Existence results and gap functions for the generalized equilibrium problem with composed functions

Radu Ioan Boţ * Adela Elisabeta Capătă†

Abstract. In this paper we provide first existence results for solutions of the generalized equilibrium problem with composed functions $(GEPC)$ under generalized convexity assumptions. Then we construct by employing some tools specific to the theory of conjugate duality two gap functions for $(GEPC)$. The importance of these gap functions is to be seen in the fact that they equivalently characterize the solutions of an equilibrium problem. We also prove that for some particular instances of $(GEPC)$ the gap functions we introduce here become among others the celebrated Auslender's and Giannessi's gap functions.

Keywords: generalized equilibrium problems, C-subconvexlike functions, existence results, gap functions, regularity conditions

AMS subject classification: 49J10, 49N15, 90C25

1 Introduction

In this paper we investigate the following *generalized equilibrium problem with composed functions*

$$(GEPC) \qquad \text{find } \bar{a} \in A \text{ such that } \varphi(\bar{a}, b) + g \circ h(b) \geq g \circ h(\bar{a}) \text{ for all } b \in A,$$

where X and Z are two topological vector spaces, the latter being partially ordered by a convex closed cone C, A is a nonempty subset of X, $h : A \to Z$ and $g : Z \to \mathbb{R}$ are given functions, while $\varphi : A \times A \to \mathbb{R}$ has the property that

*Faculty of Mathematics, Chemnitz University of Technology, D-09107 Chemnitz, Germany, e-mail: radu.bot@mathematik.tu-chemnitz.de. Research partially supported by DFG (German Research Foundation), project WA 922/1–3.

†Faculty of Mathematics and Computer Science, Babeş-Bolyai University, Cluj-Napoca, Romania, e-mail: adela.capata@math.ubbcluj.ro. Research done during the stay of the author in the winter term 2008/2009 at Chemnitz University of Technology and supported by DAAD (Deutscher Akademischer Austausch Dienst), Ref. A/07/73196.

$\varphi(a, a) = 0$ for each $a \in A$. We provide, on the one hand, weak conditions that guarantee the *existence of solutions* for this equilibrium problem and construct, on the other hand, *gap functions* which constitute a valuable tool for determining the solutions of $(GEPC)$.

In section 2 we introduce first some preliminary notions in connection to conjugate functions. Then we recall some generalized convexity notions for vector functions and discuss the relations between them. The notion of *C-lower semicontinuity* for vector functions in the sense of Penot and Théra (cf. [19]) along with some properties of it are also considered. We close the section with a result on the existence of solutions for equilibrium problems which has been recently introduced in [8].

The first result we give in section 3 is a theorem that characterizes the existence of a solution for $(GEPC)$ under weak generalized convexity assumptions and weak topological assumptions for the functions φ, g and h. A stronger version of this result, where the generalized convexity assumptions are considered only for φ and h, is also provided in case g is convex and C-increasing. This is followed by another existence result given in the classical convex setting.

The same setting is maintained in section 4, where we attach for the beginning a composed convex optimization problem to the equilibrium problem $(GEPC)$. Two conjugate dual problems (cf. [7]) of the composed convex problem are provided along with corresponding *regularity conditions* that ensure the so-called *strong duality*. We use the formulation of the optimal objective value of the dual problems in order to define the two gap functions for $(GEPC)$ (see also [2, 3] for a similar approach).

In the last section of the paper we consider, both, the *generalized equilibrium problem* and the *equilibrium problem with a basic set defined via cone-inequality constraints*. These problems can be seen as particular instances of $(GEPC)$ and therefore we derive by means of the results obtained in the sections 3 and 4 existence results, respectively, we construct corresponding gap functions for them. In this way we rediscover the gap function for the classical equilibrium problem introduced in [17]. Whenever the equilibrium problem is reduced to a variational inequality the gap functions become among others the celebrated *Auslender's* (cf. [4]) and *Giannessi's gap functions* (cf. [12]).

2 Notations and preliminaries

Consider the topological vector spaces X and its topological dual space X^*. We denote by $\langle x^*, x\rangle = x^*(x)$ the value at $x \in X$ of the continuous linear functional $x^* \in X^*$. For a set $U \subseteq X$ we denote by $\operatorname{sqri}(U)$ its *strong quasi-relative interior*, which in case U is convex represents the set of the elements $u \in U$ such that $\cup_{\lambda>0} \lambda(U - u)$ is a closed linear subspace. By $\delta_U : X \to \overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$, where $\delta_U(x) = 0$ for $x \in U$ and $\delta_U(x) = +\infty$, otherwise, we denote the *indicator* *function* of U.

Having a function $f : X \to \overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$ we use the classical notations for *domain* $\operatorname{dom}(f) = \{x \in X : f(x) < +\infty\}$ and *epigraph* $\operatorname{epi}(f) = \{(x, r) \in X \times \mathbb{R} : f(x) \leq r\}$. We call f *proper* if $f(x) > -\infty$ for all $x \in X$ and $\operatorname{dom}(f) \neq \emptyset$. The *Fenchel-Moreau conjugate function* of f is $f^* : X^* \to \overline{\mathbb{R}}$, $f^*(x^*) = \sup_{x \in X}\{\langle x^*, x\rangle - f(x)\}$. Whenever $f : U \to \mathbb{R}$ is defined on U and takes (only) real values we denote by $f_U^* : X^* \to \overline{\mathbb{R}}$, $f_U^*(x^*) = (f + \delta_U)^*(x^*) = \sup_{x \in U}\{\langle x^*, x\rangle - f(x)\}$ the so-called *conjugate function of f regarding the set U*. Between a function and its conjugate there is *Young's inequality* $f^*(x^*) + f(x) \geq \langle x^*, x\rangle$ for all $x \in X$ and all $x^* \in X^*$.

