

Duality for vector optimization problems via a general scalarization ^{*}

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Abstract. Considering a vector optimization problem to which properly efficient solutions are defined by using convex cone-monotone scalarization functions, we attach to it, by means of perturbation theory, new vector duals. When the primal problem, the scalarization function and the perturbation function are particularized, different dual vector problems are obtained, some of them already known in the literature. Weak and strong duality statements are delivered in each case.

Keywords. vector duality, conjugate functions, cone-monotone functions, cones

1 Introduction and preliminaries

One of the most used approaches to vector optimization problems is by attaching them scalar optimization problems whose optimal solutions deliver then efficient solutions to the original problems. The most used scalarization technique is the *linear scalarization*, but one can find in the literature different other scalarizations that fulfill the specific needs of certain problems better than the linear one, as it can be seen, for instance, in most of the literature we cite in this paper. Consequently, in works like [3, 11, 12, 15, 18, 19, 26] a general scalarization by using cone-monotone functions was proposed, sometimes in order to assign a vector dual to the original vector optimization problem. For a primal-dual pair of vector optimization problems one usually has weak vector duality and, under additional hypotheses, also strong vector duality.

In this paper we continue and extend our research from [3, 5], where we proposed a Fenchel-Lagrange type vector duality approach for constrained vector optimization problems. Here we assign via perturbations vector duals to a general vector optimization problem, with respect to two different classes of properly efficient solutions. The relations between the vector duals are investigated and in each case weak and strong duality statements are provided.

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Moreover, we present situations where there exist differences between the considered vector duals. Carefully choosing the scalarization functions we obtain then different other dual vector problems to the primal vector problem, in the case of linear scalarization rediscovering some older results from the literature. Nevertheless, particularizing the initial vector optimization problem to be first unconstrained, then constrained, for suitable choices of the perturbation function we deliver different vector duals for these classes of primal vector problems, in some cases rediscovering vector duals proposed in the literature.

Consider two separated locally convex vector spaces X and Y and their topological dual spaces X^* and, respectively, Y^* , and denote by $\langle x^*, x \rangle = x^*(x)$ the value at $x \in X$ of the linear continuous functional $x^* \in X^*$. A cone $K \subseteq X$ is a nonempty subset of X which fulfills $\lambda K \subseteq K$ for all $\lambda \geq 0$. A cone $K \subseteq X$ is said to be *nontrivial* if $K \neq \{0\}$ and $K \neq X$, while if $K \cap (-K) = \{0\}$ we call K *pointed*. When working in finitely dimensional spaces, all the vectors are considered as column vectors. An upper index T transposes a column vector to a row one and viceversa.

On Y we consider the partial ordering “ \leq_C ” induced by the convex cone $C \subseteq Y$, defined by $z \leq_C y \Leftrightarrow y - z \in C$ when $z, y \in Y$. We use also the notation $z \leq_C y$ to write more compactly that $z \leq_C y$ and $z \neq y$, where $z, y \in Y$. To Y we attach a greatest element with respect to “ \leq_C ”, which does not belong to Y , denoted by ∞_C and let $Y^\bullet = Y \cup \{\infty_C\}$. Then for any $y \in Y^\bullet$ one has $y \leq_C \infty_C$ and we consider on Y^\bullet the operations $y + \infty_C = \infty_C + y = \infty_C$ for all $y \in Y$ and $t \cdot \infty_C = \infty_C$ for all $t \geq 0$. Similarly we assume that $y \leq_C \infty_C$ for any $y \in Y$. When $\text{int}(C) \neq \emptyset$, by $z <_C y$ we mean $y - z \in \text{int}(C)$, where $z, y \in Y$, and we assume that $y <_C \infty_C$ for all $y \in Y$. The dual cone of C is $C^* = \{y^* \in Y^* : \langle y^*, y \rangle \geq 0 \ \forall y \in C\}$. By convention, $\langle v^*, \infty_C \rangle = +\infty$ for all $v^* \in C^*$.

Given a subset U of X , by $\text{cl}(U)$, δ_U and σ_U we denote its *closure*, *indicator function* and *support function*, respectively. In vector optimization it is often used also the *quasi interior of the dual cone* of K , $K^{*0} = \{x^* \in K^* : \langle x^*, x \rangle > 0 \text{ for all } x \in K \setminus \{0\}\}$. Moreover, we consider the *projection function* $\text{Pr}_X : X \times Y \rightarrow X$, defined by $\text{Pr}_X(x, y) = x$ for all $(x, y) \in X \times Y$.

