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Wolfe duality and Mond–Weir duality via perturbations☆

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ABSTRACT

Considering a general optimization problem, we attach to it by means of perturbation theory two dual problems having in the constraints a subdifferential inclusion relation. When the primal problem and the perturbation function are particularized different new dual problems are obtained. In the special case of a constrained optimization problem, the classical Wolfe and Mond–Weir duals, respectively, follow as particularizations of the general duals by using the Lagrange perturbation. Examples to show the differences between the new duals are given and a gate towards other generalized convexities is opened.

1. Introduction and preliminaries

The rich literature on Wolfe and Mond–Weir duality concepts has developed in the last decades especially in the differentiable case. The main direction followed in this research was the one of relaxing the convexity assumptions on the functions involved, the connections of these duality concepts to other duality types based on convex functions remaining neglected. However, as seen in different papers such as Schechter's [1], one can extend the classical Wolfe and Mond–Weir duality concepts to nondifferentiable functions, by replacing the gradients with (convex) subdifferentials. On the other hand, in most of the papers dealing with these duality concepts only finite dimensional spaces are considered.

Motivated by the huge amount of works where the Wolfe and Mond–Weir dualities are considered in various circumstances, we propose in this paper a more general approach by embedding the classical Wolfe and Mond–Weir duality concepts into a larger class of dual problems obtained via perturbation theory. Moreover, the functions involved in our investigations are defined on separated locally convex vector spaces and four types of regularity conditions that ensure strong duality are provided. In this way we hope to provide new exploration paths for the researchers interested in dealing with the Wolfe and Mond–Weir duality concepts.

Afterwards, the primal problem is specialized to be unconstrained, respectively constrained, and different perturbation functions are considered. When we take a minimization problem with both geometric and cone inequality constraints as the primal problem, for an appropriate perturbation function the duals we introduce lead to the classical Wolfe and Mond–Weir duals, respectively. Note that this is exactly the perturbation function used for obtaining the classical Lagrange dual in conjugate duality. We also investigate the inequality relations that exist between the optimal objective values of the duals we introduce in this paper to a constrained optimization problem. Nontrivial examples showing that these inequalities can be sometimes strict and that some duals are not comparable are also given.

Nevertheless, we show that strong duality for the primal–dual pairs of optimization problems we consider can be easily obtained in case the convexity is weakened to almost convexity. In this way we bring other generalized convexity notions to the attention of the community interested in working with the Wolfe and Mond–Weir duality concepts.

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Consider two separated locally convex vector spaces X and Y and their topological dual spaces X^* and Y^*, respectively, endowed with the corresponding weak* topologies. Denote by $\langle x^*, x\rangle = x^*(x)$ the value at $x \in X$ of the linear continuous functional $x^* \in X^*$.

A *cone* $K \subseteq X$ is a nonempty subset of X which fulfills $\lambda K \subseteq K$ for all $\lambda \geq 0$. On Y we consider the partial ordering "$\leq_C$" induced by the convex cone $C \subseteq Y$, defined by $z \leq_C y \Leftrightarrow y - z \in C$ when $z, y \in Y$. To Y we attach a greatest element with respect to "$\leq_C$" which does not belong to Y denoted by ∞_C and let be $Y^\bullet = Y \cup \{\infty_C\}$. Then for any $y \in Y^\bullet$ one has $y \leq_C \infty_C$ and we consider on $Y^\bullet$ the operations $y + \infty_C = \infty_C + y = \infty_C$ for all $y \in Y$ and $t \cdot \infty_C = \infty_C$ for all $t \geq 0$. The dual cone of C is $C^* = \{y^* \in Y^* : \langle y^*, y\rangle \geq 0\ \forall y \in C\}$. By convention $\langle \lambda, \infty_C\rangle = +\infty$ for all $\lambda \in C^*$.

Given a subset U of X, by $\operatorname{cl}(U)$, $\operatorname{lin}(U)$, $\operatorname{aff}(U)$, $\operatorname{cone}(U)$, $\dim(U)$, δ_U and σ_U we denote its *closure*, *linear hull*, *affine hull*, *conical hull*, *dimension*, *indicator function* and *support function*, respectively. Moreover $\operatorname{sqri}(U) = \{x \in U : \operatorname{cone}(U - x)$ is a closed linear subspace$\}$ is the *strong quasi relative interior* of U. If X is finite dimensional, then by $\operatorname{ri}(U)$ we denote the *relative interior* of $U \subseteq X$. We use also the *projection function* $\operatorname{Pr}_X : X \times Y \to X$, defined by $\operatorname{Pr}_X(x, y) = x$ for $(x, y) \in X \times Y$ and the *identity function* $\operatorname{id} : X \to X$, $\operatorname{id}(x) = x$ for $x \in X$.

Having a function $f : X \to \overline{\mathbb{R}}$ we use the classical notations for *domain* $\operatorname{dom} f = \{x \in X : f(x) < +\infty\}$, *epigraph* $\operatorname{epi} f = \{(x, r) \in X \times \mathbb{R} : f(x) \leq r\}$, *lower semicontinuous hull* $\bar{f} : X \to \overline{\mathbb{R}}$ and *conjugate function* $f^* : X^* \to \overline{\mathbb{R}}$, $f^*(x^*) = \sup\{\langle x^*, x\rangle - f(x) : x \in X\}$.

We call f *proper* if $f(x) > -\infty$ for all $x \in X$ and $\operatorname{dom} f \neq \emptyset$. For f proper, if $f(x) \in \mathbb{R}$ the *(convex) subdifferential* of f at x is $\partial f(x) = \{x^* \in X^* : f(y) - f(x) \geq \langle x^*, y - x\rangle\ \forall y \in X\}$, while if $f(x) = +\infty$ we take by convention $\partial f(x) = \emptyset$. Note that for $U \subseteq X$ we have for all $x \in U$ that $\partial \delta_U(x) = N_U(x)$, the latter being the *normal cone* of U at x. Between a function and its conjugate there is the *Young–Fenchel inequality* $f^*(x^*) + f(x) \geq \langle x^*, x\rangle$ for all $x \in X$ and $x^* \in X^*$. This inequality is fulfilled as equality if and only if $x^* \in \partial f(x)$.

Considering for each $\lambda \in \mathbb{R}$ the function $\lambda f : X \to \overline{\mathbb{R}}$, $(\lambda f)(x) = \lambda f(x)$ for $x \in X$, note that when $\lambda = 0$ we take $0f = \delta_{\operatorname{dom} f}$. Given a linear continuous mapping $A : X \to Y$, we have its *adjoint* $A^* : Y^* \to X^*$ given by $\langle A^*y^*, x\rangle = \langle y^*, Ax\rangle$ for any $(x, y^*) \in X \times Y^*$.
For a vector function $h : X \to Y^\bullet$ one has

- h is *proper* if its *domain* $\operatorname{dom} h = \{x \in X : h(x) \in Y\}$ is nonempty,
- h is C-*convex* if $h(tx + (1 - t)y) \leq_C th(x) + (1 - t)h(y)\ \forall x, y \in X\ \forall t \in [0, 1]$,
- h is C-epi-*closed* if its C-*epigraph* $\operatorname{epi}_C h = \{(x, y) \in X \times Y : y \in h(x) + C\}$ is closed,
- h is *star* C-*lower semicontinuous* if the function $(\lambda h) : X \to \overline{\mathbb{R}}$ defined by $(\lambda h)(x) = \langle \lambda, h(x)\rangle$ is lower semicontinuous whenever $\lambda \in C^*$.

For an attained infimum (supremum) instead of inf (sup) we write min (max), while the optimal objective value of the optimization problem (P) is denoted by $v(P)$.

Now let us recall the basics about the way perturbation theory is applied to conjugate duality. Consider the proper function $F : X \to \overline{\mathbb{R}}$ and the general optimization problem

$$(PG) \quad \inf_{x \in X} F(x),$$

Making use of a proper perturbation function $\Phi : X \times Y \to \overline{\mathbb{R}}$, where Y is the so-called *perturbation space*, fulfilling $\Phi(x, 0) = F(x)$ for all $x \in X$, the problem (PG) is nothing but

$$(PG) \quad \inf_{x \in X} \Phi(x, 0).$$

To it one attaches the *conjugate dual*

$$(DG) \quad \sup_{y^* \in Y^*} \{-\Phi^*(0, y^*)\},$$

and for these problems *weak duality* always holds, i.e. $v(DG) \leq v(PG)$.

