

ε - optimality conditions for composed convex optimization problems

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Abstract. The aim of the present paper is to provide a formula for the ε - subdifferential of $f + g \circ h$ different from the ones which can be found in the existent literature. Further we equivalently characterize this formula by using a so-called closedness type regularity condition expressed by means of the epigraphs of the conjugates of the functions involved. Even more, using the ε - subdifferential formula we are able to derive necessary and sufficient conditions for the ε - optimal solutions of composed convex optimization problems.

Key Words. conjugate functions, composed convex functions, ε - subdifferential, ε - optimality conditions

1 Introduction

In many practical applications it is necessary to solve an optimization problem, i.e. to find a point where the minimal or the maximal value a function can take is attained. Unfortunately this is not always possible, because an optimization problem does not necessarily have an optimal solution (such a situation can occur even if its optimal objective value can be determined). Thus we are forced sometimes to deal not with optimal solutions, but with

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approximate ones. Nevertheless, this is not a major drawback if the approximate solutions we can provide act well to our purposes. Even more, from a computational point of view it is much more advantageous to find approximate solutions as the goal of an algorithm is to deliver an approximate solution and not an optimal one (supposing that there exist an optimal solution, it is seldom possible to find it, but even in such situations this usually means a waste of time and resources). Therefore the study of the approximate solutions of an optimization problem is of great interest from many points of view and many authors have turned their attention to this topic.

It is well-known that a proper and convex function $f : \mathbb{R}^n \rightarrow \overline{\mathbb{R}}$ reaches its minimal value at $\bar{x} \in \text{dom}(f)$ if and only if $0 \in \partial f(\bar{x})$. Using this property one can easily characterize the optimal solutions of an optimization problem by means of the subdifferential. A similar property holds also for the approximate solutions of an optimization problem, which can be characterized by means of the ε - subdifferentials. From the large amount of works which deal with such a topic we mention here only some of them, namely [7, 16, 17].

Many optimization problems generated by practical fields like location and transports or economics and finance involve composed convex functions. Therefore, in order to be able to characterize the approximate solutions of an optimization problem involving composed convex functions, it is important to provide a formula for the ε - subdifferential of a composed convex function (the interested reader can consult the papers [1, 3, 4, 9–11, 13] for more information regarding the optimization problems involving composed convex functions).

The paper is organized as follows. In the second section we present some notions and results used later. The third section contains the main results of the paper. We provide a formula for the ε - subdifferential of a composed convex function of type $f + g \circ h$. The formula we give is a refinement of the one provided in Theorem 2.8.10 in [17] and, moreover, for $\varepsilon = 0$ we rediscover the subdifferential formula given in [1] and [9]. We prove that the formula we give holds if and only if a closedness type regularity condition expressed using only epigraphs of the conjugates of the functions involved is fulfilled. More on this class of regularity conditions, which has been introduced in the literature in the last years, can be found for instance in [4–8]. Further we consider an optimization problem the objective function of which is of type $f + g \circ h$. Using the connection between the ε - subdifferential of a convex function and its conjugate function we are able to point out necessary and

sufficient optimality conditions for the ε - optimal solutions of the problem. In the fourth section of the paper special instances of the functions f , g and h are considered and some special cases of our general results are provided.

2 Preliminary notions and results

Let X and Y be two separated locally convex spaces and let X^* and Y^* be their topological dual spaces endowed with the weak* topologies $w(X^*, X)$ and $w(Y^*, Y)$, respectively. Throughout the entire paper we denote by $\langle x^*, x \rangle = x^*(x)$ the value of the continuous linear functional $x^* \in X^*$ at $x \in X$. For any $K \subseteq Y$ nonempty and closed convex cone we define its *dual cone* as $K^* = \{y^* \in Y^* : \langle y^*, y \rangle \geq 0, \forall y \in K\}$. The cone K induces on Y a partial order " \leq_K " defined for $x, y \in Y$ by

$$x \leq_K y \Leftrightarrow y - x \in K.$$

To Y we attach an element $\infty_Y \notin Y$ which is the greatest element with respect to \leq_K and we denote $Y^\bullet = Y \cup \{\infty_Y\}$. Then for all $y \in Y^\bullet$ it holds $y \leq_K \infty_Y$, while the addition and multiplication with a positive real number can be naturally extended to Y^\bullet by taking

$$y + \infty_Y = \infty_Y + y = \infty_Y \quad \text{and} \quad t\infty_Y = \infty_Y$$

for all $y \in Y$ and $t \geq 0$. Moreover, for $\lambda \in K^*$ we assume that $\langle \lambda, \infty_Y \rangle = +\infty$.

For a given function $f : X \rightarrow \overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$, we denote by $\text{dom}(f) = \{x \in X : f(x) < +\infty\}$ its *effective domain* and by $\text{epi}(f) = \{(x, r) : x \in X, r \in \mathbb{R}, f(x) \leq r\}$ its *epigraph*, respectively. The function f is called *proper* if its effective domain is a nonempty set and $f(x) > -\infty$ for all $x \in X$. The *infimal convolution* of two proper functions $f, g : X \rightarrow \overline{\mathbb{R}}$, $f \square g : X \rightarrow \overline{\mathbb{R}}$, is defined as $(f \square g)(x) = \inf \{f(y) + g(x - y) : y \in X\}$. We say that $f \square g$ is *exact* if for all $x \in X$ there exists $y \in X$ such that $(f \square g)(x) = f(y) + g(x - y)$. Having a set $C \subseteq X$ we also use its *indicator function* which is defined by

$$\delta_C : X \rightarrow \overline{\mathbb{R}}, \quad \delta_C(x) = \begin{cases} 0, & x \in C, \\ +\infty, & \text{otherwise,} \end{cases}$$

For $f : X \rightarrow \overline{\mathbb{R}}$ an arbitrary function by the *conjugate function* of f we understand the function $f^* : X^* \rightarrow \overline{\mathbb{R}}$, $f^*(x^*) = \sup_{x \in X} \{\langle x^*, x \rangle - f(x)\}$.