Having a function $f : X \rightarrow \overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$ we use the classical notations for *domain* $\text{dom } f = \{x \in X : f(x) < +\infty\}$, *lower semicontinuous hull* $\text{cl } f : X \rightarrow \overline{\mathbb{R}}$ and *conjugate function* $f^* : X^* \rightarrow \overline{\mathbb{R}}$, $f^*(x^*) = \sup\{\langle x^*, x \rangle - f(x) : x \in X\}$. We call f *proper* if $f(x) > -\infty$ for all $x \in X$ and $\text{dom } f \neq \emptyset$. Between a function and its conjugate there is the *Young-Fenchel inequality* $f^*(x^*) + f(x) \geq \langle x^*, x \rangle$ for all $x \in X$ and $x^* \in X^*$. The *adjoint* of a linear continuous mapping $A : X \rightarrow Y$ is $A^* : Y^* \rightarrow X^*$ given by $\langle A^*y^*, x \rangle = \langle y^*, Ax \rangle$ for any $(x, y^*) \in X \times Y^*$.

Let $W \subseteq Y$. A function $g : Y \rightarrow \overline{\mathbb{R}}$, is said to be

- *C-increasing on W* if $g(x) \leq g(y)$ for all $x, y \in W$ such that $x \leq_C y$;
- *strongly C-increasing on W* if $g(x) < g(y)$ for all $x, y \in W$ such that $x \leq_C y$;
- *strictly C-increasing on W* if g is *C-increasing on W* , $\text{int}(C) \neq \emptyset$ and for all $x, y \in W$ fulfilling $x <_C y$ follows $g(x) < g(y)$.

When $W = Y$ we call these classes of functions *C-increasing*, *strongly C-increasing* and *strictly C-increasing*, respectively. When $Y = \mathbb{R}$, the \mathbb{R}_+ -increasing functions are actually the *increasing functions*, while the strongly and strictly \mathbb{R}_+ -increasing functions coincide, being the *strictly increasing functions*. If $\text{int}(C) \neq \emptyset$, we denote $\widehat{C} = \text{int}(C) \cup \{0\}$ and we have $\text{int}(\widehat{C}) = \text{int}(C)$. In this case the strictly *C-increasing* functions on a set $W \subseteq Y$ are strongly \widehat{C} -increasing functions on W , too.

A vector function $h : X \rightarrow Y^\bullet$ is called *proper* if its domain $\text{dom } h = \{x \in X : h(x) \in Y\}$ is nonempty and, respectively, *C-convex* if $h(tx + (1-t)y) \leq_C th(x) + (1-t)h(y)$ for all $x, y \in X$ and all $t \in [0, 1]$.

The vector optimization problems we consider in this paper consist of vector-minimizing and vector-maximizing a vector function with respect to the partial ordering induced in the image space of the vector function by a nontrivial pointed convex cone. For the primal vector optimization problems we define different types of properly efficient solutions, with respect to the considered scalarization functions, while for the vector duals we consider efficient and weakly efficient solutions.

2 Duality for a vector optimization problem via a general scalarization

Let X, Y and V be separated locally convex vector spaces, with V partially ordered by the nontrivial pointed convex cone $K \subseteq V$. Let us introduce now the minimality notions for sets we use later in order to consider different types of solutions to vector optimization problems. Let M be a nonempty subset of V and consider an arbitrary nonempty set of scalarization functions defined on V and taking values in $\overline{\mathbb{R}}$ denoted by \mathcal{M} .

Definition 1. An element $\bar{x} \in M$ is said to be a *minimal element of M* (regarding the partial ordering induced by K) if there is no $v \in M$ satisfying $v \leq_K \bar{x}$. The set of all minimal elements of M is denoted by $\text{Min}(M, K)$.

Definition 2. An element $\bar{v} \in M$ is said to be an *\mathcal{M} -properly minimal element of M* if there exists an $s \in \mathcal{M}$ such that $s(\bar{v}) \leq s(v)$ for all $v \in M$. The set of all \mathcal{M} -properly minimal elements of M is denoted by $\text{PMin}_{\mathcal{M}}(M, K)$.

Definition 3. Additionally assume that $\text{int}(K) \neq \emptyset$. An element $\bar{v} \in M$ is said to be a *weakly minimal element of M* (regarding the partial ordering induced by K) if $(\bar{v} - \text{int}(K)) \cap M = \emptyset$. The set of all weakly minimal elements of M is denoted by $\text{WMin}(M, K)$.