Note that the way Φ is defined guarantees $0 \in \operatorname{Pr}_Y(\operatorname{dom} \Phi)$. To have strong duality, different regularity conditions were proposed. We list in the following the most important of them (cf. [2,3]), namely the one involving continuity

$$(RC_1^\Phi) \quad \exists x' \in X \quad \text{such that } (x', 0) \in \operatorname{dom} \Phi \text{ and } \Phi(x', \cdot) \text{ is continuous at } 0,$$

a weak interiority type one

$$(RC_2^\Phi) \quad X \text{ and } Y \text{ are Fréchet spaces, } \Phi \text{ is lower semicontinuous and } 0 \in \operatorname{sqri}(\operatorname{Pr}_Y(\operatorname{dom} \Phi)),$$

another interiority type one which works in finite dimensional spaces

$$(RC_3^\Phi) \quad \dim(\operatorname{lin}(\operatorname{Pr}_Y(\operatorname{dom} \Phi))) < +\infty \quad \text{and} \quad 0 \in \operatorname{ri}(\operatorname{Pr}_Y(\operatorname{dom} \Phi)),$$

and finally a closedness type one

$$(RC_4^\Phi) \quad \Phi \text{ is lower semicontinuous and } \operatorname{Pr}_{X^* \times \mathbb{R}}(\operatorname{epi} \Phi^*) \text{ is closed in the topology } w(X^*, X) \times \mathbb{R}.$$

We give now the *strong duality* statement, for the primal–dual pair of problems (PG)–(DG) (cf. [2,3]).

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Lemma 1. *Assume that Φ is convex. If one of the regularity conditions (RC_i^{Φ}), $i \in \{1, 2, 3, 4\}$, is fulfilled, then $v(PG) = v(DG)$ and the dual has an optimal solution.*

Necessary and sufficient *optimality conditions* for the same pair of problems follow (see [2]).

Lemma 2. (a) *Assume that Φ is convex. Let $\bar{x} \in X$ be an optimal solution to (PG) and assume that one of the regularity conditions (RC_i^{Φ}), $i \in \{1, 2, 3, 4\}$, is fulfilled. Then there exists $\bar{y}^* \in Y^*$, an optimal solution to (DG), such that*

$$\Phi(\bar{x}, 0) + \Phi^*(0, \bar{y}^*) = 0 \tag{1}$$

or, equivalently,

$$(0, \bar{y}^*) \in \partial\Phi(\bar{x}, 0). \tag{2}$$

(b) *Assume that $\bar{x} \in X$ and $\bar{y}^* \in Y^*$ fulfill (1) or (2). Then $\bar{x}$ is an optimal solution to (PG), $\bar{y}^*$ is an optimal solution to (DG) and $v(PG) = v(DG)$.*

2. Wolfe and Mond–Weir type duals via perturbations

Besides (DG), other dual problems can be attached to (PG), by making use of the perturbation function Φ. In the following we introduce two such duals, namely a *Wolfe type* one

$$(DG_W) \quad \sup_{\substack{u \in X, y \in Y, y^* \in Y^*,\\ (0,y^*) \in \partial\Phi(u,y)}} \{-\Phi^*(0, y^*)\},$$

and a *Mond–Weir type* one

$$(DG_M) \quad \sup_{\substack{u \in X, y^* \in Y^*,\\ (0,y^*) \in \partial\Phi(u,0)}} \Phi(u, 0).$$

At a first look it may seem strange that we say that these duals have something in common with the classical duality concepts due to Wolfe and, respectively, Mond and Weir. But, as we shall see later, when one takes as primal an optimization problem consisting in minimizing a function subject to both geometric and cone inequality constraints and an appropriate perturbation function, the two dual problems introduced above lead to the classical Wolfe and Mond–Weir duals, respectively.

Next we show that weak duality holds for (PG) and these two duals, too.

Theorem 1. *One has*

$$-\infty \le v(DG_M) \le v(DG_W) \le v(DG) \le v(PG) \le +\infty. \tag{3}$$

Proof. Noting that (DG_M) can be obtained from (DG_W) by taking $y = 0$ and using the constraint involving the subdifferential, it follows that $-\infty \le v(DG_M) \le v(DG_W)$. On the other hand, (DG_W) is actually the problem (DG) with an additional constraint. Consequently, $v(DG_W) \le v(DG)$ and, taking into account the weak duality statement for (DG) and (PG), we are done. □

Situations where the last two inequalities in (3) are strict are known in the literature, while an example to have $-\infty < v(DG_M)$ can be easily constructed, too. Later, in Examples 2 and 3 we bring into attention two situations where $v(DG_M) < v(DG_W)$ and $v(DG_W) < v(DG)$, respectively.

One of the directions in which both Wolfe and Mond–Weir duality concepts were developed is towards introducing dual problems for which strong duality holds without asking the fulfillment of any regularity condition. As it can be noticed in the following observation, (DG_M) can be considered as such a dual problem.

Remark 1. Note that if the feasible set of (DG_M) is nonempty, containing for instance the element $(\bar{u}, \bar{y}^*)$, then (2) is fulfilled for $\bar{x} = \bar{u}$ and $\bar{y}^*$ and one obtains via Lemma 2(b) that $\bar{u}$ is an optimal solution to (PG), $\bar{y}^*$ is an optimal solution to (DG) and $v(DG) = v(PG) = \Phi(\bar{u}, 0) \le v(DG_M)$. Via Theorem 1 we obtain $v(DG_M) = v(DG_W) = v(DG) = v(PG)$ and moreover that $(\bar{u}, \bar{y}^*)$ is an optimal solution to (DG_M) and $(\bar{u}, 0, \bar{y}^*)$ is one to (DG_W).

Strong duality for the two duals to (PG) introduced above follows. Note that other regularity conditions can be used, too, as long as they guarantee the stability of the problem (PG).

Theorem 2. *Assume that Φ is a convex function. Let $\bar{x} \in X$ be an optimal solution to (PG) and assume that one of the regularity conditions (RC_i^{Φ}), $i \in \{1, 2, 3, 4\}$, is fulfilled. Then $v(PG) = v(DG_W) = v(DG_M)$ and there exists $\bar{y}^* \in Y^*$ for which $(\bar{x}, \bar{y}^*)$ is an optimal solution to (DG_M) and $(\bar{x}, 0, \bar{y}^*)$ is one to (DG_W).*

Proof. Lemma 2(a) guarantees that under the present hypotheses there exists $\bar{y}^* \in Y^*$, which is an optimal solution to (DG), such that $(0, \bar{y}^*) \in \partial\Phi(\bar{x}, 0)$. Thus the feasible set of (DG_M) is nonempty, containing at least the element $(\bar{x}, \bar{y}^*)$. The conclusion follows via Remark 1. □

Let us see now how do the duals arising from (DG_W) and (DG_M) look when the primal problem takes several classical particular formulations. More precisely, the primal is taken first to consist in the unconstrained minimization of a sum of a function with the composition of another function with a linear continuous mapping, then to mean finding the infimum of a function subject to both geometric and cone inequality constraints. In both situations the perturbation function is carefully chosen.

