An application of the bivariate inf-convolution formula to enlargements of monotone operators

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Abstract

Motivated by a classical result concerning the ε-subdifferential of the sum of two proper, convex and lower semicontinuous functions, we give in this paper a similar result for the enlargement of the sum of two maximal monotone operators defined on a Banach space. This is done by establishing a necessary and sufficient condition for a bivariate inf-convolution formula.

Key Words. monotone operator, Fitzpatrick function, representative function, enlargement, subdifferential

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1 Introduction and motivation

It is well known that the subdifferential of a proper, convex and lower semicontinuous function f defined on a real Banach space, denoted by ∂f, is a maximal monotone operator (cf. [28]). For $\varepsilon \geq 0$, the ε-subdifferential, introduced in [8] and denoted by $\partial_\varepsilon f$, has an important impact in convex analysis for both theoretical and practical applications. The ε-subdifferential is an enlargement of ∂f, in the sense that $\partial f(x) \subseteq \partial_\varepsilon f(x)$ for all $x \in X$ and $\varepsilon \geq 0$.

Let us mention a result concerning the ε-subdifferential of the sum of two proper, convex and lower semicontinuous functions $f, g : X \to \overline{\mathbb{R}}$ such that $\mathrm{dom}(f) \cap \mathrm{dom}(g) \neq \emptyset$, namely that $\mathrm{epi}(f^*) + \mathrm{epi}(g^*)$ is closed in $(X^*, \omega(X^*, X)) \times \mathbb{R}$ if and only if

$$\partial_\varepsilon (f+g)(x) = \bigcup_{\substack{\varepsilon_1, \varepsilon_2 \geq 0 \\ \varepsilon_1 + \varepsilon_2 = \varepsilon}} \big(\partial_{\varepsilon_1} f(x) + \partial_{\varepsilon_2} g(x)\big) \text{ for all } \varepsilon \geq 0 \text{ and for all } x \in X. \tag{1}$$

This result is proved in [10, Theorem 1] in the framework of Banach spaces, however it holds also in a real separated locally convex space (cf. [7]). The direct implication is shown in [18, Theorem 2.1]. Sufficient conditions which guarantee the equality (1) can be found in [32, Theorem 2.8.7].

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For an arbitrary monotone operator $S : X \rightrightarrows X^*$ the following enlargement can be defined for all $x \in X$ and $\varepsilon \geq 0$:

$$S^\varepsilon(x) := \{x^* \in X^* : \langle y^* - x^*, y - x\rangle \geq -\varepsilon \text{ for all } (y, y^*) \in G(S)\}.$$

Introduced in [9], the enlargement turned out to have some useful applications and properties similar to those of the ε-subdifferential (local boundedness, demiclosed graph, Lipschitz continuity, the Brøndsted-Rockafellar property), see [12, 13].

It is the aim of this paper to give a necessary and sufficient condition that guarantees in the case of enlargements of maximal monotone operators an equality similar to (1). This is achieved by giving in the next section a necessary and sufficient condition for the bivariate inf-convolution formula

$$(h_1\Box_2 h_2)^* = h_1^*\Box_1 h_2^*, \tag{2}$$

where $h_1, h_2 : X \times Y \to \overline{\mathbb{R}}$ are proper, convex and lower semicontinuous functions (see Corollary 5). On the other hand, giving conditions for (2) turns out to be useful also for the problem of establishing when the sum of two maximal monotone operators defined on a reflexive Banach space is maximal monotone (see [23, 30]).

By means of the condition for the bivariate inf-convolution formula, we give in the last section of the paper a so-called *closedness-type regularity condition* which completely characterizes the enlargement of the sum of two maximal monotone operators in terms of the enlargements of the maximal monotone operators involved (see Theorem 11). Particularizing this result to the case of subdifferential operators we obtain exactly the result established in [10].

2 A bivariate inf-convolution formula

Let us recall first some notions and results that will be used in the paper. Consider X, Y real separated locally convex spaces and X^*, Y^* their topological dual spaces, respectively. The notation $\omega(X^*, X)$ stands for the weak* topology induced by X on X^*, while by $\langle x^*, x\rangle$ we denote the value of the linear continuous functional $x^* \in X^*$ at $x \in X$. For a subset C of X we denote by $\operatorname{cl}(C)$ and ${}^{ic}C$ its *closure* and *intrinsic relative algebraic interior*, respectively. Let us note that if C is a convex set, then an element $x \in X$ belongs to ${}^{ic}C$ if and only if $\bigcup_{\lambda>0} \lambda(C - x)$ is a closed linear subspace of X (see [32]). The *indicator function* of C, $\delta_C : X \to \overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$, is defined as

$$\delta_C(x) = \begin{cases} 0, & \text{if } x \in C, \\ +\infty, & \text{otherwise.} \end{cases}$$

For a function $f : X \to \overline{\mathbb{R}}$ we denote by $\operatorname{dom}(f) = \{x \in X : f(x) < +\infty\}$ its *domain* and by $\operatorname{epi}(f) = \{(x, r) \in X \times \mathbb{R} : f(x) \leq r\}$ its *epigraph*. We call f *proper* if $\operatorname{dom}(f) \neq \emptyset$ and $f(x) > -\infty$ for all $x \in X$. By $\operatorname{cl}(f)$ we denote the *lower semicontinuous hull* of f, namely the function of which epigraph is the closure of $\operatorname{epi}(f)$ in $X \times \mathbb{R}$, that is $\operatorname{epi}(\operatorname{cl}(f)) = \operatorname{cl}(\operatorname{epi}(f))$. Having $f : X \to \overline{\mathbb{R}}$ a proper function, for $x \in \operatorname{dom}(f)$ we define the *ε-sudifferential* of f at x, where $\varepsilon \geq 0$, by

$$\partial_\varepsilon f(x) = \{x^* \in X^* : f(y) - f(x) \geq \langle x^*, y - x\rangle - \varepsilon \text{ for all } y \in X\}.$$

For $x \notin \operatorname{dom}(f)$ we take $\partial_\varepsilon f(x) := \emptyset$. The set $\partial f(x) := \partial_0 f(x)$ is then the classical *subdifferential* of f at x.

The *Fenchel-Moreau conjugate* of f is the function $f^* : X^* \to \overline{\mathbb{R}}$ defined by

$$f^*(x^*) = \sup_{x \in X}\{\langle x^*, x\rangle - f(x)\} \text{ for all } x^* \in X^*.$$

We have the so called *Young-Fenchel inequality*

$$f^*(x^*) + f(x) \geq \langle x^*, x\rangle \text{ for all } (x, x^*) \in X \times X^*.$$

We mention here some important properties of conjugate functions. If f is proper, then f is convex and lower semicontinuous if and only if $f^{**} = f$ (see [15,32]). As a consequence we have that in case f is convex and $\operatorname{cl}(f)$ is proper, then $f^{**} = \operatorname{cl}(f)$ ([32, Theorem 2.3.4]).