Take Z another topological vector space along with its topological dual space Z^* and assume that Z is partially ordered by a nonempty convex closed cone C, i.e. on Z there is the partial order "$\leq_C$", defined by $z \leq_C y \Leftrightarrow y - z \in C$ whenever $z, y \in Z$. Obviously, by $y \geq_C z$ for $y, z \in Z$ we understand $z \leq_C y$. To Z we attach a greatest element with respect to "$\leq_C$" which does not belong to Z, denoted by ∞_C, and let $Z^\bullet = Z \cup \{\infty_C\}$. Then for any $z \in Z^\bullet$ one has $z \leq_C \infty_C$. We consider on $Z^\bullet$ the following operations: $z + \infty_C = \infty_C + z = \infty_C$ and $t \cdot \infty_C = \infty_C$ for all $z \in Z$ and all $t \geq 0$. Having a set $U \subseteq X$, in analogy to the indicator function, we consider the function $\delta_U^\bullet : X \to Z^\bullet$, defined by $\delta_U^\bullet(x) = 0$ for $x \in U$ and $\delta_U^\bullet(x) = \infty_C$, otherwise, which we call the *vector indicator function* of the set U.

A function $g : Z^\bullet \to \overline{\mathbb{R}}$ is said to be *C-increasing* if $g(\infty_C) = +\infty$ and for $y, z \in Z^\bullet$ such that $z \leq_C y$ one has $g(z) \leq g(y)$. The *dual cone* of C is $C^* = \{z^* \in Z^* : \langle z^*, z\rangle \geq 0 \text{ for all } z \in C\}$. By convention, let $\langle z^*, \infty_C\rangle = +\infty$ whenever $z^* \in C^*$.

Next we extend some of the notions from above to vector functions. Let be $h : X \to Z^\bullet$. The *domain* of the function h is defined by $\operatorname{dom}(h) = \{x \in X : h(x) \in Z\}$, while its *$C$-epigraph* by $\operatorname{epi}_C(h) = \{(x, z) \in X \times Z : z \in h(x) + C\}$. We say that h is *proper* whenever $\operatorname{dom}(h) \neq \emptyset$. For $z^* \in C^*$ we consider the function $(z^*h) : X \to \overline{\mathbb{R}}$ defined by $(z^*h)(x) = \langle z^*, h(x)\rangle$. The conventions made above yield that $\operatorname{dom}(z^*h) = \operatorname{dom}(h)$ for all $z^* \in C^*$.

In the following we consider some generalized convexity notions for vector functions. Let $A \subseteq X$ be a nonempty set and $h : A \to Z$ a given function. We say that h is *C-convex on A* (cf. [10]) if A is convex and for all $t \in [0, 1]$ and $a_1, a_2 \in A$ one has

$$h(ta_1 + (1 - t)a_2) \leq_C th(a_1) + (1 - t)h(a_2).$$

We notice that h is C-convex on A if and only if $\operatorname{epi}_C(h + \delta_A^\bullet)$ is a convex set.

For the next notion we refer to the paper of Ky Fan [11]. We say that h is *C-convexlike on A* if for all $t \in [0, 1]$ and $a_1, a_2 \in A$ there exists $a \in A$ such that

$$h(a) \leq_C th(a_1) + (1 - t)h(a_2).$$

We notice that h is C-convexlike on A if and only if $h(A) + C$ is a convex set. Obviously, if h is C-convex on A, then h is C-convexlike on A.

The notion that we introduce next slightly generalizes a notion due to Jeyakumar (cf. [13]). Assume that the cone C is *solid*, i.e. $\mathrm{int}(C) \neq \emptyset$.. We say that h is *C-subconvexlike on A* if for all $c \in \mathrm{int}(C)$, $t \in [0,1]$ and $a_1, a_2 \in A$ there exists $a \in A$ such that

$$h(a) \leq_C th(a_1) + (1-t)h(a_2) + c.$$

We refer to [5] for the proof of the fact that h is C-subconvexlike on A if and only if $h(A) + \mathrm{int}(C)$ is a convex set. Obviously, when $\mathrm{int}(C) \neq \emptyset$, then h is C-convexlike on A implies h is C-subconvexlike on A. The following characterization of a C-subconvexlike function was given in [18].

Lemma 2.1. *Let $A \subseteq X$ be a nonempty set, $h : A \to Z$ a given function and C a convex solid closed cone. The following assertions are equivalent:*

(i) h is C-subconvexlike on A;

(ii) there exists $c \in C$ such that for all $\varepsilon > 0$, $t \in [0,1]$ and $a_1, a_2 \in A$ there exists $a \in A$ such that

$$f(a) \leq_C tf(a_1) + (1-t)h(a_2) + \epsilon c;$$

(iii) there exists $c \in \mathrm{int}(C)$ such that for all $\varepsilon > 0$, $t \in [0,1]$ and $a_1, a_2 \in A$ there exists $a \in A$ such that

$$f(a) \leq_C tf(a_1) + (1-t)h(a_2) + \epsilon c;$$

The function $h : A \to Z$ is called *C-concave (C-concavelike, C-subconcavelike) on A* if $-h$ is C-convex (C-convexlike, C-subconvexlike) on A.

In the following we turn our attention to a generalization of the classical lower semicontinuity to vector functions, the so-called *C-lower semicontinuity* introduced by Penot and Théra in [19] (see also [1,9]). We refer for other generalizations of the lower semicontinuity to [14,15].

Definition 2.1. A function $h : X \to Z^\bullet$ is said to be C-lower semicontinuous at $x \in X$ if for any neighborhood $V \subseteq Z$ of zero and for any $z \in Z$, satisfying $z \leq_C h(x)$, there exists a neighborhood $U \subseteq X$ of x such that

$$h(U) \subseteq z + V + C \cup \{\infty_C\}.$$

Remark 2.1. If $h(x) \in Z$, then the above definition amounts to saying that for any neighborhood $V \subseteq Z$ of zero there exists a neighborhood $U \subseteq X$ of x such that

$$h(U) \subseteq h(x) + V + C \cup \{\infty_C\}.$$

Regarding the composition of two lower semicontinuous functions one has the following result which has been proved in [19].

