TAIWANESE JOURNAL OF MATHEMATICS
Vol. 12, No. 8, pp. 1921-1942, November 2008
This paper is available online at http://www.tjm.nsysu.edu.tw/

REGULARITY CONDITIONS FOR FORMULAE OF BICONJUGATE FUNCTIONS

Radu Ioan Boţ* and Sorin-Mihai Grad

Abstract. When the dual of a normed space X is endowed with the weak* topology, the biconjugates of the proper convex lower semicontinuous functions defined on X coincide with the functions themselves. This is not the case when X^* is endowed with the strong topology. Working in the latter framework, we give formulae for the biconjugates of some functions that appear often in convex optimization, which hold provided the validity of some suitable regularity conditions. We also treat some special cases, rediscovering and improving recent results in the literature. Finally, we give a regularity condition that guarantees that the biconjugate of the supremum of a possibly infinite family of proper convex lower semicontinuous functions defined on a separated locally convex space coincides with the supremum of their biconjugates.

1. Introduction

When working with convex functions defined in infinitely dimensional spaces, one usually considers the duals of the used spaces endowed with the corresponding weak* topologies. Then the proper convex lower semicontinuous functions coincide with their biconjugates and different interesting and also "good-looking" results can be derived. We refer the reader to [11] for a comprehensive survey of convex analysis on locally convex spaces whose duals are endowed with the corresponding weak* topologies. On the other hand, there are many problems given on spaces whose duals are endowed with arbitrary topologies and not everything from the previously mentioned case can be one-to-one imported. Take for instance the maximal monotone operators. When they are defined on reflexive Banach spaces convex

Received November 4, 2007, accepted April 4, 2008.
2000 *Mathematics Subject Classification*: 42A50, 46N10, 49N15.
Key words and phrases: Conjugate functions, Regularity conditions, Dual problems, Composed convex functions.
Partially supported by DFG (German Research Foundation), project WA 922/1-2.
*Corresponding author.

analysis was successfully employed to help in dealing with different problems like sums, compositions, surjectivity etc. Such investigations were performed also for monotone operators defined on non-reflexive Banach spaces, but not all the results could be extended in this case.

Among the recent papers dealing with biconjugates of convex functions defined on normed spaces we mention [4, 6, 12]. The first two of them contain the formula for the biconjugate of the maximum of two functions, while the last one brings also formulae for the biconjugates of the functions obtained by several convexity-preserving operations. In this paper we perform similar investigations, giving formulae for the biconjugates of some functions that appear often in convex optimization. First we deal with the biconjugate of a so-called perturbation function, which satisfies a formula provided the fulfillment of some regularity conditions. Then we particularize in different ways this function, considering the perturbation functions used to obtain the Lagrange and Fenchel-Lagrange duals of a given convex constrained minimization problem, the perturbation function used for the Fenchel dual to a convex unconstrained minimization problem and perturbations of composed convex functions, respectively. For suitable values assigned to these functions we obtain further formulae for the biconjugates of some special functions, which are shown to hold under two types of regularity conditions, namely closedness-type and interiority-type. Thus we rediscover and extend several recent results in the literature. Finally, relaxing the hypothesis imposed on the space we work with in the sense that instead of normed it is taken only separated and locally convex, we give a regularity condition that guarantees that the biconjugate of the supremum of a family of arbitrarily many proper, convex and lower semicontinuous functions coincides with the supremum of the biconjugates of the mentioned functions.

Consider two separated locally convex vector spaces X and Y and their topological dual spaces X^* and Y^*. When X is a normed space with the norm $\|\cdot\|$, the norm on X^* is denoted by $\|\cdot\|_*$, and on this space we work with three topologies, namely the strong one induced by $\|\cdot\|_*$ which attaches X^{**} as dual to X^*, the weak* one induced by X on X^*, $w^*(X^*, X)$, which makes X to be the dual of X^* and the weak one induced by X^{**} on X^*, $w(X^*, X^{**})$. The topologies considered on the other spaces are taken in a similar way. We specify each time when a weak topology is used, otherwise the strong one is considered. A normed space X can be identified with a subspace of X^{**}, and we denote by $\hat{x}$ the canonical image in X^{**} of the element $x \in X$. Denote also $\widehat{U} = \{\hat{x} : x \in U\}$, for $U \subseteq X$ and by $\langle x^*, x\rangle = x^*(x)$ the value at $x \in X$ of the linear continuous functional $x^* \in X^*$. Take Y to be partially ordered by the nonempty closed convex cone C, i.e. on Y there is the partial order "$\leq_C$", defined by $z \leq_C y \Leftrightarrow y - z \in C$, $z, y \in Y$. To Y we attach a greatest element with respect to "$\leq_C$" which does not belong to Y, denoted by ∞_C and let $Y^\bullet = Y \cup \{\infty_C\}$. Then for any $y \in Y^\bullet$ one has $y \leq_C \infty_C$

and we consider on $Y^\bullet$ the following operations: $y + \infty_C = \infty_C + y = \infty_C$ and $t \cdot \infty_C = \infty_C$ for all $y \in Y$ and all $t \geq 0$. A function $g : Y \to \overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$ is said to be C-increasing if for $y, z \in Y$ such that $z \leq_C y$ one has $g(z) \leq g(y)$. The dual cone of C is $C^* = \{y^* \in Y^* : \langle y^*, y\rangle \geq 0 \ \forall y \in C\}$. We define also $\langle y^*, \infty_C\rangle = +\infty$ for all $y^* \in C^*$.

Given a subset U of X, by $\operatorname{cl}(U)$, $\operatorname{co}(U)$, δ_U and σ_U we denote its *closure*, its *convex hull*, its *indicator function* and *support function*, respectively. For sets, the closures in the weak* topologies are denoted by cl_{w^*}, while the ones in the weak topologies are denoted by cl_w. We also use the *strong quasi relative interior* of a nonempty convex set U denoted $sqri(U)$, which contains all the elements $x \in U$ for which the cone generated by $U - x$ is a closed linear subspace. Denote $\Delta_{X^n} := \{(x, \dots, x) \in X^n : x \in X\}$. We use also the *projection function* $Pr_X : X \times Y \to X$, defined by $Pr_X(x, y) = x \ \forall (x, y) \in X \times Y$ and the *identity function* on X, $id_X : X \to X$ with $id_X(x) = x \ \forall x \in X$.

For a function $f : X \to \overline{\mathbb{R}}$ we use the classical notations for *domain* $\operatorname{dom}(f) = \{x \in X : f(x) < +\infty\}$, *epigraph* $\operatorname{epi}(f) = \{(x, r) \in X \times \mathbb{R} : f(x) \leq r\}$ and *conjugate function* $f^* : X^* \to \overline{\mathbb{R}}$, $f^*(x^*) = \sup\{\langle x^*, x\rangle - f(x) : x \in X\}$. When X is normed, the conjugate function of the conjugate of a function $f : X \to \overline{\mathbb{R}}$ is said to be the *biconjugate* of f and it is denoted by $f^{**} : X^{**} \to \overline{\mathbb{R}}$, $f^{**}(x^{**}) = \sup\{\langle x^{**}, x^*\rangle - f^*(x^*) : x^* \in X^*\}$. If we consider on X^* the weak* topology, then we obtain another biconjugate for f, namely a function defined on X which takes at each $x \in X$ the value $f^{**}(\hat{x})$. We call f *proper* if $f(x) > -\infty \ \forall x \in X$ and $\operatorname{dom}(f) \neq \emptyset$. By convention, we consider $0f = \delta_{\operatorname{dom}(f)}$. Given two proper functions $f, g : X \to \overline{\mathbb{R}}$, we have the *infimal convolution* of f and g defined by $f \Box g : X \to \overline{\mathbb{R}}$, $\big(f \Box g\big)(a) = \inf\{f(x) + g(a - x) : x \in X\}$. The *convex hull* of the function $f : X \to \overline{\mathbb{R}}$ is $\operatorname{co}(f) : X \to \overline{\mathbb{R}}$, the greatest convex function everywhere less than or equal to f, while the greatest lower semicontinuous function everywhere less than or equal to f is the *lower semicontinuous hull* of f, $\operatorname{cl}(f) : X \to \overline{\mathbb{R}}$. Note that $\operatorname{epi}(\operatorname{cl}(f)) = \operatorname{cl}(\operatorname{epi}(f))$. Like for the sets, we denote for the functions the lower semicontinuous hulls obtained when we work with the weak* topologies by cl_{w^*} and the ones when we deal with the weak topologies by cl_w, too. Some of the notions introduced above can be generalized also for functions mapping into infinite dimensional spaces. For a function $h : X \to Y^\bullet$ one has

- the *domain*: $\operatorname{dom}(h) = \{x \in X : h(x) \in Y\}$,
- h is *proper*: $\operatorname{dom}(h) \neq \emptyset$,
- h is *C-convex*: $h(tx + (1-t)y) \leq_C th(x) + (1-t)h(y) \ \forall x, y \in X \ \forall t \in [0, 1]$,
- for $\lambda \in C^*$, $(\lambda h) : X \to \overline{\mathbb{R}}$, $(\lambda h)(x) = \langle \lambda, h(x)\rangle$,
- the *C-epigraph* $\operatorname{epi}_C(h) = \{(x, y) \in X \times Y : y \in h(x) + C\}$,

- h is C-epi-*closed* if $\mathrm{epi}_C(h)$ is closed,
- h is *star C-lower semicontinuous* at $x \in X$: (λh) is lower semicontinuous at x $\forall \lambda \in C^*$.