One can give the following characterizations for the subdifferential and ε-subdifferential of a proper function f by means of conjugate functions (see [15,32]):

$$x^* \in \partial f(x) \Leftrightarrow f(x) + f^*(x^*) = \langle x^*, x\rangle$$

and, respectively,

$$x^* \in \partial_\varepsilon f(x) \Leftrightarrow f(x) + f^*(x^*) \leq \langle x^*, x\rangle + \varepsilon.$$

Having $f, g : X \to \overline{\mathbb{R}}$ two proper functions we consider their *infimal convolution*, namely the function denoted by $f \Box g : X \to \overline{\mathbb{R}}$, $f \Box g(x) = \inf_{u \in X}\{f(u) + g(x-u)\}$, for all $x \in X$. We say that the infimal convolution is *exact at* $x \in X$ if the infimum in the definition is attained. Moreover, $f \Box g$ is said to be *exact* if it is exact at every $x \in X$.

For a function $f : A \times B \to \overline{\mathbb{R}}$, where A and B are nonempty sets, we denote by $f^\top$ the *transpose* of f, namely the function $f^\top : B \times A \to \overline{\mathbb{R}}, f^\top(b,a) = f(a,b)$ for all $(b,a) \in B \times A$. We consider also the *projection operator* $\operatorname{pr}_A : A \times B \to A$, $\operatorname{pr}_A(a,b) = a$ for all $(a,b) \in A \times B$. When an infimum or a supremum is attained we write min, respectively max instead of inf, respectively sup.

For M, Z two subsets of X, we say that M is *closed regarding* the set Z if $M \cap Z = \operatorname{cl}(M) \cap Z$. It is worth noting that a closed set is closed regarding any set. Several weak regularity conditions (in the theory of maximal monotone operators and convex optimization) are expressed by using this notion, see [3–6].

The result below plays an important role in the following. Let us mention that on the space $\mathbb{R}$ we consider the usual topology.

Lemma 1 *Let $\phi : X \times Y \to \overline{\mathbb{R}}$ be a proper, convex and lower semicontinuous function with $0 \in \operatorname{pr}_Y(\operatorname{dom}(\phi))$. Then*

$$\operatorname{epi}\left((\phi(\cdot,0))^*\right) = \operatorname{cl}\left(\operatorname{pr}_{X^* \times \mathbb{R}}(\operatorname{epi}(\phi^*))\right). \tag{3}$$

Remark 1 *It was observed in [33, pp. 197] and [22, pp. 628–629] that in the hypotheses of the above lemma, we have $(\phi(\cdot,0))^* = \operatorname{cl}_{w^*} h$, where $h : X^* \to \overline{\mathbb{R}}$ is defined by $h(x^*) = \inf_{y^* \in Y^*} \phi^*(x^*, y^*)$, from which Lemma 1 follows immediately (see also [6, Theorem 2]).*

Theorem 2 *Let $\phi : X \times Y \to \overline{\mathbb{R}}$ be a proper, convex and lower semicontinuous function such that $0 \in \operatorname{pr}_Y(\operatorname{dom}(\phi))$ and U be a nonempty subset of X^*. Then the following statements are equivalent:*

(i) $\sup_{x\in X}\{\langle x^*,x\rangle-\phi(x,0)\}=\min_{y^*\in Y^*}\phi^*(x^*,y^*)$ *for all* $x^*\in U$*;*

(ii) $\mathrm{pr}_{X^*\times\mathbb{R}}(\mathrm{epi}(\phi^*))$ *is closed regarding* $U\times\mathbb{R}$ *in* $(X^*,\omega(X^*,X))\times\mathbb{R}$*.*

Proof. We prove first the implication (i)⇒(ii). Take an arbitrary element $(x^*,r)\in\mathrm{cl}\left(\mathrm{pr}_{X^*\times\mathbb{R}}(\mathrm{epi}(\phi^*))\right)\cap(U\times\mathbb{R})$. Lemma 1 guarantees that $(x^*,r)\in\mathrm{epi}\left((\phi(\cdot,0))^*\right)$, which implies $(\phi(\cdot,0))^*(x^*)\le r$, that is $\sup_{x\in X}\{\langle x^*,x\rangle-\phi(x,0)\}\le r$. From (i) we obtain the existence of an element $y^*\in Y^*$ such that $\phi^*(x^*,y^*)\le r$, hence $(x^*,r)\in\mathrm{pr}_{X^*\times\mathbb{R}}(\mathrm{epi}(\phi^*))\cap(U\times\mathbb{R})$. Hence we have $\mathrm{cl}\left(\mathrm{pr}_{X^*\times\mathbb{R}}(\mathrm{epi}(\phi^*))\right)\cap(U\times\mathbb{R})\subseteq\mathrm{pr}_{X^*\times\mathbb{R}}(\mathrm{epi}(\phi^*))\cap(U\times\mathbb{R})$, and since the reverse inclusion is always satisfied, we obtain that (ii) is fulfilled.

Conversely, suppose now that (ii) is true and take $x^*\in U$ arbitrary. From the Young-Fenchel inequality we obtain

$$(\phi(\cdot,0))^*(x^*)\le\inf_{y^*\in Y^*}\phi^*(x^*,y^*). \tag{4}$$

If $(\phi(\cdot,0))^*(x^*)=+\infty$, then (i) is obviously satisfied. So we may suppose that $(\phi(\cdot,0))^*(x^*)<+\infty$. Taking into consideration that $0\in\mathrm{pr}_Y(\mathrm{dom}(\phi))$ we easily derive that $(\phi(\cdot,0))^*(x^*)\in\mathbb{R}$. We get by Lemma 1 and (ii) that $(x^*,(\phi(\cdot,0))^*(x^*))\in\mathrm{epi}\left((\phi(\cdot,0))^*\right)\cap(U\times\mathbb{R})=\mathrm{cl}\left(\mathrm{pr}_{X^*\times\mathbb{R}}(\mathrm{epi}(\phi^*))\right)\cap(U\times\mathbb{R})=\left(\mathrm{pr}_{X^*\times\mathbb{R}}(\mathrm{epi}(\phi^*))\right)\cap(U\times\mathbb{R})$. Hence there exists an element $\overline{y}^*\in Y^*$ such that $\phi^*(x^*,\overline{y}^*)\le(\phi(\cdot,0))^*(x^*)$. Combining this with (4) we obtain $(\phi(\cdot,0))^*(x^*)=\phi^*(x^*,\overline{y}^*)=\min_{y^*\in Y^*}\phi^*(x^*,y^*)$. As $x^*\in U$ was arbitrary taken, the proof is complete. ■