Proposition 2.2. *Let $h : X \to Z^\bullet$ be a C-lower semicontinuous function at $x \in X$ and $g : Z^\bullet \to \overline{\mathbb{R}}$ a C-increasing and lower semicontinuous function at $h(x)$. Then $g \circ h$ is lower semicontinuous at x.*

In the following we recall a recent result introduced in [8] which ensure the existence of a solution of the classical equilibrium problem under weak convexity assumptions. To this end we need first some notions that extend to vector functions some concepts considered in [8] for scalar functions of two variables.

Definition 2.2. (cf. [8]) Let Z be a topological vector space partially ordered by the convex solid closed cone C. The function $\varphi : A \times B \to Z$ is said to be

(a) *C-subconcavelike on A in its first variable* if for all $c \in \text{int}(C)$, $t \in [0, 1]$ and $a_1, a_2 \in A$ there exists $a \in A$ such that

$$\varphi(a, b) \geq_C t\varphi(a_1, b) + (1 - t)\varphi(a_2, b) - c \text{ for all } b \in B.$$

(b) *C-subconvexlike on B in its second variable* if for all $c \in \text{int}(C)$, $t \in [0, 1]$ and $b_1, b_2 \in B$ there exists $b \in B$ such that

$$\varphi(a, b) \leq_C t\varphi(a, b_1) + (1 - t)\varphi(a, b_2) + c \text{ for all } a \in A.$$

(c) *C-subconcavelike – C-subconvexlike* in $A \times B$ if it is C-subconcavelike on A in its first variable and C-subconvexlike on B in its second variable.

In case $Z = \mathbb{R}$ and $C = \mathbb{R}_+$ we use the terms *subconcavelike, subconvexlike* and *subconcavelike – subconvexlike* for $\mathbb{R}_+$-subconcavelike, $\mathbb{R}_+$-subconvexlike and $\mathbb{R}_+$-subconcavelike – $\mathbb{R}_+$-subconvexlike, respectively.

We can state now the result mentioned above.

Theorem 2.3. *(cf. [8]) Let A be a nonempty compact subset of X, B a given nonempty set and $\psi : A \times B \to \mathbb{R}$ a given function fulfilling:*

(i) for each $b \in B$ the function $\psi(\cdot, b) : A \to \mathbb{R}$ is upper semicontinuous on A;

(ii) ψ is subconcavelike – subconvexlike on $A \times B$;

(iii) $\sup\limits_{a \in A} \psi(a, b) \geq 0$ for all $b \in B$.

Then the equilibrium problem

$$(EP) \qquad \text{find } \bar{a} \in A \text{ such that } \psi(\bar{a}, b) \geq 0 \text{ for all } b \in B$$

admits a solution.

3 Existence results for the generalized equilibrium problem with composed functions

The aim of this section is to give some existence results for the solutions of $(GEPC)$ under under generalized convexity assumptions for the functions involved.

Theorem 3.1. *Let A be a nonempty compact set and the following conditions fulfilled:*

(i) for each $b \in A$ the function $\varphi(\cdot, b)$ is upper semicontinuous on A;

(ii) h is C-lower semicontinuous on A;

(iii) g is lower semicontinuous;

(iv) the function $(a, b) \mapsto \varphi(a, b) - g(h(a))$ is subconcavelike on A in its first variable;

(v) the function $(a, b) \mapsto \varphi(a, b) + g(h(b))$ is subconvexlike on A in its second variable.

Then the generalized equilibrium problem with composed functions $(GEPC)$ admits a solution.

Proof. The proof follows as a direct consequence of Theorem 2.3 when one takes $\psi : A \times A \to \mathbb{R}$ defined by $\psi(a, b) = \varphi(a, b) + g(h(b)) - g(h(a))$.

By Proposition 2.2 follows that $g \circ h$ is lower semicontinuous and combining this fact with the hypothesis (i) it yields that for each $b \in B$ $\psi(\cdot, b) : A \to \mathbb{R}$ is upper semicontinuous on A. That ψ is subconcavelike – subconvexlike on $A \times A$ follows by (iv) and (v) and, since $\psi(a, a) = 0$ for all $a \in A$, the hypothesis (iii) in Theorem 2.3 is also fulfilled. Thus Theorem 2.3 guarantees the existence of a solution for $(GEPC)$. $\square$

Remark 3.1. It is an easy exercise to verify that the assumptions (iv) and (v) in Theorem 3.1 are consequences of

(iv') the function $(a, b) \mapsto (\varphi(a, b), -g(h(a)))$ is $\mathbb{R}^2_+$-subconcavelike on A in its first variable

and, respectively,

(v') the function $(a, b) \mapsto (\varphi(a, b), g(h(b)))$ is $\mathbb{R}^2_+$-subconcavelike on B in its second variable.

When $g : Z \to \mathbb{R}$ is a *convex* and *C-increasing* function one can give some sufficient conditions for the hypotheses (iv') and (v') in the remark above which involve only the vector function h. To this end we consider two generalized convexity notions that can be seen in analogy to the ones introduced in Definition 2.2.

We say that $(a, b) \mapsto (\varphi(a, b), -h(a))$ is *subconcavelike – C-concavelike* on A in its first variable whenever for $\varepsilon > 0$, $t \in [0, 1]$ and $a_1, a_2 \in A$ there exists $a \in A$ such that

$$(\varphi(a, b), -h(a)) \geq_{\mathbb{R}_+ \times C} t(\varphi(a_1, b), -h(a_1)) + (1 - t)(\varphi(a_2, b), -h(a_2)) - (\varepsilon, 0)$$

for all $b \in A$.