2.1. Unconstrained optimization problems

Consider first the primal optimization problem

$$(P^A) \quad \inf_{x\in X}\{f(x)+g(Ax)\},$$

where $A : X \to Y$ is a linear continuous mapping and $f : X \to \overline{\mathbb{R}}$ and $g : Y \to \overline{\mathbb{R}}$ are proper functions fulfilling $\operatorname{dom} f \cap A^{-1}(\operatorname{dom} g) \neq \emptyset$. The perturbation function considered for assigning the Wolfe type and Mond–Weir type dual problems to (P^A) is the one which delivers from (DG) the classical Fenchel dual to it, namely (cf. [2,3])

$$\Phi^A : X\times Y \to \overline{\mathbb{R}}, \quad \Phi^A(x,y) = f(x)+g(Ax+y).$$

Since the conjugate of Φ^A is

$$(\Phi^A)^* : X^*\times Y^* \to \overline{\mathbb{R}}, \quad (\Phi^A)^*(x^*,y^*) = f^*(x^*-A^*y^*)+g^*(y^*)$$

and its subdifferential satisfies

$$(0,y^*)\in\partial\Phi(u,y) \Leftrightarrow A^*y^*\in-\partial f(u) \quad \text{and} \quad y^*\in\partial g(Au+y),$$

these duals turn out to be

$$(D_W^A) \quad \sup_{\substack{u\in X,y\in Y,y^*\in Y^*,\\ y^*\in(A^*)^{-1}(-\partial f(u))\cap\partial g(Au+y)}} \{-f^*(-A^*y^*)-g^*(y^*)\},$$

which is a *Wolfe type dual*, and, respectively a *Mond–Weir type dual*,

$$(D_M^A) \quad \sup_{\substack{u\in X,\\ 0\in(A^*)^{-1}(-\partial f(u))-\partial g(Au)}} \{f(u)+g(Au)\}.$$

When the primal problem is taken to be more particular, as happens for instance when f takes everywhere the value 0, or when A is the identity mapping of X, duals correspondingly obtained from (D_W^A) and (D_M^A) can be taken into consideration.

Theorem 1 yields weak duality for the duals just introduced, namely

$$v(D_M^A) \leq v(D_W^A) \leq v(D^A) \leq v(P^A),$$

where

$$(D^A) \quad \sup_{y^*\in Y^*}\{-f^*(-A^*y^*)-g^*(y^*)\}$$

is the classical *Fenchel dual* problem to (P^A).

For strong duality besides convexity assumptions we use regularity conditions, too, obtained by particularizing (RC_i^Φ), $i\in\{1,2,3,4\}$, namely

(RC_1^A) $\exists x'\in\operatorname{dom} f\cap A^{-1}(\operatorname{dom} g)$ such that g is continuous at Ax',

(RC_2^A) X and Y are Fréchet spaces, f and g are lower semicontinuous and $0\in\operatorname{sqri}(\operatorname{dom} g - A(\operatorname{dom} f))$,

(RC_3^A) $\dim(\operatorname{lin}(\operatorname{dom} g - A(\operatorname{dom} f))) < +\infty$ and $\operatorname{ri}(A(\operatorname{dom} f))\cap\operatorname{ri}(\operatorname{dom} g)\neq\emptyset$,

and

(RC_4^A) f and g are lower semicontinuous and $\operatorname{epi} f^* + (A^*\times\operatorname{id}_{\mathbb{R}})(\operatorname{epi} g^*)$ is closed in the topology $w(X^*,X)\times\mathbb{R}$,

where $(A^*\times\operatorname{id}_{\mathbb{R}})(\operatorname{epi} g^*) = \{(x^*,r)\in X^*\times\mathbb{R} : \exists y^*\in Y^* \text{ such that } A^*y^*=x^* \text{ and } (y^*,r)\in\operatorname{epi} g^*\}$. The strong duality statement for (P^A) and (D^A) follows directly from Theorem 2.

Theorem 3. *Assume that f and g are convex functions. Let $\bar{x}\in X$ be an optimal solution to (P^A) and assume that one of the regularity conditions (RC_i^A), $i\in\{1,2,3,4\}$, is fulfilled. Then $v(P^A)=v(D_W^A)=v(D_M^A)$ and there exists $\bar{y}^*\in Y^*$ for which $(\bar{x},\bar{y}^*)$ is an optimal solution to (DG_M) and $(\bar{x},0,\bar{y}^*)$ is one to (DG_W).*

2.2. Constrained optimization problems

Now let us turn our attention to the constrained optimization problems. Consider the nonempty set $S\subseteq X$ and the proper functions $f : X\to\overline{\mathbb{R}}$ and $g : X\to Y^\bullet$, fulfilling $\operatorname{dom} f\cap S\cap g^{-1}(-C)\neq\emptyset$. The primal problem we treat further is

$$(P^C) \quad \inf_{x\in\mathcal{A}} f(x),$$

where $\mathcal{A} = \{x\in S : g(x)\in -C\}$.

Using the perturbation functions considered in [2], we assign to (P^C) three pairs of dual problems arising from (DG_W) and (DG_M), respectively.

The *classical Lagrange dual* to (P^C)

$$(D^{C_L}) \quad \sup_{z^*\in C^*} \inf_{x\in S}\{f(x) + (z^*g)(x)\},$$

can be obtained as a special case of (DG) by using the perturbation function

$$\Phi^{C_L} : X \times Y \to \overline{\mathbb{R}}, \quad \Phi^{C_L}(x, z) = \begin{cases} f(x), & \text{if } x \in S, g(x) \in z - C, \\ +\infty, & \text{otherwise,} \end{cases}$$

whose conjugate is

$$(\Phi^{C_L})^* : X^* \times Y^* \to \overline{\mathbb{R}}, \quad (\Phi^{C_L})^*(x^*, z^*) = (f - (z^*g) + \delta_S)^*(x^*) + \delta_{C^*}(-z^*).$$

Then $(0, z^*) \in \partial\Phi^{C_L}(u, z)$ if and only if $u \in S, g(u) \in z-C$ and $(f-(z^*g)+\delta_S)^*(0)+\delta_{C^*}(-z^*)+(f+\delta_S)(u)+\delta_{-C}(g(u)-z) = \langle z^*, z\rangle$. Using the fact that $\delta^*_{-C} = \delta_{C^*}$, the latter can be equivalently rewritten as

$$\big((f - (z^*g) + \delta_S)^*(0) + (f - (z^*g) + \delta_S)(u)\big) + \big(\delta^*_{-C}(-z^*) + \delta_{-C}(g(u) - z) + \langle z^*, g(u) - z\rangle\big) = 0. \tag{4}$$

By the Young–Fenchel inequality it follows that (4) is nothing but $(f - (z^*g) + \delta_S)^*(0) + (f - (z^*g) + \delta_S)(u) = 0$ and $\delta^*_{-C}(-z^*)+\delta_{-C}(g(u)-z)+\langle z^*, g(u)-z\rangle = 0$, i.e. $0 \in \partial(f-(z^*g)+\delta_S)(u), z^* \in -C^*$ and $\delta_{-C}(g(u)-z)-\langle -z^*, g(u)-z\rangle = 0$. Thus we obtain from (DG_W) the following dual problem to (P^C)

$$(D_W^{C_L}) \quad \sup_{\substack{u\in S, z\in Y, z^*\in -C^*,\\ g(u)-z\in -C, (z^*g)(u)=\langle z^*,z\rangle,\\ 0\in\partial(f-(z^*g)+\delta_S)(u)}} \{f(u) - \langle z^*, z\rangle\},$$

which can be equivalently rewritten as

$$(D_W^{C_L}) \quad \sup_{\substack{u\in S, z^*\in C^*,\\ 0\in\partial(f+(z^*g)+\delta_S)(u)}} \{f(u) + (z^*g)(u)\}.$$

We call this the *Wolfe dual of Lagrange type* to (P^C). We shall see later that, in the particular instance where the classical Wolfe duality was considered, this dual turns into the well-known *nondifferentiable Wolfe dual* problem.

Analogously we get a dual problem to (P^C) arising from (DG_M), namely

$$(D_M^{C_L}) \quad \sup_{\substack{u\in S, z^*\in C^*,\\ g(u)\in -C, (z^*g)(u)\geq 0,\\ 0\in\partial(f+(z^*g)+\delta_S)(u)}} f(u).$$

Note that in the constraints of this dual one can replace $(z^*g)(u) \geq 0$ by $(z^*g)(u) = 0$ without altering anything. Because the classical *nondifferentiable Mond–Weir dual* to (P^C) can be obtained as a special case to $(D_M^{C_L})$ by removing the constraint $g(u) \in -C$, we consider here also the *Mond–Weir dual of Lagrange type* to (P^C)

$$(D_{MW}^{C_L}) \quad \sup_{\substack{u\in S, z^*\in C^*, (z^*g)(u)\geq 0,\\ 0\in\partial(f+(z^*g)+\delta_S)(u)}} f(u).$$

By construction it is clear that $v(D_M^{C_L}) \leq v(D_{MW}^{C_L})$. On the other hand, whenever (u, z^*) is feasible to $(D_{MW}^{C_L})$ it is feasible to $(D_W^{C_L})$, too, and moreover $(z^*g)(u) \geq 0$. This yields $f(u) \leq f(u) + (z^*g)(u) \leq v(D_W^{C_L})$. Considering the supremum over all the pairs (u, z^*) feasible to $(D_{MW}^{C_L})$ we obtain $v(D_{MW}^{C_L}) \leq v(D_W^{C_L})$. Applying the weak duality statement Theorem 1, we get

$$v(D_M^{C_L}) \leq v(D_{MW}^{C_L}) \leq v(D_W^{C_L}) \leq v(D^{C_L}) \leq v(P^C). \tag{5}$$

As can be seen in the following situations, the Wolfe dual has sometimes an indeed larger optimal objective value than the Mond–Weir one, while the classical conjugate dual can have a strictly greater one than the other mentioned two duals. Moreover, the first example we give below exhibits a situation where the first inequality in (5) is strictly fulfilled.