Remark 1.1. When a function is star C-lower semicontinuous, it is also C-epi-closed. The reverse statements do not hold in general (see [2, 8]).

Given a linear continuous mapping $A : X \to Y$, we have its *adjoint* $A^* : Y^* \to X^*$ given by $\langle A^* y^*, x\rangle = \langle y^*, Ax\rangle$ for any $(x, y^*) \in X \times Y^*$. For the proper function $f : X \to \overline{\mathbb{R}}$ we define also the *infimal function of f through A* as $Af : Y \to \overline{\mathbb{R}}$, $Af(y) = \inf\big\{f(x) : x \in X, Ax = y\big\}$, $y \in Y$.

For an attained infimum (supremum) instead of $\inf$ $(\sup)$ we write $\min$ $(\max)$. We give now some statements which play important roles in our paper. Note that the third lemma was, to the best of our knowledge, unknown until now, and it generalizes the result we give as Corollary 1.1, which was mentioned in [12], for instance.

Lemma 1.1. (cf. [1, 10, 9]). *Let $\Phi : X \times Y \to \overline{\mathbb{R}}$ be a proper convex lower semicontinuous function with $0 \in Pr_Y(\mathrm{dom}(\Phi))$. For each $x^* \in X^*$ one has*

$$(1)\qquad (\Phi(\cdot, 0))^*(x^*) = \sup_{x \in X}\{\langle x^*, x\rangle - \Phi(x, 0)\} = \mathrm{cl}_{w^*}\Big(\inf_{y^* \in Y^*} \Phi^*(\cdot, y^*)\Big)(x^*).$$

Lemma 1.2. (cf. [1, 5]). *Let $\Phi : X \times Y \to \overline{\mathbb{R}}$ be proper convex lower semicontinuous with $0 \in Pr_Y(\mathrm{dom}(\Phi))$. Then $Pr_{X^* \times \mathbb{R}}(\mathrm{epi}(\Phi^*))$ is w^*-closed if and only if*

$$(2)\qquad (\Phi(\cdot, 0))^*(x^*) = \sup_{x \in X}\{\langle x^*, x\rangle - \Phi(x, 0)\} = \min_{y^* \in Y^*} \Phi^*(x^*, y^*) \ \forall x^* \in X^*.$$

Lemma 1.3. *Let X be a normed space and let the convex function $f : X \to \overline{R}$ have a nonempty domain. To f we attach the function*

$$\hat{f} : X^{**} \to \overline{R},\ \hat{f}(x^{**}) = \begin{cases} f(x), & \textit{if } x^{**} = \hat{x}, \\ +\infty, & \textit{otherwise.} \end{cases}$$

If $\mathrm{cl}_{w^*}(\hat{f})$ is proper, then $f^{**} = \mathrm{cl}_{w^*}(\hat{f})$.

Proof. Suppose X^{**} endowed with $w^*(X^{**}, X^*)$. Then the conjugate of $\hat{f}$, $\hat{f}^* : X^* \to \overline{\mathbb{R}}$ looks for all $x^* \in X^*$ like

$$\begin{aligned}\hat{f}^*(x^*) &= \sup_{x^{**} \in \mathrm{dom}(\hat{f})}\{\langle x^{**}, x^*\rangle - \hat{f}(x^{**})\} = \sup_{x \in \mathrm{dom}(f)}\{\langle x^*, \hat{x}\rangle - f(x)\} \\ &= \sup_{x \in \mathrm{dom}(f)}\{\langle x^*, x\rangle - f(x)\} = f^*(x^*).\end{aligned}$$

Thus, by the Fenchel-Moreau Theorem, one automatically has that the conjugate of this function, which is nothing but f^{**}, is equal to $\mathrm{cl}_{w^*}(\hat{f})$. ■

Corollary 1.1. *If X is a normed space, for a nonempty convex subset U of it one has $(\delta_U)^{**} = \delta_{\mathrm{cl}_{w^*}(\widehat{U})}$.*

2. NEW FORMULAE FOR BICONJUGATE FUNCTIONS

The perturbation functions play a very important role in duality and, as one can see further, different functions used in optimization can be obtained from suitably chosen perturbation functions. For more on the importance of perturbation functions in convex analysis we refer to [1, 5, 11]. Further, unless otherwise specified, let X and Y be normed spaces. Since there is no possibility of confusion, denote the norms on X and Y by $\|\cdot\|$ and the ones on their duals by $\|\cdot\|_*$.

2.1. The biconjugate of a general perturbation function

Theorem 2.1. *Let the proper convex function $\Phi : X \times Y \to \overline{\mathbb{R}}$ fulfill the feasibility condition $0 \in Pr_Y(\mathrm{dom}(\Phi))$. Then $(\Phi(\cdot,0))^{**} \geq \Phi^{**}(\cdot,0)$. If Φ is also lower semicontinuous, then $(\Phi(\cdot,0))^{**} = \Phi^{**}(\cdot,0)$ if and only if $\mathrm{cl}_{w^*}\big(\inf_{y^*\in Y^*}\Phi^*(\cdot,y^*)\big)$ $= \mathrm{cl}\big(\inf_{y^*\in Y^*}\Phi^*(\cdot,y^*)\big)$.*

Proof. The first inequality follows from $(\Phi(\cdot,0))^* \leq \inf_{y^*\in Y^*}\Phi^*(\cdot,y^*)$, by considering the conjugates of these two functions. Take Φ moreover lower semicontinuous. Then, using Lemma 1.1, one gets

$$\begin{aligned}(\Phi(\cdot,0))^* &= \mathrm{cl}_{w^*}\Big(\inf_{y^*\in Y^*}\Phi^*(\cdot,y^*)\Big) \leq \mathrm{cl}_w\Big(\inf_{y^*\in Y^*}\Phi^*(\cdot,y^*)\Big)\\ &= \mathrm{cl}\Big(\inf_{y^*\in Y^*}\Phi^*(\cdot,y^*)\Big) \leq \inf_{y^*\in Y^*}\Phi^*(\cdot,y^*),\end{aligned}$$

from which, by considering the conjugates and taking into consideration that a function and its closure have the same conjugate, follows

$$\begin{aligned}(\Phi(\cdot,0))^{**} &= \Big(\mathrm{cl}_{w^*}\Big(\inf_{y^*\in Y^*}\Phi^*(\cdot,y^*)\Big)\Big)^* \geq \Big(\mathrm{cl}\Big(\inf_{y^*\in Y^*}\Phi^*(\cdot,y^*)\Big)\Big)^*\\ &= \Big(\inf_{y^*\in Y^*}\Phi^*(\cdot,y^*)\Big)^*\end{aligned}$$

and the last term coincides obviously with $\Phi^{**}(\cdot,0)$. It is straightforward that if the closure of $\inf_{y^*\in Y^*}\Phi^*(\cdot,y^*)$ coincides with its weak* closure one gets $(\Phi(\cdot,0))^{**}$ $= \Phi^{**}(\cdot,0)$. Assume now this equality true. Then, by the previous inequality, we have

$$\Big(\mathrm{cl}_{w^*}\Big(\inf_{y^*\in Y^*}\Phi^*(\cdot,y^*)\Big)\Big)^* = \Big(\mathrm{cl}\Big(\inf_{y^*\in Y^*}\Phi^*(\cdot,y^*)\Big)\Big)^*,$$

followed by

$$\begin{aligned}(3)\qquad &\Big(\mathrm{cl}_{w^*}\Big(\inf_{y^*\in Y^*}\Phi^*(\cdot,y^*)\Big)\Big)^{**}(x^{***})\\ &=\Big(\mathrm{cl}\Big(\inf_{y^*\in Y^*}\Phi^*(\cdot,y^*)\Big)\Big)^{**}(x^{***})\ \forall x^{***}\in X^{***}.\end{aligned}$$

