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SEQUENTIAL CHARACTERIZATION OF SOLUTIONS IN CONVEX COMPOSITE PROGRAMMING AND APPLICATIONS TO VECTOR OPTIMIZATION

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Abstract. When characterizing optimal solutions of both scalar and vector optimization problems usually constraint qualifications have to be satisfied. By considering sequential characterizations, given for the first time in vector optimization in this paper, this drawback is eliminated. In order to establish them we give first of all sequential characterizations for a convex composed optimization problem with geometric and cone constraints. Then, by means of scalarization, we extend them to the vectorial case. For exemplification we particularize the characterization in the case of linear and set scalarization.

1. **Introduction.** Vector optimization problems have received a great deal of interest from the scientific community due to their applicability in various practical areas. From the theoretical point of view comprehensive studies on the subject have been undertaken by numerous authors, among them we cite here the books of Jahn [12], Luc [16] and Sawaragi, Nakayama and Tanino [21]. Vector optimization has known a development similar to scalar optimization and one can easily notice a growing interest in the community for fields like vector duality theory (see [4, 9, 10, 22]), vector variational inequalities (see [6, 7, 15]), vector equilibrium problems (see [14]), etc.

Due to the fact that the partial order generated by a convex cone in a topological vector space is not necessarily a complete one, several notions of solutions for a vector optimization problem have been given. In analogy to the scalar optimization sufficient KKT-type optimality conditions have been given in the literature for the different types of solutions which occur in the vector case. Unfortunately, the lack of constraint qualifications makes the classical optimality conditions unusable. As

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an alternative we consider in this paper necessary and sufficient sequential optimality conditions for both scalar and vector optimization problems, which can be formulated in the absence of constraint qualifications.

In [2, 3] Boţ, Csetnek and Wanka have given sequential optimality conditions in convex programming, without any constraint qualification, via a perturbation approach, in a general framework. We consider here the convex composed optimization problem with geometric and cone constraints

$$(P_s) \inf_{\substack{x\in M\\ G(x)\in -C}} s(F(x)),$$

where X is a reflexive Banach space, Y and Z are Banach spaces partially ordered by the closed convex cones, K and C, respectively, M is a nonempty closed convex subset of X, $F : X \to Y^\bullet = Y \cup \{\infty_Y\}$ is a proper, K-convex, star K-lower semicontinuous function, $G : X \to Z^\bullet = Z \cup \{\infty_Z\}$ is a proper, C-convex, C-epi closed function and $s : Y \to \overline{\mathbb{R}}$ is a proper, convex, lower semicontinuous and K-increasing function for which we make the convention that $s(\infty_Y) = +\infty$. Using the refined version of the sequential characterizations expressed by means of the classical subdifferential in [2], we deduce sequential characterizations for the optimal solutions of the problem (P_s), with the help of an appropriate perturbation function. They are further particularized in the case when the functions involved are continuous. Then, by taking the cone $K = \{0\}$ and F the identity function on X we get improved sequential Lagrange multiplier conditions for the ordinary convex optimization problem with geometric and cone constraints

$$(P_c) \inf_{\substack{x\in M\\ G(x)\in -C}} s(x),$$

where $s : X \to \overline{\mathbb{R}}$ is a proper, convex and lower semicontinuous function.

Scalarization is probably the oldest and most studied method of characterizing optimal solutions in vector optimization. It consists in associating to a vector optimization problem a scalarized one such that making use of the solutions of the latter, one can obtain important information on the solutions of the original problem. The literature is quite rich in this respect, and we mention here several authors, such as Boţ [1], Jahn [11] and [12], Luc [16], Tammer and Göpfert [22], and the list could be continued.

A scalarized problem associated to the vector optimization problem

$$(P_v) \quad v-\min_{\substack{x\in M\\ G(x)\in -C}} F(x)$$

is actually of the form (P_s). Given a set S of scalarization functions, we introduce the so-called S-properly efficient solutions and S-weakly efficient solutions for (P_v) and characterize them by making use of the sequential optimality conditions obtained for (P_s). One can notice that each S-properly efficient solution to (P_v) is a (Pareto) efficient solution, while each S-weakly efficient solution to (P_v) is a weakly efficient one. We further particularize the sequential characterizations by taking S as a set of linear functions and as a set of K-strictly increasing function induced by a given cone, respectively.

Let us underline again the fact that this optimality conditions do not require the fulfillment of any constraint qualification.

This article is organized as follows. Section 2 contains the preliminary notions and results from convex analysis necessary to make this paper self-contained. Sequential optimality conditions for convex composed optimization problems with geometric and cone constraints are given in section 3. The general case when F and G are, among others, star K-lower semicontinuous and C-epi closed, respectively, is treated. Then, they are taken continuous and this leads to general sequential characterizations. By taking $G \equiv 0$ we obtain sequential optimality conditions for composed geometric constrained optimization. In subsection 3.2 we obtain along with a particular case of a sequential Lagrange multiplier condition, a sequential version of the well known Pshenichnyi-Rockafellar Lemma in [18], which turns out to be a refinement of Corollary 4.8 in [2] and improving thus Corollary 3.5 in [13].

Section 4 contains the results for vector optimization. In its beginning several optimality notions used in the sequel are defined and the relationship among them is described. The sequential characterizations for S-properly efficient and S-weakly efficient solutions are given in the particular cases when the scalarizing function is linear, and when it is defined with the help of a given cone, respectively.

2. **Preliminaries.** Consider X a locally convex space and X^* its topological dual space endowed with an arbitrary locally convex topology giving X as dual. The most prominent examples of such a topology are the weak* topology $\omega(X^*, X)$ or the strong topology when X is a reflexive Banach space. We denote by $\langle x^*, x\rangle$ the value of the linear continuous functional $x^* \in X^*$ at $x \in X$. For a subset $M \subseteq X$, its *indicator function*, denoted by δ_M, is defined as $\delta_M : X \to \overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$

$$\delta_M(x) = \begin{cases} 0, & \text{if } x \in M, \\ +\infty, & \text{otherwise,} \end{cases}$$

and its *support functional*, denoted by σ_M, is defined as $\sigma_M : X^* \to \overline{\mathbb{R}}$

$$\sigma_M(x^*) = \sup_{x \in M} \langle x^*, x\rangle.$$

For a function $f : X \to \overline{\mathbb{R}}$ we denote by $\operatorname{dom}(f) = \{x \in X : f(x) < +\infty\}$ its *domain* and by $\operatorname{epi}(f) = \{(x, r) \in X \times \mathbb{R} : f(x) \leq r\}$ its *epigraph*. We call f *proper* if $\operatorname{dom}(f) \neq \emptyset$ and $f(x) > -\infty$ $\forall x \in X$. For $x \in X$ such that $f(x) \in \mathbb{R}$, the *subdifferential* of f at x is defined by

$$\partial f(x) = \{x^* \in X^* : f(y) - f(x) \geq \langle x^*, y - x\rangle \ \forall y \in X\}.$$

The *normal cone* to a closed subset M of X is defined by

$$N_M(x) := \begin{cases} \partial(\delta_M)(x) = \{x^* \in X^* : \langle x^*, y - x\rangle \leq 0 \ \forall y \in M\}, & \text{if } x \in M, \\ \emptyset, & \text{otherwise.} \end{cases}$$

The *conjugate function regarding the set* $U \subseteq X$ of f is the function $f^*_U : X^* \to \overline{\mathbb{R}}$ defined by

$$f^*_U(x^*) = \sup_{x \in U} \{\langle x^*, x\rangle - f(x)\}.$$

When $U = X$ we get the classical *Fenchel-Moreau conjugate* of f denoted by f^*. The so-called *Young-Fenchel inequality* proves to be extremely useful in applications, and it reads as follows

$$f^*(x^*) + f(x) \geq \langle x^*, x\rangle \ \forall x \in X \ \forall x^* \in X^*.$$

For all $x \in \operatorname{dom}(f)$ and $x^* \in X^*$ one has

$$x^* \in \partial f(x) \Longleftrightarrow f^*(x^*) + f(x) = \langle x^*, x\rangle$$

The conjugate function of f^*, $f^{**}: X \to \overline{\mathbb{R}}, f^{**}(x) = \sup_{x^* \in X^*}\{\langle x^*, x\rangle - f^*(x^*)\}$ is said to be the *biconjugate* of f. If f is a proper function one has that f is convex and lower semicontinuous if and only if $f^{**}(x) = f(x)\ \forall x \in X$.