Remark 2 *An anonymous reviewer proposed us an alternative proof of the above theorem. Let* X *be a topological space,* $U\subseteq X$ *and* $A\subseteq X\times\mathbb{R}$*. One can prove that* $A\cap(U\times\mathbb{R})=\mathrm{cl}(A)\cap(U\times\mathbb{R})$ *if and only if* $A\cap(\{u\}\times\mathbb{R})=\mathrm{cl}(A)\cap(\{u\}\times\mathbb{R})$ *for all* $u\in U$*. Using this remark, one can deduce Theorem 2 from the corresponding statement with* U *a singleton. Indeed, for* $U=\{x^*\}$*, the statement (i) is nothing else than* $(\phi(\cdot,0))^*(x^*)=\mathrm{cl}_{w^*}h(x^*)=h(x^*)$ *and the infimum in the definition of* $h(x^*)$ *is attained, while (ii) asserts that for* $A:=(\mathrm{pr}_{X^*\times\mathbb{R}}(\mathrm{epi}(\phi^*))$ *one has* $A\cap(\{x^*\}\times\mathbb{R})=\mathrm{cl}(A)\cap(\{x^*\}\times\mathbb{R})$*. For* $x^*=0$ *the equivalence of (i) and (ii) is nothing else than [25, Theorem 4.3.1] (see also [24, pp. 6]). The statement for* $x^*\neq0$ *follows easily by a translation.*

Remark 3 *Considering in the previous theorem* $U:=X^*$ *we obtain, in the same hypotheses as in Theorem 2, that the following conditions are equivalent:*

(i) $\sup_{x\in X}\{\langle x^*,x\rangle-\phi(x,0)\}=\min_{y^*\in Y^*}\phi^*(x^*,y^*)$ *for all* $x^*\in X^*$*;*

(ii) $\mathrm{pr}_{X^*\times\mathbb{R}}(\mathrm{epi}(\phi^*))$ *is closed in* $(X^*,\omega(X^*,X))\times\mathbb{R}$*.*

This statement can be deduced from [26, Theorem 2.2] and it was proved also in [11] in the case of Banach spaces and in [6] in the framework of separated locally convex spaces. Let us notice that condition (i) is refered in the literature as ***stable strong duality*** *([6, 11, 29]).*

Corollary 3 *Let* $f,g:X\to\overline{\mathbb{R}}$ *be proper, convex and lower semicontinuous functions such that* $\mathrm{dom}(f)\cap\mathrm{dom}(g)\neq\emptyset$ *and* U *be a nonempty subset of* X^**. Then the following statements are equivalent:*

(i) $(f+g)^*(x^*)=(f^*\square g^*)(x^*)$ *and* $f^*\square g^*$ *is exact at* x^* *for all* $x^*\in U$*;*

(ii) $\mathrm{epi}(f^*) + \mathrm{epi}(g^*)$ *is closed regarding* $U \times \mathbb{R}$ *in* $(X^*, \omega(X^*, X)) \times \mathbb{R}$.

Proof. Consider the function $\phi : X \times X \to \overline{\mathbb{R}}$ defined by $\phi(x, y) = f(x) + g(x + y)$ for all $(x, y) \in X \times X$. A simple computation shows that $\phi^*(x^*, y^*) = f^*(x^* - y^*) + g^*(y^*)$ for all $(x^*, y^*) \in X^* \times X^*$. One can prove easily that the hypotheses of Theorem 2 are satisfied for this particular choice of the function ϕ. The result follows now by applying Theorem 2. ■

Remark 4 *In case $U := X^*$, the previous corollary was established also in [10, Theorem 1] and [2, Theorem 3.2]).*

The following result will lead to the bivariate inf-convolution formula.

Theorem 4 *Let $h_1, h_2 : X \times Y \to \overline{\mathbb{R}}$ be proper, convex and lower semicontinuous functions such that $\mathrm{pr}_X(\mathrm{dom}(h_1)) \cap \mathrm{pr}_X(\mathrm{dom}(h_2)) \neq \emptyset$ and V be a nonempty subset of Y^*. Consider the functions $h_1 \Box_2 h_2 : X \times Y \to \overline{\mathbb{R}}$, $(h_1 \Box_2 h_2)(x, y) = \inf\{h_1(x, u) + h_2(x, v) : u, v \in Y, u + v = y\}$ and $h_1^* \Box_1 h_2^* : X^* \times Y^* \to \overline{\mathbb{R}}$, $(h_1^* \Box_1 h_2^*)(x^*, y^*) = \inf\{h_1^*(u^*, y^*) + h_2^*(v^*, y^*) : u^*, v^* \in X^*, u^* + v^* = x^*\}$. Then the following conditions are equivalent:*

- *(i)* $(h_1 \Box_2 h_2)^*(x^*, y^*) = (h_1^* \Box_1 h_2^*)(x^*, y^*)$ *and* $h_1^* \Box_1 h_2^*$ *is exact at* (x^*, y^*) *(that is, the infimum in the definition of* $(h_1^* \Box_1 h_2^*)(x^*, y^*)$ *is attained) for all* $(x^*, y^*) \in X^* \times V$*;*

- *(ii)* $\{(a^* + b^*, u^*, v^*, r) : h_1^*(a^*, u^*) + h_2^*(b^*, v^*) \leq r\}$ *is closed regarding the set* $X^* \times \Delta_V \times \mathbb{R}$ *in* $(X^*, \omega(X^*, X)) \times (Y^*, \omega(Y^*, Y)) \times (Y^*, \omega(Y^*, Y)) \times \mathbb{R}$*, where* $\Delta_V = \{(y^*, y^*) : y^* \in V\}$*.*

Proof. Take an arbitrary $(x^*, y^*) \in X^* \times Y^*$. The following equality can be easily derived:

$$(h_1 \Box_2 h_2)^*(x^*, y^*) = \sup_{x \in X, u, v \in Y} \{\langle x^*, x\rangle + \langle y^*, u + v\rangle - h_1(x, u) - h_2(x, v)\}. \tag{5}$$

Define now the functions $F, G : X \times Y \times Y \to \overline{\mathbb{R}}$, by $F(x, u, v) = h_1(x, u)$ and $G(x, u, v) = h_2(x, v)$ for all $(x, u, v) \in X \times Y \times Y$. It holds $(h_1 \Box_2 h_2)^*(x^*, y^*) = (F + G)^*(x^*, y^*, y^*)$. One can show that for all $(x^*, u^*, v^*) \in X^* \times Y^* \times Y^*$, the conjugate functions $F^*, G^* : X^* \times Y^* \times Y^* \to \overline{\mathbb{R}}$ have the following forms: $F^*(x^*, u^*, v^*) = h_1^*(x^*, u^*) + \delta_{\{0\}}(v^*)$ and $G^*(x^*, u^*, v^*) = h_2^*(x^*, v^*) + \delta_{\{0\}}(u^*)$, respectively. Further we have $(F^* \Box G^*)(x^*, y^*, y^*) = (h_1^* \Box_1 h_2^*)(x^*, y^*)$.