The weak* closure of $\inf_{y^*\in Y^*}\Phi^*(\cdot,y^*)$ is proper and convex (cf. [1], for instance) and also weak* lower semicontinuous, which yields that it is lower semicontinuous, too. On the other hand, $\mathrm{cl}\big(\inf_{y^*\in Y^*}\Phi^*(\cdot,y^*)\big)$ is convex and lower semicontinuous and also proper because it is greater than or equal to the proper function $\mathrm{cl}_{w^*}\big(\inf_{y^*\in Y^*}\Phi^*(\cdot,y^*)\big)$ and if it would be everywhere equal to $+\infty$ then Φ^* would be everywhere $+\infty$, too. But when this happens, then Φ^{**} takes everywhere on $\widehat{X}$ the value $-\infty$, and this contradicts the properness of Φ, which is also convex and lower semicontinuous and fulfills $\Phi(x,0)=\Phi^{**}(\hat{x},0)$ whenever $x\in X$. Applying now the well-known biconjugate theorem (see [11], for instance), we obtain $\big(\mathrm{cl}_{w^*}\big(\inf_{y^*\in Y^*}\Phi^*(\cdot,y^*)\big)\big)^{**}(\widehat{x^*})=\mathrm{cl}_{w^*}\big(\inf_{y^*\in Y^*}\Phi^*(\cdot,y^*)\big)(x^*)$ and $\big(\mathrm{cl}\big(\inf_{y^*\in Y^*}\Phi^*(\cdot,y^*)\big)\big)^{**}(\widehat{x^*})=\mathrm{cl}\big(\inf_{y^*\in Y^*}\Phi^*(\cdot,y^*)\big)(x^*)\ \forall x^*\in X^*$. Finally, (3) yields $\mathrm{cl}_{w^*}\big(\inf_{y^*\in Y^*}\Phi^*(\cdot,y^*)\big)=\mathrm{cl}\big(\inf_{y^*\in Y^*}\Phi^*(\cdot,y^*)\big)$. ■

Theorem 2.2. *Let the proper convex lower semicontinuous function* $\Phi: X\times Y\to\overline{\mathbb{R}}$ *fulfill* $0\in Pr_Y(\mathrm{dom}(\Phi))$. *If one has*

$$(RC)\qquad Pr_{X^*\times\mathbb{R}}(\mathrm{epi}(\Phi^*))\ \textit{is}\ w^*\textit{-closed,}$$

or

$$(RC')\qquad X\ \textit{and}\ Y\ \textit{are Banach spaces and}\ 0\in sqri(Pr_Y(\mathrm{dom}(\Phi))),$$

then $(\Phi(\cdot,0))^{**}=\Phi^{**}(\cdot,0)$.

Proof. Lemma 1.2 says that (RC) is equivalent to (2) which is implied also by (RC') according to [11, Theorem 2.7.1]. Since the epigraph of the closure of the infimal value function of Φ^* coincides with the closure of the projection of $\mathrm{epi}(\Phi^*)$ on $X^*\times\mathbb{R}$ and the latter set is both w^*-closed and closed, the conclusion follows by the previous theorem. ■

Remark 2.1. Note that the equalities in Theorem 2.1 are further equivalent to $\mathrm{cl}_{w^*}\big(Pr_{X^*\times\mathbb{R}}(\mathrm{epi}(\Phi^*))\big)=\mathrm{cl}\big(Pr_{X^*\times\mathbb{R}}(\mathrm{epi}(\Phi^*))\big)$. The assertion in Theorem 2.1 can be also derived as a consequence of [12, Proposition 6], by taking $A: X\to X\times Y$, $A(x)=(x,0)$ and $h=\Phi$. The same applies for $(RC')\Rightarrow(\Phi(\cdot,0))^{**}=\Phi^{**}(\cdot,0)$ in Theorem 2.2. Because (RC) is equivalent to (2) and (RC') implies it, it is straightforward that (RC') yields (RC). There are examples where (RC) is valid, unlike (RC'). Therefore one can note that (RC) is indeed weaker than (RC'). Usually, the condition (RC) and the ones springing from it are said to be *closedness-type regularity conditions*, while (RC') belongs to the so-called *interiority-type regularity conditions*.

2.2 Biconjugates of the function to be minimized in a constrained convex optimization problem

Let $f : X \to \overline{\mathbb{R}}$ be a proper convex function, $h : X \to Y^\bullet$ a proper C-convex function and $S \subseteq X$ a nonempty convex set. Denote $G = S \cap h^{-1}(-C)$ and assume that $G \cap \operatorname{dom}(f)$ is nonempty.

Consider the general convex optimization problem

$$(P) \qquad \inf_{\substack{x\in S,\\ h(x)\in -C}} f(x).$$

Different dual problems can be attached to (P), some of them by means of the perturbation theory. In the following we will particularize the function Φ from the previous subsection to be the perturbation functions which are used to obtain the Lagrange and Fenchel-Lagrange dual problems to (P) (cf. [3], for instance). Recall that $\Phi : X \times Y \to \overline{\mathbb{R}}$ is a perturbation function for the problem (P) when

$$\Phi(x, 0) = f(x) + \delta_G(x) \ \forall x \in X. \tag{4}$$

The perturbation function used to attach to (P) its Lagrange dual problem is

$$\Phi_L : X \times Y \to \overline{\mathbb{R}},\ \Phi_L(x, y) = \begin{cases} f(x), & \text{if } x \in S, h(x) \in y - C, \\ +\infty, & \text{otherwise.} \end{cases}$$

One can verify that this function satisfies (4), is proper and convex, and its conjugate is

$$\Phi_L^* : X^* \times Y^* \to \overline{\mathbb{R}},\ \Phi_L^*(x^*, y^*) = \begin{cases} (f + ((-y^*)h) + \delta_S)^*(x^*), & \text{if } y^* \in -C^*, \\ +\infty, & \text{otherwise.} \end{cases}$$

Further, one can determine the second conjugate of Φ_L, which is

$$\Phi_L^{**} : X^{**} \times Y^{**} \to \overline{\mathbb{R}},\ \Phi_L^{**}(x^{**}, y^{**}) = \sup_{y^*\in C^*} \{-\langle y^{**}, y^*\rangle + (f + (y^*h) + \delta_S)^{**}(x^{**})\}.$$

We obtain the following statement.

Theorem 2.3. *One has always* $(f+\delta_G)^{**} \geq \sup_{y^*\in C^*}(f+(y^*h)+\delta_S)^{**}$. *Assuming the additional hypotheses*

(H_L) $\quad$ *f is lower semicontinuous, h is C-*epi-*closed and S is closed,*

fulfilled, it follows that the inequality above is always fulfilled as equality if and only if $\mathrm{cl}_{w^*}(\inf_{y^*\in C^*}(f+(y^*h)+\delta_S)^*) = \mathrm{cl}(\inf_{y^*\in C^*}(f+(y^*h)+\delta_S)^*)$. *Under (H_L), if one has*

(RC_L) $\quad$ $\bigcup\limits_{y^*\in C^*} \mathrm{epi}((f+(y^*h)+\delta_S)^*)$ *is w^*-closed,*

or

(RC_L') *X and Y are Banach spaces and* $0 \in sqri(h(S\cap \mathrm{dom}(f)\cap \mathrm{dom}(h))+\mathrm{C})$,

then $(f+\delta_G)^{**} = \sup_{y^*\in C^*}(f+(y^*h)+\delta_S)^{**}$.

Proof. The first part follows directly from Theorem 2.1, noting that (H_L) yields the lower semicontinuity of Φ_L. Further, the feasibility condition $0 \in Pr_Y(\mathrm{dom}(\Phi_L))$ is equivalent to saying that $G\cap\mathrm{dom}(f)$ is nonempty, which is true. Noting that (RC) becomes (RC_L) for Φ_L and (RC') turns into (RC_L'), we can apply Theorem 2.2 and we are done. ∎

If we want to separate f from h in the formula of $(f+\delta_G)^{**}$, a good choice is to consider the perturbation function which leads to the Fenchel-Lagrange dual problem to (P). It is

$$\Phi_{FL}: X\times X\times Y \to \overline{\mathbb{R}},\ \Phi_{FL}(x,z,y) = \begin{cases} f(x+z), & \text{if } x\in S, h(x)\in y-C,\\ +\infty, & \text{otherwise.}\end{cases}$$

This function satisfies the feasibility condition and (4), being also proper and convex. Simple calculations show that its conjugate is $\Phi_{FL}^*: X^*\times X^*\times Y^* \to \overline{\mathbb{R}}$,

$$\Phi_{FL}^*(x^*,z^*,y^*) = \begin{cases} f^*(z^*) + (((-y^*)h)+\delta_S)^*(x^*-z^*), & \text{if } y^*\in -C^*,\\ +\infty, & \text{otherwise,}\end{cases}$$

while its second conjugate of turns out to be $\Phi_{FL}^{**}: X^{**}\times X^{**}\times Y^{**}\to\overline{\mathbb{R}}$,

$$\Phi_{FL}^{**}(x^{**},z^{**},y^{**}) = f^{**}(x^{**}+z^{**}) + \sup_{y^*\in C^*}\{-\langle y^{**},y^*\rangle + ((y^*h)+\delta_S)^{**}(x^{**})\}.$$

This perturbation function leads to another formula for the biconjugate of $f+\delta_G$.