In case X is a nonreflexive Banach space the biconjugate of f is defined as $f^{**}: X^{**} \to \overline{\mathbb{R}}, f^{**}(x^{**}) = \sup_{x^* \in X^*}\{\langle x^{**}, x^*\rangle - f^*(x^*)\}$. If $f : X \to \overline{\mathbb{R}}$ is a proper, convex and lower semicontinuous function, then for all $x \in X$ it holds $f(x) = f^{**}(\hat{x})$, where $\hat{x}$ is the canonical image of x. One can easily show that the reverse is also true, namely if $f : X \to \overline{\mathbb{R}}$ is a proper function such that $f(x) = f^{**}(\hat{x})\ \forall x \in X$, then f is convex and lower semicontinuous.

Having a non-empty cone $C \subseteq X$, we denote by $C^+ = \{x^* \in X^* : \langle x^*, x\rangle \geq 0 \text{ for all } x \in C\}$ its *dual cone*, and by $C^{+0} = \{x^* \in X^* : \langle x^*, x\rangle > 0 \text{ for all } x \in C\backslash\{0\}\}$ the *quasi-interior* of the dual cone. We denote by $\leq_C$ the partial order defined by $x \leq_C y$ if $y - x \in C$ for all $x, y \in X$.

Definition 2.1. (see also [16]) Let $C \subseteq X$ be a nonempty convex cone. A function $s : X \to \overline{\mathbb{R}}$ is called C-increasing if for $x, y \in X$ such that $x \leq_C y$ it follows that $s(x) \leq s(y)$. A C-increasing function is called C-strongly increasing whenever $x \leq_C y$, $x \neq y$ implies $s(x) < s(y)$. In the case when $\operatorname{int}(C) \neq \emptyset$ a C-increasing function is called C-strictly increasing if for $y - x \in \operatorname{int}(C)$ it holds $s(x) < s(y)$.

There are notions given for functions with extended real values that can be extended also for functions having their ranges in infinite dimensional spaces. Thus, let us consider Y a locally convex space partially ordered by the nonempty convex cone K. To Y we attach a greatest element with respect to $\leq_K$, which does not belong to Y, denoted by ∞_Y, and let us consider $Y^\bullet = Y \cup \{\infty_Y\}$. Then, for each $y \in Y^\bullet$, $y \leq_K \infty_Y$ and we consider on $Y^\bullet$ the following operations: $y + \infty_Y = \infty_Y$, $t \cdot \infty_Y = \infty_Y$ and $\langle \lambda, \infty_Y\rangle = +\infty$ for all $y \in Y$, $t \geq 0$ and $\lambda \in K^+$.

For a function $F : X \to Y^\bullet$ its *domain* is defined by $\operatorname{dom}(F) = \{x \in X : F(x) \in Y\}$ and F is said to be *proper* if $\operatorname{dom}(F) \neq \emptyset$. The most common extension of the classical notion of convexity for extended real valued functions is the notion of cone-convexity, thus, F is said to be $K-$*convex* if

$$F(tx + (1-t)y) \leq_K tF(x) + (1-t)F(y)\ \forall x, y \in X\ \forall t \in [0, 1].$$

Denoting by $\operatorname{epi}_K(F) = \{(x, y) \in X \times Y : F(x) \leq_K y\}$ the *K-epigraph of F* we have that F is K-convex if and only if $\operatorname{epi}_K(F)$ is convex.

For each $\lambda \in K^+$ we consider the function $(\lambda F) : X \to \overline{\mathbb{R}}$ defined by $(\lambda F)(x) = \langle \lambda, F(x)\rangle$ for all $x \in X$. We say that F is *star K-lower semicontinuous* if (λF) is lower semicontinuous for all $\lambda \in K^+$ and that F is *K-epi closed* if $\operatorname{epi}_K(F)$ is a closed set. The notion of a K-epi closed function was introduced by Luc in [16]. By [16, Theorem 5.9] it follows that every star K-lower semicontinuous function is K-epi closed. One can easily observe, that when $Y = \mathbb{R}$ and $K = \mathbb{R}_+$ the notions star K-lower semicontinuous and K-epi closed coincide, as they collapse into the the classical lower semicontinuity. An example of a K-epi closed function which is not star K-lower semicontinuous was given in [19, Example 1.2]. Nevertheless, this function fails to be K-convex. An example of a K-convex and K-epi closed function which is not star K-lower semicontinuous was recently given in [3].

3. **Sequential Optimality Conditions for Convex Composed Optimization Problems with Geometric and Cone Constraints.** The general framework used within this section is described in the following. Let us consider $(X, \|\cdot\|)$ a reflexive Banach space,$(Y, \|\cdot\|)$ and $(Z, \|\cdot\|)$ be Banach spaces, with $(X^*, \|\cdot\|_*)$,

$(Y^*, \|\cdot\|_*)$, $(Z, \|\cdot\|_*)$, respectively, their topological dual spaces. Although the spaces X, Y, Z and X^*, Y^*, Z^*, are endowed with different norms, respectively, we use the same notation for the norm, as there will be no danger of confusion. Let $\{x_n^* : n \in \mathbb{N}\}$ be a sequence in X^*. We write $x_n^* \xrightarrow{\omega^*} 0$ ($x_n^* \xrightarrow{\|\cdot\|_*} 0$) for the case when x_n^* converges to 0 in the weak* (strong) topology. We make the following convention: if in a certain property we write $x_n^* \to 0$, we understand that the property holds no matter which of the two topologies (weak* or strong) is used. The following property will be frequently used in the paper:

$$\text{if } x_n^* \to 0 \text{ and } x_n \to a, \text{ then } \langle x_n^*, x_n \rangle \to 0,$$

where $\{x_n : n \in \mathbb{N}\} \subseteq X$, $a \in X$ and $x_n \to a$ means $\|x_n - a\| \to 0$, that is the convergence in the topology induced by the norm on X.

Furthermore, on Y and Z we consider the partial orders induced by the nonempty closed convex cones $K \subseteq Y$ and $C \subseteq Z$, respectively, denoted by $\leq_K$ and $\leq_C$, respectively.