Example 1. Let $X = \mathbb{R}$, $Y = \mathbb{R}$, $C = \mathbb{R}_+$, $Y^\bullet = \mathbb{R} \cup \{\infty_{\mathbb{R}_+}\}$, $S = [0, +\infty)$, $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x$, and $g : \mathbb{R} \to \mathbb{R} \cup \{\infty_{\mathbb{R}_+}\}$,

$$g(x) = \begin{cases} -x, & \text{if } x > 0, \\ 2, & \text{if } x = 0, \\ \infty_{\mathbb{R}_+}, & \text{if } x < 0, \end{cases}$$

where we note that $\infty_{\mathbb{R}_+}$ can be actually identified with $+\infty$. We have $0 \in \partial(f+(0g)+\delta_S)(0) = (-\infty, 1]$ and $(0g)(0) = 0$, thus $(0, 0)$ is feasible to $(D_{MW}^{C_L})$. So $v(D_{MW}^{C_L}) \geq 0$ and since $v(P^C) = 0$ one gets $v(D_{MW}^{C_L}) = 0$. On the other hand, $g(0) = 2 > 0$, thus there is no $z^* \in \mathbb{R}_+$ for which $(0, z^*)$ is feasible to $(D_M^{C_L})$. Noting that $g(u) \neq 0$ for all $u \in \mathbb{R}$, from the constraint $(z^*g)(u) = 0$ (see the discussion after introducing $(D_M^{C_L})$) one obtains that whenever (u, z^*) were feasible to $(D_M^{C_L})$ there should be $z^* = 0$. Since for every $u > 0$ we have $\partial(f + (0g) + \delta_S)(u) = \{1\}$, it follows that $(D_M^{C_L})$ has no feasible points, consequently $v(D_M^{C_L}) = -\infty$. Therefore, $v(D_M^{C_L}) < v(D_{MW}^{C_L})$ for this setting.

Example 2. Let $X = \mathbb{R}$, $Y = \mathbb{R}^2$, $C = \mathbb{R}^2_+$, $S = \mathbb{R}_+$, $f : \mathbb{R} \to \overline{\mathbb{R}}$,

$$f(x) = \begin{cases} x, & \text{if } x > 0, \\ +\infty, & \text{otherwise,} \end{cases}$$

and $g : \mathbb{R} \to \mathbb{R}^2$, $g(x) = (x-1, -x)^T$. When, for $u > 0$ and $z^* = (z_1^*, z_2^*)^T \geqq 0$ it holds $0 \in \partial(f + (z^*g) + \delta_S)(u)$, one obtains $z_1^* - z_2^* = -1$. Thus $(D_W^{C_L})$ has feasible points and, for some $u > 0$ we obtain $v(P) = 0 \geq v(D_W^{C_L}) \geq \sup\{f(u) + (z^*g)(u) : z^* = (z_1^*, z_2^*)^T \in \mathbb{R}^2_+, z_1^* - z_2^* = -1\} = \sup\big\{u + z_1^*(u-1) - z_2^* u : z^* = (z_1^*, z_2^*)^T \in \mathbb{R}^2_+, z_1^* - z_2^* = -1\big\} = \sup\big\{u(1 + z_1^* - z_2^*) - z_2^* : z^* = (z_1^*, z_2^*)^T \in \mathbb{R}^2_+, z_1^* - z_2^* = -1\big\} = \sup_{z_2^* \geq 0}\{-z_2^*\} = 0$. Then obviously $v(D_W^{C_L}) = 0$. On the other hand, $(z^*g)(u) \geq 0$ yields $u(z_1^* - z_2^*) - z_1^* \geq 0$, i.e. $-u - z_1^* \geq 0$. But $u > 0$ and $z_1^* \geq 0$, thus we obtained a contradiction, consequently $(D_{MW}^{C_L})$ has no feasible points. Employing also (5), we obtain $v(D_M^{C_L}) = v(D_{MW}^{C_L}) = -\infty$. Therefore, $v(D_M^{C_L}) = v(D_{MW}^{C_L}) < v(D_W^{C_L})$ in this situation.

Example 3. Let $X = \mathbb{R}^2$, $Y = \mathbb{R}$, $C = \mathbb{R}_+$, $S = \mathbb{R}^2$, $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x, y) = y$, and $g : \mathbb{R}^2 \to \mathbb{R}$, $g(x, y) = e^x - y$. For $u = (u_1, u_2)^T \in \mathbb{R}^2$ and $z^* \geq 0$ we have, taking into account the continuity of f and g, $\partial(f + (z^*g) + \delta_S)(u) = \partial f(u) + \partial(z^*g)(u) = \{(0, 1)^T\} + \{(z^*e^{u_1}, -z^*)^T\} = \{(z^*e^{u_1}, 1 - z^*)^T\}$. Then $(0, 0) \in \partial(f + (z^*g) + \delta_S)(u)$ if and only if concomitantly $z^* = 0$ and $z^* = 1$, that is impossible. Consequently, via (5), $v(D_M^{C_L}) = v(D_{MW}^{C_L}) = v(D_W^{C_L}) = -\infty$. On the other hand, $v(D^{C_L}) = \sup_{z^* \geq 0} \inf_{(u_1,u_2) \in \mathbb{R}^2}[u_2 + z^*e^{u_1} - z^*u_2] = \sup_{z^* \geq 0}\big\{\inf_{u_1 \in \mathbb{R}}[z^*e^{u_1}] + \inf_{u_2 \in \mathbb{R}}[u_2(1 - z^*)]\big\} = 0$. Therefore, $v(D_M^{C_L}) = v(D_{MW}^{C_L}) = v(D_W^{C_L}) < v(D^{C_L})$ in this setting.

To obtain strong duality we take into account again the regularity conditions considered in the general case particularized for the framework we deal now with, namely

$$(RC_1^{C_L}) \quad \exists x' \in \operatorname{dom} f \cap S \quad \text{such that } g(x') \in -\operatorname{int}(C),$$

which is the classical *Slater constraint qualification*,

$$(RC_2^{C_L}) \quad \begin{array}{l} X \text{ and } Y \text{ are Fréchet spaces, } S \text{ is closed, } f \text{ is lower semicontinuous,} \\ g \text{ is } C\text{-epi-closed and } 0 \in \operatorname{sqri}\big(g(\operatorname{dom} f \cap S \cap \operatorname{dom} g) + C\big), \end{array}$$

$$(RC_3^{C_L}) \quad \dim\left(\operatorname{lin}\left(g(\operatorname{dom} f \cap S \cap \operatorname{dom} g) + C\right)\right) < +\infty \quad \text{and} \quad 0 \in \operatorname{ri}\left(g(\operatorname{dom} f \cap S \cap \operatorname{dom} g) + C\right),$$

and

$$(RC_4^{C_L}) \quad \begin{array}{l} S \text{ is closed, } f \text{ is lower semicontinuous, } g \text{ is } C\text{-epi-closed and} \\ \displaystyle\bigcup_{z^* \in C^*} \operatorname{epi}(f + (z^*g) + \delta_S)^* \text{ is closed in the topology } w(X^*, X) \times \mathbb{R}. \end{array}$$

Particularizing Theorem 2 for the present context we obtain a strong duality statement for (P^C) and its Lagrange type duals considered above.