Theorem 2.4. *One always has* $(f+\delta_G)^{**} \geq f^{**}+\sup_{y^*\in C^*}((y^*h)+\delta_S)^{**}$. *Assuming the additional hypotheses* (H_L) *fulfilled, it follows that the inequality above is always fulfilled as equality if and only if* $\operatorname{cl}_{w^*}(\inf_{y^*\in C^*}(f^*\Box((y^*h)+\delta_S)^*) = \operatorname{cl}(\inf_{y^*\in C^*}(f^*\Box((y^*h)+\delta_S)^*)$. *Under* (H_L), *if one has*

(RC_{FL}) $\qquad \operatorname{epi}(f^*) + \bigcup_{y^*\in C^*} \operatorname{epi}(((y^*h)+\delta_S)^*)$ *is* w^**-closed,*

or

(RC'_{FL}) $\quad$ *X and Y are Banach spaces and*
$0 \in sqri(\operatorname{dom}(f) \times \mathrm{C} - (\operatorname{epi}_{-\mathrm{C}}(-h)) \cap (\mathrm{S} \times \mathrm{Y}))$,

then $(f+\delta_G)^{**} = f^{**} + \sup_{y^*\in C^*}((y^*h)+\delta_S)^{**}$.

Proof. The first part follows directly from Theorem 2.1, noting that (H_L) yields the lower semicontinuity of Φ_{FL}, too. Further, noting that (RC) becomes (RC_{FL}) for Φ_{FL} and (RC') turns into (RC'_{FL}), we can apply Theorem 2.2, which yields the conclusion. ■

2.3. Biconjugates of the sum of functions to be minimized in an unconstrained convex optimization problem

In the previous subsection we gave formulae for the biconjugate of the function to be minimized in a constrained convex optimization problem, where usually Lagrange duality is considered. In order to do similar things for unconstrained convex optimization problems to which Fenchel dual problems are usually attached, consider the proper convex functions $f : X \to \overline{\mathbb{R}}$ and $g : Y \to \overline{\mathbb{R}}$ and the linear continuous operator $A : X \to Y$ fulfilling the feasibility condition $A(\operatorname{dom}(f)) \cap \operatorname{dom}(g) \neq \emptyset$. We are interested to give the biconjugate of the function $f + g \circ A$. In order to do this, let the perturbation function

$$\Phi_A : X \times Y \to \overline{\mathbb{R}},\ \Phi_A(x, y) = f(x) + g(Ax + y).$$

By construction this is a proper convex function which satisfies the feasibility condition and also $\Phi_A(x, 0) = f(x) + g(Ax)\ \forall x \in X$. The conjugate of Φ_A is

$$\Phi_A : X^* \times Y^* \to \overline{\mathbb{R}},\ \Phi_A(x^*, y^*) = f^*(x^* - A^*y^*) + g^*(y^*),$$

and its biconjugate turns out to be

$$\Phi_A^{**} : X^{**} \times Y^{**} \to \overline{\mathbb{R}},\ \Phi_A^{**}(x^{**}, y^{**}) = f^{**}(x^{**}) + g^{**}(A^{**}x^{**} + y^{**}).$$

We have the following statement.

Theorem 2.5. *One always has* $(f + g \circ A)^{**} \geq f^{**} + g^{**} \circ A^{**}$. *Assuming the additional hypotheses*

$$(H_A) \qquad f \text{ and } g \text{ are lower semicontinuous,}$$

fulfilled, it follows that the inequality above is always fulfilled as equality if and only if $\mathrm{cl}_{w^*}(f^* \square A^* g^*) = \mathrm{cl}(f^* \square A^* g^*)$. *Under* (H_A), *if one has*

$$(RC_A) \qquad \mathrm{epi}(f^*) + (A^* \times id_{\mathbb{R}})(\mathrm{epi}(g^*)) \text{ is } w^*\text{-closed,}$$

or

$$(RC'_A) \qquad X \text{ and } Y \text{ are Banach spaces and } 0 \in sqri(\mathrm{dom}(g) - \mathrm{A}(\mathrm{dom}(f))),$$

then $(f + g \circ A)^{**} = f^{**} + g^{**} \circ A^{**}$.

Proof. The first part follows directly from Theorem 2.1, noting that (H_A) yields the lower semicontinuity of Φ_A. Further, noting that (RC) becomes (RC_A) for Φ_A and (RC') turns into (RC'_A), we can apply Theorem 2.2, which yields the conclusion. ■

When $f(x) = 0 \; \forall x \in X$, the feasibility condition becomes $A(X) \cap \mathrm{dom}(g) \neq \emptyset$ and the result given above collapses into the following statement.

Theorem 2.6. *One always has* $(g \circ A)^{**} \geq g^{**} \circ A^{**}$. *Assuming the additional hypothesis*

$$(H_0) \qquad g \text{ is lower semicontinuous,}$$

fulfilled, it follows that the inequality above is always fulfilled as equality if and only if $\mathrm{cl}_{w^*}(A^* g^*) = \mathrm{cl}(A^* g^*)$. *Under* (H_0), *if one has*

$$(RC_0) \qquad (A^* \times id_{\mathbb{R}})(\mathrm{epi}(g^*)) \text{ is } w^*\text{-closed,}$$

or

$$(RC'_0) \qquad X \text{ and } Y \text{ are Banach spaces and } 0 \in sqri(\mathrm{dom}(g) - A(X)),$$

then $(g \circ A)^{**} = g^{**} \circ A^{**}$.

When taking $Y = X^n$, $f_i : X \to \overline{\mathbb{R}}$, $i = 1, \ldots, n$, proper convex functions,

$g(x) = \sum_{i=1}^n f_i(x_i)$, where $x = (x_1, \ldots, x_n) \in X^n$ and $Ax = (x, \ldots, x)$, the feasibility condition becomes $\cap_{i=1}^n \mathrm{dom}(f_i) \neq \emptyset$ and we obtain the following assertions.

Theorem 2.7. *One always has* $(\sum_{i=1}^n f_i)^{**} \geq \sum_{i=1}^n f_i^{**}$. *Assuming the additional hypotheses*

(H_n) $\quad$ f_i *is lower semicontinuous,* $i = 1, \ldots, n,$

fulfilled, it follows that the inequality above is always fulfilled as equality if and only if $\mathrm{cl}_{w^*}(\Box_{i=1}^n f_i^*) = \mathrm{cl}(\Box_{i=1}^n f_i^*)$. *Under* (H_n)*, if one has*

$$(RC_n) \qquad \sum_{i=1}^n \mathrm{epi}(f_i^*) \text{ is } w^*\text{-closed,}$$

or

$$(RC_n') \qquad X \text{ is a Banach space and } 0 \in sqri\Big(\prod_{i=1}^n \mathrm{dom}(f_i) - \Delta_{\mathrm{X^n}}\Big),$$

then $(\sum_{i=1}^n f_i)^{**} = \sum_{i=1}^n f_i^{**}$.

Taking in this statement $n = 2$ or from Theorem 2.5 in case $X = Y$ and $A = id_X$, the feasibility condition turns into $\mathrm{dom}(f) \cap \mathrm{dom}(g) \neq \emptyset$ and we get the following statement.

Theorem 2.8. *One always has* $(f+g)^{**} \geq f^{**} + g^{**}$. *Assuming the additional hypotheses* (H_A) *fulfilled, it follows that the inequality above is always fulfilled as equality if and only if* $\mathrm{cl}_{w^*}(f^* \Box g^*) = \mathrm{cl}(f^* \Box g^*)$. *Under* (H_A)*, if one has*

$$(RC_S) \qquad \mathrm{epi}(f^*) + \mathrm{epi}(g^*) \text{ is } w^*\text{-closed,}$$

or

$$(RC_S') \qquad X \text{ is a Banach space and } 0 \in sqri(\mathrm{dom}(g) - \mathrm{dom}(f)),$$

then $(f+g)^{**} = f^{**} + g^{**}$.

Remark 2.2. In the last three theorems we rediscover and partially extend some recent results given in [12].

Using Theorem 2.8 and Corollary 1.1 we can give another formula for the biconjugate of the function $f + \delta_U$, where $U \subseteq X$ is a nonempty convex set.

Theorem 2.9. *One always has* $(f + \delta_U)^{**} \geq f^{**} + \delta_{\mathrm{cl}_{w^*}(\widehat{U})}$. *Assuming the additional hypotheses*

(H_F) *f is lower semicontinuous and U is closed,*

fulfilled, it follows that the inequality above is always fulfilled as equality if and only if $\mathrm{cl}_{w^*}(f^*\Box\sigma_U) = \mathrm{cl}(f^*\Box\sigma_U)$. *Under* (H_F)*, if one has*

(RC_F) $\mathrm{epi}(f^*) + \mathrm{epi}\sigma_U$ *is* w^**-closed,*

or

(RC'_F) *X is a Banach space and* $0 \in sqri(\mathrm{dom}(f) - \mathrm{U})$,

then $(f + \delta_U)^{**} = f^{**} + \delta_{\mathrm{cl}_{w^*}(\widehat{U})}$.