We are going to give sequential optimality conditions for the following convex composed optimization problem with geometric and cone constraints

$$(P_s) \inf_{\substack{x \in M \\ G(x) \in -C}} s(F(x)),$$

where M is a nonempty closed convex subset of X, $F : X \to Y^\bullet$ is a proper, K-convex, star K-lower semicontinuous function, $G : X \to Z^\bullet$ is a proper, C-convex, C-epi closed function and $s : Y \to \overline{\mathbb{R}}$ is a proper, convex, lower semicontinuous and K-increasing function, for which we make the convention that $s(\infty_Y) = +\infty$. Furthermore, the following feasibility condition is required

$$(FC) \qquad F(M \cap \mathrm{dom}(F) \cap G^{-1}(-C)) \cap \mathrm{dom}(s) \neq \emptyset.$$

Remark 1. One should notice that in the framework stated above, since s is K-increasing on Y, one has that $s^*(y^*) = +\infty$ for all $y^* \notin K^+$.

Recently, in [2, 3], Boţ, Csetnek and Wanka have given sequential optimality conditions in convex optimization via perturbation approach improving several preexisting results. Such characterizations in optimization prove to be of major importance due to the fact that optimality conditions are given without requiring the fulfillment of any constraint qualification, the case encountered when dealing with optimality conditions obtained by other means, say for example by duality. We extend the results in [2, 3] to the case of convex composed optimization problems with geometric and cone constraints rediscovering some of the results in the above mentioned papers as particular cases of the ones given here. Furthermore, they prove to be helpful in giving sequential characterizations of solutions for vector optimization problems. In section 5 of [2] the following theorem is given.

Theorem 3.1. *Let $\Phi : X \times Y \to \overline{\mathbb{R}}$ be a proper convex and lower semicontinuous function such that $\inf_{x \in X} \Phi(x, 0) < +\infty$. For each $a \in \mathrm{dom}(\Phi(\cdot, 0))$ the following statements are equivalent:*

a) a is a minimizer of $\Phi(\cdot, 0)$ on X;
b) there exist sequences $(x_n, y_n) \in \mathrm{dom}(\Phi)$ and $(x_n^*, y_n^*) \in \partial\Phi(x_n, y_n)$ such that

$$x_n^* \to 0, x_n \to a, y_n \to 0, \langle y_n^*, y_n \rangle \to 0, \Phi(x_n, y_n) - \Phi(a, 0) \to 0.$$

We are going to rewrite (P_s) equivalently as

$$(P_0) \qquad \inf_{x\in X} \Phi(x,0,0),$$

where $\Phi : X \times Y \times Z \to \overline{\mathbb{R}}$ is the perturbation function

$$\Phi(x,y,z) = \begin{cases} s(F(x)+y), & \text{if } x \in M, G(x) - z \in -C, \\ +\infty, & \text{otherwise.} \end{cases}$$

In order to be able to apply Theorem 3.1 to (P_0) we first need to prove that Φ is a proper convex and lower semicontinuous function.

Lemma 3.2. *The function Φ is proper, convex and lower semicontinuous.*

Proof. First we notice that for all $(x,y,z) \in X \times Y \times Z$ it holds $\Phi(x,y,z) = s(F(x)+y) + \delta_M(x) + \delta_{\mathrm{epi}_C(G)}(x,z)$. We only have to prove that the function $\Psi : X \times Y \to \overline{\mathbb{R}}$, $\Psi(x,y) = s(F(x)+y)$ is convex and lower semicontinuous and this will lead to the desired conclusion.

To this end we show that $\Psi = \Psi^{**}|_{X\times\hat{Y}}$, where $\hat{Y}$ is the image of Y through the canonical embedding into Y^{**}. We start by computing the conjugate function of Ψ. For $(x^*,y^*) \in X^* \times Y^*$ we have

$$\begin{aligned} \Psi^*(x^*,y^*) &= \sup_{x\in X, y\in Y} \{\langle x^*,x\rangle + \langle y^*,y\rangle - s(F(x)+y)\} \\ &= \sup_{x\in X, t\in Y} \{\langle x^*,x\rangle + \langle y^*, t - F(x)\rangle - s(t)\} \\ &= \sup_{x\in X} \{\langle x^*,x\rangle - \langle y^*, F(x)\rangle\} + s^*(y^*). \end{aligned}$$

From Remark 1 we have $s^*(y^*) = +\infty$ for all $y^* \notin K^+$ and therefore

$$\Psi^*(x^*,y^*) = \begin{cases} (y^*F)^*(x^*) + s^*(y^*), & y^* \in K^+, \\ +\infty, & \text{otherwise.} \end{cases}$$

By calculating the value of the biconjugate function of Ψ, $\Psi^{**} : X \times Y^{**} \to \overline{\mathbb{R}}$, on $(x,\hat{y}) \in X \times Y^{**}$ for $(x,y) \in X \times Y$, one gets

$$\begin{aligned} \Psi^{**}(x,\hat{y}) &= \sup_{x^*\in X^*, y^*\in K^+} \{\langle x^*,x\rangle + \langle y^*,y\rangle - (y^*F)^*(x^*) - s^*(y^*)\} \\ &= \sup_{y^*\in K^+} \{\langle y^*,y\rangle - s^*(y^*) + (y^*F)^{**}(x)\}. \end{aligned}$$

The function F is proper, K-convex and star K-lower semicontinuous, therefore since $y^* \in K^+$ the function (y^*F) is proper, convex and lower semicontinuous. Thus $(y^*F)^{**}(x) = (y^*F)(x)$ for all $x \in X$. Using this we obtain that

$$\begin{aligned} \Psi^{**}(x,\hat{y}) &= \sup_{y^*\in K^+} \{\langle y^*,y\rangle - s^*(y^*) + (y^*F)(x)\} \\ &= \sup_{y^*\in Y^*} \left\{\langle y^*, \widehat{y+F(x)}\rangle - s^*(y^*)\right\} = s^{**}(\widehat{y+F(x)}) \\ &= s(y+F(x)) = \Psi(x,y). \end{aligned}$$

In conclusion $\Psi^{**}|_{X\times\hat{Y}} = \Psi$ and this implies that Ψ is convex and lower semicontinuous. This concludes the proof. □

The following theorem gives sequential optimality conditions for (P_0), which are actually optimality conditions for (P_s).

Theorem 3.3. *An element $a \in M \cap G^{-1}(-C) \cap \mathrm{dom}(F)$ is an optimal solution of problem (P_s) if and only if*

$$\left\{\begin{array}{l} \exists (x_n, y_n, z_n) \in (M \cap \mathrm{dom}(F)) \times \mathrm{dom}(s) \times -C, \exists (x_n^*, y_n^*, z_n^*) \in X^* \times K^+ \times C^+, \\ x_n^* \in \partial\left((y_n^* F) + (z_n^* G) + \delta_M\right)(x_n), y_n^* \in \partial s(y_n), \langle z_n^*, z_n\rangle = 0 \ \forall n \in \mathbb{N}, \\ x_n^* \to 0, x_n \to a, y_n - F(x_n) \to 0, z_n - G(x_n) \to 0, \\ \langle y_n^*, y_n - F(a)\rangle - \langle z_n^*, G(a)\rangle \to 0, \\ \langle y_n^*, F(x_n) - F(a)\rangle + \langle z_n^*, G(x_n) - G(a)\rangle \to 0, s(y_n) - s(F(a)) \to 0. \end{array}\right. \tag{1}$$

