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ON AN OPEN PROBLEM REGARDING TOTALLY FENCHEL UNSTABLE FUNCTIONS

RADU IOAN BOŢ AND ERNÖ ROBERT CSETNEK

ABSTRACT. We give an answer to the Problem 11.5 posed in Stephen Simons's book "From Hahn-Banach to Monotonicity".

1. INTRODUCTION AND PROBLEM FORMULATION

Before introducing the problem proposed by Stephen Simons, we recall some preliminary notions and results. Throughout this note, E denotes a nontrivial real Banach space, E^* its topological dual space and E^{**} its bidual space. The canonical embedding of E into E^{**} is defined by $\widehat{\ }: E \to E^{**}$, $\langle x^*, \widehat{x} \rangle := \langle x, x^* \rangle$, for all $x \in E$ and $x^* \in E^*$, where $\langle x, x^* \rangle$ denotes the value of the linear continuous functional x^* at x. For $D \subseteq E$, we denote by $\widehat{D}$ the image of the set D through the canonical embedding, that is $\widehat{D} = \{\widehat{x} : x \in D\}$.

The *indicator function* of $D \subseteq E$, denoted by δ_D, is defined as $\delta_D : E \to \overline{\mathbb{R}}$,

$$\delta_D(x) = \begin{cases} 0, & \text{if } x \in D, \\ +\infty, & \text{otherwise,} \end{cases}$$

where $\overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$. For a function $f : E \to \overline{\mathbb{R}}$ we denote by $\operatorname{dom}(f) = \{x \in E : f(x) < +\infty\}$ its *domain* and by $\operatorname{epi}(f) = \{(x,r) \in E \times \mathbb{R} : f(x) \leq r\}$ its *epigraph*. We call f *proper* if $\operatorname{dom}(f) \neq \emptyset$ and $f(x) > -\infty$ for all $x \in E$. The *Fenchel-Moreau conjugate* of f is the function $f^* : E^* \to \overline{\mathbb{R}}$ defined by $f^*(x^*) = \sup_{x \in E}\{\langle x, x^* \rangle - f(x)\}$ for all $x^* \in E^*$.

Consider $f, g : E \to \overline{\mathbb{R}}$ two arbitrary convex functions. We say that f and g satisfy *stable Fenchel duality* if for all $x^* \in E^*$, there exists $z^* \in E^*$ such that

$$(f+g)^*(x^*) = f^*(x^* - z^*) + g^*(z^*).$$

If this property holds for $x^* = 0$, then f and g satisfy the classical *Fenchel duality*. The pair f, g is *totally Fenchel unstable* (see [10]) if f and g satisfy Fenchel duality but

$$y^*, z^* \in E^* \text{ and } (f+g)^*(y^* + z^*) = f^*(y^*) + g^*(z^*) \Longrightarrow y^* + z^* = 0.$$

We refer the reader to [1] for a geometric characterization of these concepts.

Obviously, stable Fenchel duality implies Fenchel duality, but the converse is not true (see the example in [1], pp. 2798-2799 and Example 11.1 in [10]). Nevertheless, each of these examples (which are given in $\mathbb{R}^2$) fails when one tries to verify total Fenchel unstability. Surprisingly, in the finite dimensional case, it is still an open question if there exists a pair of functions which is totally Fenchel unstable (see

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Problem 11.6 in [10]). In the infinite dimensional setting this problem receives an answer, due to the existence of extreme points which are not support points of certain convex sets. Recall that if C is a convex subset of E, then $x \in C$ is a *support point* of C if there exists $x^* \in E^*$, $x^* \neq 0$ such that $\langle x, x^* \rangle = \sup \langle C, x^* \rangle$. We give below an example, proposed in [10], of a pair f, g which is totally Fenchel unstable.

Example 1.1. Let C be a nonempty, bounded, closed and convex subset of E such that there exists an extreme point x_0 of C which is not a support point of C (an example of a set C and a point x_0 with the above mentioned properties was given in the space l_2, following an idea due to Jonathan Borwein, see [10]). Take $A := x_0 - C$, $B := C - x_0$, $f := \delta_A$ and $g := \delta_B$. One can prove that the pair f, g is totally Fenchel unstable (see Example 11.3 in [10]).