Hence the condition (i) is fulfilled if and only if $(F+G)^*(x^*, y^*, y^*) = (F^* \Box G^*)(x^*, y^*, y^*)$ and $(F^* \Box G^*)(x^*, y^*, y^*)$ is exact at (x^*, y^*, y^*) for all $(x^*, y^*, y^*) \in X^* \times \Delta_V$. In view of Corollary 3, the last condition is equivalent to $\mathrm{epi}(F^*) + \mathrm{epi}(G^*)$ is closed regarding the set $X^* \times \Delta_V \times \mathbb{R}$ in $(X^*, \omega(X^*, X)) \times (Y^*, \omega(Y^*, Y)) \times (Y^*, \omega(Y^*, Y)) \times \mathbb{R}$. Finally the equality $\mathrm{epi}(F^*) + \mathrm{epi}(G^*) = \{(a^* + b^*, u^*, v^*, r) : h_1^*(a^*, u^*) + h_2^*(b^*, v^*) \leq r\}$, the proof of which presents no difficulty, gives the desired result. ■

We give now a necessary and sufficient condition for the bivariate inf-convolution formula. For the particular case when $V := Y^*$ we obtain from the previous result the following corollary.

Corollary 5 *Let $h_1, h_2 : X \times Y \to \overline{\mathbb{R}}$ be proper, convex and lower semicontinuous functions such that $\mathrm{pr}_X(\mathrm{dom}(h_1)) \cap \mathrm{pr}_X(\mathrm{dom}(h_2)) \neq \emptyset$. The following statements are equivalent:*

(i) $(h_1\Box_2 h_2)^* = h_1^*\Box_1 h_2^*$ *and* $h_1^*\Box_1 h_2^*$ *is exact;*

(ii) $\{(a^* + b^*, u^*, v^*, r) : h_1^*(a^*, u^*) + h_2^*(b^*, v^*) \leq r\}$ *is closed regarding the subspace* $X^* \times \Delta_{Y^*} \times \mathbb{R}$ *in* $(X^*, \omega(X^*, X)) \times (Y^*, \omega(Y^*, Y)) \times (Y^*, \omega(Y^*, Y)) \times \mathbb{R}$.

Remark 5 *A generalized interior point condition which guarantees relation (i) in Corollary 5 was given in [30, Theorem 4.2], namely:*

$$(CQ^{SZ}) \qquad 0 \in {}^{ic}\big(\mathrm{pr}_X(\mathrm{dom}(h_1)) - \mathrm{pr}_X(\mathrm{dom}(h_2))\big).$$

Nevertheless, unlike the condition (ii), which is necessary and sufficient for (i), the condition (CQ^{SZ}) *is only sufficient, as the following example, which can be found in [3], shows.*

Example 6 *Take* $X = Y = \mathbb{R}^2$*, equipped with the Euclidean norm* $\|\cdot\|_2$*,* $f, g : \mathbb{R}^2 \to \overline{\mathbb{R}}$*,* $f = \|\cdot\|_2 + \delta_{\mathbb{R}^2_+}$*,* $g = \delta_{-\mathbb{R}^2_+}$*,*

$$h_1(x, x^*) = f(x) + f^*(x^*) \text{ for all } (x, x^*) \in \mathbb{R}^2 \times \mathbb{R}^2$$

and, respectively,

$$h_2(x, x^*) = g(x) + g^*(x^*) \text{ for all } (x, x^*) \in \mathbb{R}^2 \times \mathbb{R}^2.$$

One can see that $g^* = \delta_{\mathbb{R}^2_+}$ *and* $f^* = \delta_{\overline{B}(0,1) - \mathbb{R}^2_+}$*, where* $\overline{B}(0, 1)$ *is the closed unit ball of* $\mathbb{R}^2$*. We have*

$$\{(x^* + y^*, x, y, r) : f(x) + f^*(x^*) + g(y) + g^*(y^*) \leq r\} =$$

$$\mathbb{R}^2 \times \{(x, y, r) : x \in \mathbb{R}^2_+, y \in -\mathbb{R}^2_+, \|x\|_2 \leq r\},$$

which is closed, hence closed regarding the set $\mathbb{R}^2 \times \Delta_{\mathbb{R}^2} \times \mathbb{R}$*. Thus, by Corollary 5, (i) is fulfilled. However, condition* (CQ^{SZ}) *becomes:* $\mathbb{R}^2_+$ *is a closed linear subspace of* $\mathbb{R}^2$*, and thus it fails in this case.*

By taking in Theorem 4 $Y := X^*$ and $V := X$, where X is supposed to be a normed space, so that $V = X$ can be seen as a subspace of $Y^* = X^{**}$, we obtain the following result.

Corollary 7 *Let* $h_1, h_2 : X \times X^* \to \overline{\mathbb{R}}$ *be proper, convex and lower semicontinuous functions in the strong topology of* $X \times X^*$ *such that* $\mathrm{pr}_X(\mathrm{dom}(h_1)) \cap \mathrm{pr}_X(\mathrm{dom}(h_2)) \neq \emptyset$*. The following statements are equivalent:*

(i) $(h_1\Box_2 h_2)^*(x^*, x) = (h_1^*\Box_1 h_2^*)(x^*, x)$ *and* $h_1^*\Box_1 h_2^*$ *is exact at* (x^*, x) *for all* $(x^*, x) \in X^* \times X$*;*

(ii) $\{(a^* + b^*, u^{**}, v^{**}, r) : h_1^*(a^*, u^{**}) + h_2^*(b^*, v^{**}) \leq r\}$ *is closed regarding the subspace* $X^* \times \Delta_X \times \mathbb{R}$ *in* $(X^*, \omega(X^*, X)) \times (X^{**}, \omega(X^{**}, X^*)) \times (X^{**}, \omega(X^{**}, X^*)) \times \mathbb{R}$.

3 Monotone operators and enlargements

In the following we recall some notions and results concerning monotone operators. For the rest of the paper we assume that X is a nonzero real Banach space. A set-valued operator $S : X \rightrightarrows X^*$ is said to be *monotone* if

$$\langle y^* - x^*, y - x\rangle \geq 0, \text{ whenever } x^* \in S(x) \text{ and } y^* \in S(y).$$

The monotone operator S is called *maximal monotone* if its graph

$$G(S) = \{(x, x^*) : x^* \in S(x)\} \subseteq X \times X^*$$

is not properly contained in the graph of any other monotone operator $S' : X \rightrightarrows X^*$. The classical example of a maximal monotone operator is the subdifferential of a proper, convex and lower semicontinuous function (see [28]). However, there exist maximal monotone operators which are not subdifferentials (see [29]).

An element $(x_0, x_0^*) \in X \times X^*$ is said to be *monotonically related* to the graph of S if

$$\langle y^* - x_0^*, y - x_0\rangle \geq 0 \text{ for all } (y, y^*) \in G(S).$$

One can show that a monotone operator S is maximal monotone if and only if the set of monotonically related elements to $G(S)$ is exactly $G(S)$.