We say that $(a, b) \mapsto (\varphi(a, b), h(b))$ is *subconvexlike – C-convexlike* on A in its second variable whenever for $\varepsilon > 0$, $t \in [0, 1]$ and $b_1, b_2 \in A$ there exists $b \in A$ such that

$$(\varphi(a, b), h(b)) \leq_{\mathbb{R}_+ \times C} t(\varphi(a, b_1), h(b_1)) + (1 - t)(\varphi(a, b_2), h(b_2)) + (\varepsilon, 0)$$

for all $a \in A$.

Now we can state a second result on the existence of solutions for $(GEPC)$.

Theorem 3.2. *Let A be a nonempty compact set and the following conditions fulfilled:*

(i) for each $b \in A$ the function $\varphi(\cdot, b)$ is upper semicontinuous on A;

(ii) h is C-lower semicontinuous on A;

(iii) g is convex, lower semicontinuous and C-increasing;

(iv) the function $(a, b) \mapsto (\varphi(a, b), -h(a))$ is subconcavelike – C-concavelike on A in its first variable;

(v) the function $(a, b) \mapsto (\varphi(a, b), h(b))$ is subconvexlike – C-convexlike on A in its second variable.

Then the generalized equilibrium problem with composed functions $(GEPC)$ admits a solution.

Proof. The desired conclusion will follow if we prove that the assumptions (iv) and (v) in Theorem 3.1 are fulfilled.

We start with (iv) and to this aim we consider arbitrary $\varepsilon > 0$, $t \in [0, 1]$ and $a_1, a_2 \in A$. Then there exists $a \in A$ such that

$$(\varphi(a, b), -h(a)) \geq_{\mathbb{R}_+ \times C} t(\varphi(a_1, b), -h(a_1)) + (1 - t)(\varphi(a_2, b), -h(a_2)) - (\varepsilon, 0)$$

for all $b \in B$. This yields that

$$\varphi(a, b) \geq t\varphi(a_1, b) + (1 - t)\varphi(a_2, b) - \varepsilon \text{ for all } b \in B$$

and, since g is convex and C-increasing,

$$g(h(a)) \leq tg(h(a_1)) + (1-t)g(h(a_2)).$$

The relations above allow us to write for all $b \in A$

$$\varphi(a,b) - g(h(a)) \geq t(\varphi(a_1,b) - g(h(a_1))) + (1-t)(\varphi(a_2,b) - g(h(a_2))) - \varepsilon,$$

which actually means that $(a,b) \mapsto \varphi(a,b) - g(h(a))$ is subconcavelike on A in its first variable.

Analogously one can prove that hypothesis (v) imply that $(a,b) \mapsto \varphi(a,b) + g(h(b))$ is subconvexlike on A in its second variable. In this way the existence of a solution for $(GEPC)$ follows from Theorem 3.1. □

The next corollary shows that when considering classical assumptions for the sets and functions involved in the formulation of the equilibrium problem $(GEPC)$, then this has a solution. Recall that the function $\varphi : A \times A \to \mathbb{R}$ is said to be *concave-convex* if $\varphi(\cdot, b)$ is concave for all $b \in A$ and $\varphi(a, \cdot)$ is convex for all $a \in A$.

Corollary 3.3. *Let A be a nonempty convex and compact set and the following conditions fulfilled:*

(i) for each $b \in A$ the function $\varphi(\cdot, b)$ is upper semicontinuous on A;

(ii) h is C-convex and C-lower semicontinuous on A;

(iii) g is convex, lower semicontinuous and C-increasing;

(iv) the function φ is concave-convex on $A \times A$.

Then the generalized equilibrium problem with composed functions $(GEPC)$ admits a solution.

Proof. Consider some arbitrary $\varepsilon > 0$, $t \in [0,1]$ and $a_1, a_2 \in A$. Then for $a := ta_1 + (1-t)a_2 \in A$ it holds

$$(\varphi(a,b), -h(a)) \geq_{\mathbb{R}_+ \times C} t(\varphi(a_1,b), -h(a_1)) + (1-t)(\varphi(a_2,b), -h(a_2)) - (\varepsilon, 0)$$

for all $b \in A$. This means that $(a,b) \mapsto (\varphi(a,b), -h(a))$ is subconcavelike – C-concavelike on A in its first variable. Analogously one can prove that the function $(a,b) \mapsto (\varphi(a,b), h(b))$ is subconvexlike – C-convexlike on A in its second variable. The conclusion follows now via Theorem 3.2. □

4 Two gap functions for $(GEPC)$

In this section we provide two gap functions for $(GEPC)$ for the construction of which we employ the conjugate duality for composed convex optimization problems. For the notion of gap function for an equilibrium problem we refer to [17], where this notion has been introduce as an extension of the similar notion for *variational inequalities* (cf. [4]). A function $\gamma : X \to \overline{\mathbb{R}}$ is said to be *gap function* for the generalized equilibrium problem with composed functions $(GEPC)$ if

(i) $\gamma(a) \geq 0$ for all $a \in A$;

(ii) $\gamma(\bar{a}) = 0$ for $\bar{a} \in A$ if and only if $\bar{a}$ is a solution of $(GEPC)$.

Throughout the section we assume that X and Z are separated locally convex spaces, $A \subseteq X$ is a nonempty convex and closed set, $\varphi : A \times A \to \mathbb{R}$ is convex in its second variable, i.e. $\varphi(a, \cdot)$ is convex for all $a \in A$, $h : A \to Z$ is C-convex and C-lower semicontinuous on A and $g : Z \to \mathbb{R}$ is convex and C-increasing. In Corollary 3.3 we proved that $(GEPC)$ admits a solution if, additionally, A is compact and φ is concave and upper semicontinuous in its first variable.