The following relation is the well-known *Fenchel - Young inequality*

$$f^*(x^*) + f(x) \geq \langle x^*, x \rangle, \forall x \in X, \forall x^* \in X^*. \quad (1)$$

If f is a proper function the ε - *subdifferential* of f at $\bar{x} \in \text{dom}(f)$ is the set

$$\partial_\varepsilon f(\bar{x}) = \left\{ x^* \in X^* : f(x) - f(\bar{x}) \geq \langle x^*, x - \bar{x} \rangle - \varepsilon, \forall x \in X \right\},$$

where $\varepsilon \geq 0$ is a non - negative real number. For $\varepsilon = 0$ we denote $\partial f(\bar{x}) = \partial_0 f(\bar{x})$ and we say that the function f is *subdifferentiable* at $\bar{x} \in \text{dom}(f)$ if $\partial f(\bar{x}) \neq \emptyset$. It can be easily proved (see, for instance, [17]) that for all $\bar{x} \in \text{dom}(f)$ and $x^* \in X^*$ we have

$$f(\bar{x}) + f^*(x^*) \leq \langle x^*, \bar{x} \rangle + \varepsilon \Leftrightarrow x^* \in \partial_\varepsilon f(\bar{x}). \quad (2)$$

Definition 1 A function $g : Y \rightarrow \overline{\mathbb{R}}$ is called *K - increasing* if for all $x, y \in Y$ fulfilling $x \leq_K y$ the inequality $g(x) \leq g(y)$ holds.

Definition 2 The function $h : X \rightarrow Y^\bullet$ is called *K - convex* if for all $x, y \in Y$ and $\lambda \in [0, 1]$ it fulfills the property

$$h(\lambda x + (1 - \lambda)y) \leq_K \lambda h(x) + (1 - \lambda)h(y).$$

Without fear of confusion we say that the function $h : X \rightarrow Y^\bullet$ is *proper* if its *effective domain* $\text{dom}(h) = \{x \in X : h(x) \in Y\}$ is a non - empty set. Moreover, for all $\lambda \in K^*$ the function $(\lambda h) : X \rightarrow \overline{\mathbb{R}}$ is defined as $(\lambda h)(x) = \langle \lambda, h(x) \rangle$.

Definition 3 A function $h : X \rightarrow Y^\bullet$ is called *star K - lower semicontinuous* if the function (λh) is lower semicontinuous for all $\lambda \in K^*$.

During the last decades various generalizations of the notion of lower semicontinuity have been given (we mention here only two papers, namely [9] and [14], where the authors introduce *K - lower semicontinuous* and *K - sequentially lower semicontinuous* functions). We use the *star K - lower semicontinuity* since it is a weak topological assumption (one can prove that a *K - lower semicontinuous* function is *star K - lower semicontinuous*, but the reverse implications fails in general), but nevertheless enough for our aim.

If $A : X \rightarrow Y$ is a linear continuous operator, then $A^* : Y^* \rightarrow X^*$ defined such that

$$\langle A^*y^*, x \rangle = \langle y^*, Ax \rangle \quad \forall x \in X, \forall y^* \in Y^*$$

is called its adjoint. We consider the identity function over the space \mathbb{R} defined as follows $\text{id}_{\mathbb{R}} : \mathbb{R} \rightarrow \mathbb{R}$, $\text{id}_{\mathbb{R}}(r) = r$ for all $r \in \mathbb{R}$. We define also the function $A^* \times \text{id}_{\mathbb{R}} : Y^* \times \mathbb{R} \rightarrow X^* \times \mathbb{R}$, $A^* \times \text{id}_{\mathbb{R}}(y^*, r) = (A^*y^*, r)$ for all $(y^*, r) \in Y^* \times \mathbb{R}$.

We also mention that everywhere in the present paper we write \min (\max) instead of \inf (\sup) when the infimum (supremum) is attained.

3 The general case

The functions $f : X \rightarrow \overline{\mathbb{R}}$, $g : Y \rightarrow \overline{\mathbb{R}}$ and $h : X \rightarrow Y^\bullet$ are taken such that f is proper, convex and lower semicontinuous, g is proper, convex, lower semicontinuous and K - increasing, while h is proper, K - convex and star K - lower semicontinuous, respectively. The function g will be extended to Y^\bullet by taking $g(\infty_Y) = +\infty$. Moreover, throughout the entire section we assume that the condition $h(\text{dom}(f) \cap \text{dom}(h)) \cap \text{dom}(g) \neq \emptyset$ is fulfilled (one can easily prove that this condition secures the properness of the function $f + g \circ h$).

Consider an arbitrary $p^* \in X^*$. By using the Fenchel-Young inequality one can easily show (see, for instance, (2.68) in [17]) that for all $\lambda \in K^*$ and for all $x^* \in X^*$ the inequality

$$(f + g \circ h)^*(p^*) \leq g^*(\lambda) + f^*(x^*) + (\lambda h)^*(p^* - x^*) \quad (3)$$

is always fulfilled. Under some circumstances (see also [1] for details) the existence of some $\lambda \in K^*$ and $x^* \in X^*$ such that equality holds in (3) is secured. A necessary and sufficient condition for this is given in the following theorem (see [2] for the proof).

Theorem 1 *The regularity condition*

$$(RC) \quad \text{epi}(f^*) + \bigcup_{\lambda \in \text{dom}(g^*)} \left(\text{epi}((\lambda h)^*) + (0, g^*(\lambda)) \right) \text{ is closed}$$

is fulfilled if and only if

$$(f + g \circ h)^*(p^*) = \min_{\substack{\lambda \in K^*, \\ x^* \in X^*}} \left\{ g^*(\lambda) + f^*(x^*) + (\lambda h)^*(p^* - x^*) \right\}, \forall p^* \in X^*. \quad (4)$$

The equivalence in Theorem 1 follows by applying for instance Theorem 3.1 in [8] or Lemma 2 in [2] (see also [15]) for the following so-called perturbation function $\Phi : X \times Y \times X \rightarrow \overline{\mathbb{R}}$, $\Phi(x, y, z) = f(x+z) + g(h(x) - y)$. The hypotheses made in the beginning of this section ensure that Φ is proper, convex and lower semicontinuous. The lower semicontinuity of Φ is guaranteed even in the situation when h is (only) star- K lower semicontinuous.

The next theorem gives a general formula for the ε - subdifferential of the function $f + g \circ h$, which holds in case (RC) is fulfilled. We would like to mention that the formula we give is a refinement of the one proved in Theorem 2.8.10 in [17]. Moreover, for $\varepsilon = 0$ we rediscover the subdifferential formula given in [1, 2, 9].