Remark 2.3. When U coincides with G, the feasible set of the problem (P) considered earlier, the statement from above gives another characterization of the biconjugate of $f + \delta_G$ which corresponds to the perturbation function used to attach to (P) its Fenchel dual problem, namely

$$\Phi_F : X \times X \to \overline{\mathbb{R}},\ \Phi_F(x,z) = \begin{cases} f(x+z), & \text{if } x \in G, \\ +\infty, & \text{otherwise.} \end{cases}$$

Remark 2.4. A sufficient hypothesis to have G closed is to take S closed and h C-epi-closed. Note also that under these hypotheses one has (cf. [2]) $\mathrm{epi}\sigma_G = \mathrm{cl}_{w^*}(\mathrm{epi}(\sigma_S) + \cup_{y^* \in C^*} \mathrm{epi}((y^*h)^*))$.

3. Other Important Biconjugates

In this section we give formulae for the biconjugates of other important functions in optimization, namely composed convex functions and indicators of constraint sets.

3.1. Biconjugates of the composed convex functions

Consider the proper convex function $f : X \to \overline{\mathbb{R}}$, the proper convex C-increasing function $g : Y \to \overline{\mathbb{R}}$ with the convention $g(\infty_C) = +\infty$ and the proper C-convex function $h : X \to Y^\bullet$. We impose moreover the feasibility condition $\big(h(\mathrm{dom}(h) \cap \mathrm{dom}(f)) + \mathrm{C}\big) \cap \mathrm{dom}(g) \neq \emptyset$. We give two formulae for the biconjugate of the function $f + g \circ h$, by considering different perturbation functions (see also [1]).

Let first the perturbation function

$$\Phi_1 : X \times Y \to \overline{\mathbb{R}},\ \Phi_1(x,y) = f(x) + g(h(x) + y).$$

It is proper and convex, fulfills the feasibility condition required in Theorem 2.1 and $\Phi_1(x,0) = (f + g \circ h)(x)\ \forall x \in X$, and its conjugate function turns out to be

$$\Phi_1^* : X^* \times Y^* \to \overline{\mathbb{R}},\ \Phi_1^*(x^*, y^*) = \begin{cases} g^*(y^*) + (f + (y^*h))^*(x^*), & \text{if } y^* \in C^*, \\ +\infty, & \text{otherwise.} \end{cases}$$

Here we used the fact that $g^*(y^*) = +\infty$ whenever $y^* \notin C^*$. The biconjugate of Φ_1 is $\Phi_1^{**} : X^{**} \times Y^{**} \to \overline{\mathbb{R}}$,

$$\begin{aligned} \Phi_1^{**}(x^{**}, y^{**}) &= \sup_{\substack{x^* \in X^*, \\ y^* \in C^*}} \{\langle x^{**}, x^* \rangle + \langle y^{**}, y^* \rangle - g^*(y^*) - (f + (y^*h))^*(x^*)\} \\ &= \sup_{y^* \in C^*} \{(f + (y^*h))^{**}(x^{**}) + \langle y^{**}, y^* \rangle - g^*(y^*)\}. \end{aligned}$$

We are ready now to give the first formula for the biconjugate of $f + g \circ h$.

Theorem 3.1. *One always has* $(f + g \circ h)^{**} \geq \sup_{y^* \in C^*}\{(f + (y^*h))^{**}(\cdot) - g^*(y^*)\}$. *Assuming the additional hypotheses*

(H_{CC}) *f and g are lower semicontinuous and h is star C-lower semicontinuous,*

fulfilled, it follows that the inequality above is always fulfilled as equality if and only if $\mathrm{cl}_{w^*}(\inf_{y^* \in C^*}\{(f+(y^*h))^*(\cdot)+g^*(y^*)\} = \mathrm{cl}(\inf_{y^* \in C^*}\{(f+(y^*h))^*(\cdot)+g^*(y^*)\}$. *Under (H_{CC}), if one has*

(RC_1) $\qquad \bigcup_{y^* \in \mathrm{dom}(g^*)} \big((0, g^*(y^*)) + \mathrm{epi}((f + (y^*h))^*)\big)$ *is w^*-closed,*

or

(RC_1') *X and Y are Banach spaces and* $0 \in sqri(\mathrm{dom}(g) - \mathrm{h}(\mathrm{dom}(f) \cap \mathrm{dom}(h)))$,

then $(f + g \circ h)^{**} = \sup_{y^* \in \mathrm{dom}(g^*)}\{(f + (y^*h))^{**}(\cdot) - g^*(y^*)\}$.

Proof. The first part follows directly from Theorem 2.1, since (H_{CC}) guarantees (cf. [1]) that the proper convex function Φ_1 coincides at each $(x, y) \in X \times Y$ with $\Phi_1^{**}(\hat{x}, \hat{y})$, thus it is lower semicontinuous. Further, noting that (RC) becomes (RC_1) for Φ_1 and (RC') turns into (RC_1'), we can apply Theorem 2.2, which yields the conclusion. ■

If we want to have f separated from h in the formula of the biconjugate, the following perturbation function can be considered

$$\Phi_2 : X \times X \times Y \to \overline{\mathbb{R}},\ \Phi_2(x, z, y) = f(x + z) + g(h(x) + y).$$

It is proper and convex, fulfills the feasibility condition required in Theorem 2.1 and $\Phi_2(x,0,0) = (f + g \circ h)(x)\ \forall x \in X$. Using again that $g^*(y^*) = +\infty$ whenever $y^* \notin C^*$, the conjugate function of Φ_2 turns out to be $\Phi_2^* : X^* \times X^* \times Y^* \to \overline{\mathbb{R}}$,

$$\Phi_2^*(x^*, z^*, y^*) = \begin{cases} f^*(z^*) + g^*(y^*) + (y^*h)^*(x^* - z^*), & \text{if } y^* \in C^*, \\ +\infty, & \text{otherwise,} \end{cases}$$

while its biconjugate is $\Phi_2^{**} : X^{**} \times X^{**} \times Y^{**} \to \overline{\mathbb{R}}$,

$$\Phi_2^{**}(x^{**}, z^{**}, y^{**}) = \sup_{x^*, z^* \in X^*, y^* \in C^*} \{\langle x^{**}, x^*\rangle + \langle z^{**}, z^*\rangle + \langle y^{**}, y^*\rangle - f^*(z^*) - g^*(y^*)$$
$$-(y^*h)^*(x^* - z^*)\} = f^{**}(x^{**} + z^{**}) + \sup_{y^* \in C^*} \{(y^*h)^{**}(x^{**}) + \langle y^{**}, y^*\rangle - g^*(y^*)\}.$$

We are ready now to give the second formula for the biconjugate of $f + g \circ h$, where f and h are separated.

Theorem 3.2. *One always has* $(f + g \circ h)^{**} \geq f^{**} + \sup_{y^* \in C^*}\{(y^*h)^{**}(\cdot) - g^*(y^*)\}$. *Assuming the additional hypotheses* (H_{CC}) *fulfilled, the inequality above is always fulfilled as equality if and only if* $\operatorname{cl}_{w^*}(\inf_{y^* \in C^*}\{(f^* \Box (y^*h)^*)(\cdot) + g^*(y^*)\})$ $= \operatorname{cl}(\inf_{y^* \in C^*}\{(f^* \Box (y^*h)^*)(\cdot) + g^*(y^*)\})$. *Under* (H_{CC}), *if one has*

(RC_2) $\qquad \operatorname{epi}(f^*) + \bigcup_{y^* \in \operatorname{dom}(g^*)} \big((0, g^*(y^*)) + \operatorname{epi}((y^*h))^*)\big) \text{is } w^* - \text{closed},$

or

(RC_2') *X and Y are Banach spaces and* $0 \in sqri(\operatorname{dom}(f) \times \operatorname{dom}(g) - \operatorname{epi}_C(h))$,

then $(f + g \circ h)^{**} = f^{**} + \sup_{y^* \in \operatorname{dom}(g^*)} \{(y^*h)^{**}(\cdot) - g^*(y^*)\}$.

Proof. The first part follows directly from Theorem 2.1, because (H_{CC}) guarantees (cf. [1]) that the proper convex function Φ_2 coincides at each $(x, z, y) \in X \times X \times Y$ with $\Phi_2^{**}(\hat{x}, \hat{z}, \hat{y})$, thus it is lower semicontinuous. Further, noting that (RC) becomes (RC_2) for Φ_2 and (RC') turns into (RC_2'), we can apply Theorem 2.2, which yields the conclusion. ■

Remark 3.1. When (H_{CC}) is valid, one has $(f + g \circ h)^{**}(\hat{x}) = f^{**}(\hat{x}) + g^{**}(\widehat{h(x)})\ \forall x \in X$.