Corresponding maximality notions are defined by using the definitions from above. The elements of the set $\text{Max}(M, K) := \text{Min}(M, -K)$ are called *maximal elements of M* , while the set $\text{WMax}(M, K) := \text{WMin}(M, -K)$ contains the *weakly maximal elements of M* .

Now we formulate the vector optimization problem we shall work with. Let

$F : X \rightarrow V^\bullet$ be a proper vector function and consider the general vector-minimization problem

$$(PVG) \quad \underset{x \in X}{\text{Min}} F(x).$$

The solution concepts we consider for (PVG) follow from the ones introduced above for sets. An element $\bar{x} \in X$ fulfilling $\bar{x} \in \text{dom } F$ is said to be an *efficient solution* to the vector optimization problem (PVG) if $F(\bar{x}) \in \text{Min}(F(\text{dom } F), K)$ and, respectively, when $\text{int}(K) \neq \emptyset$, a *weakly efficient solution* to the same problem if $F(\bar{x}) \in \text{WMin}(F(\text{dom } F), K)$.

Consider now the set of *scalarization functions*

$$\mathcal{S} \subseteq \left\{ s : V \rightarrow \overline{\mathbb{R}} : F(\text{dom } F) + K \subseteq \text{dom } s \text{ and } s \text{ is proper, convex} \right. \\ \left. \text{and strongly } K\text{-increasing on } F(\text{dom } F) + K \right\}.$$

By convention, we extend every $s \in \mathcal{S}$ with the value $s(\infty_K) = +\infty$. An element $\bar{x} \in X$ is said to be an *\mathcal{S} -properly efficient solution* to (PVG) if $F(\bar{x}) \in \text{PMin}_{\mathcal{S}}(F(\text{dom } F), K)$.

Remark 1. Every \mathcal{S} -properly efficient solution to (PVG) belongs to $\text{dom } F$ and it is also an efficient solution to the same vector optimization problem and, if $\text{int}(K) \neq \emptyset$, each efficient solution to (PVG) is a weakly efficient one, too.

In order to deal with (PVG) via duality, consider now the vector perturbation function $\Phi : X \times Y \rightarrow V^\bullet$ which fulfills $\Phi(x, 0) = F(x)$ for all $x \in X$. We call Y the *perturbation space* and its elements *perturbation variables*. Then $0 \in \text{Pr}_Y(\text{dom } \Phi)$ and thus Φ is proper. The primal vector optimization problem introduced above can be reformulated as

$$(PVG) \quad \underset{x \in X}{\text{Min}} \Phi(x, 0).$$

Inspired by the way conjugate dual problems are attached to a given scalar primal problem via perturbations, we attach to (PVG) the following dual vector problems with respect to \mathcal{S} -properly efficient solutions

$$(DVG_1^{\mathcal{S}}) \quad \underset{(s, v^*, y^*, v) \in \mathcal{B}_1^{\mathcal{S}}}{\text{Max}} h_1^{G\mathcal{S}}(s, v^*, y^*, v),$$

where

$$\mathcal{B}_1^{\mathcal{S}} = \left\{ (s, v^*, y^*, v) \in \mathcal{S} \times K^* \times Y^* \times V : s(v) \leq -s^*(v^*) - (v^* \Phi)^*(0, y^*) \right\}$$

and

$$h_1^{G\mathcal{S}}(s, v^*, y^*, v) = v,$$

and, respectively,

$$(DVG_2^{\mathcal{S}}) \quad \underset{(s, y^*, v) \in \mathcal{B}_2^{\mathcal{S}}}{\text{Max}} h_2^{G\mathcal{S}}(s, y^*, v),$$

where

$$\mathcal{B}_2^{\mathcal{S}} = \left\{ (s, y^*, v) \in \mathcal{S} \times Y^* \times V : s(v) \leq -(s \circ \Phi)^*(0, y^*) \right\}$$

and

$$h_2^{GS}(s, y^*, v) = v.$$

Without resorting to the vector perturbation function Φ one can also attach to (PVG) another vector dual, inspired by the vector dual (DV^{CP}) from [5, Section 4.3.3], namely

$$(DVG_3^S) \quad \text{Max}_{(s,v) \in \mathcal{B}_3^{GS}} h_3^{GS}(s, v),$$

where

$$\mathcal{B}_3^{GS} = \left\{ (s, v) \in \mathcal{S} \times V : s(v) \leq \inf_{x \in X} s(F(x)) \right\}$$