For a fixed $a \in A$ let us consider the composed optimization problem

$$(P(a)) \qquad \inf_{b \in A} \{\varphi(a, b) + g(h(b))\}.$$

When one succeeds in attaching to $(P(a))$ a dual problem, let this be called $(D(a))$, and in guaranteeing the existence of strong duality for this primal-dual pair, then a gap function $\gamma : X \to \overline{\mathbb{R}}$ for $(GEPC)$ can be defined as being $\gamma(a) := -v(D(a)) + g(h(a))$, where by $v(D(a))$ we denote the optimal objective value of the optimization problem $(D(a))$. Let us extend the functions involved in the formulation of $(GEPC)$ to the whole space X in the following way. We define $\widetilde{\varphi}_a : X \to \overline{\mathbb{R}}$, $\widetilde{\varphi}_a = \varphi(a, \cdot) + \delta_A$, $\widetilde{h} : X \to Z^\bullet$, $\widetilde{h} = h + \delta_A^\bullet$ and assume, by convention, that $g(\infty_C) = +\infty$. Thus $\widetilde{\varphi}_a$ is a convex function with $\mathrm{dom}(\widetilde{\varphi}_a) = A$ and $\widetilde{h}$ is a C-convex and C-lower semicontinuous (as h is C-lower semicontinuous on A and A is closed) vector function with $\mathrm{dom}(\widetilde{h}) = A$. Now one can write $(P(a))$, equivalently, as an optimization problem over the whole space X

$$(P(a)) \qquad \inf_{b \in X} \{\widetilde{\varphi}_a(b) + g(\widetilde{h}(b))\}.$$

For the first conjugate dual problem to $(P(a))$ introduced here

$$(D_1(a)) \qquad \sup_{z^* \in C^*} \{-g^*(z^*) - (\widetilde{\varphi}_a + (z^*\widetilde{h}))^*(0)\}$$

we refer to [9] (see also [7, 20]). For this primal-dual pair one always has *weak duality*, i.e. $v(D_1(a)) \leq v(P(a))$, and, under the fulfilment of a so-called *regularity condition*, *strong duality*, which is the situation when $v(P(a)) = v(D_1(a))$ and

$(D_1(a))$ has an optimal solution. In the literature one can find, both, *interior point* and *closedness-type* regularity conditions for composed convex optimization problems and for the relations between these two classes of conditions we refer to [7]. In this paper we deal with two regularity conditions of the first type, which are stated in the following:

$$(RC_1^1(a)) \quad \left| \begin{array}{l} \text{there exists } b \in \operatorname{dom}(\widetilde{\varphi}_a) \cap \operatorname{dom}(\widetilde{h}) \cap \widetilde{h}^{-1}(\operatorname{dom}(g)) \text{ such that} \\ g \text{ is continuous at } \widetilde{h}(b) \end{array} \right.$$

and, respectively,

$$(RC_1^2(a)) \quad \left| \begin{array}{l} X \text{ and } Z \text{ are Fréchet spaces, } \widetilde{\varphi}_a \text{ and } g \text{ are lower semicontinuous,} \\ \text{and } 0 \in \operatorname{sqri}\Big(\operatorname{dom}(g) - h(\operatorname{dom}(\widetilde{\varphi}_a) \cap \operatorname{dom}(\widetilde{h}))\Big). \end{array} \right.$$

Let $\gamma_1 : X \to \overline{\mathbb{R}}$ be defined by

$$\begin{aligned} \gamma_1(a) := -v(D_1(a)) + g(h(a)) &= \inf_{z^* \in C^*} \{g^*(z^*) + (\varphi(a,\cdot) + (z^*h))^*_A(0)\} + g(h(a)) \\ &= \inf_{z^* \in C^*} \sup_{b \in A} \{g^*(z^*) + g(h(a)) - \varphi(a,b) - \langle z^*, h(b)\rangle\}. \end{aligned}$$

Theorem 4.1. *If the regularity condition*

$$(RC_{\gamma_1}^1) \quad \left| \ \textit{there exists } b \in A \textit{ such that } g \textit{ is continuous at } h(b) \right.$$

or the regularity condition

$$(RC_{\gamma_1}^2) \quad \left| \begin{array}{l} X \textit{ and } Z \textit{ are Fréchet spaces and } \varphi(a,\cdot) \textit{ and } g \textit{ are} \\ \textit{lower semicontinuous for all } a \in A \end{array} \right.$$

is fulfilled, then γ_1 is a gap function for $(GEPC)$.

Proof. We show that γ_1 is a gap function for $(GEPC)$ by verifying the conditions $(i) - (ii)$ from the definition.

Consider first an arbitrary $a \in A$. Then, by weak duality,

$$\begin{aligned} \gamma_1(a) = -v(D_1(a)) + g(h(a)) \geq -v(P(a)) + g(h(a)) = \\ \sup_{b \in A}\{g(h(a)) - g(h(b)) - \varphi(a,b)\} \geq 0. \end{aligned}$$

Here we used the fact that $\varphi(a,a) = 0$ for all $a \in A$. This proves statement (i) in the definition of the gap function.

Consider now $\bar{a} \in X$ such that $\bar{a}$ is a solution for $(GEPC)$. This is equivalent to $\bar{a} \in A$ and $g(h(\bar{a})) = \inf_{b \in A}\{\varphi(\bar{a},b) + g(h(b))\} = v(P(\bar{a}))$. On the other hand one can notice that $(RC_{\gamma_1}^1)$ is fulfilled if and only if $(RC_1^1(a))$ is fulfilled for all $a \in A$. Coming now to $(RC_{\gamma_1}^2)$, it is easy to see that this condition guarantees the fulfilment of $(RC_1^2(a))$ for all $a \in A$. With respect to this statement, let us notice that as $\operatorname{dom}(g) = X$, the condition $0 \in \operatorname{sqri}\Big(\operatorname{dom}(g) - h(\operatorname{dom}(\widetilde{\varphi}_a) \cap \operatorname{dom}(\widetilde{h}))\Big)$ is automatically fulfilled. Therefore, by [7, Theorem 4.1], one has that $v(P(a)) = v(D(a))$ for all $a \in A$. Consequently, $g(h(\bar{a})) = v(D(\bar{a}))$ or, equivalently, $\gamma_1(\bar{a}) = 0$. $\square$