Theorem 4. *Assume that S is a convex set, f is a convex function and g is a C-convex vector function. Let $\bar{x} \in X$ be an optimal solution to (P^C) and assume that one of the regularity conditions $(RC_i^{C_L})$, $i \in \{1, 2, 3, 4\}$, is fulfilled. Then $v(P^C) = v(D_W^{C_L}) = v(D_M^{C_L}) = v(D_{MW}^{C_L})$ and there exists $\bar{z}^* \in C^*$ for which $(\bar{x}, \bar{z}^*)$ is an optimal solution to all three duals.*

Remark 2. One can notice that in the situation considered in Example 3 the convexity hypotheses of Theorem 4 and the Slater constraint qualification $(RC_1^{C_L})$ are valid, but strong duality fails for the Wolfe and Mond–Weir type duals. This happens because the infimal objective value of (P^C) is not attained, the primal problem having no optimal solutions.

Remark 3. Assume that S is a convex set, f is a convex function and g is a C-convex vector function. Denote $\Delta_{X^3} = \{(x, x, x) : x \in X\}$. When one of the following conditions

(i) f and g are continuous at a point in $\operatorname{dom} f \cap \operatorname{dom} g \cap S$;
(ii) $\operatorname{dom} f \cap \operatorname{int}(S) \cap \operatorname{dom} g \neq \emptyset$ and f or g is continuous at a point in $\operatorname{dom} f \cap \operatorname{dom} g$;
(iii) X is a Fréchet space, S is closed, f is lower semicontinuous, g is star C-lower semicontinuous and $0 \in \operatorname{sqri}(\operatorname{dom} f \times S \times \operatorname{dom} g - \Delta_{X^3})$;
(iv) $\dim(\operatorname{lin}(\operatorname{dom} f \times S \times \operatorname{dom} g - \Delta_{X^3})) < +\infty$ and $0 \in \operatorname{ri}(\operatorname{dom} f \times S \times \operatorname{dom} g - \Delta_{X^3})$;

is satisfied, then (see [2,3])

$$\partial f(x) + \partial(z^*g)(x) + N_S(x) = \partial(f + (z^*g) + \delta_S)(x) \quad \forall x \in S\ \forall z^* \in C^*$$

Consequently, when one of these situations occurs, the constraint involving the subdifferential in $(D_W^{C_L})$, $(D_M^{C_L})$ and $(D_{MW}^{C_L})$ can be correspondingly modified.

Remark 4. If $X = \mathbb{R}^n$, $Y = \mathbb{R}^m$, $C = \mathbb{R}^m_+$, $f : \mathbb{R}^n \to \mathbb{R}$ and $g = (g_1, \ldots, g_m)^T : \mathbb{R}^n \to \mathbb{R}^m$, and the functions f and g_j, $j = 1, \ldots, m$, are convex, then $(D_W^{C_L})$ turns out to be the classical nondifferentiable Wolfe dual problem mentioned in the literature. See for instance [1,4]. Meanwhile, $(D_{MW}^{C_L})$ is the classical nondifferentiable Mond–Weir dual problem to (P^C). In case the functions f and $g_j, j = 1, \ldots, m$, are moreover differentiable on S, which is taken to be open, and the subdifferentials are replaced by gradients in the constraints, $(D_W^{C_L})$ turns out to be the classical Wolfe dual problem (see [5]), while $(D_{MW}^{C_L})$ is nothing but the classical Mond–Weir dual problem from [6].

To (P^C) one can attach a *Fenchel dual* problem, too, namely

$$(D^{C_F}) \qquad \sup_{y^* \in X^*} \{-f^*(y^*) - \sigma_{\mathcal{A}}(-y^*)\},$$

by using the perturbation function

$$\Phi^{C_F} : X \times X \to \overline{\mathbb{R}}, \quad \Phi^{C_F}(x, y) = \begin{cases} f(x + y), & \text{if } x \in \mathcal{A}, \\ +\infty, & \text{otherwise,} \end{cases}$$

having as conjugate the function

$$(\Phi^{C_F})^* : X^* \times X^* \to \overline{\mathbb{R}}, \quad (\Phi^{C_F})^*(x^*, y^*) = \sigma_{\mathcal{A}}(x^* - y^*) + f^*(y^*).$$

Regarding its subdifferential, one has $(0, y^*) \in \partial\Phi^{C_F}(u, y)$ if and only if it holds $\sigma_{\mathcal{A}}(-y^*) + f^*(y^*) + f(u+y) + \delta_{\mathcal{A}}(u) = \langle y^*, y\rangle$, which is equivalent to

$$\big(f^*(y^*) + f(u + y) - \langle y^*, u + y\rangle\big) + \big(\sigma_{\mathcal{A}}(-y^*) + \delta_{\mathcal{A}}(u) - \langle -y^*, u\rangle\big) = 0,$$

i.e. $y^* \in \partial f(u + y)$ and $y^* \in -\partial\delta_{\mathcal{A}}(u) = -N_{\mathcal{A}}(u)$. Thus, the dual problem to (P^C) obtained from (DG_W) is in this case

$$(D_W^{C_F}) \qquad \sup_{\substack{u \in S, y \in X, y^* \in X^*, \\ y^* \in \partial f(u+y) \cap (-N_{\mathcal{A}}(u))}} \{\langle y^*, u\rangle - f^*(y^*)\},$$

the *Wolfe dual of Fenchel type* to (P^C), while the *Mond–Weir dual of Fenchel type* to (P^C) that arises via (DG_M) is

$$(D_M^{C_F}) \qquad \sup_{u \in S, 0 \in \partial f(u) + N_{\mathcal{A}}(u)} f(u).$$

Note that these two dual problems can be obtained also directly from (D_W^A) and (D_M^A) taking there $A := \mathrm{id}_X, f := f$ and $g := \delta_{\mathcal{A}}$. From Theorem 1 we get

$$v(D_M^{C_F}) \le v(D_W^{C_F}) \le v(D^{C_F}) \le v(P^C).$$

For strong duality the following regularity conditions are obtained from the general case

$$(RC_1^{C_F}) \quad \exists x' \in \operatorname{dom} f \cap \mathcal{A} \quad \text{such that } f \text{ is continuous at } x',$$

$$(RC_2^{C_F}) \quad X \text{ is a Fréchet space, } \mathcal{A} \text{ is closed, } f \text{ is lower semicontinuous and } 0 \in \operatorname{sqri}(\operatorname{dom} f - \mathcal{A}),$$

$$(RC_3^{C_F}) \quad \dim(\operatorname{lin}(\operatorname{dom} f - \mathcal{A})) < +\infty \quad \text{and} \quad \operatorname{ri}(\operatorname{dom} f) \cap \operatorname{ri}(\mathcal{A}) \neq \emptyset,$$

and

$$(RC_4^{C_F}) \quad \mathcal{A} \text{ is closed, } f \text{ is lower semicontinuous and } \operatorname{epi} f^* + \operatorname{epi} \sigma_{\mathcal{A}} \text{ is closed in the topology } w(X^*, X) \times \mathbb{R}.$$

Particularizing Theorem 2 for the present framework we obtain a strong duality statement for (P^C) and its Fenchel type duals introduced above.