Theorem 2 *Suppose that the regularity condition (RC) is fulfilled. Then for all $x \in \text{dom}(f) \cap \text{dom}(h) \cap h^{-1}(\text{dom}(g))$ and for all $\varepsilon \geq 0$ we have*

$$\partial_\varepsilon(f + g \circ h)(x) = \bigcup_{\substack{\varepsilon_1, \varepsilon_2, \varepsilon_3 \geq 0, \\ \varepsilon_1 + \varepsilon_2 + \varepsilon_3 = \varepsilon}} \left\{ \partial_{\varepsilon_1} f(x) + \partial_{\varepsilon_2}(\lambda h)(x) : \lambda \in K^* \cap \partial_{\varepsilon_3} g(h(x)) \right\}. \quad (5)$$

Proof. " \subseteq " Let $x \in \text{dom}(f) \cap \text{dom}(h) \cap h^{-1}(\text{dom}(g))$ and $\varepsilon \geq 0$ be arbitrary chosen. Take $x^* \in \partial_\varepsilon(f + g \circ h)(x)$. According to relation (2) it holds

$$(f + g \circ h)^*(x^*) + (f + g \circ h)(x) \leq \langle x^*, x \rangle + \varepsilon.$$

Taking into consideration Theorem 1, there exist $\lambda \in K^*$ and $x_1^*, x_2^* \in X^*$, $x_1^* + x_2^* = x^*$, such that

$$g^*(\lambda) + f^*(x_1^*) + (\lambda h)^*(x_2^*) + (f + g \circ h)(x) \leq \langle x_1^* + x_2^*, x \rangle + \varepsilon.$$

The last inequality can be equivalently written as

$$\begin{aligned} & [f^*(x_1^*) + f(x) - \langle x_1^*, x \rangle] + [(\lambda h)^*(x_2^*) + (\lambda h)(x) - \langle x_2^*, x \rangle] \\ & + [g^*(\lambda) + g(h(x)) - \langle \lambda, h(x) \rangle] \leq \varepsilon. \end{aligned}$$

Further we take $\tilde{\varepsilon}_1 := f^*(x_1^*) + f(x) - \langle x_1^*, x \rangle$, $\bar{\varepsilon}_2 := (\lambda h)^*(x_2^*) + (\lambda h)(x) - \langle x_2^*, x \rangle$, and $\bar{\varepsilon}_3 := g^*(\lambda) + g(h(x)) - \langle \lambda, h(x) \rangle$. Relation (1) ensures that $\tilde{\varepsilon}_1$, $\bar{\varepsilon}_2$ and $\bar{\varepsilon}_3$ are non - negative real numbers. Moreover, taking into consideration the previous inequality, one can easily see that $\tilde{\varepsilon}_1 + \bar{\varepsilon}_2 + \bar{\varepsilon}_3 \leq \varepsilon$. Then for $\bar{\varepsilon}_1 := \varepsilon - \bar{\varepsilon}_2 - \bar{\varepsilon}_3 \geq \tilde{\varepsilon}_1$ it holds $\bar{\varepsilon}_1 + \bar{\varepsilon}_2 + \bar{\varepsilon}_3 = \varepsilon$ and, moreover, the inequalities $f^*(x_1^*) + f(x) \leq \langle x_1^*, x \rangle + \bar{\varepsilon}_1$, $(\lambda h)^*(x_2^*) + (\lambda h)(x) \leq \langle x_2^*, x \rangle + \bar{\varepsilon}_2$ and $g^*(\lambda) + g(h(x)) \leq \langle \lambda, h(x) \rangle + \bar{\varepsilon}_3$ hold, too. According to (2) we have $x_1^* \in \partial_{\bar{\varepsilon}_1} f(x)$, $x_2^* \in \partial_{\bar{\varepsilon}_2}(\lambda h)(x)$, and $\lambda \in \partial_{\bar{\varepsilon}_3} g(h(x))$. Thus

$$\begin{aligned} x^* &= x_1^* + x_2^* \in \partial_{\bar{\varepsilon}_1} f(x) + \partial_{\bar{\varepsilon}_2}(\lambda h)(x) \\ &\subseteq \bigcup_{\substack{\varepsilon_1, \varepsilon_2, \varepsilon_3 \geq 0, \\ \varepsilon_1 + \varepsilon_2 + \varepsilon_3 = \varepsilon}} \left\{ \partial_{\varepsilon_1} f(x) + \partial_{\varepsilon_2}(\lambda h)(x) : \lambda \in K^* \cap \partial_{\varepsilon_3} g(h(x)) \right\} \end{aligned}$$

and the first part of the proof is finished.

" \supseteq " In order to prove the reverse inclusion let

$$x^* \in \bigcup_{\substack{\varepsilon_1, \varepsilon_2, \varepsilon_3 \geq 0, \\ \varepsilon_1 + \varepsilon_2 + \varepsilon_3 = \varepsilon}} \left\{ \partial_{\varepsilon_1} f(x) + \partial_{\varepsilon_2}(\lambda h)(x) : \lambda \in K^* \cap \partial_{\varepsilon_3} g(h(x)) \right\}$$

be arbitrarily taken. Then there exist $\varepsilon_1, \varepsilon_2, \varepsilon_3 \geq 0$, $\lambda \in K^* \cap \partial_{\varepsilon_3} g(h(x))$, $x_1^* \in \partial_{\varepsilon_1} f(x)$ and $x_2^* \in \partial_{\varepsilon_2}(\lambda h)(x)$ such that $\varepsilon_1 + \varepsilon_2 + \varepsilon_3 = \varepsilon$ and $x^* = x_1^* + x_2^*$. This implies further $f^*(x_1^*) + f(x) \leq \langle x_1^*, x \rangle + \varepsilon_1$, $(\lambda h)^*(x_2^*) + (\lambda h)(x) \leq \langle x_2^*, x \rangle + \varepsilon_2$ and $g^*(\lambda) + g(h(x)) \leq \langle \lambda, h(x) \rangle + \varepsilon_3$. By summing up we acquire

$$\begin{aligned} &f^*(x_1^*) + f(x) + (\lambda h)^*(x_2^*) + (\lambda h)(x) + g^*(\lambda) + g(h(x)) \\ &\leq \langle x_1^*, x \rangle + \varepsilon_1 + \langle x_2^*, x \rangle + \varepsilon_2 + \langle \lambda, h(x) \rangle + \varepsilon_3 \\ &= \langle \lambda, h(x) \rangle + \langle x_1^* + x_2^*, x \rangle + \varepsilon_1 + \varepsilon_2 + \varepsilon_3 = \langle \lambda, h(x) \rangle + \langle x^*, x \rangle + \varepsilon. \end{aligned}$$