Proof. We notice first that Φ is a proper convex and lower semicontinuous function such that $\inf_{x\in X} \Phi(x, 0, 0) < +\infty$. By Theorem 3.1 and taking (FC) into consideration we have that $a \in M \cap G^{-1}(-C) \cap \mathrm{dom}(F)$ is an optimal solution of problem (P_s) if and only if it is also an optimal solution of (P_0). So, a is an optimal solution of (P_s) if and only if there exist the sequences $(x_n, u_n, v_n) \in \mathrm{dom}(\Phi)$, $(x_n^*, u_n^*, -v_n^*) \in \partial\Phi(x_n, u_n, v_n)$ such that

$$x_n^* \to 0, x_n \to a, (u_n, v_n) \to 0, \langle (u_n^*, -v_n^*), (u_n, v_n)\rangle \to 0, \text{ and}$$

$$\Phi(x_n, u_n, v_n) - \Phi(a, 0, 0) \to 0.$$

For all $n \in \mathbb{N}$ from $(x_n, u_n, v_n) \in \mathrm{dom}(\Phi)$ we get that $x_n \in M \cap \mathrm{dom}(F), F(x_n) + u_n \in \mathrm{dom}(s)$ and $G(x_n) - v_n \in -C$. For $(x_n^*, u_n^*, -v_n^*) \in \partial\Phi(x_n, u_n, v_n)$ we use the characterization

$$\Phi^*(x_n^*, u_n^*, -v_n^*) + \Phi(x_n, u_n, v_n) = \langle x_n^*, x_n\rangle + \langle u_n^*, u_n\rangle - \langle v_n^*, v_n\rangle.$$

Thus, for all $n \in \mathbb{N}$, we obtain equivalently that $u_n^* \in K^+, v_n^* \in C^+$ and

$$((u_n^* F) + (v_n^* G) + \delta_M)^*(x_n^*) + s^*(u_n^*) + s(F(x_n) + u_n) = \langle x_n^*, x\rangle + \langle u_n^*, u_n\rangle - \langle v_n^*, v_n\rangle. \tag{2}$$

As $v_n^* \in C^+$ and $G(x_n) - v_n \in -C$ we have $\langle v_n^*, G(x_n) - v_n\rangle \leq 0$. Using the Fenchel-Young inequality we also get $((u_n^* F) + (v_n^* G) + \delta_M)^*(x_n^*) \geq \langle x_n^*, x_n\rangle - (u_n^* F)(x_n) - (v_n^* G)(x_n) - \delta_M(x_n) \geq \langle x_n^*, x_n\rangle - (u_n^* F)(x_n) - \langle v_n^*, v_n\rangle$ and $s^*(u_n^*) + s(F(x_n) + u_n) \geq \langle u_n^*, u_n\rangle + (u_n^* F)(x_n)\ \forall n \in \mathbb{N}$. Thus, by (2) we obtain equality in all the inequalities above. Hence $(x_n^*, u_n^*, -v_n^*) \in \partial\Phi(x_n, u_n, v_n)$ is equivalent to $u_n^* \in K^+$, $v_n^* \in C^+$, $x_n^* \in \partial((u_n^* F) + (v_n^* G) + \delta_M)(x_n)$, $u_n^* \in \partial s(F(x_n) + u_n)$ and $\langle v_n^*, G(x_n) - v_n\rangle = 0$. Consequently, $a \in M \cap G^{-1}(-C) \cap \mathrm{dom}(F)$ is an optimal solution of problem (P_s) if and only if

$$\left\{\begin{array}{l} \exists (x_n, u_n, v_n) \in (M \cap \mathrm{dom}(F)) \times Y \times Z, \\ F(x_n) + u_n \in \mathrm{dom}(s), G(x_n) - v_n \in -C, \\ \exists (x_n^*, u_n^*, v_n^*) \in X^* \times K^+ \times C^+, x_n^* \in \partial\left((u_n^* F) + (v_n^* G) + \delta_M\right)(x_n), \\ u_n^* \in \partial s(F(x_n) + u_n), \langle v_n^*, G(x_n) - v_n\rangle = 0 \ \forall n \in \mathbb{N}, \\ x_n^* \to 0, x_n \to a, u_n \to 0, v_n \to 0, \\ \langle u_n^*, u_n)\rangle - \langle v_n^*, v_n\rangle \to 0, s(F(x_n) + u_n) - s(F(a)) \to 0. \end{array}\right. \tag{3}$$

Making the following notations: $y_n := F(x_n) + u_n$, $z_n := G(x_n) - v_n$, $y_n^* := u_n^*$, $z_n^* := v_n^*$ for all $n \in \mathbb{N}$, (3) becomes

$$\left\{\begin{array}{l} \exists (x_n, y_n, z_n) \in (M \cap \mathrm{dom}(F)) \times \mathrm{dom}(s) \times -C, \exists (x_n^*, y_n^*, z_n^*) \in X^* \times K^+ \times C^+, \\ x_n^* \in \partial\left((y_n^* F) + (z_n^* G) + \delta_M\right)(x_n), y_n^* \in \partial s(y_n), \langle z_n^*, z_n\rangle = 0 \ \forall n \in \mathbb{N}, \\ x_n^* \to 0, x_n \to a, y_n - F(x_n) \to 0, z_n - G(x_n) \to 0, \\ \langle y_n^*, y_n - F(x_n)\rangle - \langle z_n^*, G(x_n)\rangle \to 0, s(y_n) - s(F(a)) \to 0. \end{array}\right. \tag{4}$$

Further, we improve the relations in (4), by proving that under the hypotheses

$$\left\{\begin{array}{l}(x_n,y_n,z_n)\in(M\cap\operatorname{dom}(F))\times\operatorname{dom}(s)\times -C,(x_n^*,y_n^*,z_n^*)\in X^*\times K^+\times C^+,\\ x_n^*\in\partial\left((y_n^*F)+(z_n^*G)+\delta_M\right)(x_n),y_n^*\in\partial s(y_n),\langle z_n^*,z_n\rangle=0\ \forall n\in\mathbb{N},\\ x_n^*\to 0,x_n\to a,y_n-F(x_n)\to 0,z_n-G(x_n)\to 0,s(y_n)-s(F(a))\to 0\end{array}\right. \tag{5}$$

for the following three real sequences defined as:

$$b_n:=\langle y_n^*,y_n-F(a)\rangle+\langle z_n^*,-G(a)\rangle\ \forall n\in\mathbb{N},$$

$$c_n:=\langle y_n^*,F(x_n)-F(a)\rangle+\langle z_n^*,G(x_n)-G(a)\rangle\ \forall n\in\mathbb{N}$$

and

$$a_n:=\langle y_n^*,y_n-F(x_n)\rangle-\langle z_n^*,G(x_n)\rangle=b_n-c_n\ \forall n\in\mathbb{N},$$

we have

$$a_n\to 0\ \text{ if and only if } b_n\to 0 \text{ and } c_n\to 0.$$

The sufficiency follows at once. Therefore, it remains to prove the necessity. Assume now that $a_n\to 0$. For all $n\in\mathbb{N}$, since $y_n^*\in\partial s(y_n)$, it holds $\langle y_n^*,y_n-F(a)\rangle\geq s(y_n)-s(F(a))$. From $G(a)\in -C$ and $z_n^*\in C^+$ we have $\langle z_n^*,-G(a)\rangle\geq 0$. Hence

$$b_n\geq s(y_n)-s(F(a))+\langle z_n^*,-G(a)\rangle\geq s(y_n)-s(F(a))\ \forall n\in\mathbb{N}. \tag{6}$$

Furthermore,

$$\begin{aligned}-c_n&=\left((y_n^*F)+(z_n^*G)\right)(a)-\left((y_n^*F)+(z_n^*G)\right)(x_n)\\ &=\left((y_n^*F)+(z_n^*G)+\delta_M\right)(a)-\left((y_n^*F)+(z_n^*G)+\delta_M\right)(x_n)\geq\langle x_n^*,a-x_n\rangle,\end{aligned}$$

as $x_n^*\in\partial\left((y_n^*F)+(z_n^*G)+\delta_M\right)(x_n)\ \ \forall n\in\mathbb{N}$. Thus,

$$-c_n\geq\langle x_n^*,a-x_n\rangle\ \forall n\in\mathbb{N}. \tag{7}$$

From (6) and (7) we get

$$s(y_n)-s(F(a))\leq b_n=a_n+c_n\leq a_n+\langle x_n^*,x_n-a\rangle\ \forall n\in\mathbb{N}. \tag{8}$$

For $n\to+\infty$, since $a_n\to 0$ and taking into consideration that, by (5), $s(y_n)-s(F(a))\to 0$, and $x_n^*\to 0$ and $x_n-a\to 0$, which imply $\langle x_n^*,x_n-a\rangle\to 0$, we obtain from (8) that $b_n=a_n+c_n\to 0$ and thus also $c_n\to 0$.