3.2. Biconjugates of indicators of constraint sets

Let $h : X \to Y^\bullet$ be a proper C-convex function with the property that $h^{-1}(-C) \neq \emptyset$. The conjugate of the function $\delta_{h^{-1}(-C)}$ plays important roles in many recent papers on duality in convex optimization. Thus the question how does its biconjugate look when the duals of the spaces X and Y are endowed with the strong topology arises naturally. We give in the following a statement for this biconjugate function. It can be obtained from Theorem 2.3 or Theorem 2.4 for $f(x) = 0$ $\forall x \in X$ and $S = X$ or from Theorem 3.1 or Theorem 3.2 for $f(x) = 0$ $\forall x \in X$ and $g = \delta_{-C}$.

Theorem 3.3. *One always has* $\delta^{**}_{h^{-1}(-C)} \geq \sup_{y^* \in C^*}(y^*h)^{**}$. *Assuming the additional hypothesis*

(H_C) $\qquad$ *h is C-epi-closed,*

fulfilled, it follows that the inequality above is always fulfilled as equality if and only if $\mathrm{cl}_{w^*}(\inf_{y^* \in C^*}(y^*h)^*) = \mathrm{cl}(\inf_{y^* \in C^*}(y^*h)^*)$. *Under (H_C), if one has*

(RC_C) $\qquad$ $\bigcup\limits_{y^* \in C^*} \mathrm{epi}((y^*h)^*)$ *is w^*-closed,*

or

(RC'_C) $\qquad$ *X and Y are Banach spaces and* $0 \in sqri(h(\mathrm{dom}(h)) + \mathrm{C})$,

then $\delta^{**}_{h^{-1}(-C)} = \sup_{y^* \in C^*}(y^*h)^{**}$.

Remark 3.2. One may wonder how can we apply Theorem 3.1 or Theorem 3.2 to obtain the statement above, since in their additional hypotheses (H_{CC}) h needs to be taken star C-lower semicontinuous, not C-epi-closed as in (H_C). Fortunately, in the special case $f(x) = 0$ $\forall x \in X$ and $g = \delta_{-C}$ the perturbation functions Φ_1 and Φ_2 are lower semicontinuous even when we weaken the topological assumption on h from star C-lower semicontinuity to (H_C).

4. The Biconjugate of the Supremum of a Family of Convex Functions

In the recent papers [4, 6, 12] the biconjugate of the maximum of two proper convex functions defined on a normed space is determined via different approaches. The natural question which arises is how would the formula of the biconjugate of the supremum of a possibly infinite family of proper convex functions look like and under which regularity conditions. In the following we give an answer to this question and we show that the formulae we give hold even in locally convex spaces.

Consider further a dual system formed by a separated locally convex vector space X and its topological dual X^*. According to [7], with a family $\mathcal{M}$ of totally saturated bounded subsets of X one can induce on X^* the so-called *uniform* *convergence topology on* $\mathcal{M}$ denoted $\tau_{\mathcal{M}}$. Consider further X^* endowed with $\tau_{\mathcal{M}}$. Restricting the sets in $\mathcal{M}$ to satisfy certain properties, one obtains different classical locally convex topologies on X^*. For instance, when all the elements of $\mathcal{M}$ are finite sets $\tau_{\mathcal{M}}$ becomes the weak* topology, when they are absolutely convex and weakly compact it coincides with the Mackey topology τ_k, while when all these sets are weakly bounded we get the strong topology on X^*, τ_b. Note that τ_b coincides, when X is a normed space, with the strong topology induced by the norm on X^*. The weak* topology is the weakest uniform convergence topology that can be considered on X^*, while τ_b is the strongest. The dual of X^* is X if and only if $\tau_{\mathcal{M}}$ is weaker than τ_k, but stronger than weak*. On the other hand, when $\tau_{\mathcal{M}}$ is strictly stronger than τ_k, but weaker than τ_b, the dual of X^*, denoted X^{**} and referred to also as the *bidual* of X, does not coincide anymore with X. Note also that by endowing X^* with any $\tau_{\mathcal{M}}$ stronger than weak* its dual has X among its linear subspaces.

In this section we take X to be a separated locally convex space and its dual X^* to be endowed with an arbitrary uniform convergence topology $\tau_{\mathcal{M}}$ which is strictly stronger than τ_k, but weaker than τ_b. Then X^* has a dual X^{**} which does not coincide with X. In this case the biconjugate of a function $f : X \to \overline{\mathbb{R}}$ is defined analogously to the case when X is normed. Let the proper convex functions $f_t : X \to \overline{\mathbb{R}}$, $t \in T$, where T is an arbitrary index set, possibly uncountable, such that $\mathrm{dom}\big(\sup_{t\in T} f_t\big) \neq \emptyset$. This yields the nonemptiness of the set $\cap_{t\in T}\mathrm{dom}(f_t)$. $\mathbb{R}^T$ is the space of all functions $y : T \to \mathbb{R}$, endowed with the product topology and with the operations being the usual pointwise ones. For simplicity, denote $y_t = y(t)$ $\forall y \in \mathbb{R}^T$ $\forall t \in T$. Let $\Delta_{\mathbb{R}^T}$ be the subset of the constant functions $y \in \mathbb{R}^T$. The dual space of $\mathbb{R}^T$ is $(\mathbb{R}^T)^*$, the so-called *space of generalized finite sequences* $\lambda = (\lambda_t)_{t\in T}$ such that $\lambda_t \in \mathbb{R}$ $\forall t \in T$, and with only finitely many λ_t different from zero. The positive cone in $\mathbb{R}^T$ is $\mathbb{R}^T_+ = \{y \in \mathbb{R}^T : y_t = y(t) \geq 0 \ \forall t \in T\}$, and its dual is the positive cone in $(\mathbb{R}^T)^*$, namely $(\mathbb{R}^T_+)^* = \{y^* = (y^*_t)_{t\in T} \in (\mathbb{R}^T)^* : y^*_t \geq 0$ $\forall t \in T\}$. Denote also $\mathcal{B} = \{y^* \in (\mathbb{R}^T_+)^* : \sum_{t\in T} y^*_t = 1\}$. Using previous results from this paper we can formulate the following statement for the biconjugate of $\sup_{t\in T} f_t$.

Theorem 4.1. *One always has* $\big(\sup_{t\in T} f_t\big)^{**} \geq \sup_{y^*\in\mathcal{B}} \big(\sum_{t\in T} y^*_t f_t\big)^{**}$. *Assuming the additional hypotheses*

(H_T) *f_t is lower semicontinuous whenever $t \in T$,*

fulfilled, there is always

$$\Big(\sup_{t\in T} f_t\Big)^{**} = \Big(\mathrm{cl}_{w^*}\Big(\inf_{y^*\in\mathcal{B}}\Big(\sum_{t\in T} y^*_t f_t\Big)^*\Big)\Big)^* \geq \sup_{y^*\in\mathcal{B}}\Big(\sum_{t\in T} y^*_t f_t\Big)^{**}. \tag{5}$$

When X is a normed space the inequality in (5) *is always fulfilled as equality if* *and only if* $\mathrm{cl}_{w^*}(\inf_{y^*\in\mathcal{B}}(\sum_{t\in T} y_t^*\ f_t)^*) = \mathrm{cl}(\inf_{y^*\in\mathcal{B}}(\sum_{t\in T} y_t^* f_t)^*)$.

Under (H_T)*, if one has*

$$(RC_T) \qquad \mathrm{co}\Big(\bigcup_{t\in T}\mathrm{epi}(f_t^*)\Big) \text{ is } w^*\text{-closed,}$$

or

(RC'_T) T *is at most countable,* X *is a Fréchet space and*
$$0 \in sqri\Big(\prod_{t\in T} f_t\Big(\bigcap_{t\in T}\mathrm{dom}(f_t)\Big) - \Delta_{\mathbb{R}^{\mathrm{T}}} + \mathbb{R}_+^{\mathrm{T}}\Big),$$

or

$$(RC''_T) \qquad T \text{ is finite,}$$

then

$$\Big(\sup_{t\in T} f_t\Big)^{**} = \sup_{y^*\in\mathcal{B}}\Big(\sum_{t\in T} y_t^* f_t\Big)^{**}. \tag{6}$$

Proof. For each $y^* \in \mathcal{B}$ one has $\sup_{t\in T} f_t \geq \sum_{t\in T} y_t^* f_t$ and so $\big(\sup_{t\in T} f_t\big)^{**} \geq \big(\sum_{t\in T} y_t^* f_t\big)^{**}$. This yields the first inequality.