Thus we get

$$g^*(\lambda) + f^*(x_1^*) + (\lambda h)^*(x_2^*) + (f + g \circ h)(x) \leq \langle x^*, x \rangle + \varepsilon,$$

and using (3) follows

$$(f + g \circ h)^*(x^*) + (f + g \circ h)(x) \leq \langle x^*, x \rangle + \varepsilon.$$

But this implies $x^* \in \partial_\varepsilon(f + g \circ h)(x)$ and the proof of the theorem is complete. \square

In the following we prove that (RC) is not just a sufficient, but also a necessary condition for having (5) fulfilled for all $x \in \text{dom}(f) \cap \text{dom}(h) \cap h^{-1}(\text{dom}(g))$ and all $\varepsilon \geq 0$.

Theorem 3 *If relation (5) holds for all $x \in \text{dom}(f) \cap \text{dom}(h) \cap h^{-1}(\text{dom}(g))$ and all $\varepsilon \geq 0$, then*

$$\text{epi}(f^*) + \bigcup_{\lambda \in \text{dom}(g^*)} \left(\text{epi}((\lambda h)^*) + (0, g^*(\lambda)) \right) = \text{epi}((f + g \circ h)^*).$$

Thus the regularity condition (RC) is fulfilled.

Proof. Take first an arbitrary

$$(x^*, r) \in \text{epi}(f^*) + \bigcup_{\lambda \in \text{dom}(g^*)} \left(\text{epi}((\lambda h)^*) + (0, g^*(\lambda)) \right).$$

Then there exist $\lambda \in \text{dom}(g^*) \subseteq K^*$ and the tuples $(x_1^*, r_1) \in \text{epi}(f^*)$ and $(x_2^*, r_2) \in \text{epi}((\lambda h)^*)$ such that

$$(x^*, r) = (x_1^*, r_1) + (x_2^*, r_2) + (0, g^*(\lambda)).$$

This equality implies $x^* = x_1^* + x_2^*$ and, taking into consideration the properties of the epigraph, we acquire $r = r_1 + r_2 + g^*(\lambda) \geq g^*(\lambda) + f^*(x_1^*) + (\lambda h)^*(x_2^*)$. Since the inequality (3) is always satisfied, we get $(f + g \circ h)^*(x^*) \leq r$ and so $(x^*, r) \in \text{epi}((f + g \circ h)^*)$. As no additional assumptions are imposed regarding the tuple (x^*, r) we conclude that

$$\text{epi}(f^*) + \bigcup_{\lambda \in K^* \cap \text{dom}(g^*)} \left(\text{epi}((\lambda h)^*) + (0, g^*(\lambda)) \right) \subseteq \text{epi}((f + g \circ h)^*). \quad (6)$$

Let us prove now that relation (5) secures the reverse inclusion in (6). Take an arbitrary tuple $(x^*, r) \in \text{epi}((f + g \circ h)^*)$. Then $(f + g \circ h)^*(x^*) \leq r$ and for some arbitrary $x \in \text{dom}(f) \cap \text{dom}(h) \cap h^{-1}(\text{dom}(g))$ we get

$$(f + g \circ h)^*(x^*) + (f + g \circ h)(x) \leq \langle x^*, x \rangle + [r + (f + g \circ h)(x) - \langle x^*, x \rangle].$$

Further we take

$$\varepsilon := r + (f + g \circ h)(x) - \langle x^*, x \rangle.$$

Using the Fenchel - Young inequality one can see that since $(f+g \circ h)^*(x^*) \leq r$ one has $\varepsilon \geq r - (f + g \circ h)^*(x^*) \geq 0$. Moreover, as $(f + g \circ h)^*(x^*) + (f + g \circ h)(x) \leq \langle x^*, x \rangle + \varepsilon$, relation (2) implies $x^* \in \partial_\varepsilon(f + g \circ h)(x)$. According to relation (5) there exist $\varepsilon_1, \varepsilon_2, \varepsilon_3 \geq 0$, $\lambda \in K^* \cap \partial_{\varepsilon_3}g(h(x))$, $x_1^* \in \partial_{\varepsilon_1}f(x)$ and $x_2^* \in \partial_{\varepsilon_2}(\lambda h)(x)$ such that $\varepsilon_1 + \varepsilon_2 + \varepsilon_3 = \varepsilon$ and $x_1^* + x_2^* = x^*$. Making use of relation (2) we get $f^*(x_1^*) + f(x) \leq \langle x_1^*, x \rangle + \varepsilon_1$, $(\lambda h)^*(x_2^*) + (\lambda h)(x) \leq \langle x_2^*, x \rangle + \varepsilon_2$ and $g^*(\lambda) + g(h(x)) \leq \langle \lambda, h(x) \rangle + \varepsilon_3$. Thus $\lambda \in \text{dom}(g^*)$ and by summing up the three inequalities given above we get

$$\begin{aligned} & f^*(x_1^*) + f(x) + (\lambda h)^*(x_2^*) + (\lambda h)(x) + g^*(\lambda) + g(h(x)) \\ & \leq \langle x_1^*, x \rangle + \varepsilon_1 + \langle x_2^*, x \rangle + \varepsilon_2 + \langle \lambda, h(x) \rangle + \varepsilon_3 = \langle x^*, x \rangle + \langle \lambda, h(x) \rangle + \varepsilon, \end{aligned}$$

which implies

$$g^*(\lambda) + f^*(x_1^*) + (\lambda h)^*(x_2^*) + (f + g \circ h)(x) \leq \langle x^*, x \rangle + \varepsilon.$$