Regarding the functions defined in the above example, Stephen Simons asks whether, denoting $E^* \setminus \{0\}$ with $\{0\}^c$, the following representation of the Minkowski sum of the sets $\operatorname{epi}(f^*)$ and $\operatorname{epi}(g^*)$ is true:

$$\operatorname{epi}(f^*) + \operatorname{epi}(g^*) = (\{0\} \times [0, \infty)) \cup (\{0\}^c \times (0, \infty)). \tag{1.1}$$

The justification of this question comes from a similar representation of the set $\operatorname{epi}(f_0^*) + \operatorname{epi}(g_0^*)$, proved in [10] for a pair of functions f_0, g_0 defined on the space $\mathbb{R}^2$ in a similar way as in Example 1.1 above (see Example 11.1 and Example 11.2 in [10]).

We give in the following a reformulation of this problem (as in [10]). The conjugates of the functions f and g are

$$f^*(y^*) = \langle x_0, y^* \rangle - \inf \langle C, y^* \rangle \geq 0 \text{ for all } y^* \in E^* \text{ and}$$
$$g^*(y^*) = \sup \langle C, y^* \rangle - \langle x_0, y^* \rangle \geq 0 \text{ for all } y^* \in E^*.$$

One can use the boundedness of the set C to conclude that f^* and g^* are continuous functions. The inclusion " $\subseteq$ " in (1.1) holds and, since $(0,0) = (0,0) + (0,0) \in \operatorname{epi}(f^*) + \operatorname{epi}(g^*)$, relation (1.1) is equivalent to

$$\operatorname{epi}(f^*) + \operatorname{epi}(g^*) \supset E^* \times (0, \infty). \tag{1.2}$$

Let us mention that for the implication (1.2)$\Rightarrow$(1.1) the assumption that x_0 is not a support point of C is decisive.

In case E is reflexive, this question has a positive answer. Although the proof is given in [10] (Example 11.3), we give the details for the reader's convenience. Let $y^* \in E^*$ be arbitrary. Consider the functions $h : E^* \to \mathbb{R}$ and $k : E^* \to \mathbb{R}$ defined by $h(z^*) := f^*(z^*)$ and $k(z^*) := g^*(y^* - z^*)$ for all $z^* \in E^*$. Since h and k are continuous, it follows that h and k satisfy Fenchel duality (see Theorem 2.8.7 in [11]). This and the reflexivity of the space E gives

$$-\inf_{E^*}[h + k] = (h + k)^*(0) = \min_{z \in E}[h^*(z) + k^*(-z)].$$

A simple computation shows that $h^*(z) = f(z)$ and $k^*(-z) = g(z) - \langle z, y^* \rangle$, for all $z \in E$. Hence, since x_0 is an extreme point of C,

$$-\inf_{E^*}[h + k] = \min_{E}[f + g - y^*] = \min_{E}[\delta_{\{0\}} - y^*] = 0,$$

so, for all $\varepsilon > 0$, there exists $z^* \in E^*$ such that $h(z^*) + k(z^*) \leq \varepsilon$, that is $f^*(z^*) + g^*(y^* - z^*) \leq \varepsilon$. This means exactly that $(y^*, \varepsilon) \in \operatorname{epi}(f^*) + \operatorname{epi}(g^*)$, hence the proof of (1.2) is complete.

Remark 1.2. Regarding the proof given above, one can easily notice that relation (1.1) is fulfilled if and only if for all $y^* \in E^*$ and for all $\varepsilon > 0$ there exists $z^* \in E^*$ such that $f^*(z^*) + g^*(y^* - z^*) \leq \varepsilon$. This is equivalent to the statement that there exists $z^* \in E^*$ such that for all $x, y \in E$, $f(x) + g(y) - \langle x - y, z^* \rangle \geq \langle y, y^* \rangle - \varepsilon$. Using the Hahn-Banach-Lagrange theorem (see Theorem 1.11 in [10]), this is equivalent to the following: there exists $M \geq 0$ such that for all $x, y \in E$, $f(x) + g(y) + M\|x - y\| \geq \langle y, y^* \rangle - \varepsilon$, that is to say there exists $M \geq 0$ such that for all $u, v \in C$, $M\|u + v - 2x_0\| \geq \langle v - x_0, y^* \rangle - \varepsilon$.

