continuous mapping and take $h(x) = Ax$ for all $x \in X$. Moreover let $K = \{0\}$, so h is K -convex and K -epi-closed as required and $K^* = Y^*$. The condition regarding the domains of the functions involved is in this case $A(\mathrm{dom}(f)) \cap \mathrm{dom}(g) \neq \emptyset$. The constraint qualification derived from (CQ) would be in this case

(CQ_2) $\{0\} \times \mathrm{epi}(g^*) + \bigcup\limits_{\lambda \in Y^*} \big\{(a, -\lambda, r) : (a, r) \in \mathrm{epi}((f + (\lambda A))^*)\big\}$ is closed regarding the subspace $X^* \times \{0\} \times \mathbb{R}$,

while $(\overline{CQ})$ turns into

$(\overline{CQ_2})$ $\{0\} \times \mathrm{epi}(g^*) + \big\{(p, 0, r) : (p, r) \in \mathrm{epi}(f^*)\big\} + \bigcup\limits_{\lambda \in Y^*} \big\{(p, -\lambda, r) : (p, r) \in \mathrm{epi}((\lambda A)^*)\big\}$ is closed regarding the subspace $X^* \times \{0\} \times \mathbb{R}$.

We prove the equivalence of these constraint qualifications by using the conjugate functions

$$(\lambda A)^*(p) = \begin{cases} 0, & \text{if } A^*\lambda = p, \\ +\infty, & \text{otherwise,} \end{cases}$$

and

$$\begin{aligned} \big(f + (\lambda A)\big)^*(p) &= \sup_{x \in X} \big\{\langle p, x\rangle - f(x) - \langle \lambda, Ax\rangle\big\} = \sup_{x \in X} \big\{\langle p, x\rangle - f(x) - \langle A^*\lambda, x\rangle\big\} \\ &= \sup_{x \in X} \big\{\langle p - A^*\lambda, x\rangle - f(x)\big\} = f^*\big(p - A^*\lambda\big), \end{aligned}$$

for any $\lambda \in Y^*$ and any $p \in X^*$. One has $\cup_{\lambda \in Y^*} \big\{(a, -\lambda, r) : (a, r) \in \mathrm{epi}((f + (\lambda A))^*)\big\} = \cup_{\lambda \in Y^*} \big\{(a, -\lambda, r) : (a - A^*\lambda, r) \in \mathrm{epi}(f^*)\big\} = \cup_{\lambda \in Y^*} \big\{(p + A^*\lambda, -\lambda, r) : (p, r) \in \mathrm{epi}(f^*)\big\} = \big\{(p, 0, r) : (p, r) \in \mathrm{epi}(f^*)\big\} + \big\{(A^*\lambda, -\lambda, 0) : \lambda \in Y^*\big\}$. On the other hand, $\cup_{\lambda \in Y^*} \big\{(p, -\lambda, r) : (p, r) \in \mathrm{epi}((\lambda A)^*)\big\} = \cup_{\lambda \in Y^*} \big\{(A^*\lambda, -\lambda, r) : 0 \leq r\big\} = \big\{(A^*\lambda, -\lambda, 0) : \lambda \in Y^*\big\} + \{(0, 0)\} \times \mathbb{R}_+$ and $\{0\} \times \mathrm{epi}(g^*) + \{(0, 0)\} \times \mathbb{R}_+ = \{0\} \times \mathrm{epi}(g^*)$. Therefore, $\{0\} \times \mathrm{epi}(g^*) + \cup_{\lambda \in Y^*} \big\{(a, -\lambda, r) : (a, r) \in \mathrm{epi}((f + (\lambda A))^*)\big\} = \{0\} \times \mathrm{epi}(g^*) + \big\{(p, 0, r) : (p, r) \in$

$\operatorname{epi}(f^*)\big\} + \big\{(A^*\lambda, -\lambda, 0) : \lambda \in Y^*\big\} = \{0\} \times \operatorname{epi}(g^*) + \big\{(p, 0, r) : (p, r) \in \operatorname{epi}(f^*)\big\} + \cup_{\lambda \in Y^*} \big\{(p, -\lambda, r) : (p, r) \in \operatorname{epi}((\lambda A)^*)\big\}$.