Taking into consideration the way the constant ε was defined we obtain

$$\begin{aligned} & g^*(\lambda) + f^*(x_1^*) + (\lambda h)^*(x_2^*) + (f + g \circ h)(x) \\ & \leq \langle x^*, x \rangle + r + (f + g \circ h)(x) - \langle x^*, x \rangle = r + (f + g \circ h)(x) \end{aligned}$$

and from here the inequality

$$g^*(\lambda) + f^*(x_1^*) + (\lambda h)^*(x_2^*) \leq r$$

can be easily deduced. Let $t \geq 0$ be such that

$$g^*(\lambda) + f^*(x_1^*) + (\lambda h)^*(x_2^*) + t = r.$$

Then

$$\begin{aligned} (x^*, r) &= (x_1^*, f^*(x_1^*) + t) + (x_2^*, (\lambda h)^*(x_2^*)) + (0, g^*(\lambda)) \\ &\in \text{epi}(f^*) + \bigcup_{\lambda \in \text{dom}(g^*)} \left(\text{epi}((\lambda h)^*) + (0, g^*(\lambda)) \right) \end{aligned}$$

and the proof is over. □

Let us consider the optimization problem

$$(P) \quad \inf_{x \in X} \left\{ f(x) + (g \circ h)(x) \right\}.$$

For $\varepsilon \geq 0$ we say that $\bar{x} \in X$ is an ε - optimal solution to (P) if one has that

$$f(\bar{x}) + (g \circ h)(\bar{x}) \leq \inf_{x \in X} \left\{ f(x) + (g \circ h)(x) \right\} + \varepsilon.$$

One can easily see that \bar{x} is an ε - optimal solution to (P) if and only if $0 \in \partial_\varepsilon(f + g \circ h)(\bar{x})$. The results presented above allow us to provide necessary and sufficient optimality conditions for ε - optimal solutions of the problem (P). The next theorem is devoted to that matter.

Theorem 4 (a) *Suppose that the condition (RC) is fulfilled. If $\bar{x} \in X$ is an ε - optimal solution of the problem (P) for some $\varepsilon \geq 0$, then there exist $\varepsilon_1, \varepsilon_2, \varepsilon_3 \geq 0$, $\bar{\lambda} \in K^*$ and $\bar{x}^* \in X^*$ such that*

- (i) $0 \leq g^*(\bar{\lambda}) + g(h(\bar{x})) - \langle \bar{\lambda}, h(\bar{x}) \rangle \leq \varepsilon_3;$
- (ii) $0 \leq f^*(\bar{x}^*) + f(\bar{x}) - \langle \bar{x}^*, \bar{x} \rangle \leq \varepsilon_1;$
- (iii) $0 \leq (\bar{\lambda}h)^*(-\bar{x}^*) + (\bar{\lambda}h)(\bar{x}) + \langle \bar{x}^*, \bar{x} \rangle \leq \varepsilon_2;$
- (iv) $\varepsilon_1 + \varepsilon_2 + \varepsilon_3 = \varepsilon.$

(b) *If there exist $\varepsilon_1, \varepsilon_2, \varepsilon_3 \geq 0$, $\bar{\lambda} \in K^*$ and $\bar{x}^* \in X^*$ such that the relations (i) – (iv) hold for some $\bar{x} \in X$, then \bar{x} is an ε - optimal solution of the problem (P).*

Proof. (a) As \bar{x} is an ε - optimal solution of the problem (P) we know that $0 \in \partial_\varepsilon(f + g \circ h)(\bar{x})$. By relation (5) there exist $\varepsilon_1, \varepsilon_2, \varepsilon_3 \geq 0$ and $\bar{\lambda} \in K^*$, such that $\varepsilon_1 + \varepsilon_2 + \varepsilon_3 = \varepsilon$, $\bar{\lambda} \in K^* \cap \partial_{\varepsilon_3}g(h(\bar{x}))$ and $0 \in \partial_{\varepsilon_1}f(\bar{x}) + \partial_{\varepsilon_2}(\bar{\lambda}h)(\bar{x})$. As $\bar{\lambda} \in \partial_{\varepsilon_3}g(h(\bar{x}))$ the assertion (i) is a direct consequence of (2). Moreover, there exists some $\bar{x}^* \in X^*$ such that $\bar{x}^* \in \partial_{\varepsilon_1}f(\bar{x})$ and $-\bar{x}^* \in \partial_{\varepsilon_2}(\bar{\lambda}h)(\bar{x})$ and, using once more relation (2), the assertions (ii) and, respectively, (iii), can be easily deduced.

(b) By summing up the relations (i) – (iii) and taking into consideration (iv) we acquire

$$\begin{aligned} & g^*(\bar{\lambda}) + g(h(\bar{x})) - \langle \bar{\lambda}, h(\bar{x}) \rangle + f^*(\bar{x}^*) + f(\bar{x}) - \langle \bar{x}^*, \bar{x} \rangle \\ & + (\bar{\lambda}h)^*(-\bar{x}^*) + (\bar{\lambda}h)(\bar{x}) + \langle \bar{x}^*, \bar{x} \rangle \leq \varepsilon_1 + \varepsilon_2 + \varepsilon_3 = \varepsilon. \end{aligned}$$

By (3) we get

$$(f + g \circ h)^*(0) + (f + g \circ h)(\bar{x}) \leq \varepsilon$$

and this is nothing else than $0 \in \partial_\varepsilon(f + g \circ h)(\bar{x})$. Thus \bar{x} is an ε -optimal solution of (P) and the proof is complete. \square

Remark 1 (a) *Inspired by the statement of Theorem 1 one can introduce to (P) the following conjugate dual problem*

$$(D) \quad \sup_{\lambda \in K^*, x^* \in X^*} \left\{ -g^*(\lambda) - f^*(x^*) - (\lambda h)^*(-x^*) \right\}.$$

Taking in (4) $p^* = 0$ it follows that (RC) is a sufficient condition for strong duality between (P) and (D), namely the situation when the optimal objective values of both problems are equal and the dual has an optimal solution.

(b) *If (RC) is fulfilled and $\bar{x} \in X$ is an ε -optimal solution of the problem (P) for some $\varepsilon \geq 0$, then the element $(\bar{\lambda}, \bar{x}^*) \in K^* \times X^*$ provided by Theorem 4 (a) satisfies*

$$\begin{aligned} \sup_{\lambda \in K^*, x^* \in X^*} \left\{ -g^*(\lambda) - f^*(x^*) - (\lambda h)^*(-x^*) \right\} - \varepsilon &\leq f(\bar{x}) + (g \circ h)(\bar{x}) - \varepsilon \leq \\ &-g^*(\bar{\lambda}) - f^*(\bar{x}^*) - (\bar{\lambda} h)^*(-\bar{x}^*), \end{aligned}$$

which means that it is an ε -optimal solution for the dual problem (D).