and

$$h_3^{GS}(s, v) = v.$$

Remark 2. It is a simple verification to show that in general it holds $(s \circ \Phi)^* \leq \inf_{v^* \in K^*} [s^*(v^*) + (v^* \Phi)^*]$, thus whenever $(s, v^*, y^*, v) \in \mathcal{B}_1^{GS}$ we have $s(v) \leq -(s \circ \Phi)^*(0, y^*)$, which yields $(s, y^*, v) \in \mathcal{B}_2^{GS}$. Consequently, $h_1^{GS}(\mathcal{B}_1^{GS}) \subseteq h_2^{GS}(\mathcal{B}_2^{GS})$. Sufficient conditions for having equality in this inclusion can be found in [5, Theorem 3.5.2]; we mention here only one, namely that, provided that Φ is proper and K -convex, for each $s \in \mathcal{S}$ there exists a point $\bar{x} \in \text{dom } F$ such that s is continuous at $F(\bar{x})$. Since the scalarization functions most used in the literature (see [3, 5]) are also continuous at least over the sets they are strongly K -increasing on, we note that when the mentioned condition is satisfied the first two vector duals introduced above have the same images of their feasible sets through their objective vector functions, thus it is not necessary to particularize both of them when dealing with concrete scalarization functions from the literature in Section 3. However, we treat them in the general case since the scalarization functions need not be continuous. In Example 1 one can find a possible scalarization function of this kind, while in Example 2 we deliver another one, in a situation where (DVG_1^S) and (DVG_2^S) do not coincide.

Example 1. Let $X = \mathbb{R}$, $V = \mathbb{R}^2$, $K = \mathbb{R}_+^2$, $\mathcal{S} = \{\bar{s}\}$, $\bar{s} : \mathbb{R}^2 \rightarrow \bar{\mathbb{R}}$, $\bar{s}(x, y) = x^2 + y^2 + \delta_{\mathbb{R}_+^2}(x, y)$ and $F : \mathbb{R} \rightarrow (\mathbb{R}^2)^\bullet$, $F(x) = (x, 0)^T$ if $x \in (0, 1)$ and $F(x) = \infty_{\mathbb{R}_+^2}$ otherwise. One can easily see that $F(\text{dom } F) = (0, 1) \times \{0\}$ and $\text{dom } \bar{s} = \mathbb{R}_+^2$, thus the condition $F(\text{dom } F) + \mathbb{R}_+^2 \subseteq \text{dom } \bar{s}$ is satisfied. Moreover, the scalarization function \bar{s} is proper, convex and strongly \mathbb{R}_+^2 -increasing on \mathbb{R}_+^2 , but it is not continuous on $F(\text{dom } F)$.

Example 2. Let $X = \mathbb{R}$, $Y = \mathbb{R}$, $V = \mathbb{R}^2$, $K = \{(0, 0)\}$, $\mathcal{S} = \{\bar{s}\}$,

$$\bar{s} : \mathbb{R}^2 \rightarrow \bar{\mathbb{R}}, \quad \bar{s}(x, y) = \begin{cases} x \ln x - x + \frac{y^2}{2}, & \text{if } x > 0, y \leq 0, \\ \frac{y^2}{2}, & \text{if } x = 0, y \leq 0, \\ +\infty, & \text{otherwise,} \end{cases}$$

and

$$\Phi : \mathbb{R}^2 \rightarrow (\mathbb{R}^2)^\bullet, \quad \Phi(x, y) = \begin{cases} (x, x), & \text{if } x = y = 0 \text{ or } (x \neq 0 \text{ and } y \neq 0), \\ \infty_{\{(0,0)\}}, & \text{otherwise.} \end{cases}$$

Then the scalarization function \bar{s} is proper, convex and strongly K -increasing on its domain and $(\Phi(\cdot, 0))(\text{dom } \Phi(\cdot, 0)) + K = \{(0, 0)\} \subseteq [0, +\infty) \times (-\infty, 0] = \text{dom } \bar{s}$. Regarding the conjugates that appear in the formulation of (DVG_1^S) and (DVG_2^S) , we have

$$\bar{s}^*(v_1^*, v_2^*) = \begin{cases} e^{v_1^*} + \frac{(v_2^*)^2}{2}, & \text{if } v_1^* \in \mathbb{R}, v_2^* \leq 0, \\ e^{v_1^*}, & \text{if } v_1^* \in \mathbb{R}, v_2^* > 0, \end{cases}$$

$$((v_1^*, v_2^*)^T \Phi)^*(0, y^*) = \begin{cases} 0, & \text{if } v_1^* + v_2^* = y^* = 0, \\ +\infty, & \text{otherwise,} \end{cases}$$