To any monotone operator $S : X \rightrightarrows X^*$ we associate the *Fitzpatrick function* $\varphi_S : X \times X^* \to \overline{\mathbb{R}}$, defined by

$$\varphi_S(x, x^*) = \sup\{\langle y^*, x\rangle + \langle x^*, y\rangle - \langle y^*, y\rangle : y^* \in S(y)\},$$

which is obviously convex and weak-weak* $(w - w^*)$ lower semicontinuous. Introduced by Fitzpatrick (cf. [16]), it proved to be very important in the theory of maximal monotone operators, revealing some connections between convex analysis and monotone operators (see [1,3–5,14,23,30] and the references therein). Considering the functions $c : X \times X^* \to \mathbb{R}$, $c(x, x^*) = \langle x^*, x\rangle$ for all $(x, x^*) \in X \times X^*$ and $c_S : X \times X^* \to \overline{\mathbb{R}}$, $c_S = c + \delta_{G(S)}$, we get the equality $\varphi_S(x, x^*) = (c_S)^{*\top}(x, x^*)$ for all $(x, x^*) \in X \times X^*$, where we consider the natural injection $X \subseteq X^{**}$. The function $\psi_S = \operatorname{cl}(\operatorname{co} c_S)$, where the closure is taken in the strong topology of $X \times X^*$, is well-linked to the Fitzpatrick function. Let us mention that on $X \times X^*$ we have $\psi_S^{*\top} = \varphi_S$ and, in the framework of reflexive Banach spaces the equality $\varphi_S^{*\top} = \psi_S$ holds (cf. [13, Remark 5.4]).

Lemma 8 *([16]) Let S be a maximal monotone operator. Then*

(i) $\varphi_S(x, x^*) \geq \langle x^*, x\rangle$ *for all* $(x, x^*) \in X \times X^*$;

(ii) $G(S) = \{(x, x^*) \in X \times X^* : \varphi_S(x, x^*) = \langle x^*, x\rangle\}$.

Motivated by these properties of the Fitzpatrick function, the notion of *representative function* of a monotone operator was introduced and studied in the literature.

Definition 1 *For $S : X \rightrightarrows X^*$ a monotone operator, we call* ***representative function*** *of S a convex and strong lower semicontinuous function $h_S : X \times X^* \to \overline{\mathbb{R}}$ fulfilling*

$$h_S \geq c \text{ and } G(S) \subseteq \{(x, x^*) \in X \times X^* : h_S(x, x^*) = \langle x^*, x\rangle\}.$$

We observe that if $G(S) \neq \emptyset$ (in particular if S is maximal monotone), then every representative function of S is proper. It follows immediately from Lemma 8, that the Fitzpatrick function associated to a maximal monotone operator is a representative function of the operator. The following proposition is a direct consequence of the results from [13] (see also [19, Proposition 1.2, Theorem 4.2 (1)]).

Proposition 9 *Let $S : X \rightrightarrows X^*$ be a maximal monotone operator and h_S be a representative function of S. Then:*

(i) $\varphi_S(x, x^*) \leq h_S(x, x^*) \leq \psi_S(x, x^*)$ *for all* $(x, x^*) \in X \times X^*$*;*

(ii) the canonical restriction of $h_S^{*\top}$ to $X \times X^*$ is also a representative function of S;

(iii) $\{(x, x^*) \in X \times X^* : h_S(x, x^*) = \langle x^*, x\rangle\} = \{(x, x^*) \in X \times X^* : h_S^{*\top}(x, x^*) = \langle x^*, x\rangle\} = G(S)$.

Remark 6 *These properties of representative functions are well-known in the framework of reflexive Banach spaces (see [23]). For more on the properties of representative functions we refer to [3, 20, 23] and the references therein.*

The following concept of *enlargement* of an arbitrary monotone operator $S : X \rightrightarrows X^*$ was introduced in [9]: given $\varepsilon \geq 0$, let $S^\varepsilon : X \rightrightarrows X^*$ be defined for all $x \in X$ by

$$S^\varepsilon(x) := \{x^* \in X^* : \langle y^* - x^*, y - x\rangle \geq -\varepsilon \text{ for all } (y, y^*) \in G(S)\}.$$

This notion was intensively studied in [9, 12–14, 17, 27, 31]. Due to the monotonicity of S, we have $S(x) \subseteq S^\varepsilon(x)$ for all $x \in X$ and $\varepsilon \geq 0$. The operator S^0 need not to be monotone. It is worth noting that $G(S^0)$ is exactly the set of monotonically related elements to $G(S)$, hence S is maximal monotone if and only if $S = S^0$ ([9, Proposition 2] and [27, Proposition 3.1]). The enlargement S^ε can be characterized via the Fitzpatrick function associated to S:

$$x^* \in S^\varepsilon(x) \Leftrightarrow \varphi_S(x, x^*) \leq \varepsilon + \langle x^*, x\rangle.$$

Motivated by this characterization, the following enlargement for the monotone operator S can be considered (cf. [13, 14]): for a representative function h_S of S we define for all $x \in X$ and $\varepsilon \geq 0$

$$S^\varepsilon_{h_S}(x) := \{x^* \in X^* : h_S(x, x^*) \leq \varepsilon + \langle x^*, x\rangle\}.$$

Obviously, $S^\varepsilon_{\varphi_S} = S^\varepsilon$ and $G(S) \subseteq G(S^\varepsilon_{h_S})$ for all $\varepsilon \geq 0$. Moreover, $S^\varepsilon_{h_S}$ has convex closed values and it holds $S^{\varepsilon_1}_{h_S}(x) \subseteq S^{\varepsilon_2}_{h_S}(x)$, provided that $0 \leq \varepsilon_1 \leq \varepsilon_2$. Further, if S is maximal monotone, then, in view of Proposition 9, we have

$$S(x) \subseteq S^\varepsilon_{\psi_S}(x) \subseteq S^\varepsilon_{h_S}(x) \subseteq S^\varepsilon_{\varphi_S}(x) = S^\varepsilon(x)$$

and

$$S(x) \subseteq S^\varepsilon_{\psi_S}(x) \subseteq S^\varepsilon_{h_S^*}(x) \subseteq S^\varepsilon_{\varphi_S}(x) = S^\varepsilon(x),$$

where $S^\varepsilon_{h_S^*}(x) = \{x^* \in X^* : h_S^*(x^*, x) \leq \varepsilon + \langle x^*, x\rangle\}$, as well as $S = S^0_{\psi_S} = S^0_{h_S} = S^0_{h_S^*} = S^0_{\varphi_S} = S^0$. Let us notice that in case S is a monotone operator and $S = S^0_{h_S}$, where $h_S \neq \varphi_S$, we do not necessarily have that S is maximal monotone.

Remark 7 *If $S = \partial f$, where f is a proper, convex and lower semicontinuous function, then*

$$\partial f(x) \subseteq \partial_\varepsilon f(x) \subseteq \partial^\varepsilon f(x) := (\partial f)^\varepsilon(x)$$

and the inclusions can be strict (see [9, 21]). Moreover, taking $h : X \times X^* \to \overline{\mathbb{R}}$, $h(x, x^*) = f(x) + f^*(x^*)$ for all $(x, x^*) \in X \times X^*$, which is a representative function of ∂f, we see that $(\partial f)^\varepsilon_h(x) = \partial_\varepsilon f(x)$.