We show that the results delivered by the main statement of the paper in this case are actually the ones given by Boţ and Wanka in Theorem 3.1 in [3], under some other constraint qualification, namely

$$(RC_A) \qquad \operatorname{epi}(f^*) + A^* \times \operatorname{id}_{\mathbb{R}} \big(\operatorname{epi}(g^*)\big) \text{ is closed in the product topology of } \big(X^*, w(x^*, X)\big) \times \mathbb{R}.$$

Let us show the equivalence of (CQ_2) and (RC_A). First, by Lemma 2.9 we know that $\operatorname{epi}(f + g \circ A)^* = \operatorname{cl}(\operatorname{epi}(f^*) + \operatorname{epi}(g \circ A)^*)$. As $\operatorname{epi}(g \circ A)^*) = \operatorname{cl}(A^* \times \operatorname{id}_{\mathbb{R}}(\operatorname{epi}(g^*)))$ (see for instance [3]), we get $\operatorname{epi}(f + g \circ A)^* = \operatorname{cl}(\operatorname{epi}(f^*) + \operatorname{cl}(A^* \times \operatorname{id}_{\mathbb{R}}(\operatorname{epi}(g^*))))$, further writable as

$$\operatorname{epi}(f + g \circ A)^* = \operatorname{cl}(\operatorname{epi}(f^*) + (A^* \times \operatorname{id}_{\mathbb{R}}(\operatorname{epi}(g^*)))). \tag{5.1}$$

We have that (CQ_2) is equivalent to the fact that for any $(p, 0, r) \in \operatorname{epi}(F + G)^*$ it follows $(p, 0, r) \in \operatorname{epi}(F^*) + \operatorname{epi}(G^*)$, and, by Proposition 3.2 and using the calculation of the conjugate of $f + (\lambda A)$, to the implication $[\forall (p, r) \in \operatorname{epi}(f + g \circ A)^* \Rightarrow (p, 0, r) \in \{0\} \times \operatorname{epi}(g^*) + \cup_{\lambda \in Y^*} \big\{(a, -\lambda, t) : (a - A^*\lambda, t) \in \operatorname{epi} f^*\big\}]$. This is further equivalent to saying that $\forall (p, r) \in \operatorname{epi}(f + g \circ A)^*$ there is some $\lambda \in Y^*$ such that $f^*(p - A^*\lambda) \leq r - g^*(\lambda)$, which means, denoting $q = p - A^*\lambda$ and $s = r - g^*(\lambda)$, that for any $(p, r) \in \operatorname{epi}(f + g \circ A)^*$ there is some $\lambda \in Y^*$ such that $(p, r) = (q, s) + (A^*\lambda, g^*(\lambda))$, where $(q, s) \in \operatorname{epi}(f^*)$. Noticing that $(A^*\lambda, g^*(\lambda)) \in A^* \times \operatorname{id}_{\mathbb{R}} \big(\operatorname{epi}(g^*)\big)$, we conclude that (CQ_2) is equivalent to the fact that any $(p, r) \in \operatorname{epi}(f + g \circ A)^*$ belongs also to $\operatorname{epi}(f^*) + A^* \times \operatorname{id}_{\mathbb{R}} \big(\operatorname{epi}(g^*)\big)$. By (5.1) we know that $\operatorname{epi}(f^*) + A^* \times \operatorname{id}_{\mathbb{R}} \big(\operatorname{epi}(g^*)\big)$ is a subset of $\operatorname{epi}(f + g \circ A)^* = \operatorname{cl}(\operatorname{epi}(f^*) + (A^* \times \operatorname{id}_{\mathbb{R}}(\operatorname{epi}(g^*))))$, so we get that (CQ_2) is equivalent to the relation $\operatorname{cl}(\operatorname{epi}(f^*) + (A^* \times \operatorname{id}_{\mathbb{R}}(\operatorname{epi}(g^*)))) = \operatorname{epi}(f^*) + A^* \times \operatorname{id}_{\mathbb{R}} \big(\operatorname{epi}(g^*)\big)$, i.e. (CQ_2) is equivalent to (RC_A).