(c) *In case $\varepsilon = 0$ by means of (i) – (iii) we rediscover the optimality conditions given in the past for characterizing the (exact) optimal solutions of the primal problem (P) and its dual problem (D) (see, for example, [3])*

- (i) $g^*(\bar{\lambda}) + g(h(\bar{x})) - \langle \bar{\lambda}, h(\bar{x}) \rangle = 0$;
- (ii) $f^*(\bar{x}^*) + f(\bar{x}) - \langle \bar{x}^*, \bar{x} \rangle = 0$;
- (iii) $(\bar{\lambda} h)^*(-\bar{x}^*) + (\bar{\lambda} h)(\bar{x}) + \langle \bar{x}^*, \bar{x} \rangle = 0$.

Remark 2 *If the variable x does not cover the whole space X , but a non-empty closed convex subset $C \subseteq X$, then some similar results can be easily provided if we replace the function h with the function*

$$h_C : X \rightarrow Y^\bullet, h_C(x) = \begin{cases} h(x), & x \in \text{dom}(h) \cap C, \\ +\infty_Y, & \text{otherwise.} \end{cases}$$

Before going further we would like to mention that the regularity condition (*RC*) is weaker than the conditions imposed in [9–11] for composed convex optimization problems (see [1] for an elaborate discussion on this topic).

4 Special cases

4.1 Composition with a linear operator

Let us consider $A : X \rightarrow Y$ a linear continuous operator and take

$$h : X \rightarrow Y, \quad h(x) = Ax, \quad \forall x \in X.$$

Taking $K = \{0\} \subset Y$ one has that h and g are K - convex and K - increasing, respectively, and (*P*) is nothing else than

$$(P^A) \quad \inf_{x \in X} (f + g \circ A)(x).$$

Further, one can easily prove that for all $\lambda \in Y^* = K^*$ we have

$$(\lambda h)^*(x^*) = \begin{cases} 0, & x^* = A^* \lambda, \\ +\infty, & \text{otherwise.} \end{cases}$$

For this choice formula (4) becomes

$$\begin{aligned} (f + g \circ A)^*(p^*) &= \min_{\substack{\lambda \in K^*, \\ x^* \in X^*}} \left\{ g^*(\lambda) + f^*(x^*) + (\lambda h)^*(p^* - x^*) \right\} \\ &= \min_{\substack{\lambda \in Y^*, \\ x^* \in X^*, \\ p^* - x^* = A^* \lambda}} \left\{ g^*(\lambda) + f^*(x^*) \right\} = \min_{\lambda \in Y^*} \left\{ g^*(\lambda) + f^*(p^* - A^* \lambda) \right\}, \forall p^* \in X^*. \quad (7) \end{aligned}$$

Moreover, using only the definition of the epigraph and the special form of the function $(\lambda h)^*$ the equality

$$\bigcup_{\lambda \in \text{dom}(g^*)} \left(\text{epi}((\lambda h)^*) + (0, g^*(\lambda)) \right) = A^* \times \text{id}_{\mathbb{R}}(\text{epi}(g^*))$$

can be easily proved (see also [1, 4]). Thus the regularity condition (*RC*) becomes in this special case

(RC^A) $\text{epi}(f^*) + A^* \times \text{id}_{\mathbb{R}}(\text{epi}(g^*))$ is closed

and according to Theorem 1 it is fulfilled if and only if relation (7) holds.

Now let $\varepsilon \geq 0$ be arbitrarily taken. According to relation (2) we have $x^* \in \partial_\varepsilon(\lambda h)(x)$ if and only if $(\lambda h)^*(x^*) + (\lambda h)(x) \leq \langle x^*, x \rangle + \varepsilon$. Because of the special form of the function $(\lambda h)^*$ it is binding to have $x^* = A^*\lambda$. Thus $x^* \in \partial_\varepsilon(\lambda h)(x)$ if and only if $x^* = A^*\lambda$ and $\langle \lambda, Ax \rangle \leq \langle A^*\lambda, x \rangle + \varepsilon$. As the last inequality is always fulfilled (see the definition of the adjoint operator), we get $\partial_\varepsilon(\lambda h)(x) = \{A^*\lambda\}$. Relation (5) becomes

$$\partial_\varepsilon(f + g \circ A)(x) = \bigcup_{\substack{\varepsilon_1, \varepsilon_2, \varepsilon_3 \geq 0, \\ \varepsilon_1 + \varepsilon_2 + \varepsilon_3 = \varepsilon}} \left\{ \partial_{\varepsilon_1} f(x) + A^*\lambda : \lambda \in \partial_{\varepsilon_3} g(Ax) \right\}$$

and taking into consideration Theorem 2 and Theorem 3 the following result can be easily proved.

Theorem 5 *The regularity condition (RC^A) is fulfilled if and only if for all $x \in \text{dom}(f) \cap A^{-1}(\text{dom}(g))$ and for all $\varepsilon \geq 0$ we have*

$$\partial_\varepsilon(f + g \circ A)(x) = \bigcup_{\substack{\varepsilon_1, \varepsilon_3 \geq 0, \\ \varepsilon_1 + \varepsilon_3 = \varepsilon}} \left\{ \partial_{\varepsilon_1} f(x) + A^*\partial_{\varepsilon_3} g(Ax) \right\}. \quad (8)$$

Remark 3 (a) *The closedness type regularity condition (RC^A) , which we have rediscovered here as a particular case of (RC) , has been considered in [8] and [4] as a necessary and sufficient condition for the existence of the so-called stable strong duality (this is how the property in 7 is called in the literature) for the problem (P^A) and its Fenchel-type dual.*

(b) *In case $X = Y$ and $A = \text{id}_X$ the equivalence in Theorem 5 is nothing else than (ii) \Leftrightarrow (iii) in Theorem 1 in [7]. These assertions are further equivalent to $\text{epi}((f+g)^*) = \text{epi}(f^*) + \text{epi}(g^*)$ and, further, to $(f+g)^* = f^* \square g^*$ and the infimal convolution is exact (cf. Proposition 2.2 in [4]). In this way we get the equivalence (i) \Leftrightarrow (ii) in Theorem 1 in [7].*

The following theorem, which provides necessary and sufficient optimality conditions for ε - optimal solutions of the problem (P^A) , is a consequence of Theorem 4.