Following this remark, Stephen Simons proposed the following problem (Problem 11.5 in [10]):

Problem 1.3. Let C be a nonempty, bounded, closed and convex subset of a nonreflexive Banach space E, x_0 be an extreme point of C, $y^* \in E^*$ and $\varepsilon > 0$. Then does there always exist $M \geq 0$ such that, for all $u, v \in C$, $M\|u + v - 2x_0\| \geq \langle v - x_0, y^* \rangle - \varepsilon$? If the answer to this question is positive, then $\operatorname{epi}(f^*) + \operatorname{epi}(g^*) \supset E^* \times (0, \infty)$.

2. The solution to Problem 1.3

We give in this section an answer to Problem 1. We show that in the nonreflexive case the answer depends on whether x_0 is a weak*-extreme point of C or not. We recall that x_0 is a *weak*-extreme* point of the nonempty, bounded, closed and convex set $C \subseteq E$ if $\widehat{x_0}$ is an extreme point of $\operatorname{cl} \widehat{C}$, where the closure is taken with respect to the weak* topology $\omega(E^{**}, E^*)$ (see [6]). One can show that if x_0 is a weak*-extreme point of C, then x_0 is an extreme point of C. The history of this notion goes back to the paper of Phelps (see [8]), where the author asked the following: must the image $\widehat{x}$ of an extreme point of $x \in B_E$ (the unit ball of E) be an extreme point of $B_{E^{**}}$ (the unit ball of the bidual)? We recall that by the Goldstine theorem, the closure of $\widehat{B_E}$ in the weak* topology $\omega(E^{**}, E^*)$ is $B_{E^{**}}$ (hence the generalization to a nonempty, bounded, closed and convex set is natural). Several papers from the literature deal with this notion, see [2–4, 6–8]. In the spaces $C(X)$ and $L^p (1 \leq p \leq \infty)$ all the extreme points of the corresponding unit balls are weak*-extreme points (see [7]). The first example of a Banach space of which unit ball contains elements which are not weak*-extreme was suggested by K. de Leeuw and proved by Y. Katznelson (see the note added at the end of [8]). If E is a separable Banach space containing an isomorphic copy of c_0, then E is isomorphic to a strictly convex space F such that B_F has no weak*-extreme points (see [7]). For the general case when C is a bounded, closed and convex set, we refer to [2] and [6] for more on this subject. We recall from [2] the following result: a Banach space E has the Radon-Nikodým property if and only if every bounded, closed and convex subset C of E has a weak*-extreme point. Of course, in a Radon-Nikodým space it is possible that some of the extreme points are not weak*-extreme points (see [5] for other equivalent formulations of the Radon-Nikodým property).

Theorem 2.1. *We have $E^* \times (0, \infty) \subset \operatorname{epi}(f^*) + \operatorname{epi}(g^*)$ if and only if x_0 is a weak*-extreme point of C.*

Proof. Let $y^* \in E^*$ and $\varepsilon > 0$ be arbitrary. In view of Remark 1.2, the condition $(y^*, \varepsilon) \in \operatorname{epi}(f^*) + \operatorname{epi}(g^*)$ is equivalent to the statement that there exists $z^* \in E^*$ such that for all $x, y \in E$, $f(x) + g(y) - \langle x - y, z^* \rangle \geq \langle y, y^* \rangle - \varepsilon$, which is nothing else than there exists $z^* \in E^*$ such that for all $u, v \in C$, $\langle u + v - 2x_0, z^* \rangle + \langle x_0 - v, y^* \rangle \geq -\varepsilon$. Hence the inclusion $E^* \times (0, \infty) \subset \operatorname{epi}(f^*) + \operatorname{epi}(g^*)$ is fulfilled if and only if

$$\inf_{y^* \in E^*} \sup_{z^* \in E^*} \inf_{(u,v) \in C \times C} [\langle u + v - 2x_0, z^* \rangle + \langle x_0 - v, y^* \rangle] \geq 0. \tag{2.1}$$