Regarding the subdifferentials, Theorem 3.3 says that under (CQ_2) one has for any $x \in A(\operatorname{dom}(f)) \cap \operatorname{dom}(g)$

$$\partial\big(f + g \circ A\big)(x) = \bigcup_{\lambda \in \partial g(Ax)} \partial\big(f + (\lambda A)\big)(x),$$

while Theorem 3.1 in [3] asserts that (RC_A), which is equivalent to (CQ_2), yields

$$\partial\big(f + g \circ A\big)(x) = \partial f(x) + A^* \partial g(Ax).$$

Taking $p \in \cup_{\lambda \in \partial g(Ax)} \partial(f + (\lambda A))(x)$ is the same as asserting that there is some $\lambda \in \partial g(Ax)$ and we also have $p \in \partial\big(f + (\lambda A)\big)(x)$. This is equivalent to saying that there is some $\lambda \in \partial g(Ax)$ such that $\langle p, x\rangle = \big(f + (\lambda A)\big)(x) + \big(f + (\lambda A)\big)^*(p) = f(x) + \langle \lambda, Ax\rangle + f^*(p - A^*\lambda)$. The last relation may be rewritten $f^*(p - A^*\lambda) + f(x) = \langle p - A^*\lambda, x\rangle$, which is actually $p - A^*\lambda \in \partial f(x)$. Therefore we have proved that $p \in \cup_{\lambda \in \partial g(Ax)} \partial(f + (\lambda A))(x)$ is equivalent to the existence of some $\lambda \in \partial g(Ax)$ such that $p - A^*\lambda \in \partial f(x)$, i.e. $\partial(f + (\lambda A))(x) = \partial f(x) + A^* \partial g(Ax)$. Thus we obtain Theorem 3.1 in [3] as a special case of our Theorem 3.3, as follows.

Theorem 5.4 *We have*

(a) (RC_A) *is fulfilled if and only if for any* $p \in X^*$

$$(f + g \circ A)^*(p) = \min_{\lambda \in Y^*} \big[g^*(\lambda) + f^*(p - A^*\lambda)\big].$$

(b) *If* (RC_A) *is fulfilled, then for any* $x \in \operatorname{dom}(f) \cap A^{-1}(\operatorname{dom}(g))$

$$\partial(f + g \circ A)(x) = \partial f(x) + A^* \partial g(Ax).$$

Remark 5.5 The constraint qualification (RC_A) is weaker than some well-known generalized interior-point regularity conditions given in the literature. We do not mention all of them here, though we refer the reader to [9] and [15] (Theorem 2.8.3), to see how they look and how they imply one another, and to [3], where (RC_A) is proved to be weaker than the weakest of them.

Like in the general case we give also a constraint qualification inspired from (CQ_2) that guarantees the validity of the formula given in Theorem 5.4(a) holds at 0. We are interested to investigate the connections between what does (CQD) mean in the present configuration and the condition (FRC_A) in [3],

(FRC_A) $f^*\square A^*g^*$ is lower-semicontinuous and $\operatorname{epi}(f^*\square A^*g^*) \cap (\{0\} \times \mathbb{R}) = (\operatorname{epi}(f^*) + A^* \times \mathrm{id}_{\mathbb{R}}(\operatorname{epi}(g^*))) \cap (\{0\} \times \mathbb{R})$.

We have, for $(p,q) \in X^* \times Y^*$,

$$(F^*\square G^*)(p,q) = \inf_{\beta \in Y^*+q} \big\{g^*(\beta) + (f + ((\beta - q)A))^*(p)\big\} = \inf_{\beta \in Y^*+q} \big[g^*(\beta) + f^*(p - A^*(\beta - q))\big].$$

Thus (CQD) is in this case

(CQD_2) $(p,q) \mapsto \inf_{\beta \in Y^*+q} \big[g^*(\beta) + f^*(p - A^*(\beta - q))\big]$ is lower-semicontinuous regarding the subspace $X^* \times \{0\}$ and the infimum is attained at (0, 0).

Let us start by proving that $f^*\square A^*g^*$ is lower-semicontinuous if and only if $F^*\square G^*$ is lower-semicontinuous regarding the subspace $X^* \times \{0\}$.

For any $p \in X^*$ we have $(f^*\square A^*g^*)(p) = \inf_{w \in X^*} \big[f^*(p - w) + A^*g^*(w)\big] = \inf_{w \in X^*} \big[f^*(p - w) + \inf_{A^*\lambda = w} g^*(\lambda)\big] = \inf_{w \in X^*} \big[f^*(p - w) + g^*(\lambda) : A^*\lambda = w\big] = \inf_{\lambda \in Y^*} \big[f^*(p - A^*\lambda) + g^*(\lambda)\big] = (F^*\square G^*)(p, 0)$.