Theorem 10 *Let $S, T : X \rightrightarrows X^*$ be two maximal monotone operators with representative functions h_S and h_T, respectively, such that $\mathrm{pr}_X(\mathrm{dom}(h_S)) \cap \mathrm{pr}_X(\mathrm{dom}(h_T)) \neq \emptyset$ and consider the function $h : X \times X^* \to \overline{\mathbb{R}}$, $h(x, x^*) = (h_S \Box_2 h_T)^*(x^*, x)$ for all $(x, x^*) \in X \times X^*$. If*

$\{(a^* + b^*, u^{**}, v^{**}, r) : h_S^*(a^*, u^{**}) + h_T^*(b^*, v^{**}) \leq r\}$ is closed regarding the subspace $X^* \times \Delta_X \times \mathbb{R}$ in $(X^*, \omega(X^*, X)) \times (X^{**}, \omega(X^{**}, X^*)) \times (X^{**}, \omega(X^{**}, X^*)) \times \mathbb{R}$,

then h is a representative function of the monotone operator $S + T$. If additionally X is reflexive, then $S + T$ is a maximal monotone operator.

Proof. The function h is obviously convex and strong-weak* lower semicontinuous, hence lower semicontinuous in the strong topology of $X \times X^*$. Applying Corollary 7 we obtain $h(x, x^*) = (h_S^* \Box_1 h_T^*)(x^*, x)$ and $h_S^* \Box_1 h_T^*$ is exact at (x^*, x) for all $(x^*, x) \in X^* \times X$. By using Proposition 9 we have for all $(x, x^*) \in X \times X^*$ that $h(x, x^*) = (h_S^* \Box_1 h_T^*)(x^*, x) = \inf\{h_S^*(u^*, x) + h_T^*(v^*, x) : u^*, v^* \in X^*, u^* + v^* = x^*\} \geq \inf\{\langle u^*, x\rangle + \langle v^*, x\rangle : u^*, v^* \in X^*, u^* + v^* = x^*\} = \langle x^*, x\rangle$, hence $h \geq c$.

It remains to show that $G(S + T) \subseteq \{(x, x^*) : h(x, x^*) = \langle x^*, x\rangle\}$. Take an arbitrary $(x, x^*) \in G(S + T)$. There exist $u^* \in S(x)$ and $v^* \in T(x)$ such that $x^* = u^* + v^*$. Employing once more Proposition 9 we obtain

$$\langle x^*, x\rangle \leq h(x, x^*) = (h_S^* \Box_1 h_T^*)(x^*, x)$$

$$\leq h_S^*(u^*, x) + h_T^*(v^*, x) = \langle u^*, x\rangle + \langle v^*, x\rangle = \langle x^*, x\rangle,$$

thus $G(S + T) \subseteq \{(x, x^*) : h(x, x^*) = \langle x^*, x\rangle\}$.

Actually, we prove that in this case

$$G(S + T) = \{(x, x^*) : h(x, x^*) = \langle x^*, x\rangle\}. \tag{6}$$

Take an arbitrary (x, x^*) such that $h(x, x^*) = \langle x^*, x\rangle$. Since we have that $h(x, x^*) = (h_S^* \Box_1 h_T^*)(x^*, x)$ and $h_S^* \Box_1 h_T^*$ is exact at (x^*, x), there exist $u^*, v^* \in X^*$, $u^* + v^* = x^*$ such that

$$h_S^*(u^*, x) + h_T^*(v^*, x) = \langle u^*, x\rangle + \langle v^*, x\rangle. \tag{7}$$

The function h_S and h_T being representative, from Proposition 9 we have $h_S^*(u^*, x) \geq \langle u^*, x\rangle$ and $h_T^*(v^*, x) \geq \langle v^*, x\rangle$, hence, in view of (7), the inequalities above must be fulfilled as equalities, thus by Proposition 9 we get $u^* \in S(x)$ and $v^* \in T(x)$, so $x^* = u^* + v^* \in S(x) + T(x) = (S + T)(x)$ and (6) is fulfilled.

Suppose now that X is a reflexive Banach space. Since in this case the weak* topology coincides with the weak topology and the weak closure of a convex set is exactly the strong closure of the same set, the regularity condition becomes

$\{(a^*+b^*,u,v,r): h_S^*(a^*,u)+h_T^*(b^*,v)\le r\}$ is closed regarding the subspace $X^*\times\Delta_X\times\mathbb{R}$ in the strong topology of $X^*\times X\times X\times\mathbb{R}$,

which is exactly the condition given in [3] for the maximal monotonicity of the operator $S+T$. However, we give in the following a different proof of this result.

Since h_S and h_T are representative functions we have $h_S\Box_2 h_T\ge c$. As the duality product is continuous, it follows $\mathrm{cl}_{\|\cdot\|\times\|\cdot\|_*}(h_S\Box_2 h_T)\ge c$. From the definition of h we obtain $h^{*\top}=\mathrm{cl}_{\|\cdot\|\times\|\cdot\|_*}(h_S\Box_2 h_T)\ge c$. The conclusion follows now by combining [23, Proposition 2.1] with relation (6). ∎

Remark 8 *In case of reflexive Banach spaces, the condition in the above theorem is the weakest one given so far for the maximality of the sum of two maximal monotone operators (see [3–5] for a discussion regarding several conditions given in the literature on this topic).*

Let us state now the main result of the paper.

Theorem 11 *Let $S,T:X\rightrightarrows X^*$ be two maximal monotone operators with representative functions h_S and h_T, respectively, such that $\mathrm{pr}_X(\mathrm{dom}(h_S))\cap\mathrm{pr}_X(\mathrm{dom}(h_T))\neq\emptyset$ and consider again the function h defined as in the previous theorem. Then the following statements are equivalent:*

(i) $\{(a^*+b^*,u^{**},v^{**},r): h_S^*(a^*,u^{**})+h_T^*(b^*,v^{**})\le r\}$ is closed regarding the subspace $X^*\times\Delta_X\times\mathbb{R}$ in $(X^*,\omega(X^*,X))\times(X^{**},\omega(X^{**},X^*))\times(X^{**},\omega(X^{**},X^*))\times\mathbb{R}$;

(ii) $(S+T)_h^{\varepsilon}(x)=\bigcup\limits_{\substack{\varepsilon_1,\varepsilon_2\ge 0\\ \varepsilon_1+\varepsilon_2=\varepsilon}}\left(S_{h_S^*}^{\varepsilon_1}(x)+T_{h_T^*}^{\varepsilon_2}(x)\right)$ for all $\varepsilon\ge 0$ and for all $x\in X$,

where $(S+T)_h^{\varepsilon}(x):=\{x^*\in X^*: h(x,x^*)\le\varepsilon+\langle x^*,x\rangle\}$ for every $x\in X$ and $\varepsilon\ge 0$.