From [1, Section 4.3] we know that, provided the fulfillment of (H_T), one always has $(\sup_{t\in T} f_t)^* = \mathrm{cl}_{w^*}\big(\inf_{y^*\in\mathcal{B}}\big(\sum_{t\in T} y_t^* f_t\big)^*\big)$. Conjugating in both sides we obtain $(\sup_{t\in T} f_t)^{**} = \big(\mathrm{cl}_{w^*}\big(\inf_{y^*\in\mathcal{B}}\big(\sum_{t\in T} y_t^* f_t\big)^*\big)\big)^*$. On the other hand, $\mathrm{cl}_{w^*}\big(\inf_{y^*\in\mathcal{B}}\big(\sum_{t\in T} y_t^* f_t\big)^*\big) \leq \inf_{y^*\in\mathcal{B}}\big(\sum_{t\in T} y_t^* f_t\big)^*$, which yields

$$\Big(\mathrm{cl}_{w^*}\Big(\inf_{y^*\in\mathcal{B}}\Big(\sum_{t\in T} y_t^* f_t\Big)^*\Big)\Big)^* \geq \Big(\inf_{y^*\in\mathcal{B}}\Big(\sum_{t\in T} y_t^* f_t\Big)^*\Big)^* = \sup_{y^*\in\mathcal{B}}\Big(\sum_{t\in T} y_t^* f_t\Big)^{**}, \tag{7}$$

and using the equality obtained before we get (5). If $\mathrm{cl}_{w^*}(\inf_{y^*\in\mathcal{B}}(\sum_{t\in T} y_t^* f_t)^*) = \mathrm{cl}(\inf_{y^*\in\mathcal{B}}(\sum_{t\in T} y_t^* f_t)^*)$, the inequality in (7) is fulfilled as equality, since a function and its lower semicontinuous hull have the same conjugate, and this means that the inequality in (5) turns into an equality. So far we did no need the additional assumption on the space X to be normed. But this is needed to show the other implication in the desired equivalence regarding the equality case in the inequality from (5). If this inequality is fulfilled as equality, then $\big(\mathrm{cl}_{w^*}(\inf_{y^*\in\mathcal{B}}(\sum_{t\in T} y_t^* f_t)^*)\big)^* = \big(\mathrm{cl}(\inf_{y^*\in\mathcal{B}}(\sum_{t\in T}\ y_t^* f_t)^*)\big)^*$ and, using a similar argumentation to the one in the proof of Theorem 2.1, we obtain $\mathrm{cl}_{w^*}(\inf_{y^*\in\mathcal{B}}\ (\sum_{t\in T} y_t^* f_t)^*) = \mathrm{cl}(\inf_{y^*\in\mathcal{B}}(\sum_{t\in T} y_t^* f_t)^*)$. This was the only place in this subsection where X had to be taken normed.

To prove the second part of the theorem we treat each regularity condition separately. First, note that there is always

$$\mathrm{co}\Big(\bigcup_{t\in T}\mathrm{epi}(f_t^*)\Big) \subseteq \bigcup_{y^*\in\mathcal{B}}\mathrm{epi}\Big(\Big(\sum_{t\in T} y_t^* f_t\Big)^*\Big)$$
$$\subseteq \mathrm{epi}\Big(\inf_{y^*\in\mathcal{B}}\Big(\sum_{t\in T} y_t^* f_t\Big)^*\Big) \subseteq \mathrm{epi}\Big(\Big(\sup_{t\in T} f_t\Big)^*\Big),$$

and the weak* closures of all these sets coincide with $\mathrm{epi}\big(\big(\sup_{t\in T} f_t\big)^*\big)$. Assuming the validity of (RC_T), the sets in the chain of inclusions from above become equal, therefore one gets

$$\Big(\sup_{t\in T} f_t\Big)^* = \min_{y^*\in\mathcal{B}}\Big(\sum_{t\in T} y^* f_t\Big)^*. \tag{8}$$

Now we deal with (RC'_T). Note that for any arbitrary $x^*\in X^*$, we have

$$-\Big(\sup_{t\in T} f_t\Big)^*(x^*) = \inf_{x\in X}\Big\{-\langle x^*,x\rangle + \sup_{t\in T} f_t(x)\Big\} = \inf_{\substack{x\in X,u\in\mathbb{R},\\ f_t(x)\le u\forall t\in T}}\{u-\langle x^*,x\rangle\}. \tag{9}$$

Let us take a closer look to the term in the right-hand side of (9). Removing the term $\langle x^*,x\rangle$, we can see $\inf\{u : x\in X, u\in\mathbb{R}, f_t(x)\le u\ \forall t\in T\}$ as a convex minimization problem like (P), for $X = X\times\mathbb{R}$, $Y=\mathbb{R}^T$, $S = X\times\mathbb{R}$, $f(x,u)=u$ and

$$h(x,u) = \begin{cases} (f_t(x)-u)_{t\in T}, & \text{if } x\in\bigcap_{t\in T}\mathrm{dom}(f_t),\\ \infty_{\mathbb{R}^T_+}, & \text{otherwise.}\end{cases}$$

One can easily notice that S is nonempty, convex and closed, f is proper, convex and lower semicontinuous and it can be proven that h is proper, $\mathbb{R}^T_+$-convex and $\mathbb{R}^T_+$-epi-closed. Note moreover that $S\cap h^{-1}(-\mathbb{R}^T_+)\cap\mathrm{dom}(f)\neq\emptyset$. It is known (cf. [1] for instance) that (RC_L) guarantees in this case the stable strong duality for the problem (P) and its Lagrange dual. Since (RC'_L) yields (RC_L), the mentioned stable strong duality holds under the fulfillment of (RC'_L), too. Giving to S, f and h the values assigned above, it is not hard to see that in this case (RC'_L) turns into (RC'_T). Assuming this condition fulfilled, one obtains (see also [2]) then that there is stable strong duality for the special case of the problem (P) taken above and its Lagrange dual, which yields that for all $x^*\in X^*$ one has

$$\inf_{\substack{x\in X,u\in\mathbb{R},\\ f_t(x)\le u\forall t\in T}}\{u-\langle x^*,x\rangle\} = \max_{y^*\in(\mathbb{R}^T_+)^*}\inf_{\substack{x\in X,\\ u\in\mathbb{R}}}\{u-\langle x^*,x\rangle + (y^*h)(x,u)\}. \tag{10}$$

Rewriting the term in the right-hand side, we obtain further

$$\inf_{\substack{x\in X,u\in\mathbb{R},\\ f_t(x)\le u\forall t\in T}}\{u-\langle x^*,x\rangle\} = \max_{y^*\in(\mathbb{R}^T_+)^*}\inf_{\substack{x\in X,\\ u\in\mathbb{R}}}\Big\{u-\langle x^*,x\rangle + \sum_{t\in T} y_t^*(f_t(x)-u)\Big\}$$

$$= \max_{y^* \in (\mathbb{R}^T_+)^*} \Big\{ \inf_{x \in X} \Big\{ -\langle x^*, x\rangle + \sum_{t\in T} y_t^* f_t(x) \Big\} + \inf_{u\in\mathbb{R}} u\Big(1 - \sum_{t \in T} y_t^*\Big)\Big\}$$

$$= \max_{y^* \in \mathcal{B}} -\Big(\sum_{t\in T} y_t^* f_t\Big)^*(x^*) = -\min_{y^*\in\mathcal{B}} \Big(\sum_{t\in T} y_t^* f_t\Big)^*(x^*).$$

Using (9) the equalities above yield (8).

If (RC''_T) is valid, let $T = \{1, \ldots, n\}$. In this case $\mathbb{R}^T$ becomes $\mathbb{R}^n$ and we use the fact that $int(\mathbb{R}^n_+) \neq \emptyset$. For any $\bar{x} \in \cap_{t=1}^n \mathrm{dom}(f_t)$ there is always an $\bar{u} > \max_{t=1,\ldots,n} f_t(\bar{x})$ such that $(f_t(\bar{x}) - \bar{u})_{t=1,\ldots,n} \in -int(\mathbb{R}^n_+)$. This means actually that for each $x^* \in X^*$ the classical Slater Constraint Qualification for the convex minimization problem considered in the left-hand side of (10) is valid. Then there is strong duality for this problem and its Lagrange dual, which lies in the right-hand side of (10), whenever $x^* \in X^*$. Relation (8) follows analogously to the case of the validity of (RC'_T). Alternatively, when (RC''_T) holds one can apply [11, Corollary 2.8.11] to obtain $(\sup_{1\leq t\leq n} f_t)^* = \min\{(\sum_{t=1}^n y_t^* f_t)^* : y_t^* \geq 0,\ \sum_{t=1}^n y_t^* = 1\}$, which is nothing but (8).