Theorem 5. *Assume that $\mathcal{A}$ is a convex set and f is a convex function. Let $\bar{x} \in X$ be an optimal solution to (P^C) and assume that one of the regularity conditions $(RC_i^{C_F})$, $i \in \{1, 2, 3, 4\}$, is fulfilled. Then $v(P^C) = v(D_W^{C_F}) = v(D_M^{C_F})$, $\bar{x}$ is an optimal solution to $(D_M^{C_F})$ and there exist $(\bar{y}, \bar{y}^*) \in X \times X^*$ for which $(\bar{x}, \bar{y}, \bar{y}^*)$ is an optimal solution to $(D_W^{C_F})$.*

Note that to ensure the convexity of the set $\mathcal{A}$ it is sufficient to take S to be a convex set and g a C-convex vector function. In [2] was considered the *Fenchel–Lagrange dual* problem to (P^C), namely

$$(D^{C_{FL}}) \qquad \sup_{y^* \in X^*, z^* \in C^*} \big\{-f^*(y^*) - ((z^*g) + \delta_S)^*(-y^*)\big\},$$

obtained from (DG) via the perturbation function

$$\Phi^{C_{FL}}: X \times X \times Y \to \overline{\mathbb{R}}, \quad \Phi^{C_{FL}}(x, y, z) = \begin{cases} f(x+y), & \text{if } x \in S, g(x) \in z - C, \\ +\infty, & \text{otherwise.} \end{cases}$$

This has as conjugate the function $(\Phi^{C_{FL}})^*: X^* \times X^* \times Y^* \to \overline{\mathbb{R}}$,

$$(\Phi^{C_{FL}})^*(x^*, y^*, z^*) = f^*(y^*) + (-(z^*g) + \delta_S)^*(x^* - y^*) + \delta_{-C^*}(z^*)$$

and $(0, y^*, z^*) \in \partial \Phi^{C_{FL}}(u, y, z)$ if and only if $u \in S, g(u) \in z - C$ and $f^*(y^*) + (-(z^*g) + \delta_S)^*(-y^*) + \delta_{-C^*}(z^*) + f(u + y) + \delta_{-C}(g(u) - z) + \delta_S(u) = \langle y^*, y\rangle + \langle z^*, z\rangle$, which is nothing but $u \in S, g(u) \in z - C$ and

$$\big(f^*(y^*) + f(u+y) - \langle y^*, u+y\rangle\big) + \big((-(z^*g) + \delta_S)^*(-y^*) + (-(z^*g) + \delta_S)(u) - \langle -y^*, u\rangle\big) + \big(\delta^*_{-C}(-z^*) + \delta_{-C}(g(u) - z) - \langle -z^*, g(u) - z\rangle\big) = 0,$$

i.e. $u \in S, -z^* \in C^*, g(u) - z \in -C, y^* \in \partial f(u+y), -y^* \in \partial(-(z^*g) + \delta_S)(u)$ and $(z^*g)(u) = \langle z^*, z\rangle$. Thus we obtain from (DG_W) the following dual to (P^C)

$$(D_W^{C_{FL}}) \quad \sup_{\substack{u\in S, y\in X, z\in Y, y^*\in X^*, z^*\in -C^*,\\ g(u)-z\in -C, (z^*g)(u)=\langle z^*, z\rangle,\\ y^*\in\partial f(u+y)\cap(-\partial((z^*g)+\delta_S)(u))}} \{f(u+y) - \langle y^*, y\rangle - \langle z^*, z\rangle\},$$

which can be equivalently turned into

$$(D_W^{C_{FL}}) \quad \sup_{\substack{u\in S, y\in X, y^*\in X^*, z^*\in C^*,\\ y^*\in\partial f(u+y)\cap(-\partial((z^*g)+\delta_S)(u))}} \{\langle y^*, u\rangle + (z^*g)(u) - f^*(y^*)\},$$

further referred to as the *Wolfe dual of Fenchel–Lagrange type* to (P^C). Analogously, the dual problem to (P^C) arising from (DG_M) is

$$(D_M^{C_{FL}}) \quad \sup_{\substack{u\in S, z^*\in C^*,\\ (z^*g)(u)\geq 0, g(u)\in -C,\\ 0\in\partial f(u)+\partial((z^*g)+\delta_S)(u)}} f(u).$$

Note that in the constraints of this dual one can replace $(z^*g)(u) \geq 0$ by $(z^*g)(u) = 0$ without altering anything. Removing from it the constraint $g(u) \in -C$, we obtain the *Mond–Weir dual of Fenchel–Lagrange type* to (P^C)

$$(D_{MW}^{C_{FL}}) \quad \sup_{\substack{u\in S, z^*\in C^*, (z^*g)(u)\geq 0,\\ 0\in\partial f(u)+\partial((z^*g)+\delta_S)(u)}} f(u).$$

Applying the weak duality statement and using similar arguments to the ones used concerning $(D_{MW}^{C_L})$, we obtain the following weak duality inequality

$$v(D_M^{C_{FL}}) \leq v(D_{MW}^{C_{FL}}) \leq v(D_W^{C_{FL}}) \leq v(D^{C_{FL}}) \leq v(P^C).$$

For strong duality we obtain from the general case the following regularity conditions (cf. [2])

$(RC_1^{C_{FL}})$ $\exists x' \in \operatorname{dom} f \cap S$ such that f is continuous at x' and $g(x') \in -\operatorname{int}(C)$,

$(RC_2^{C_{FL}})$ X and Y are Fréchet spaces, S is closed, f is lower semicontinuous, g is C-epi-closed and $0 \in \operatorname{sqri}\big(\operatorname{dom} f \times C - \operatorname{epi}_{(-C)}(-g) \cap (S \times Y)\big)$,

$(RC_3^{C_{FL}})$ $\dim\big(\operatorname{lin}\big(\operatorname{dom} f \times C - \operatorname{epi}_{(-C)}(-g) \cap (S \times Z)\big)\big) < +\infty$ and $0 \in \operatorname{ri}\big(\operatorname{dom} f \times C - \operatorname{epi}_{(-C)}(-g) \cap (S \times Z)\big)$.

and

$(RC_4^{C_{FL}})$ S is closed, f is lower semicontinuous, g is C-epi-closed and $\operatorname{epi} f^* + \bigcup_{z^*\in C^*} \operatorname{epi}((z^*g) + \delta_S)^*$ is closed in the topology $w(X^*, X) \times \mathbb{R}$.

To give the strong duality statement for (P^C) and its Fenchel–Lagrange type duals introduced above we can simply apply Theorem 2 for the present framework.

Theorem 6. *Assume that S is a convex set, f is a convex function and g is C-convex vector function. Let $\bar{x} \in X$ be an optimal solution to (P^C) and assume that one of the regularity conditions $(RC_i^{C_{FL}})$, $i \in \{1, 2, 3, 4\}$, is fulfilled. Then $v(P^C) = v(D_W^{C_{FL}}) = v(D_M^{C_{FL}}) = v(D_{MW}^{C_{FL}})$ and there exist $\bar{y}^* \in X^*$ and $\bar{z}^* \in C^*$ for which $(\bar{x}, 0, \bar{y}^*, \bar{z}^*)$ is an optimal solution to $(D_W^{C_{FL}})$ and $(\bar{x}, \bar{z}^*)$ is an optimal solution to $(D_M^{C_{FL}})$ and $(D_{MW}^{C_{FL}})$.*

Remark 5. Results analogous to the one from Remark 3 can be given for the Fenchel–Lagrange type duals, too. We refer the reader to [2,3] for such sufficient conditions.

From [2] it is known that one has

$$v(D^{C_{FL}}) \leq \begin{matrix} v(D^{C_L}) \\ v(D^{C_F}) \end{matrix} \leq v(P^C). \tag{6}$$

A natural question is if similar inequalities exist also for the dual problems introduced in this subsection. First we deal with the ones that are particular instances of (DG_M).

Theorem 7. *One has*

(i) $v(D_M^{C_{FL}}) \leq \begin{matrix} v(D_M^{C_L}) \\ v(D_M^{C_F}) \end{matrix} \leq v(P^C)$;

(ii) $v(D_{MW}^{C_{FL}}) \leq v(D_{MW}^{C_L}) \leq v(P^C)$.

Proof. If the feasible set of $(D_M^{C_{FL}})$ is empty, there is nothing to prove. Let (u, z^*) be feasible to $(D_M^{C_{FL}})$. Then $u \in S, z^* \in C^*$, $(z^*g)(u) \geq 0, g(u) \in -C$ and $0 \in \partial f(u) + \partial((z^*g) + \delta_S)(u)$. The last relation implies $0 \in \partial(f + (z^*g) + \delta_S)(u)$, consequently (u, z^*) is feasible to $(D_M^{C_L})$, too. As both $(D_M^{C_{FL}})$ and $(D_M^{C_L})$ have f as objective function, it is clear that $v(D_M^{C_{FL}}) \leq v(D_M^{C_L})$. Analogously one can show that $v(D_{MW}^{C_{FL}}) \leq v(D_{MW}^{C_L})$. Note that an alternative proof which uses Remark 1 can be given, too.