Remark 9 *In view of Theorem 10, when the condition (i) is fulfilled, then h is a representative function of the operator $S+T$, hence the notation $(S+T)_h^{\varepsilon}(x):=\{x^*\in X^*: h(x,x^*)\le\varepsilon+\langle x^*,x\rangle\}$ is justified. Conversely, when condition (ii) is true, then (i) is also fulfilled (see the proof below), thus also in this case the use of this notation makes sense.*

Proof. Let us suppose that (i) is fulfilled and take $x\in X$ and $\varepsilon\ge 0$. We show first the inclusion

$$\bigcup_{\substack{\varepsilon_1,\varepsilon_2\ge 0\\ \varepsilon_1+\varepsilon_2=\varepsilon}}\left(S_{h_S^*}^{\varepsilon_1}(x)+T_{h_T^*}^{\varepsilon_2}(x)\right)\subseteq(S+T)_h^{\varepsilon}(x). \tag{8}$$

Indeed, take $\varepsilon_1,\varepsilon_2\ge 0$, $\varepsilon_1+\varepsilon_2=\varepsilon$, $u^*\in S_{h_S^*}^{\varepsilon_1}(x)$ and $v^*\in T_{h_T^*}^{\varepsilon_2}(x)$. Then $h(x,u^*+v^*)=(h_S\Box_2 h_T)^*(u^*+v^*,x)\le(h_S^*\Box_1 h_T^*)(u^*+v^*,x)\le h_S^*(u^*,x)+h_T^*(v^*,x)\le\varepsilon_1+\langle u^*,x\rangle+\varepsilon_2+\langle v^*,x\rangle=\varepsilon+\langle u^*+v^*,x\rangle$, hence $u^*+v^*\in(S+T)_h^{\varepsilon}(x)$, that is, the inclusion (8) is true. Let us mention that this inclusion is always fulfilled, as there is no need of (i) to prove (8).

However, to show the opposite inclusion, we use condition (i). Take $x^*\in(S+T)_h^{\varepsilon}(x)$. We have $(h_S\Box_2 h_T)^*(x^*,x)\le\varepsilon+\langle x^*,x\rangle$. After applying Corollary 7, we get $(h_S^*\Box_1 h_T^*)(x^*,x)\le\varepsilon+\langle x^*,x\rangle$ and the infimum in the definition of $(h_S^*\Box_1 h_T^*)(x^*,x)$ is attained. Hence, there exist $u^*,v^*\in X^*$ such that $u^*+v^*=x^*$ and

$$h_S^*(u^*,x)+h_T^*(v^*,x)\le\varepsilon+\langle u^*,x\rangle+\langle v^*,x\rangle. \tag{9}$$

Take $\varepsilon_1 := h_S^*(u^*, x) - \langle u^*, x\rangle$ and $\varepsilon_2 := \varepsilon - \varepsilon_1$. By using Proposition 9 and the inequality (9) we obtain $\varepsilon_1 \geq 0$, and $\varepsilon_2 \geq h_T^*(v^*, x) - \langle v^*, x\rangle \geq 0$. Thus $u^* \in S_{h_S^*}^{\varepsilon_1}(x)$ and $v^* \in T_{h_T^*}^{\varepsilon_2}(x)$, that is $x^* = u^* + v^* \in \bigcup_{\substack{\varepsilon_1,\varepsilon_2 \geq 0 \\ \varepsilon_1+\varepsilon_2=\varepsilon}} \big(S_{h_S^*}^{\varepsilon_1}(x) + T_{h_T^*}^{\varepsilon_2}(x)\big)$, so (ii) is fulfilled.

Conversely, assume that (ii) is true. We start by proving that

$$h(x, x^*) \geq \langle x^*, x\rangle \text{ for all } (x, x^*) \in X \times X^*. \tag{10}$$

Take an arbitrary $(x_0, x_0^*) \in X \times X^*$ such that $h(x_0, x_0^*) \leq \langle x_0^*, x_0\rangle$. By using the condition (ii) for $\varepsilon := 0$ we obtain $x_0^* \in (S+T)_h^0(x_0) = S_{h_S^*}^0(x_0) + T_{h_T^*}^0(x_0) = S(x_0) + T(x_0)$. Hence there exist $u_0^* \in S(x_0)$ and $v_0^* \in T(x_0)$ such that $x_0^* = u_0^* + v_0^*$. From Proposition 9 we obtain $h_S(x_0, u_0^*) = \langle u_0^*, x_0\rangle$ and $h_T(x_0, v_0^*) = \langle v_0^*, x_0\rangle$. Like in (5) we get

$$\begin{aligned} h(x_0, x_0^*) &= \sup_{x \in X, u^*, v^* \in X^*} \{\langle x_0^*, x\rangle + \langle u^*, x_0\rangle + \langle v^*, x_0\rangle - h_S(x, u^*) - h_T(x, v^*)\} \\ &\geq \langle x_0^*, x_0\rangle + \langle u_0^*, x_0\rangle + \langle v_0^*, x_0\rangle - h_S(x_0, u_0^*) - h_T(x_0, v_0^*) = \langle x_0^*, x_0\rangle, \end{aligned}$$

thus (10) is fulfilled.

In view of Corollary 7, it is sufficient to show that $h(x, x^*) = (h_S^* \Box_1 h_T^*)(x^*, x)$ and that $h_S^* \Box_1 h_T^*$ is exact at (x^*, x) for all $(x^*, x) \in X^* \times X$. Take an arbitrary $(x^*, x) \in X^* \times X$. The inequality

$$h(x, x^*) \leq (h_S^* \Box_1 h_T^*)(x^*, x) \tag{11}$$

is always true. In case when $h(x, x^*) = +\infty$, there is nothing to be proved. The condition $\operatorname{pr}_X(\operatorname{dom}(h_S)) \cap \operatorname{pr}_X(\operatorname{dom}(h_T)) \neq \emptyset$ ensures that $h(x, x^*) > -\infty$, so we may suppose that $h(x, x^*) \in \mathbb{R}$. Let us denote by $r := h(x, x^*)$. We have $h(x, x^*) = \langle x^*, x\rangle + (r - \langle x^*, x\rangle)$. With $\varepsilon := r - \langle x^*, x\rangle \geq 0$ (cf. (10)), we obtain that $x^* \in (S+T)_h^\varepsilon(x)$. Since (ii) is true, there exist $\varepsilon_1, \varepsilon_2 \geq 0$, $\varepsilon_1 + \varepsilon_2 = \varepsilon$ and $u^* \in S_{h_S^*}^{\varepsilon_1}(x)$ and $v^* \in T_{h_T^*}^{\varepsilon_2}(x)$, respectively, such that $x^* = u^* + v^*$. Further, adding the two inequalities

$$h_S^*(u^*, x) \leq \varepsilon_1 + \langle u^*, x\rangle$$

and

$$h_T^*(v^*, x) \leq \varepsilon_2 + \langle v^*, x\rangle$$

we obtain

$$h_S^*(u^*, x) + h_T^*(v^*, x) \leq \varepsilon_1 + \varepsilon_2 + \langle u^* + v^*, x\rangle = r = h(x, x^*),$$

hence, in view of (11) we get $h(x, x^*) = h_S^*(u^*, x) + h_T^*(v^*, x) = (h_S^* \Box_1 h_2^*)(x^*, x)$ and the proof is complete. ■

One can give also an interior point condition in order to guarantee the equality (ii) in the previous result. The following corollary is a direct consequence of Theorem 11 combined with Remark 5.