The second gap function that we introduce in this section for $(GEPC)$ can be seen as a refinement of γ_1 and its construction is based on the following conjugate dual problem one can also attach to $(P(a))$ (cf. [7])

$$(D_2(a)) \quad \sup_{b^*\in X^*, z^*\in C^*} \left\{-g^*(z^*) - (\widetilde{\varphi}_a)^*(b^*) - (z^*\widetilde{h})^*(-b^*)\right\}.$$

For the two interior point regularity conditions which ensure strong duality for the primal-dual pair $(P(a)) - (D_2(a))$

$$(RC_2^1(a)) \quad \left| \begin{array}{l} \text{there exists } b \in \operatorname{dom}(\widetilde{\varphi}_a) \cap \operatorname{dom}(\widetilde{h}) \cap \widetilde{h}^{-1}(\operatorname{dom}(g)) \text{ such that } \widetilde{\varphi}_a \\ \text{is continuous at } b \text{ and } g \text{ is continuous at } \widetilde{h}(b) \end{array} \right.$$

and, respectively,

$$(RC_2^2(a)) \quad \left| \begin{array}{l} X \text{ and } Z \text{ are Fréchet spaces, } \widetilde{\varphi}_a \text{ and } g \text{ are lower semicontinuous,} \\ \text{and } 0 \in \operatorname{sqri}\left(\operatorname{dom}(\widetilde{\varphi}_a) \times \operatorname{dom}(g) - \operatorname{epi}_C(\widetilde{h})\right) \end{array} \right.$$

we refer again to [7].

Let $\gamma_2 : X \to \overline{\mathbb{R}}$ be defined by

$$\gamma_2(a) := -v(D_2(a)) + g(h(a)) =$$
$$\inf_{b^*\in X^*, z^*\in C^*} \{g(h(a)) + g^*(z^*) + (\varphi(a,\cdot))_A^*(b^*) + (z^*h)_A^*(-b^*)\}.$$

Theorem 4.2. *If the regularity condition*

$$(RC_{\gamma_2}^1) \quad \left| \begin{array}{l} \textit{for all } a \in A \textit{ there exists } b \in A \textit{ such that } \varphi(a,\cdot) \textit{ is continuous} \\ \textit{at } b \textit{ and } g \textit{ is continuous at } h(b) \end{array} \right.$$

or the regularity condition

$$(RC_{\gamma_2}^2) \quad \left| \begin{array}{l} X \textit{ and } Z \textit{ are Fréchet spaces, } \varphi(a,\cdot) \textit{ and } g \textit{ are} \\ \textit{lower semicontinuous for all } a \in A \textit{ and } 0 \in \operatorname{sqri}(A - A) \end{array} \right.$$

is fulfilled, then γ_2 is a gap function for $(GEPC)$.

Proof. The proof follows in the lines of the proof of Theorem 4.1. With respect to this one should notice that in case $(RC_{\gamma_2}^1)$ is fulfilled, then $(RC_2^1(a))$ is also fulfilled for all $a \in A$. On the other hand, condition $(RC_{\gamma_2}^2)$ guarantees the fulfilment of $(RC_2^2(a))$ for all $a \in A$. This fact is obvious when one notices that condition $0 \in \operatorname{sqri}\left(\operatorname{dom}(\widetilde{\varphi}_a) \times \operatorname{dom}(g) - \operatorname{epi}_C(\widetilde{h})\right)$ is nothing else than $0 \in \operatorname{sqri}\left(A \times X - \operatorname{epi}_C(h + \delta_A^\bullet)\right)$ or, equivalently, $0 \in \operatorname{sqri}((A-A) \times Z)$. As $\operatorname{sqri}((A-A) \times Z) = \operatorname{sqri}(A-A) \times Z$, this is further equivalent to $0 \in \operatorname{sqri}(A-A)$. $\square$

Remark 4.1. By employing some conjugate calculus it is easy to see that for all $a \in X$ it holds $\gamma_2(a) \geq \gamma_1(a)$. On the other hand, as follows from the investigations made in [7, section 4], for guaranteeing that γ_1 and γ_2 are gap functions one can replace the C-lower semicontinuity for h with the weaker assumption that h is *star C-lower semicontinuous*, namely that the function $(z^*h) : X \to \overline{\mathbb{R}}$ is lower semicontinuous for all $z^* \in C^*$.

5 Particular cases

In this section we discuss some particular instances of the generalized equilibrium problem with composed functions $(GEPC)$. We provide existence results and show that the gap functions introduced in section 4 collapse in these special situations among others with some celebrated gap functions from the literature.

5.1 The classical generalized equilibrium problem

Consider X a topological vector space, $A \subseteq X$ a nonempty set, $h : A \to \mathbb{R}$ and $\varphi : A \times A \to \mathbb{R}$ given functions such that $\varphi(a, a) = 0$ for all $a \in A$. The so-called *generalized equilibrium problem*

$$(GEP) \qquad \text{find } \bar{a} \in A \text{ such that } \varphi(\bar{a}, b) + h(b) \geq h(\bar{a}) \text{ for all } b \in A$$

can be seen as a particular instance of $(GEPC)$ whenever one takes $Z = \mathbb{R}$, $C = \mathbb{R}_+$ and $g : \mathbb{R} \to \mathbb{R}$ defined by $g(z) = z$ for all $z \in \mathbb{R}$, which is a convex, continuous and C-increasing function. For recent investigations on (GEP), where the authors propose a so-called *dual equilibrium problem* to it, we refer to [6].