On the other hand, $\partial((z^*g) + \delta_S)(u) \subseteq N_{\mathcal{A}}(u)$ since $u \in \mathcal{A}$, thus u is feasible to $(D_M^{C_F})$. Since both $(D_M^{C_{FL}})$ and $(D_M^{C_F})$ have f as objective function, it is clear that $v(D_M^{C_{FL}}) \leq v(D_M^{C_F})$. $\square$

Situations where the inequalities from Theorem 7 are strictly fulfilled can be found in Examples 4–6.

Remark 6. When the feasible set of $(D_M^{C_L})$ or $(D_M^{C_F})$ is empty, then so is the feasible set of $(D_M^{C_{FL}})$, too. In general, the fact that $(D_M^{C_{FL}})$ has no feasible points does not imply the emptiness of any of the feasible sets of $(D_M^{C_L})$ and $(D_M^{C_F})$.

However, a result similar to (6) or Theorem 7(i) does not hold for the Wolfe type duals to (P^C). Even if the primal problem is convex, the optimal objective values of $(D_W^{C_{FL}})$, $(D_W^{C_L})$ and $(D_W^{C_F})$ are not comparable. In the following we sustain this claim by several examples. First we deal with the Wolfe duals of types Lagrange and Fenchel–Lagrange, respectively.

Example 4. Let $X = \mathbb{R}^2, Y = \mathbb{R}, C = \mathbb{R}_+$,

$$S = \left\{ (x_1, x_2)^T \in \mathbb{R}^2 : 0 \leq x_1 \leq 2, \begin{matrix} 3 \leq x_2 \leq 4, & \text{if } x_1 = 0, \\ 1 \leq x_2 \leq 4, & \text{if } x_1 \in (0, 2] \end{matrix} \right\},$$

$$f : \mathbb{R}^2 \to \overline{\mathbb{R}}, \quad f(x_1, x_2) = \begin{cases} x_2, & \text{if } x_1 \leq 0, \\ +\infty, & \text{otherwise,} \end{cases}$$

and $g : \mathbb{R}^2 \to \mathbb{R}, g(x_1, x_2) = 0$. Then

$$(f + \delta_S)(u_1, u_2) = \begin{cases} u_2, & \text{if } u_1 = 0, u_2 \in [3, 4], \\ +\infty, & \text{otherwise.} \end{cases}$$

For any $z^* \in \mathbb{R}_+$ we get $(0, 0) \in \partial(f + (z^*g) + \delta_S)(0, 3)$, thus $v(D_W^{C_L}) \geq 3$. Since it can be seen that $v(P^C) = 3$, we get $v(D_W^{C_L}) = 3$. On the other hand, taking without loss of generality $z^* = 1$, we have $y^* \in \partial f(u + y) \cap (-N_S(u))$. From $y^* \in \partial f(u + y)$ we obtain that $y^* = (y_1^*, y_2^*)^T \in \mathbb{R}_+ \times \{1\}$. Consequently, $y_2^* = 1$. Let us see now for what $y_1^* \in \mathbb{R}_+$ one has $(-y_1^*, -1) \in N_S(u_1, u_2)$ for some $(u_1, u_2) \in S$. We have $(-y_1^*, -1) \in N_S(u_1, u_2)$ if and only if $\sigma_S(-y_1^*, -1) = -y_1^*u_1 - u_2$. Since this can take place only if $y_1^* = 0$, it follows that $u_1 \in (0, 2], u_2 = 1$ and $(0, 1)^T$ is the only possible value for y^*. Consequently, $v(D_W^{C_{FL}}) = \sup\{u_2 : u_1 \in (0, 2], u_2 = 1\} = 1$. Therefore, $v(D_W^{C_{FL}}) < v(D_W^{C_L})$ in this case.

Example 5. Let $X = \mathbb{R}, Y = \mathbb{R}, C = \mathbb{R}_+, S = \mathbb{R}$,

$$f : \mathbb{R} \to \overline{\mathbb{R}}, \quad f(x) = \begin{cases} x, & \text{if } x > 0, \\ +\infty, & \text{otherwise,} \end{cases}$$

and

$$g : \mathbb{R} \to \mathbb{R}, \quad g(x) = \begin{cases} -x, & \text{if } x \leq 0, \\ 0, & \text{otherwise.} \end{cases}$$

Then $\mathcal{A} = [0, +\infty)$ and $v(P) = 0$. Note that for all $z^* \geq 0$ one has $f + (z^*g) + \delta_S \equiv f$. Thus, for all $z^* \geq 0$,

$$\partial(f + (z^*g) + \delta_S)(u) = \partial f(u) = \begin{cases} \{1\}, & \text{if } u > 0, \\ \emptyset, & \text{otherwise.} \end{cases}$$

Consequently, $(D_W^{C_L})$ has no feasible points, therfore $v(D_W^{C_L}) = -\infty$. On the other hand, taking $u = 0, y = 1, y^* = 1$ and $z^* = 1$, we have $(z^*g) + \delta_S \equiv g$, $1 \in \partial f(1)$ and $-1 \in \partial g(0)$. Thus $(0, 1, 1, 1)$ is feasible to $(D_W^{C_{FL}})$. Then $v(D_W^{C_{FL}}) \geq 0 = v(P)$, thus $v(D_W^{C_{FL}}) = 0$. Obviously, $v(D_W^{C_{FL}}) > v(D_W^{C_L})$ for this setting.

Now let us see two examples where the values Wolfe duals of types Fenchel and Fenchel–Lagrange surpass each other, respectively.

Example 6. Let $X = \mathbb{R}^2$, $Y = \mathbb{R}$, $C = \mathbb{R}_+$,

$$S = \left\{ (x_1, x_2)^T \in \mathbb{R}^2 : 0 \leq x_1 \leq 2, \begin{array}{ll} 3 \leq x_2 \leq 4, & \text{if } x_1 = 0, \\ 1 \leq x_2 \leq 4, & \text{if } x_1 \in (0, 2] \end{array} \right\},$$

$f : \mathbb{R}^2 \to \mathbb{R}$, $f(x_1, x_2) = x_2$ and $g : \mathbb{R}^2 \to \mathbb{R}$, $g(x_1, x_2) = x_1$. Since $(0, 1)^T \in \partial f(0, 3) \cap (-N_{\mathcal{A}}(0, 3))$, $v(D_W^{C_F}) \geq 3$. As $v(P^C) = 3$, $v(D_W^{C_F}) = 3$. On the other hand, for $u \in S$, $y, y^* \in \mathbb{R}^2$ and $z^* \geq 0$, $y^* \in \partial f(u + y) \cap (-\partial((z^*g) + \delta_S)(u))$ if and only if $y^* = (0, 1)^T$, $z^* = 0$ and $u \in (0, 2] \times \{1\}$. Then $v(D_W^{C_{FL}}) = \sup\{u_2 : (u_1, u_2)^T \in (0, 2] \times \{1\}\} = 1$. Therefore, $v(D_W^{C_{FL}}) < v(D_W^{C_F})$ in this case.

Example 7. Consider again the situation from Example 1. We have $\mathcal{A} = (0, +\infty)$, $N_{\mathcal{A}}(u) = \{0\}$ for all $u \in \mathcal{A}$, $v(P^C) = 0$ and $\partial f(u) = \{1\}$ for all $u \in \mathbb{R}$, thus $\partial f(u + y) \cap (-N_{\mathcal{A}}(u)) = \emptyset$ for all $u \in S$ and all $y \in \mathbb{R}$. Consequently, $v(D_W^{C_F}) = -\infty$. On the other hand, taking $z^* = 0$ we get $\partial((z^*g) + \delta_S)(u) = N_{\mathbb{R}_+}(u)$ for all $u \geq 0$ and it can be shown that $N_{\mathbb{R}_+}(0) = \mathbb{R}_-$. Thus $1 \in \partial f(y) \cap \partial(-((z^*g) + \delta_S)(0))$ for all $y \in \mathbb{R}$, i.e. the quadruple $(0, y, 1, 0)$ is feasible to $(D_W^{C_{FL}})$ whenever $y \in \mathbb{R}$. This yields $v(D_W^{C_{FL}}) \geq 0 = v(P^C)$, thus $v(D_W^{C_{FL}}) = 0$. Therefore, $v(D_W^{C_{FL}}) > v(D_W^{C_F})$ for this setting.