Corollary 12 *Let $S, T : X \rightrightarrows X^*$ be two maximal monotone operators with representative functions h_S and h_T, respectively, such that $\operatorname{pr}_X(\operatorname{dom}(h_S)) \cap \operatorname{pr}_X(\operatorname{dom}(h_T)) \neq \emptyset$. If the following condition is satisfied*

(CQ^{SZ}) $$0 \in {}^{ic}\big(\mathrm{pr}_X(\mathrm{dom}(h_S)) - \mathrm{pr}_X(\mathrm{dom}(h_T))\big),$$

then $(S+T)^{\varepsilon}_h(x) = \bigcup_{\substack{\varepsilon_1,\varepsilon_2\geq 0\\ \varepsilon_1+\varepsilon_2=\varepsilon}} \big(S^{\varepsilon_1}_{h_S^*}(x) + T^{\varepsilon_2}_{h_T^*}(x)\big)$, *for all* $\varepsilon \geq 0$, *for all* $x \in X$.

Remark 10 *The condition* (CQ^{SZ}) *in the above result is only sufficient for the equality (ii), as it can be seen by taking* $X = \mathbb{R}^2$, $S := \partial f$, $T := \partial g$, *where* f, g, h_S *and* h_T *are defined as in Example 6.*

We show in the following how the result given in [10, Theorem 1] for ε-subdifferentials can be derived from Theorem 11.

Corollary 13 *Let* $f, g : X \to \overline{\mathbb{R}}$ *be proper, convex and lower semicontinuous such that* $\mathrm{dom}(f) \cap \mathrm{dom}(g) \neq \emptyset$. *The following statements are equivalent:*

(i) $\mathrm{epi}(f^*) + \mathrm{epi}(g^*)$ *is closed in* $(X^*, \omega(X^*, X)) \times \mathbb{R}$*;*

(ii) $\partial_\varepsilon(f+g)(x) = \bigcup_{\substack{\varepsilon_1,\varepsilon_2\geq 0\\ \varepsilon_1+\varepsilon_2=\varepsilon}} \big(\partial_{\varepsilon_1} f(x) + \partial_{\varepsilon_2} g(x)\big)$ *for all* $\varepsilon \geq 0$ *and for all* $x \in X$.

Proof. Consider the functions $h_1, h_2 : X \times X^* \to \overline{\mathbb{R}}$, $h_1(x, x^*) = f(x) + f^*(x^*)$ and $h_2(x, x^*) = g(x) + g^*(x^*)$ for all $(x, x^*) \in X \times X^*$. We have $h_1^*(x^*, x^{**}) = f^{**}(x^{**}) + f^*(x^*)$ and $h_2^*(x^*, x^{**}) = g^{**}(x^{**}) + g^*(x^*)$ for all $(x^*, x^{**}) \in X^* \times X^{**}$. Further, the condition $(h_1 \Box_2 h_2)^*(x^*, x) = (h_1^* \Box_1 h_2^*)(x^*, x)$ and $h_1^* \Box_1 h_2^*$ is exact at (x^*, x) for all $(x^*, x) \in X^* \times X$ is fulfilled if and only if $(f+g)^* = f^* \Box g^*$ and $f^* \Box g^*$ is exact. Applying Corollary 3 for $U = X^*$, (i) is fulfilled if and only if $(f+g)^* = f^* \Box g^*$ and $f^* \Box g^*$ is exact, which is equivalent to $(h_1 \Box_2 h_2)^*(x^*, x) = (h_1^* \Box_1 h_2^*)(x^*, x)$ and $h_1^* \Box_1 h_2^*$ is exact at (x^*, x) for all $(x^*, x) \in X^* \times X$. The later one is equivalent to (see Corollary 7)

$\{(a^* + b^*, u^{**}, v^{**}, r) : h_1^*(a^*, u^{**}) + h_2^*(b^*, v^{**}) \leq r\}$ is closed regarding the subspace $X^* \times \Delta_X \times \mathbb{R}$ in $(X^*, \omega(X^*, X)) \times (X^{**}, \omega(X^{**}, X^*)) \times (X^{**}, \omega(X^{**}, X^*)) \times \mathbb{R}$.

Since h_1 and h_2 are representative functions of the maximal monotone operators ∂f and ∂g, respectively, we obtain, by Theorem 11, applied to the operators $S := \partial f$ and $T := \partial g$, that (i) is fulfilled if and only if for all $\varepsilon \geq 0$ and for all $x \in X$ we have

$$(\partial f + \partial g)^{\varepsilon}_h(x) = \bigcup_{\substack{\varepsilon_1,\varepsilon_2\geq 0\\ \varepsilon_1+\varepsilon_2=\varepsilon}} \big((\partial f)^{\varepsilon_1}_{h_1^*}(x) + (\partial g)^{\varepsilon_2}_{h_2^*}(x)\big),$$

where $h : X \times X^* \to \overline{\mathbb{R}}$, $h(x, x^*) = (h_1 \Box_2 h_2)^*(x^*, x) = (f+g)(x) + (f+g)^*(x^*)$ for all $(x, x^*) \in X \times X^*$. Taking into consideration that $(\partial f + \partial g)^{\varepsilon}_h(x) = \{x^* \in X^* : (f+g)(x) + (f+g)^*(x^*) \leq \varepsilon + \langle x^*, x\rangle\} = \partial_\varepsilon(f+g)(x)$ and $(\partial f)^{\varepsilon_1}_{h_1^*}(x) = \partial_{\varepsilon_1} f(x)$, respectively, $(\partial g)^{\varepsilon_2}_{h_2^*}(x) = \partial_{\varepsilon_2} g(x)$ (cf. Remark 7) we get the desired result. ■

Remark 11 *(a) In reflexive Banach spaces one can deduce the equivalence in Corollary 13 by using the results presented in [14] for enlargements of monotone operators (see [14, Theorem 6.9]).*

(b) Following the approach presented above, one can give a similar result to Theorem 11, where, instead of $S+T$ one can consider the operator $S+A^*TA$, where $S : X \rightrightarrows X^*$ and $T : Y \rightrightarrows Y^*$ are maximal monotone operators, X, Y are Banach spaces and $A : X \to Y$ is a linear and continuous operator and $A^* : Y^* \to X^*$ is its adjoint operator. We preferred to consider the case $S + T$ in order to make the presentation more clear.

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