Take $y^* \in E^*$. For $z^* \in E^*$, we have

$$\inf_{(u,v) \in C \times C} [\langle u + v - 2x_0, z^* \rangle + \langle x_0 - v, y^* \rangle] = \inf_{(u,v) \in \widehat{C} \times \widehat{C}} [\langle z^*, u + v - 2\widehat{x_0} \rangle + \langle y^*, \widehat{x_0} - v \rangle]$$

$$= \inf_{(u,v) \in \operatorname{cl} \widehat{C} \times \operatorname{cl} \widehat{C}} [\langle z^*, u + v - 2\widehat{x_0} \rangle + \langle y^*, \widehat{x_0} - v \rangle],$$

where the first equality follows by the definition of the canonical embedding and the second one is a consequence of the continuity (in the weak* topology $\omega(E^{**}, E^*)$) of the functions $\langle x^*, \cdot\rangle : E^{**} \to \mathbb{R}$, for all $x^* \in E^*$. The set C being bounded, we use the celebrated Banach-Alaoglu theorem to conclude that the set $\operatorname{cl}\widehat{C}$ is weak*-compact. We apply a minimax theorem (see for example Theorem 3.1 in [9]) and obtain that

$$\sup_{z^*\in E^*} \inf_{(u,v)\in C\times C} [\langle u+v-2x_0, z^*\rangle + \langle x_0 - v, y^*\rangle] =$$

$$\sup_{z^*\in E^*} \inf_{(u,v)\in \operatorname{cl}\widehat{C}\times \operatorname{cl}\widehat{C}} [\langle z^*, u+v-2\widehat{x_0}\rangle + \langle y^*, \widehat{x_0} - v\rangle] =$$

$$\inf_{(u,v)\in \operatorname{cl}\widehat{C}\times \operatorname{cl}\widehat{C}} \sup_{z^*\in E^*} [\langle z^*, u+v-2\widehat{x_0}\rangle + \langle y^*, \widehat{x_0} - v\rangle] = \inf_{\substack{(u,v)\in \operatorname{cl}\widehat{C}\times \operatorname{cl}\widehat{C}\\ u+v=2\widehat{x_0}}} \langle y^*, \widehat{x_0} - v\rangle.$$

Thus

$$\inf_{y^*\in E^*} \sup_{z^*\in E^*} \inf_{(u,v)\in C\times C} [\langle u+v-2x_0, z^*\rangle + \langle x_0 - v, y^*\rangle] =$$

$$\inf_{y^*\in E^*} \inf_{\substack{(u,v)\in \operatorname{cl}\widehat{C}\times \operatorname{cl}\widehat{C}\\ u+v=2\widehat{x_0}}} \langle y^*, \widehat{x_0} - v\rangle = \inf_{\substack{(u,v)\in \operatorname{cl}\widehat{C}\times \operatorname{cl}\widehat{C}\\ u+v=2\widehat{x_0}}} \inf_{y^*\in E^*} \langle y^*, \widehat{x_0} - v\rangle =$$

$$\inf_{\substack{(u,v)\in \operatorname{cl}\widehat{C}\times \operatorname{cl}\widehat{C}\\ u+v=2\widehat{x_0}}} -\delta_{\{\widehat{x_0}\}}(v).$$

Since this has the value 0 if x_0 is a weak*-extreme point of C, and the value $-\infty$ otherwise, this completes the proof of (2.1). □

Remark 2.2. The above result gives the solution to Problem 1.3 (see Remark 1.2), namely the answer is positive if and only if x_0 is a weak*-extreme point of C. Let us mention that the closedness of the set C, requested in [10], is not needed anymore for this result.

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Faculty of Mathematics, Chemnitz University of Technology, D-09107 Chemnitz, Germany
E-mail address: radu.bot@mathematik.tu-chemnitz.de

Faculty of Mathematics, Chemnitz University of Technology, D-09107 Chemnitz, Germany
E-mail address: robert.csetnek@mathematik.tu-chemnitz.de