We have proven that under any of the three regularity conditions we considered (8) holds. Taking the conjugates of the terms in both sides of it, (6) follows. ■

Remark 4.1. Note that when T is uncountable $\mathbb{R}^T$ is not a Fréchet space, thus no interiority-type regularity conditions originating from $(\mathbb{R}C'_L)$ can be considered in the statement above in this case.

We proved that provided the validity (H_T) and of each of (RC_T), (RC'_T) and (RC''_T) one has (6). Since the regularity conditions given in [4, 6, 12] ensure that the biconjugate of the maximum of two functions coincides with the maximum of their biconjugates, it is natural to look for conditions which guarantee the formula $\big(\sup_{t\in T} f_t\big)^{**} = \sup_{t\in T} f_t^{**}$. In the following we show that (RC_T) guarantees this formula, too.

Theorem 4.2. *One always has* $\big(\sup_{t\in T} f_t\big)^{**} \geq \sup_{t\in T} f_t^{**}$. *Assuming the additional hypothesis* (H_T) *and the condition* (RC_T) *fulfilled, it follows* $\big(\sup_{t\in T} f_t\big)^{**} = \sup_{t\in T} f_t^{**}$.

Proof. Conjugating twice both sides of the obvious inequality $f_j \leq \sup_{t\in T} f_t$ $\forall j \in T$ we get $f_j^{**} \leq \big(\sup_{t\in T} f_t\big)^{**}$ $\forall j \in T$. Taking the supremum regarding $j \in T$ in the left-hand side we obtain the desired inequality, $\sup_{t\in T} f_t^{**} \leq \big(\sup_{t\in T} f_t\big)^{**}$.

In [1, Section 4.3] we have shown that there is

$$\text{(11)} \quad \mathrm{epi}\Big(\Big(\sup_{t\in T} f_t\Big)^*\Big) = \mathrm{cl}_{w^*}\Big(\mathrm{co}\Big(\mathrm{epi}\Big(\inf_{t\in T} f_t^*\Big)\Big)\Big) = \mathrm{cl}_{w^*}\Big(\mathrm{co}\Big(\bigcup_{t\in T} \mathrm{epi}(f_t^*)\Big)\Big).$$

Since $\bigcup_{t\in T}\operatorname{epi}(f_t^*)$ is a subset of $\operatorname{epi}(\inf_{t\in T} f_t^*)$, one has

$$\operatorname{co}\Big(\bigcup_{t\in T}\operatorname{epi}(f_t^*)\Big)\subseteq \operatorname{co}\Big(\operatorname{epi}\Big(\inf_{t\in T} f_t^*\Big)\Big). \tag{12}$$

Let $(x,r)\in \operatorname{co}\big(\operatorname{epi}\big(\inf_{t\in T} f_t^*\big)\big)$. Using the definition of the convex hull of a function it follows immediately that $(x,r)\in \operatorname{epi}\big(\operatorname{co}\big(\inf_{t\in T} f_t^*\big)\big)$, therefore

$$\operatorname{co}\Big(\operatorname{epi}\Big(\inf_{t\in T} f_t^*\Big)\Big)\subseteq \operatorname{epi}\Big(\operatorname{co}\Big(\inf_{t\in T} f_t^*\Big)\Big). \tag{13}$$

As

$$\operatorname{cl}_{w^*}\Big(\operatorname{epi}\Big(\operatorname{co}\Big(\inf_{t\in T} f_t^*\Big)\Big)\Big)=\operatorname{epi}\Big(\operatorname{cl}_{w^*}\Big(\operatorname{co}\Big(\inf_{t\in T} f_t^*\Big)\Big)\Big)=\operatorname{cl}_{w^*}\Big(\operatorname{co}\Big(\operatorname{epi}\Big(\inf_{t\in T} f_t^*\Big)\Big)\Big),$$

from (11), (12) and (13) we obtain, using also the fulfillment of (RC_T), $\operatorname{epi}((\sup_{t\in T} f_t)^*) = \operatorname{epi}(\operatorname{co}\ (\inf_{t\in T} f_t^*))$, which is nothing but $(\sup_{t\in T} f_t)^* = \operatorname{co}(\inf_{t\in T} f_t^*)$. Considering the conjugates of these functions one gets $(\sup_{t\in T} f_t)^{**} = (\operatorname{co}(\inf_{t\in T} f_t^*))^*$. Since the convex hull of a function is greater than or equal to the lower semicontinuous convex hull of the same function and it is less than or equal to the function itself and [11, Theorem 2.3.1(iv)] asserts that the conjugates of $\operatorname{cl}(\operatorname{co}(f))$ and f coincide for each function f, it follows that $(\operatorname{co}(\inf_{t\in T} f_t^*))^* = (\inf_{t\in T} f_t^*)^*$. Therefore, we have $(\sup_{t\in T} f_t)^{**} = (\inf_{t\in T} f_t^*)^*$. Since $(\inf_{t\in T} f_t^*)^* = \sup_{t\in T} f_t^{**}$, one gets $(\sup_{t\in T} f_t)^{**} = \sup_{t\in T} f_t^{**}$. ■

Remark 4.2. When T contains only two elements, say $T=\{1,2\}$, we obtain from the statement given above a new regularity condition that, provided the lower semicontinuity of the proper convex functions $f_1, f_2 : X \to \overline{\mathbb{R}}$, ensures the formula $(\sup\{f_1, f_2\})^{**} = \sup\{f_1^{**}, f_2^{**}\}$, namely

$$(RC_M) \qquad \operatorname{co}\big(\operatorname{epi}(f_1^*)\bigcup\operatorname{epi}(f_2^*)\big) \text{ is } w^*\text{-closed.}$$

Regularity conditions of interiority-type which guarantee the same formula can be found also in [6, Theorem 6], [4, Theorem 3.1] and [12, Proposition 7]. In all these papers X was considered a normed space, and this fact plays a decisive role in the proofs of the mentioned formula, while our results are given for X locally convex. Of course, we can formulate them also in the framework used in Sections 2 and 3 and in [4, 6, 12] by taking X normed and X^* endowed with τ_b.

Remark 4.3. Taking into consideration the discussion from the beginning of this section, one can expect to extend in a natural way the results given in the previous two sections from normed to locally convex spaces, in which case the regularity conditions arising from (RC') would impose the spaces to be Fréchet, not Banach.

ACKNOWLEDGMENT

The authors are grateful to their colleague E. R. Csetnek for pertinent comments on an earlier draft of the paper. The authors are indebted to an anonymous reviewer for carefully reading the paper, valuable comments and suggestions and providing alternative proofs for some of the results in this article.

REFERENCES

1. R. I. Boţ, S.-M. Grad and G. Wanka, *Generalized Moreau-Rockafellar results for composed convex functions*, Preprint **16**, Chemnitz University of Technology, 2007.
2. R. I. Boţ, S.-M. Grad and G. Wanka, On strong and total Lagrange duality for convex optimization problems, *J. Math. Anal. Appl.*, **337** (2008), 1315-1325.
3. R. I. Boţ and G. Wanka, An alternative formulation for a new closed cone constraint qualification, *Nonlinear Anal.*, **64** (2006), 1367-1381.
4. R. I. Boţ and G. Wanka, The conjugate of the pointwise maximum of two convex functions revisited, *J. Global Optim.*, **41** (2008), 625-632.
5. R. S. Burachik, V. Jeyakumar and Z.-Y. Wu, Necessary and sufficient conditions for stable conjugate duality, *Nonlinear Anal.* **64** (2006), 1998-2006.
6. S. P. Fitzpatrick and S. Simons, On the pointwise maximum of convex functions, *Proc. Amer. Math. Soc.*, **128** (2000), 3553-3561.
7. G. Köthe, *Topologische lineare Räume I*, Die Grundlehren der mathematischen Wissenschaften **107**, Springer-Verlag, Berlin-Göttingen-Heidelberg, 1960.
8. J.-P. Penot and M. Théra, Semi-continuous mappings in general topology, *Arch. Math.* (*Basel*), **38** (1982), 158-166.
9. J.-P. Penot and C. Zălinescu, Bounded convergence for perturbed minimization problems, *Optimization*, **53** (2004), 625-640.
10. T. Precupanu, Closedness conditions for the optimality of a family of nonconvex optimization problems, *Math. Operationsforsch. Statist. Ser. Optim.*, **15** (1984), 339-346.
11. C. Zălinescu, *Convex analysis in general vector spaces*, World Scientific, River Edge, 2002.
12. C. Zălinescu, On the second conjugate of several convex functions in general normed vector spaces, *J. Global Optim.*, **40** (2007), 475-487.

Radu Ioan Boţ
Faculty of Mathematics,
Chemnitz University of Technology,
D-09107 Chemnitz, Germany
E-mail: radu.bot@mathematik.tu-chemnitz.de

Sorin-Mihai Grad
Faculty of Mathematics,
Chemnitz University of Technology,
D-09107 Chemnitz, Germany
E-mail: sorin-mihai.grad@mathematik.tu-chemnitz.de