Therefore (1) is equivalent to (4) and the theorem is proved. □

Theorem 3.3 is stated under rather weak assumptions imposed on the sets and the functions involved in its formulation. By specializing some of these objects, we are able to obtain as particular cases not only other important results we use later, but also sequential characterizations of optimality previously given in the literature.

3.1. The Case when F and G are Continuous.

The framework we work within remains basically the same as in the beginning of this section. Imposing stronger conditions on the functions involved in defining the optimization problem (P_s) the sequential characterization of the optimal solutions in Theorem 3.3 can be substantially refined.

Take $F:X\to Y$ and $G:X\to Z$ continuous, thus $\operatorname{dom}(F)=\operatorname{dom}(G)=X$ and the feasibility condition (FC) becomes

$$(FC_C)\qquad\qquad F(M\cap G^{-1}(-C))\cap\operatorname{dom}(s)\neq\emptyset.$$

With these hypotheses we get the following result.

Theorem 3.4. *An element $a \in M \cap G^{-1}(-C)$ is an optimal solution of problem (P_s) if and only if*

$$\left\{ \begin{array}{l} \exists (x_n, y_n, z_n) \in M \times \operatorname{dom}(s) \times -C, \\ \exists (u_n^*, v_n^*, t_n^*, y_n^*, z_n^*) \in X^* \times X^* \times X^* \times K^+ \times C^+, \\ u_n^* \in \partial(y_n^* F)(x_n), v_n^* \in \partial(z_n^* G)(x_n), t_n^* \in N_M(x_n), y_n^* \in \partial s(y_n), \\ \langle z_n^*, z_n \rangle = 0 \ \forall n \in \mathbb{N}, u_n^* + v_n^* + t_n^* \to 0, x_n \to a, y_n \to F(a), z_n \to G(a), \\ \langle y_n^*, y_n - F(a) \rangle - \langle z_n^*, G(a) \rangle \to 0, \\ \langle y_n^*, F(x_n) - F(a) \rangle + \langle z_n^*, G(x_n) - G(a) \rangle \to 0, s(y_n) - s(F(a)) \to 0. \end{array} \right. \tag{9}$$

Proof. From Theorem 3.3 we have that $a \in M \cap G^{-1}(-C)$ is an optimal solution of problem (P_s) if and only if (1) is satisfied. Using Theorem 2.8.7 in [23], for each $n \in \mathbb{N}$, from the fact that F and G are continuous functions we have

$$\partial \left((y_n^* F) + (z_n^* G) + \delta_M \right)(x_n) = \partial(y_n^* F)(x_n) + \partial(z_n^* G)(x_n) + N_M(x_n).$$

Thus for all $x_n^* \in \partial \left((y_n^* F) + (z_n^* G) + \delta_M \right)(x_n), n \in \mathbb{N}$, taking into consideration the equality above, one obtains the existence of other three sequences such that $u_n^*, v_n^*, t_n^* \in X^*$ and

$$x_n^* = u_n^* + v_n^* + t_n^*, \ \ u_n^* \in \partial(y_n^* F)(x_n), \ \ v_n^* \in \partial(z_n^* G)(x_n), \ \ t_n^* \in N_M(x_n).$$

Since $x_n^* \to 0$, we have $u_n^* + v_n^* + t_n^* \to 0$. As F and G are continuous and $x_n \to a$ when $n \to +\infty$, it holds $F(x_n) \to F(a)$ and $G(x_n) \to G(a)$ when $n \to +\infty$. Then $y_n - F(x_n) \to 0$ and $z_n - G(x_n) \to 0$ is the same with

$$y_n \to F(a) \text{ and } z_n \to G(a).$$

Changing in (1) the sequences according to the facts listed above, we obtain exactly (9). □

The aforementioned particular case can be further specialized when considering only geometric constrained problems, i.e in the case when the function $G \equiv 0$. In this situation we obtain the following result which is a natural consequence of Theorem 3.4, therefore its proof is omitted.

Corollary 1. *An element $a \in M$ is an optimal solution of the optimization problem*

$$(P_g) \qquad \inf_{x \in M} s(F(x))$$

if and only if

$$\left\{ \begin{array}{l} \exists (x_n, y_n) \in M \times \operatorname{dom}(s), \exists (u_n^*, t_n^*, y_n^*) \in X^* \times X^* \times K^+, \\ u_n^* \in \partial(y_n^* F)(x_n), t_n^* \in N_M(x_n), y_n^* \in \partial s(y_n) \ \forall n \in \mathbb{N}, \\ u_n^* + t_n^* \to 0, x_n \to a, y_n \to F(a), \\ \langle y_n^*, y_n - F(a) \rangle \to 0, \langle y_n^*, F(x_n) - F(a) \rangle \to 0, s(y_n) - s(F(a)) \to 0. \end{array} \right. \tag{10}$$

3.2. **Sequential Lagrange Multiplier Conditions.** The convex composed optimization problem (P_s) can be reduced to an ordinary one, by taking $X = Y$, the function F as the identity function on X, i.e. $F : X \to X, F(x) = x$ for all $x \in X$, and the cone $K = \{0\}$. Using again Theorem 3.4 we are able to develop sequential optimality conditions for the convex optimization problem with geometric and cone constraints

$$(P_c) \ \inf_{\substack{x \in M \\ G(x) \in -C}} s(x),$$

where $s : X \to \overline{\mathbb{R}}$ is a proper, convex and lower semicontinuous function, $G : X \to Z$ is a C-convex and continuous function and the feasibility condition

$$(FC_{CL}) \qquad M \cap G^{-1}(-C) \cap \operatorname{dom}(s) \neq \emptyset$$

is satisfied. Then the following sequential Lagrange multiplier condition can be given. It actually turns out to be a refinement of the result in Theorem 4.10 in [2].