Theorem 6 (a) Suppose that (RC^A) holds. If $\bar{x} \in X$ is an ε - optimal solution of the problem (P^A) for some $\varepsilon \geq 0$, then there exist $\varepsilon_1, \varepsilon_3 \geq 0$ and $\bar{\lambda} \in Y^*$ such that

$$(i^A) \quad 0 \leq g^*(\bar{\lambda}) + g(A\bar{x}) - \langle \bar{\lambda}, A\bar{x} \rangle \leq \varepsilon_3;$$

$$(ii^A) \quad 0 \leq f^*(-A^*\bar{\lambda}) + f(\bar{x}) + \langle A^*\bar{\lambda}, \bar{x} \rangle \leq \varepsilon_1;$$

$$(iii^A) \quad \varepsilon_1 + \varepsilon_3 = \varepsilon.$$

(b) If there exist $\varepsilon_1, \varepsilon_3 \geq 0$ and $\bar{\lambda} \in Y^*$ such that the relations $(i^A) - (iii^A)$ hold for some $\bar{x} \in X$, then \bar{x} is an ε - optimal solution of the problem (P^A) .

Proof. (a) Since the hypotheses of Theorem 4 are fulfilled there exist $\varepsilon_1, \varepsilon_2, \varepsilon_3 \geq 0$, $\bar{\lambda} \in Y^*$ and $\bar{x}^* \in X^*$ such that the inequalities $(i) - (iv)$ hold. It is easy to see that in this case the assertions (i) and (iv) are equivalent to the assertions (i^A) and (iii^A) , respectively. Since $(\bar{\lambda}h)^*(-\bar{x}^*) = +\infty$ for all $-\bar{x}^* \neq A^*\bar{\lambda}$, relation (iii) implies $\bar{x}^* = -A^*\bar{\lambda}$ and now it is easy to see that the assertion (iii^A) is equivalent to (ii) .

(b) For $\varepsilon_2 = 0$ and $\bar{x}^* = -A^*\bar{\lambda}$ it can be easily proved that the assertions $(i) - (iv)$ of Theorem 4 are fulfilled. \square

4.2 The case $f \equiv 0$

Consider the function $f : X \rightarrow \mathbb{R}$ fulfilling $f(x) = 0$ for all $x \in X$. Thus the optimization problem (P) becomes

$$(P_0) \quad \inf_{x \in X} (g \circ h)(x).$$

Since

$$f^*(x^*) = \begin{cases} 0, & \text{if } x^* = 0, \\ +\infty, & \text{otherwise,} \end{cases}$$

relation (4) has the following formulation

$$(g \circ h)^*(p^*) = \min_{\lambda \in K^*} \left\{ g^*(\lambda) + (\lambda h)^*(p^*) \right\}, \forall p^* \in X^*. \quad (9)$$

As $\text{epi}(f^*) = \{0\} \times \mathbb{R}_+$ we get

$$\begin{aligned} & \text{epi}(f^*) + \bigcup_{\lambda \in \text{dom}(g^*)} \left(\text{epi}((\lambda h)^*) + (0, g^*(\lambda)) \right) \\ &= \{0\} \times \mathbb{R}_+ + \bigcup_{\lambda \in \text{dom}(g^*)} \left(\text{epi}((\lambda h)^*) + (0, g^*(\lambda)) \right) \\ &= \bigcup_{\lambda \in \text{dom}(g^*)} \left(\text{epi}((\lambda h)^*) + (0, g^*(\lambda)) \right) \end{aligned}$$

and the regularity condition (RC) becomes

$$(RC_0) \quad \bigcup_{\lambda \in \text{dom}(g^*)} \left(\text{epi}((\lambda h)^*) + (0, g^*(\lambda)) \right) \text{ is closed.}$$

Moreover, according to Theorem 1 the condition (RC_0) is fulfilled if and only if relation (9) holds.

Taking into consideration relation (2) and the form of the function f^* it is not hard to prove that $\partial_\varepsilon f(x) = \{0\} \subset X^*$ for all $x \in X$ and all $\varepsilon \geq 0$. The next result, which provides a formula for the ε - subdifferential of the function $g \circ h$, is a straightforward consequence of Theorem 2 and Theorem 3.

Theorem 7 *The regularity condition (RC_0) is fulfilled if and only if for all $x \in \text{dom}(h) \cap h^{-1}(\text{dom}(g))$ and for all $\varepsilon \geq 0$ we have*

$$\partial_\varepsilon(g \circ h)(x) = \bigcup_{\substack{\varepsilon_2, \varepsilon_3 \geq 0, \\ \varepsilon_2 + \varepsilon_3 = \varepsilon}} \left\{ \partial_{\varepsilon_2}(\lambda h)(x) : \lambda \in K^* \cap \partial_{\varepsilon_3} g(h(x)) \right\}. \quad (10)$$

The next theorem provides necessary and sufficient optimality conditions for the ε - optimal solutions of the problem (P_0) .

Theorem 8 (a) *Suppose that (RC_0) is fulfilled. If $\bar{x} \in X$ is an ε - optimal solution of the problem (P_0) for some $\varepsilon \geq 0$, then there exist $\varepsilon_2, \varepsilon_3 \geq 0$ and $\bar{\lambda} \in K^*$ such that*

$$(i_0) \quad 0 \leq g^*(\bar{\lambda}) + g(h(\bar{x})) - \langle \bar{\lambda}, h(\bar{x}) \rangle \leq \varepsilon_3;$$

$$(ii_0) \quad 0 \leq (\bar{\lambda}h)^*(0) + (\bar{\lambda}h)(\bar{x}) \leq \varepsilon_2;$$

$$(iii_0) \quad \varepsilon_2 + \varepsilon_3 = \varepsilon.$$

(b) If there exist $\varepsilon_2, \varepsilon_3 \geq 0$ and $\bar{\lambda} \in K^*$ such that the relations $(i_0) - (iii_0)$ hold for some $\bar{x} \in X$, then \bar{x} is an ε - optimal solution of the problem (P_0) .

Proof. (a) By Theorem 4 there exist $\varepsilon_1, \varepsilon_2, \varepsilon_3 \geq 0$, $\bar{\lambda} \in K^*$ and $\bar{x}^* \in X^*$ such that the assertions $(i) - (iv)$ are fulfilled. Obviously (i_0) and (iii_0) are implied by (i) and (iv) , respectively. Moreover, since the relation (ii) implies $\bar{x}^* = 0$, (ii_0) can be easily derived from (iii) .