By means of the existence theorems given for $(GEPC)$ in section 3, one can easily provide for (GEP) existence results under generalized convexity assumptions. We do this only in the case when the sets and functions involved fulfill classical convexity assumptions, since this is the setting that we consider below when constructing gap functions for (GEP). The following result is an easy consequence of Corollary 3.3

Theorem 5.1. *Let A be a nonempty convex and compact set and the following conditions fulfilled:*

(i) for each $b \in A$ the function $\varphi(\cdot, b)$ is upper semicontinuous on A;

(ii) h is convex and lower semicontinuous on A;

(iii) the function φ is concave-convex on $A \times A$.

Then the generalized equilibrium problem (GEP) admits a solution.

The gap functions that we propose for (GEP) are particularizations of γ_1 and γ_2. To this aim let us assume that X is a separated locally convex spaces, A is convex and closed, $\varphi : A \times A \to \mathbb{R}$ is convex in its second variable and $h : A \to \mathbb{R}$ is convex and lower semicontinuous on A. Since $g^* = \delta_{\{1\}}$, γ_1 and γ_2 turn out to be

$$\gamma_1^{GEP}(a) = \sup_{b \in A} \{h(a) - h(b) - \varphi(a, b)\}$$

and

$$\gamma_2^{GEP}(a) = \inf_{b^* \in X^*} \{h(a) + (\varphi(a, \cdot))^*_A(b^*) + h^*_A(-b^*)\},$$

respectively.

As the regularity condition $(RC^1_{\gamma_1})$ is automatically fulfilled, it follows that γ_1^{GEP} is always a gap function for (GEP), while γ_2^{GEP} has this property if the regularity condition

$$(RC^1_{\gamma_2^{GEP}}) \quad \left| \begin{array}{l} \text{for all } a \in A \text{ there exists } b \in A \text{ such that} \\ \varphi(a, \cdot) \text{ is continuous at } b \end{array} \right.$$

or the regularity condition

$$(RC^2_{\gamma_2^{GEP}}) \quad \left| \begin{array}{l} X \text{ is a Fréchet space, } \varphi(a, \cdot) \text{ is lower semicontinuous} \\ \text{for all } a \in A \text{ and } 0 \in \operatorname{sqri}(A - A) \end{array} \right.$$

is fulfilled.

Assuming that $h(a) = 0$ for all $a \in A$, (GEP) becomes the classical *equilibrium problem*

$$(EP) \qquad \text{find } \bar{a} \in A \text{ such that } \varphi(\bar{a}, b) \geq 0 \text{ for all } b \in A.$$

A dual formulation for (EP) via conjugate calculus has been given in [16]. For (EP) the two gap functions γ_1^{GEP} and γ_2^{GEP} become equal and collapse into $\gamma^{EP} : X \to \overline{\mathbb{R}}$

$$\gamma^{EP}(a) = \gamma_1^{GEP}(a) = \gamma_2^{GEP}(a) = \sup_{b \in A}\{-\varphi(a, b)\} \text{ for all } a \in X,$$

which always fulfills the properties (i) and (ii) in the definition of the gap function. In this way we rediscover the gap function for the equilibrium problem (EP) introduced in [17]. Let us notice that in case $\varphi(a, b) := \langle F(a), b - a\rangle$, where $F : A \to X^*$, the equilibrium problem (EP) becomes the celebrated Stampacchia's *variational inequality* and γ^{EP} is nothing else than the corresponding so-called *Auslender's gap function* studied in [4].

5.2 The equilibrium problem with a basic set defined via cone-inequality constraints

Consider again X and Z two topological vector spaces, the latter being partially ordered by the convex closed cone C, $A \subseteq X$ a nonempty set and $h : A \to Z$ such that $B := \{a \in A : h(a) \in -C\}$ is nonempty and $\varphi : A \times A \to \mathbb{R}$ is a given functions with the property that $\varphi(a, a) = 0$ for all $a \in A$. The equilibrium problem that we investigate in this subsection

$$(EPIC) \qquad \text{find } \bar{a} \in B \text{ such that } \varphi(\bar{a}, b) \geq 0 \text{ for all } b \in B.$$

This class of equilibrium problems has been considered in finite dimensional spaces, for instance, in [12]. Choosing $g : X \to \overline{\mathbb{R}}$, $g := \delta_{-C}$, one can equivalently formulate $(EPIC)$ as an equilibrium problem of type $(GEPC)$, namely

$$(EPIC) \qquad \text{find } \bar{a} \in A \text{ such that } \varphi(\bar{a}, b) + g(h(b)) \geq g(h(\bar{a})) \text{ for all } b \in A.$$

In order to state existence results and gap functions for $(EPIC)$ one can use the corresponding results for the generalized equilibrium problem with composed functions obtained in the sections 3 and 4. Different to the investigations made there, we notice that $\mathrm{dom}(g) = -C$ and this means that the domain of g is now not necessary equal to the whole space X. This fact has no influence on the existence results, but on the regularity conditions introduced in connection to the gap functions.

Noticing that g is a convex, lower semicontinuous and C-increasing function, we get via Corollary 3.3 the following existence result for $(EPIC)$.

Theorem 5.2. *Let A be a nonempty convex and compact set and the following conditions fulfilled:*

(i) for each $b \in A$ the function $\varphi(\cdot, b)$ is upper semicontinuous on A;

(ii) h is C-convex and C-lower semicontinuous on A;

(iii) the function φ is concave-convex on $A \times A$.

Then the equilibrium problem $(EPIC)$ admits a solution.