Theorem 3.5. *An element $a \in M \cap G^{-1}(-C) \cap \operatorname{dom}(s)$ is a solution of the optimization problem (P_c) if and only if*

$$\begin{cases} \exists (x_n, y_n, z_n) \in M \times \operatorname{dom}(s) \times -C, \exists (v_n^*, t_n^*, y_n^*, z_n^*) \in X^* \times X^* \times X^* \times C^+, \\ v_n^* \in \partial(z_n^* G)(x_n), t_n^* \in N_M(x_n), y_n^* \in \partial s(y_n), \langle z_n^*, z_n \rangle = 0 \ \forall n \in \mathbb{N}, \\ y_n^* + v_n^* + t_n^* \to 0, x_n \to a, y_n \to a, z_n \to G(a), \\ \langle y_n^*, y_n - a \rangle - \langle z_n^*, G(a) \rangle \to 0, \\ \langle y_n^*, x_n - a \rangle + \langle z_n^*, G(x_n) - G(a) \rangle \to 0, s(y_n) - s(a) \to 0. \end{cases} \tag{11}$$

Proof. One can easily see that (P_c) is nothing but a particularization of (P_s). Since $K = \{0\}$, its dual is $K^+ = X^*$ and s is K-increasing. Using Theorem 3.4 we have that $a \in M \cap G^{-1}(-C) \cap \operatorname{dom}(s)$ is a solution of (P_c) if and only if (9) holds. Since F is the identity on X and $u_n^* \in \partial(y_n^* F)(x_n)$ it is easy to see that $u_n^* = y_n^*$ for $n \in \mathbb{N}$, i.e. (9) becomes (11). $\square$

Remark 2. It is well known that for all $x \in M \cap G^{-1}(-C) \cap \operatorname{dom}(s)$ the following relations hold

$$\begin{aligned} \partial\left(s + \delta_{\{u \in M : G(u) \in -C\}}\right)(x) &\supseteq \bigcup_{\substack{z^* \in C^+, \\ (z^*G)(x) = 0}} \partial(s + (z^*G) + \delta_M)(x) \\ &\supseteq \partial s(x) + \bigcup_{\substack{z^* \in C^+, \\ (z^*G)(x) = 0}} \partial((z^*G) + \delta_M)(x). \end{aligned}$$

Thus for a given $a \in M \cap G^{-1}(-C) \cap \operatorname{dom}(s)$, if one has that

$$0 \in \bigcup_{\substack{z^* \in C^+, \\ (z^*G)(a) = 0}} \partial(s + (z^*G) + \delta_M)(a)$$

or

$$0 \in \partial s(a) + \bigcup_{\substack{z^* \in C^+, \\ (z^*G)(a) = 0}} \partial((z^*G) + \delta_M)(a),$$

then a is an optimal solution to (P_c). In other words, this is nothing else than asking for the classical KKT optimality conditions for the optimization problem (P_c) to be fulfilled. Let us also notice that, in case some regularity conditions are satisfied (for more details, see [5]), the inclusion relations above turn into equalities. Nevertheless, there are situations when the KKT optimality conditions are not fulfilled, unlike the sequential optimality conditions given in Theorem 3.5. This is also the case for the problem considered in the example below.

Example 1. *Let $X = \mathbb{R}$, $Z = \mathbb{R}^2$, $C = \mathbb{R}^2_+$, $M = \mathbb{R}$, $s : \mathbb{R} \to \overline{\mathbb{R}}$be defined by $s(x) = -\sqrt{x} + \delta_{\mathbb{R}_+}(x)$, $x \in \mathbb{R}$, and $G = (G_1, G_2)^T : \mathbb{R} \to \mathbb{R}^2$ be defined by $G_1(x) = -1 - x$ and $G_2(x) = x$, $x \in \mathbb{R}$. Then s is a proper, convex and lower semicontinuous function, G is $\mathbb{R}^2_+$-convex and continuous function and the feasibility* *condition (FC_{CL}) is fulfilled. The element $a = 0$ is the (unique) optimal solution of the problem (P_c). Since*

$$\bigcup_{\substack{z^* \in C^+, \\ (z^*G)(0)=0}} \partial(s + (z^*G) + \delta_M)(0) = \partial s(0) + \bigcup_{\substack{z^* \in C^+, \\ (z^*G)(0)=0}} \partial((z^*G) + \delta_M)(0) = \emptyset,$$

the classical KKT optimality conditions fail. Nevertheless, as we show in the following, the sequential optimality conditions in (11) are satisfied. To this end it is enough to take $x_n = 0$, $y_n = 1/n$, $z_n = (1/n - 1, 0)^T$, $v_n^* = \sqrt{n}/2$, $t_n^* = 0$, $y_n^* = -\sqrt{n}/2$ and $z_n^* = (0, \sqrt{n}/2)^T$ for all $n \in \mathbb{N}$. Thus, for all $n \in \mathbb{N}$, $\sqrt{n}/2 \in \partial(z_n^*G)(0)$, $0 \in N_M(0)$, $-\sqrt{n}/2 \in \partial s(1/n)$ and $z_n^{*T} z_n = 0$. Further, $y_n^* + v_n^* + t_n^* = 0$, $x_n \to 0$, $y_n \to 0$ and $z_n \to (-1, 0)^T = G(0)$. Moreover, $\langle y_n^*, y_n - a\rangle - \langle z_n^*, G(a)\rangle = -1/(2\sqrt{n}) \to 0$, $\langle y_n^*, x_n - a\rangle + \langle z_n^*, G(x_n) - G(a)\rangle = 0$ and $s(1/n) \to s(0)$.

A sequential generalization of the well-known Pshenichnyi-Rockafellar Lemma can be given, by taking in Theorem 3.5 only geometric constraints. It is stated bellow.

Theorem 3.6. *Let $s : X \to \overline{\mathbb{R}}$ be a proper, convex and lower semicontinuous function such that $M \cap \operatorname{dom}(s) \neq \emptyset$. Then $a \in M \cap \operatorname{dom}(s)$ is an optimal solution of the problem*

$$(P_{PR}) \qquad \inf_{x \in M} s(x)$$

if and only if

$$\begin{cases} \exists (x_n, y_n) \in M \times \operatorname{dom}(s), \exists (t_n^*, y_n^*,) \in X^* \times X^*, \\ y_n^* \in \partial s(y_n), t_n^* \in N_M(x_n)\ \forall n \in \mathbb{N}, \\ y_n^* + t_n^* \to 0, x_n \to a, y_n \to a, \\ \langle y_n^*, y_n - a\rangle \to 0, \langle y_n^*, x_n - a\rangle \to 0, s(y_n) - s(a) \to 0. \end{cases} \tag{12}$$

Proof. Problem (P_{PR}) is nothing else than (P_c) when $G \equiv 0$. Relation (12) is nothing else than (11) in Theorem 3.5 in this particular case. □

Theorem 3.6 is also a refinement of Corollary 4.8 in [2] and, consequently, a generalization of Corollary 3.5 in [13].

4. **Sequential Optimality Conditions in Vector Optimization.**

Vector optimization problems are thoroughly studied due to their utility in various practical areas. In the literature, several approaches in defining and studying optimal solutions of vector optimization problems have been undertaken.

Let us consider the following vector optimization problem

$$(P_v) \quad v - \min_{\substack{x \in M \\ G(x) \in -C}} F(x)$$

where X is a reflexive Banach space, Y and Z are Banach spaces, $K \subseteq Y$ and $C \subseteq Z$ are closed convex cones which define partial orders on Y and Z denoted by $\leq_K$ and $\leq_C$, respectively. We also assume that K is pointed ($K \cap -K = \{0\}$). Let M be a nonempty closed convex subset of X, $F : X \to Y$ be a K-convex continuous function and $G : X \to Z$ be a C-convex continuous function such that the following feasibility condition is satisfied

$$M \cap G^{-1}(-C) \neq \emptyset.$$

Definition 4.1. (see [12]) An element $a \in M \cap G^{-1}(-C)$ is called a (Pareto) efficient solution to (P_v) if from $F(x) \leq_K F(a)$ for an $x \in M \cap G^{-1}(-C)$ it follows that $F(x) = F(a)$.