The lower-semicontinuity of $F^*\square G^*$ regarding the subspace $X^* \times \{0\}$ means, by Lemma 2.9, $\operatorname{epi}(F^*\square G^*) \cap (X^* \times \{0\} \times \mathbb{R}) = \operatorname{epi}((F+G)^*) \cap (X^* \times \{0\} \times \mathbb{R})$. This is the same as $(p, 0, r) \in \operatorname{epi}(F^*\square G^*)$ is equivalent to $(p, 0, r) \in \operatorname{epi}((F+G)^*)$ and further, by Proposition 3.2, to $(p, r) \in \operatorname{epi}(f + g \circ A)^*$. As $(p, 0, r) \in \operatorname{epi}(F^*\square G^*)$ means $\inf_{\lambda \in Y^*} \big[g^*(\lambda) + f^*(p - A^*\lambda)\big] \leq r$, which is exactly $(p, r) \in \operatorname{epi}((f^*\square A^*g^*))$, it follows that the lower-semicontinuity of $F^*\square G^*$ regarding the subspace $X^* \times \{0\}$ is equivalent to $\operatorname{epi}(f + g \circ A)^* \subseteq \operatorname{epi}(f^*\square A^*g^*)$.

Using Lemma 2.9, we have $\operatorname{cl}(\operatorname{epi}(f^*\square A^*g^*)) = \operatorname{cl}(\operatorname{epi}(f^*) + \operatorname{epi}(A^*g^*)) = \operatorname{cl}\,(\operatorname{epi}(f^*) + \operatorname{cl}(\operatorname{epi}(A^*g^*)))$. By Theorem 2.3 in [3] we get $\operatorname{cl}(\operatorname{epi}(f^*\square A^*g^*)) = \operatorname{cl}(\operatorname{epi}(f^*) + \operatorname{epi}((g \circ A)^*)) = \operatorname{epi}(f + g \circ A)^*$.

Therefore, $F^*\square G^*$ is lower-semicontinuous regarding the subspace $X^* \times \{0\}$ if and only if $\operatorname{epi}(f^*\square A^*g^*)$ is closed, i.e. $f^*\square A^*g^*$ is lower-semicontinuous.

By Theorem 2.3 in [3] and Lemma 2.9, $\operatorname{epi}(f^*\square A^*g^*) \subseteq \operatorname{cl}(\operatorname{epi}(f^*\square A^*g^*)) = \operatorname{cl}(\operatorname{epi}(f^*) + \operatorname{epi}(A^*g^*)) = \operatorname{cl}(\operatorname{epi}(f^*) + \operatorname{cl}(\operatorname{epi}(A^*g^*))) = \operatorname{cl}(\operatorname{epi}(f^*) + \operatorname{epi}((g \circ A)^*)) = \operatorname{epi}(f + g \circ A)^*$.

The exactness of $F^*\square G^*$ at $(0, 0)$ means that there is some $\bar{\lambda} \in Y^*$ such that $\inf_{\lambda \in Y^*} \big[g^*(\lambda) + f^*(-A^*(\lambda))\big] = g^*(\bar{\lambda}) + f^*(-A^*(\bar{\lambda}))$.

Assume (FRC_A) valid. This means that $\operatorname{epi}(f^*\square A^*g^*)$ is closed and $\operatorname{epi}(f^*\ \square A^*g^*) \cap (\{0\} \times \mathbb{R}) = (\operatorname{epi}(f^*) + A^* \times \mathrm{id}_{\mathbb{R}}(\operatorname{epi}(g^*))) \cap (\{0\} \times \mathbb{R})$. The closedness yields $\operatorname{epi}(f^*\square A^*g^*) \cap (\{0\} \times \mathbb{R}) = \operatorname{epi}(f + g \circ A)^* \cap (\{0\} \times \mathbb{R})$, so the second condition is equivalent to

$$(\operatorname{epi}(f^*) + A^* \times \mathrm{id}_{\mathbb{R}}(\operatorname{epi}(g^*))) \cap (\{0\} \times \mathbb{R}) = \operatorname{epi}(f + g \circ A)^* \cap (\{0\} \times \mathbb{R}),$$

i.e. for any $(0, (f + g \circ A)^*(0)) \in \operatorname{epi}(f + g \circ A)^*$ there is some $\lambda \in Y^*$ such that $f^*(-A^*\lambda) \leq (f + g \circ A)^*(0) - g^*(\lambda)$. As $(f + g \circ A)^*(0) \leq f^*(-A^*\lambda) + g^*(\lambda)$ for any $\lambda \in Y^*$, the latter inequality means actually the exactness of $F^*\square G^*$ at $(0, 0)$. Thus (CQD_2) stands if and only if (FRC_A) is valid. Using the facts and observations from above we give the following statement, derived from Theorem 4.3.