Further we introduce for $(EPIC)$ two gap functions which are nothing else than particularizations of γ_1 and, respectively, γ_2. To this end we assume that X is a separated locally convex space, A is convex and closed, $\varphi : A \times A \to \mathbb{R}$ is convex in its second variable and $h : A \to Z$ is C-convex and C-lower semicontinuous on A. Since $g^* = \delta^*_{-C} = \delta_{C^*}$, γ_1 and γ_2 turn out to be

$$\gamma_1^{EPIC}(a) = \inf_{z^* \in C^*} \sup_{b \in A} \{g(h(a)) - \varphi(a, b) - \langle z^*, h(b) \rangle\}$$

$$= \begin{cases} \inf\limits_{z^* \in C^*} \sup\limits_{b \in A} \{-\varphi(a, b) - \langle z^*, h(b) \rangle\}, & \text{if } a \in A, h(a) \in -C, \\ +\infty, & \text{otherwise} \end{cases}$$

and

$$\gamma_2^{EPIC}(a) = \inf_{z^* \in C^*, b^* \in X^*} \{g(h(a)) + (\varphi(a, \cdot))^*_A(b^*) + (z^*h)^*_A(-b^*)\}$$

$$= \begin{cases} \inf\limits_{z^* \in C^*, b^* \in X^*} \{(\varphi(a, \cdot))^*_A(b^*) + (z^*h)^*_A(-b^*)\}, & \text{if } a \in A, h(a) \in -C, \\ +\infty, & \text{otherwise,} \end{cases}$$

respectively. Via the general regularity conditions introduced in section 4 one can easily deduce that γ_1^{EPIC} if a gap function for $(EPIC)$ if the regularity condition

$$(RC^1_{\gamma_1^{EPIC}}) \quad \Big| \quad 0 \in h(A) + \mathrm{int}(C)$$

or the regularity condition

$$(RC^2_{\gamma_1^{EPIC}}) \quad \Bigg| \quad \begin{array}{l} X \text{ and } Z \text{ are Fréchet spaces, } \varphi(a, \cdot) \text{ is lower semicontinuous} \\ \text{for all } a \in A \text{ and } 0 \in \mathrm{sqri}(h(A) + C) \end{array}$$

is fulfilled. On the other hand, γ_2^{EPIC} if a gap function for $(EPIC)$ if the regularity condition

$$(RC^1_{\gamma_2^{EPIC}}) \left| \begin{array}{l} \text{for all } a \in A \text{ there exists } b \in A \text{ such that} \\ \varphi(a, \cdot) \text{ is continuous at } b \text{ and } h(b) \in -\operatorname{int}(C) \end{array} \right.$$

or the regularity condition

$$(RC^2_{\gamma_2^{EPIC}}) \left| \begin{array}{l} X \text{ and } Z \text{ are Fréchet spaces, } \varphi(a, \cdot) \text{ is lower semicontinuous} \\ \text{for all } a \in A \text{ and } 0 \in \operatorname{sqri}\left(A \times (-C) - \operatorname{epi}_C(h + \delta_A^\bullet)\right). \end{array} \right.$$

We close the paper by noticing that in case $X = \mathbb{R}^n$, $Z = \mathbb{R}^m$, $C = \mathbb{R}^m_+$, $A \subseteq \mathbb{R}^n$, $h : A \to \mathbb{R}^m$ and $\varphi(a, b) := \langle F(a), b - a \rangle$, where $F : A \to \mathbb{R}^n$, the gap function γ_1^{EPIC} is nothing else than the so-called *Giannesi's gap function* introduced and investigated in [12].

References

[1] Aït Mansour M., Metrane A., Théra M., *Lower semicontinuous regularization for vector-valued mappings*, Journal of Global Optimization 35(2), 2006, 283-309.

[2] Altangerel L., Boţ R.I., Wanka G., *On gap functions for equilibrium problems via Fenchel duality*, Pacific Journal of Optimization 2(3), 2006, 667-678

[3] Altangerel L., Boţ R.I., Wanka G., *On the construction of gap functions for variational inequalities via conjugate duality*, Asia-Pacific Journal of Operational Research 24(3), 2007, 353-371

[4] Auslender A., *Optimisation: Méthodes Numérique*, Masson, Paris, 1976.

[5] Borwein J.M., Jeyakumar V., *On convexlike Lagrangian and minimax theorems*, Research Report 24, University of Waterloo, 1988.

[6] Bigi G., Castellani M., Kassay G., *A dual view of equilibrium problems*, Journal of Mathematical Analysis and Applications 342(1), 2008, 17-26.

[7] Boţ, R.I., *Conjugate duality in convex optimization*, Chemnitz University of Technology, Faculty of Mathematics, Habilitation Thesis, 2008.

[8] Capătă A., Kassay G., *On vector equilibrium problems and applications*, submitted to Taiwanese Journal of Mathematics, 2008.

[9] Combari C., Laghdir M., Thibault L., *Sous-différentiels de fonctions convexes composées*, Annales des Sciences Mathématiques du Québec 18(2), 1994, 119-148.

[10] Craven B.D., *Mathematical programming and control theory*, Chapman and Hall, London, 1978.

[11] Fan K., *Minimax theorems*, Proceedings of the National Academy of Sciences of the USA 39, 1953, 42-47.

[12] Giannessi F., *On some connections among variational inequalities, combinatorial and continuous optimization*, Annals of Operations Research 58, 1995, 181-200.

[13] Jeyakumar V., *Convexlike alternative theorems and mathemathical programming*, Optimization 16, 1985, 643-652.

[14] Jeyakumar V., Song W., Dinh N., Lee G.M., *Stable strong duality in convex optimization*, Applied Mathematics Report AMR 05/22, University of New South Wales, 2005.

[15] Luc D.T., *Theory of vector optimization*, Springer-Verlag, Berlin, 1989.

[16] Martinez-Legaz J.E., Sosa W., *Duality for equilibrium problems*, Journal of Global Optimization 36(2), 2006, 311-319.

[17] Mastroeni, G., *Gap functions for equilibrium problems*, Journal of Global Optimization 27(4), 2003, 411-426.

[18] Paeck S., *Convexlike and concavelike conditions in alternative, minimax and minimization theorems*, Journal of Optimization Theory and Applications 74(2), 1992, 317-332.

[19] Penot J.P., Théra M., *Semi-continuous mapping in general topology*, Archiv der Mathematik 38(1), 1982, 158-166.

[20] Zălinescu C., *Convex analysis in general vector spaces*, World Scientific, River Edge, 2002.