Let us consider the set of convex and K-strongly increasing functions on Y

$$S = \{s : Y \to \mathbb{R} : s \text{ is convex and } K\text{-strongly increasing}\}.$$

Definition 4.2. (see [9, 10]) An element $a \in M \cap G^{-1}(-C)$ is said to be a S-properly efficient solution to (P_v) if there exists a function $s \in S$ such that $s(F(a)) \leq s(F(x))$ for all $x \in M \cap G^{-1}(-C)$, i.e. if it is an optimal solution of the problem

$$(P_s) \quad \inf_{\substack{x \in M \\ G(x) \in -C}} s(F(x)).$$

Remark 3. Each S-properly efficient solution to (P_v) is also an efficient one.

If $\operatorname{int}(K) \neq \emptyset$ one can also introduce another efficiency notion, the so-called weakly efficient solution.

Definition 4.3. (see [12]) An element $a \in M \cap G^{-1}(-C)$ is said to be a weakly efficient solution to (P_v) if there exists no $x \in M \cap G^{-1}(-C)$ such that $F(x) - F(a) \in -\operatorname{int}(K)$.

Considering the set of convex and K-strictly increasing functions on Y

$$T = \{s : Y \to \mathbb{R} : s \text{ is convex and } K\text{-strictly increasing}\},$$

one can define the following new class of efficient solutions.

Definition 4.4. An element $a \in M \cap G^{-1}(-C)$ is said to be a T-weakly efficient solution to (P_v) if there exists a function $s \in T$ such that $s(F(a)) \leq s(F(x))$ for all $x \in M \cap G^{-1}(-C)$, i.e. if it is an optimal solution of the problem (P_s).

Remark 4. As an easy consequence of the last definition one deduces that each T-weakly efficient solution to (P_v) is also a weakly efficient one.

Using the results from the previous section we can give sequential optimality conditions for both S-properly and T-weakly efficient solutions to (P_v). One must acknowledge the fact that such conditions hold without any other constraint qualification, therefore they represent an improvement for the optimality conditions given so far in the literature (see [4]). They can also be stated in the case when the functions F and G are only star K-lower semicontinuous and C-epi closed, respectively, but for the simplicity of the presentation we have chosen to express them in the continuous case.

Sequential characterization of optimal solutions for problem (P_s) have already been given in Theorem 3.4. Since they look identical both for S-properly efficient solutions and T-weakly efficient solutions, the only difference being that for the first one the function s is K-strongly increasing while for the second the function is K-strictly increasing, they are not repeated here at this point. Nevertheless, in the following subsections we give two particular cases by specializing the scalarizing function. For them, we state explicitly the sequential optimality conditions.

It is worth mentioning that our theory can be applied to a wider area of scalarizing functions. Nevertheless, we restrict ourselves to the afore mentioned particular cases since besides being representative they also suffice as examples of obtaining sequential optimality conditions for vector optimization by means of scalarization.

4.1. **Linear Scalarization.** The most famous and used scalarization in vector optimization is the one with K-strongly increasing linear functionals. Let us start by noticing that for each $\lambda \in K^{+0}$ the function $s_\lambda : Y \to \mathbb{R}$ defined by

$$s_\lambda(y) = \langle \lambda, y \rangle \ \forall y \in Y$$

is K-strongly increasing, continuous and convex. Then, considering the set

$$S_l = \{s_\lambda : Y \to \mathbb{R} : s_\lambda(y) = \langle \lambda, y \rangle \ \forall y \in Y, \lambda \in K^{+0}\},$$

an element $a \in M \cap G^{-1}(-C)$ is a *S_l-properly efficient solution* to (P_v) if there exists a $\lambda \in K^{+0}$ such that $\langle \lambda, F(a) \rangle \leq \langle \lambda, F(y) \rangle$ for all $y \in M \cap G^{-1}(-C)$. The following sequential optimality condition can be given.

Theorem 4.5. *An element $a \in M \cap G^{-1}(-C)$ is a S_l- properly efficient solution to (P_v) if and only if there exists a $\lambda \in K^{+0}$ such that*

$$\left\{ \begin{array}{l} \exists (x_n, z_n) \in M \times -C, \exists (u_n^*, v_n^*, t_n^*, z_n^*) \in X^* \times X^* \times X^* \times C^+, \\ u_n^* \in \partial(\lambda F)(x_n), v_n^* \in \partial(z_n^* G)(x_n), t_n^* \in N_M(x_n), \langle z_n^*, z_n \rangle = 0 \ \forall n \in \mathbb{N}, \\ u_n^* + v_n^* + t_n^* \to 0, x_n \to a, z_n \to G(a), \langle z_n^*, G(a) \rangle \to 0, \langle z_n^*, G(x_n) \rangle \to 0. \end{array} \right. \tag{13}$$

Proof. An element $a \in M \cap G^{-1}(-C)$ is a S_l-properly efficient solution to (P_v) if and only if there exists a $\lambda \in K^{+0}$ such that it is an optimal solution to the problem

$$(P_\lambda) \quad \inf_{\substack{x \in M \\ G(x) \in -C}} \langle \lambda, F(x) \rangle,$$

which is nothing but a reformulation of problem (P_s) in this framework. Since all the function involved are continuous and s_λ is K-increasing, we can use Theorem 3.4 where the sequential characterization of the optimal solutions is given by system (9). Due to the particular form of s_λ the following changes are made. First of all $\mathrm{dom}(s_\lambda) = Y$ while $y_n^* \in \partial s_\lambda(y_n)$ means actually that $y_n^* = \lambda$ for all $n \in \mathbb{N}$. By replacing y_n^* with λ in (9) we obtain exactly (13), taking also into account that s_λ and F are continuous, and $y_n \to F(a)$ for $n \to \infty$, thus becoming superfluous for us to write the conditions $s_\lambda(y_n) - s_\lambda(F(a)) \to 0$, $\langle \lambda, y_n - F(a) \rangle \to 0$ and $\langle \lambda, F(x_n) - F(a) \rangle \to 0$. □

Remark 5. By taking into consideration Remark 2 one can easily see that if there exists, for a fixed $a \in M \cap G^{-1}(-C)$, an element $\lambda \in K^{+0}$ such that

$$0 \in \bigcup_{\substack{z^* \in C^+, \\ (z^*G)(a) = 0}} \partial((\lambda F) + (z^*G) + \delta_M)(a)$$

or

$$0 \in \partial(\lambda F)(a) + \bigcup_{\substack{z^* \in C^+, \\ (z^*G)(a) = 0}} \partial((z^*G) + \delta_M)(a),$$

then a is a S_l-properly efficient solution to (P_v). For more details on KKT-type optimality conditions for vector optimization problems we refer the reader to [1]. As in the scalar case, there are situations, like in the example below, where these conditions fail, while the sequential optimality conditions are satisfied.