Theorem 5.6 *If (FRC_A) is fulfilled, then*

$$\inf_{x \in X} \big[f(x) + g(Ax)\big] = \max_{\lambda \in Y^*} \big\{ - f^*(-A^*\lambda) - g^*(\lambda)\big\}.$$

Remark 5.7 As proved in [3], but also according to our general case, we know that (RC_A) implies (FRC_A). The reverse is not always true and Example 5.11 provides a counter-example.

When A is the identity mapping of X, i.e. $A = \mathrm{id}_X$, the constraint qualification (CQ_2), equivalent to (RC_A) becomes, as in [3],

(CQ_3) $\qquad$ $\mathrm{epi}(f^*) + \mathrm{epi}(g^*)$ is closed in the product topology of $(X^*, w(X^*, X)) \times \mathbb{R}$.

The conditions regarding domains becomes $\mathrm{dom}(f) \cap \mathrm{dom}(g) \neq \emptyset$ and we have the following statement.

Theorem 5.8 (a) (CQ_3) *is fulfilled if and only if for any* $p \in X^*$

$$(f+g)^*(p) = \min_{\lambda \in X^*} \big[g^*(\lambda) + f^*(p-\lambda)\big].$$

(b) *If* (CQ_3) *is fulfilled, then for any* $x \in \mathrm{dom}(f) \cap \mathrm{dom}(g)$

$$\partial(f+g)(x) = \partial f(x) + \partial g(x).$$

The constraint qualification (CQD_2) becomes in this case, as in [3],

(CQD_3) $\qquad$ $f^* \Box g^*$ is a lower-semicontinuous function and is exact at 0,

and we give the following result, which is actually the classical Fenchel duality statement, but under weaker assumptions.

Theorem 5.9 *If* (CQD_3) *is valid, then*

$$\inf_{x \in X} \big[f(x) + g(x)\big] = \max_{\lambda \in X^*} \big\{ -f^*(-\lambda) - g^*(\lambda)\big\}.$$

Remark 5.10 We have that (CQ_3) implies (CQD_3), while the reverse implication does not always hold, as proved by the following example.

Example 5.11 Consider in $\mathbb{R}^2$ the unit ball B. Let $f, g : \mathbb{R}^2 \to \overline{\mathbb{R}}$, $f = \delta_B$ and $g = \delta_{[1,+\infty)\times\mathbb{R}}$. Then, for $y_1, y_2 \in \mathbb{R}$, $f^*(y_1, y_2) = \|(y_1, y_2)\|$ and

$$g^*(y_1, y_2) = \begin{cases} y_1, & \text{if } y_1 \leq 0, y_2 = 0, \\ +\infty, & \text{otherwise.} \end{cases}$$

For any $(y_1, y_2) \in \mathbb{R}^2$ we have

$$\begin{aligned} (f^* \Box g^*)(y_1, y_2) &= \inf_{(a_1,a_2)\in\mathbb{R}^2} \big[f^*(a_1, a_2) + g^*(y_1 - a_1, y_2 - a_2)\big] \\ &= \inf_{\substack{(a_1,a_2)\in\mathbb{R}^2, \\ a_1 \geq y_1, a_2 = y_2}} \big[\|(a_1, a_2)\| + y_1 - a_1\big] \\ &= y_1 + \inf_{a_1 \geq y_1} \Big[\sqrt{a_1^2 + y_2^2} - a_1\Big] = y_1. \end{aligned}$$

It is clear that $f^* \Box g^*$ is lower-semicontinuous and, moreover, the infimum within is attained only when $y_2 = 0$. Thus (CQD_3) is valid for this choice of functions, while (CQ_3) is violated, as it is equivalent to saying that $f^* \Box g^*$ is lower-semicontinuous and exact everywhere. At $(0, 1)$, for instance, $f^* \Box g^*$ is not exact as the infimum from its formula is not attained. $\Box$

Remark 5.12 To the best of our knowledge (CQD_3) is the weakest constraint qualification considered so far in the literature that guarantees Fenchel duality.

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