It is known that in general $v(D^{C_L})$ and $v(D^{C_F})$ are not comparable. This happens for $v(D_W^{C_L})$ and $v(D_W^{C_F})$, too. In Example 1 we have $v(D_W^{C_L}) = 0 > v(D_W^{C_F}) = -\infty$, while in the situation presented in Example 5 $v(D_W^{C_L}) = -\infty < v(D_W^{C_F}) = 0$.

3. A glimpse into generalized convexity

A characteristic of many results in the literature concerning Wolfe duality and Mond–Weir duality in the differential case is the usage of different generalized convexity hypotheses, like quasiconvexity, pseudoconvexity or invexity, for the functions involved. However, generalized convexity hypotheses can be taken into account in the nondifferentiable case, too, as it can be seen in the following. To proceed, let $X = \mathbb{R}^n$ and $Y = \mathbb{R}^m$ and take the convex cone $C \subseteq \mathbb{R}^m$. We consider here only the notion of *almost convexity* (for properties we refer to [7–9]) for both sets and functions. Nevertheless, other generalized convexity notions successfully used in conjugate duality, like *near convexity* or *convexlikeness* can be employed here, too.

A set $U \subseteq \mathbb{R}^n$ is called *almost convex* if $\text{cl}(U)$ is convex and $\text{ri}(\text{cl}(U)) \subseteq U$. A function $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ is said to be *almost convex* if $\bar{f}$ is convex and $\text{ri}(\text{epi}\,\bar{f}) \subseteq \text{epi}\,f$. Moreover, a vector function $g : \mathbb{R}^n \to \mathbb{R}^m$ is said to be C-*almost convex* if $\text{epi}_C g$ is an almost convex set. First we give the following strong duality statement involving (PG) and the duals we introduced for it.

Theorem 8. *Assume that* $\Phi : \mathbb{R}^n \times \mathbb{R}^m \to \overline{\mathbb{R}}$ *is a proper and almost convex function, with its domain fulfilling* $0 \in \text{Pr}_{\mathbb{R}^m}(\text{dom}\,\Phi)$. *Let* $\bar{x} \in \mathbb{R}^n$ *be an optimal solution to* (PG) *and assume that the regularity condition*

$$0 \in \text{ri}(\text{Pr}_Y(\text{dom}\,\Phi))$$

is fulfilled. Then $v(PG) = v(DG_W) = v(DG_M)$ *and there exists* $\bar{y}^* \in \mathbb{R}^m$ *for which* $(\bar{x}, \bar{y}^*)$ *is an optimal solution to* (DG_M) *and* $(\bar{x}, 0, \bar{y}^*)$ *is one to* (DG_W).

Proof. From [8, Corollary 3.1] it is known that under these hypotheses one has $v(PG) = v(DG)$ with the latter attained at some $\bar{y}^* \in \mathbb{R}^m$. Then the optimality condition (2) holds for $\bar{x}$ and $\bar{y}^*$ and this means that $(\bar{x}, 0, \bar{y}^*)$ is feasible to (DG_W) and $(\bar{x}, \bar{y}^*)$ is feasible to (DG_M). The conclusion follows via Remark 1. □

Note that the regularity condition used in Theorem 8 is nothing but (RC_3^{Φ}) written in the framework considered in this section. Of course this statement can be particularized for the duals considered in Sections 2.1 and 2.2, too, as follows.

Theorem 9. *Assume that* $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ *and* $g : \mathbb{R}^m \to \overline{\mathbb{R}}$ *are proper and almost convex functions and* $A : \mathbb{R}^n \to \mathbb{R}^m$ *is a linear continuous mapping fulfilling* $\text{dom}\,f \cap A^{-1}(\text{dom}\,g) \neq \emptyset$. *Let* $\bar{x} \in \mathbb{R}^n$ *be an optimal solution to* (P^A) *and assume that the regularity condition*

$$A(\text{ri}(\text{dom}\,f)) \cap \text{ri}(\text{dom}\,g) \neq \emptyset$$

is fulfilled. Then $v(P^A) = v(D_W^A) = v(D_M^A)$ *and there exists* $\bar{y}^* \in \mathbb{R}^m$ *for which* $(\bar{x}, \bar{y}^*)$ *is an optimal solution to* (DG_M) *and* $(\bar{x}, 0, \bar{y}^*)$ *is one to* (DG_W).

Theorem 10. *Assume that* S *is a nonempty and almost convex set,* $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ *is a proper and almost convex function and* $g : \mathbb{R}^n \to \mathbb{R}^m$ *is* C-*almost convex vector function fulfilling the feasibility condition* $\text{dom}\,f \cap S \cap g^{-1}(-C) \neq \emptyset$. *Let* $\bar{x} \in \mathbb{R}^n$ *be an* *optimal solution to* (P^C) *and assume that the regularity condition*

$$0 \in \operatorname{ri}\big(g(\operatorname{dom} f \cap S) + C\big)$$

is fulfilled. Then $v(P^C) = v(D_W^{C_L}) = v(D_M^{C_L}) = v(D_{MW}^{C_L})$ *and there exists* $\bar{z}^* \in C^*$ *for which* $(\bar{x}, \bar{z}^*)$ *is an optimal solution to all three duals.*

Theorem 11. *Assume that* S *is a nonempty and almost convex set,* $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ *is a proper and almost convex function and* $g : \mathbb{R}^n \to \mathbb{R}^m$ *is* C*-almost convex vector function fulfilling the feasibility condition* $\operatorname{dom} f \cap S \cap g^{-1}(-C) \neq \emptyset$*. Let* $\bar{x} \in \mathbb{R}^n$ *be an optimal solution to* (P^C) *and assume that the regularity condition*

$$0 \in \operatorname{ri}\Big(\operatorname{dom} f \times C - \operatorname{epi}_{(-C)}(-g) \cap (S \times \mathbb{R}^m)\Big)$$

is fulfilled. Then $v(P^C) = v(D_W^{C_{FL}}) = v(D_M^{C_{FL}}) = v(D_{MW}^{C_{FL}})$ *and there exist* $\bar{y}^* \in \mathbb{R}^n$ *and* $\bar{z} \in C^*$ *for which* $(\bar{x}, 0, \bar{y}^*, \bar{z}^*)$ *is an optimal solution to* $(D_W^{C_{FL}})$ *and* $(\bar{x}, \bar{z}^*)$ *is an optimal solution to* $(D_M^{C_{FL}})$ *and* $(D_{MW}^{C_{FL}})$*.*

4. Conclusions and further challenges

Conclusions. We successfully embedded the classical Wolfe duality in a general duality scheme based on the perturbation theory, showing moreover that a similar approach is available for the classical Mond–Weir duality theory, too. When the perturbation function is the one that delivers in conjugate duality the Lagrange dual to a constrained optimization problem, the classical Wolfe and Mond–Weir duals are rediscovered, respectively. On the other hand, when other perturbation functions are considered, new Wolfe type and Mond–Weir type duals are obtained. The relations between the dual problems assigned to a constrained optimization problem are investigated, several not trivial examples of problems where the optimal objective values of these duals do not agree being given. As a byproduct of the general perturbational approach proposed here, one can see that these classical duality concepts, considered so far only for constrained optimization problems, can be taken into account also when dealing with unconstrained ones. For all the duals considered weak duality holds automatically, while strong duality is given under convexity assumptions, provided the fulfillment of different regularity conditions. However, since one of the main development directions of both Wolfe and Mond–Weir duality concepts involves generalized convexity assumptions for the functions involved, we opened them the gate towards other generalized convexity properties. We employed here the almost convexity, but at least also near convexity or convexlikeness can be taken into account.

Further challenges. A problem which can be posed in connection to this paper is how can be formulated a dual problem to (PG) which becomes $(D_{MW}^{C_L})$ in a particular case. Moreover, it can be interesting to study how can one give weak and strong duality statements for the primal–dual pairs of optimization problems considered in this paper when the functions involved are differentiable on an open set S and the subdifferentials are replaced by gradients in the duals by using generalized convexity notions like quasiconvexity, pseudoconvexity, even invexity.

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