Example 2. *Let $X = \mathbb{R}$, $Y = \mathbb{R}^2$, $Z = \mathbb{R}$, $K = \mathbb{R}^2_+$, $C = \mathbb{R}_+$, $M = \mathbb{R}$, $F = (F_1, F_2)^T : \mathbb{R} \to \mathbb{R}^2$, be defined by $F_1(x) = x$ and $F_2(x) = x^2$, $x \in \mathbb{R}$, and $G : \mathbb{R} \to \mathbb{R}$ be defined by $G(x) = x^2$, $x \in \mathbb{R}$. The function F is $\mathbb{R}^2_+$-convex and continuous, while G is $\mathbb{R}_+$-convex and continuous. Moreover, the feasibility condition $M \cap G^{-1}(-C) \neq$*

$\emptyset$ is satisfied. Obviously, $a = 0$ is a S_l-properly efficient solution to (P_v), but there is no $\lambda \in \operatorname{int}(\mathbb{R}^2_+)$ such that one of the optimality conditions in Remark 5 is fulfilled. This is not the case for the sequential optimality conditions given in (13), as follows from the following considerations. For $\lambda = (1,1)^T$ we take $x_n = -1/n$, $z_n = 0$, $u_n^* = 1-2/n$, $v_n^* = -1$, $t_n^* = 0$ and $z_n^* = n/2$ for all $n \in \mathbb{N}$. Thus, for all $n \in \mathbb{N}$, one has $1 - 2/n \in \partial(\lambda F)(1/n)$, $-1 \in \partial(n/2G)(-1/n)$, $0 \in N_M(-1/n)$ and $z_n^* z_n = 0$. Moreover, $u_n^* + v_n^* + t_n^* = -2/n \to 0$, $x_n \to 0$, $z_n \to 0 = G(0)$, $\langle z_n^*, G(a)\rangle = 0$ and $\langle z_n^*, G(x_n)\rangle = 1/(2n) \to 0$.

4.2. **Set Scalarization.** Some quite recent scalarization methods are based on already given or constructed sets which have to satisfy some conditions. The scalarization function we use in the following is attributed to Gerth and Weidner (cf. [8]), but it was used before, for example, by Rubinov in [20] and Pascoletti and Serafini in the context of vector optimization in [17].

To the general framework from the beginning of the section we add the assumption that the convex closed cone K fulfills the assumption $\operatorname{int}(K) \neq \emptyset$. For each $\mu \in \operatorname{int}(K)$ we consider the function

$$s_\mu : Y \to \mathbb{R}, s_\mu(y) = \inf\{t \in \mathbb{R} : y \in t\mu - K\},$$

which is K-strictly increasing, convex and continuous, according to [8]. Let us consider the following set of K-strictly increasing, convex and continuous functions

$$T_s = \{s_\mu : \mu \in \operatorname{int}(K)\}.$$

Then an element $a \in M \cap G^{-1}(-C)$ is a T_s-*weakly efficient solution* to (P_v) if there exists a $\mu \in \operatorname{int}(K)$ such that a is an optimal solution to problem

$$(P_\mu) \inf_{\substack{x \in M \\ G(x) \in -C}} s_\mu(F(x)).$$

The following sequential characterization of T_s-weakly efficient solutions can be given.

Theorem 4.6. *An element $a \in M \cap G^{-1}(-C)$ is a T_s-weakly efficient solution to (P_v) if and only if there exists a $\mu \in \operatorname{int}(K)$ such that*

$$\begin{cases} \exists (x_n, y_n, z_n) \in M \times Y \times -C, \exists (u_n^*, v_n^*, t_n^*, y_n^*, z_n^*) \in X^* \times X^* \times X^* \times K^+ \times C^+, \\ u_n^* \in \partial(y_n^* F)(x_n), v_n^* \in \partial(z_n^* G)(x_n), t_n^* \in N_M(x_n), \langle z_n^*, z_n\rangle = 0, \\ \langle y_n^*, \mu\rangle = 1, \sigma_{\{\lambda \in K^+ : \langle \lambda, \mu\rangle = 1\}}(y_n) = \langle y_n^*, y_n\rangle \ \forall n \in \mathbb{N} \\ u_n^* + v_n^* + t_n^* \to 0, x_n \to a, y_n \to F(a), z_n \to G(a), \\ \langle y_n^*, y_n - F(a)\rangle - \langle z_n^*, G(a)\rangle \to 0, \\ \langle y_n^*, F(x_n) - F(a)\rangle + \langle z_n^*, G(x_n) - G(a)\rangle \to 0. \end{cases} \tag{14}$$

Proof. For $\mu \in \operatorname{int}(K)$ problem (P_μ) is nothing but a reformulation of problem (P_s) in this particular framework. From Theorem 3.4 we know that a is an optimal solution of (P_μ) if and only if (9) holds. In order to be able to reexpress it, we need to establish the conjugate of s_μ, $s_\mu^* : Y^* \to \overline{\mathbb{R}}$. For each $y^* \in K^+$ one has

$$
\begin{aligned}
s_\mu^*(y^*) &= \sup_{y\in Y}\left\{\langle y^*, y\rangle - \inf_{\substack{t\in\mathbb{R}\\ y\in t\mu-K}} t\right\} = \sup_{\substack{t\in\mathbb{R}\\ y\in t\mu-K}}\{\langle y^*, y\rangle - t\} \\
&= \sup_{t\in\mathbb{R}}\left\{-t + \sup_{u\in -K}\langle y^*, u+t\mu\rangle\right\} = \sup_{t\in\mathbb{R}}\{t(\langle y^*,\mu\rangle - 1)\} + \sup_{u\in -K}\langle y^*, u\rangle \\
&= \begin{cases} 0, & \langle y^*,\mu\rangle = 1 \\ +\infty, & \text{otherwise.}\end{cases}
\end{aligned}
$$

Thus, for $y_n \in Y$ and $y_n^* \in K^+$, $y_n^* \in \partial s_\mu(y_n)$ is equivalent to

$$
\langle y_n^*, \mu\rangle = 1 \text{ and } \inf_{t\in\mathbb{R}, y_n\in t\mu-K} t = \langle y_n^*, y_n\rangle\ \forall n\in\mathbb{N}. \tag{15}
$$

The above expression can be further refined thus obtaining a more interesting formulation of it. Let $y_n \in Y, n\in\mathbb{N}$, be fixed. Since $\mu\in\operatorname{int}(K)$ it is easy to verify the Slater condition for the optimization problem

$$
(P_I)\ \inf_{t\in\mathbb{R}, y_n - t\mu\in -K} t,
$$

i.e. there exists $t'\in\mathbb{R}$ such that $t'\mu - y_n \in \operatorname{int}(K)$. Thus, strong duality holds between (P_I) and its Lagrange dual problem. Therefore

$$
\begin{aligned}
\inf_{t\in\mathbb{R}, y_n\in t\mu-K} t &= \sup_{\lambda\in K^+}\inf_{t\in\mathbb{R}}\{t + \langle\lambda, y_n - t\mu\rangle\} = \sup_{\lambda\in K^+}\left\{\langle\lambda, y_n\rangle + \inf_{t\in\mathbb{R}}\{t - \langle\lambda, t\mu\rangle\}\right\} \\
&= \sup_{\lambda\in K^+, \langle\lambda,\mu\rangle=1}\langle\lambda, y_n\rangle = \sigma_{\{\lambda\in K^+, \langle\lambda,\mu\rangle=1\}}(y_n).
\end{aligned} \tag{16}
$$

From (15) and (16) we obtain that, for $y_n\in Y$ and $y_n^*\in K^+$, $y_n^*\in\partial s_\mu(y_n)$ if and only if

$$
\langle y_n^*, \mu\rangle = 1 \text{ and } \sigma_{\{\lambda\in K^+, \langle\lambda,\mu\rangle=1\}}(y_n) = \langle y_n^*, y_n\rangle\ \forall n\in\mathbb{N}.
$$

Taking into account that s_μ is continuous and $y_n\to F(a)$ for $n\to\infty$, it is superfluous to write the condition $s_\mu(y_n) - s_\mu(F(a))\to 0$. By replacing in (9) everything according to the discussion above, we obtain exactly (14). □

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