(b) In this special case the assertions $(i) - (iv)$ of Theorem 4 are fulfilled for $\varepsilon_1 = 0$ and $\bar{x}^* = 0$. \square

4.3 The ordinary convex optimization problem

Consider the function

$$g : Y \rightarrow \bar{\mathbb{R}}, \quad g(y) = \delta_{-K}(y) = \begin{cases} 0, & \text{if } y \in -K, \\ +\infty, & \text{otherwise.} \end{cases}$$

One can prove that the function g is convex and K - increasing and that $g^* = \delta_{K^*}$. In this subsection we deal with the ordinary convex optimization problem with geometric and cone constraints

$$(P^0) \quad \inf_{\substack{x \in X, \\ h(x) \leq_{K^0} 0}} f(x).$$

This problem can be seen as a particularization of (P) since it can be rewritten as

$$(P^0) \quad \inf_{x \in X} (f + \delta_{-K} \circ h)(x).$$

One can easily observe that relation (4) is nothing else than

$$(f + \delta_{-K} \circ h)^*(p^*) = \min_{\substack{\lambda \in K^*, \\ x^* \in X^*}} \left\{ f^*(x^*) + (\lambda h)^*(p^* - x^*) \right\}, \forall p^* \in X^*. \quad (11)$$

Moreover, according to Theorem 1 the relation (11) holds if and only if the regularity condition

(RC^O) $\text{epi}(f^*) + \bigcup_{\lambda \in K^*} \text{epi}((\lambda h)^*)$ is closed

is fulfilled.

Taking into consideration the way g^* looks like and relation (2) one can easily show that for all $y \in -K$ we have $\lambda \in \partial_\varepsilon \delta_{-K}(y)$ if and only if $\lambda \in K^*$ and $0 \leq \langle \lambda, y \rangle + \varepsilon$. Relation (5) states in this case

$$\partial_\varepsilon(f + \delta_{-K} \circ h)(x) = \bigcup_{\substack{\varepsilon_1, \varepsilon_2, \varepsilon_3 \geq 0, \\ \varepsilon_1 + \varepsilon_2 + \varepsilon_3 = \varepsilon}} \left\{ \partial_{\varepsilon_1} f(x) + \partial_{\varepsilon_2}(\lambda h)(x) : \lambda \in K^*, \right. \\ \left. 0 \leq \langle \lambda, h(x) \rangle + \varepsilon_3 \right\}.$$

Making use of Theorem 2 and Theorem 3 the subsequent result can be easily proved.

Theorem 9 *The regularity condition (RC^O) is fulfilled if and only if for all $x \in \text{dom}(f) \cap \text{dom}(h) \cap h^{-1}(-K)$ and for all $\varepsilon \geq 0$ we have*

$$\partial_\varepsilon(f + \delta_{-K} \circ h)(x) = \bigcup_{\substack{\varepsilon_1, \varepsilon_2 \geq 0, \\ \varepsilon_1 + \varepsilon_2 - \varepsilon \leq \langle \lambda, h(x) \rangle}} \left\{ \partial_{\varepsilon_1} f(x) + \partial_{\varepsilon_2}(\lambda h)(x) : \lambda \in K^* \right\}. \quad (12)$$

Before going further we would like that the regularity condition (RC^O) belongs to the class of *closed cone constraint qualifications* considered for the first time in [12] (the reader is invited to consult [5] for an equivalent formulation s).

Necessary and sufficient conditions for ε -optimal solutions of the problem (P^O) can be derived from (12).

Theorem 10 (a) *Suppose that (RC^O) is fulfilled. If $\bar{x} \in X$ is an ε -optimal solution of the problem (P^O) for some $\varepsilon \geq 0$, then there exist $\varepsilon_1, \varepsilon_2 \geq 0$, $\bar{\lambda} \in K^*$ and $\bar{x}^* \in X^*$ such that*

$$(i^O) \quad 0 \leq f^*(\bar{x}^*) + f(\bar{x}) - \langle \bar{x}^*, \bar{x} \rangle \leq \varepsilon_1;$$

$$(ii^O) \quad 0 \leq (\bar{\lambda} h)^*(-\bar{x}^*) + (\bar{\lambda} h)(\bar{x}) + \langle \bar{x}^*, \bar{x} \rangle \leq \varepsilon_2;$$

$$(iii^O) \quad 0 \leq -\langle \bar{\lambda}, h(\bar{x}) \rangle \leq \varepsilon - \varepsilon_1 - \varepsilon_2.$$

(b) If there exist $\varepsilon_1, \varepsilon_2 \geq 0$, $\bar{\lambda} \in K^*$ and $\bar{x}^* \in X^*$ such that the relations $(i^O) - (iii^O)$ hold for some $\bar{x} \in X$, then \bar{x} is an ε - optimal solution of the problem (P^O) .

Proof. (a) Once again we apply Theorem 4. Thus there exist $\bar{\lambda} \in K^*$ and $\bar{x}^* \in X^*$ such that the relations $(i) - (iv)$ are fulfilled. Since (i^O) and (ii^O) are direct consequences of the assertions (ii) and (iii) , respectively, it remains to prove (iii^O) . As assertion (i) implies $0 \leq -\langle \bar{\lambda}, h(\bar{x}) \rangle \leq \varepsilon_3$, the last inequality and (iv) are enough to ensure the desired conclusion.

(b) It is straightforward to see that for $\varepsilon_3 = \varepsilon - \varepsilon_1 - \varepsilon_2$ the assertions $(i) - (iv)$ of Theorem 4 are fulfilled. \square

5 Conclusions

In this paper we give a new formula for the ε - subdifferential of the sum of a function and the composition of another convex function which is K - increasing with a K - convex function (we suppose that K is a closed convex cone). Using the epigraphs of the conjugates of the functions involved we give a closedness type regularity condition which turns out to be equivalent to the formula mentioned above. Using the strong connection between the ε - subdifferential of a function and its conjugate function we provide necessary and sufficient optimality conditions for composed convex optimization problems. We also consider some special cases and get corresponding ε - subdifferential sum formulae. In the case of the composition with a linear operator our result is an extension of Theorem 1 in [7].

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