and $(\bar{s} \circ \Phi)^*(0, y^*) = 0$ for all $y^* \in \mathbb{R}$. It is straightforward to see that $\bar{s}(0, 0) = 0 = -(\bar{s} \circ \Phi)^*(0, y^*)$ for all $y^* \in \mathbb{R}$, thus $(0, 0) \in h_2^{Gs}(\mathcal{B}_2^{Gs})$. On the other hand, $\bar{s}^*(v_1^*, v_2^*) > 0$ for all $v_1^*, v_2^* \in \mathbb{R}$, thus $-\bar{s}^*(v_1^*, v_2^*) - ((v_1^*, v_2^*)^T \Phi)^*(0, y^*) < 0$ whenever $v_1^*, v_2^*, y^* \in \mathbb{R}$. As $\bar{s}(0, 0) = 0$, it is obvious that $(0, 0) \notin h_1^{Gs}(\mathcal{B}_1^{Gs})$. Consequently, (DVG_1^S) and (DVG_2^S) do not coincide in this situation.

Remark 3. Note also that we have $\inf_{x \in X} s(F(x)) \geq -(s \circ \Phi)^*(0, y^*)$ for all $(s, y^*, v) \in \mathcal{B}_2^{Gs}$, thus $h_2^{Gs}(\mathcal{B}_2^{Gs}) \subseteq h_3^{Gs}(\mathcal{B}_3^{Gs})$. To show that the opposite inclusion does not always hold we consider the situation presented in Example 3. However, as it will be seen further in Theorem 4, for (DVG_3^S) strong duality holds whenever (PVG) has an \mathcal{S} -properly efficient element, so we will not insist much on this vector dual.

Example 3. Let $X = \mathbb{R}^2$, $Y = \mathbb{R}$, $C = \mathbb{R}_+$, $V = \mathbb{R}^2$, $K = \mathbb{R}_+^2$,

$$U = \left\{ (x, y)^T \in \mathbb{R}^2 : 0 \leq x \leq 2, \begin{array}{l} 3 \leq y \leq 4, \text{ if } x = 0, \\ 1 \leq y \leq 4, \text{ if } x \in (0, 2] \end{array} \right\},$$

$$F : \mathbb{R}^2 \rightarrow (\mathbb{R}^2)^\bullet, F(x, y) = \begin{cases} \begin{pmatrix} y \\ y \end{pmatrix}, & \text{if } (x, y)^T \in U, x \leq 0, \\ \infty_{\mathbb{R}_+^2}, & \text{otherwise,} \end{cases}$$

$$\Phi : \mathbb{R}^2 \times \mathbb{R} \rightarrow (\mathbb{R}^2)^\bullet, \Phi(x, y, z) = \begin{cases} \begin{pmatrix} y \\ y \end{pmatrix}, & \text{if } (x, y)^T \in U, x - z \leq 0, \\ \infty_{\mathbb{R}_+^2}, & \text{otherwise,} \end{cases}$$

and (see also Subsection 3.1)

$$\mathcal{S} = \{s : \mathbb{R}^2 \rightarrow \mathbb{R}, s(x, y) = ax + by : (a, b) \in \text{int}(\mathbb{R}_+^2)\}.$$

Note first that $F(x, y) = (y, y)^T$ if $x = 0$ and $3 \leq y \leq 4$, while otherwise $F(x, y) = \infty_{\mathbb{R}_+^2}$. Whenever $s \in \mathcal{S}$ there exist $(v_1^*, v_2^*)^T \in \text{int}(\mathbb{R}_+^2)$ such that $s \circ F = (v_1^*, v_2^*)^T F$. We have $(v_1^*, v_2^*)^T F(x, y) \geq 3(v_1^* + v_2^*)$ for all $(x, y)^T \in \mathbb{R}^2$. Consequently, $(3, 3)^T \in h_3^{Gs}(\mathcal{B}_3^{Gs})$. Assuming that $(3, 3)^T \in h_2^{Gs}(\mathcal{B}_2^{Gs})$, it follows that there exist $(v_1^*, v_2^*)^T \in \text{int}(\mathbb{R}_+^2)$ and $y^* \in \mathbb{R}$ such that $3(v_1^* +$

$v_2^*) \leq -((v_1^*, v_2^*)^T \Phi)^*(0, y^*)$, i.e. $((v_1^*, v_2^*)^T \Phi)^*(0, y^*) \leq -3(v_1^* + v_2^*)$. For all $(v_1^*, v_2^*)^T \in \text{int}(\mathbb{R}_+^2)$ and all $y^* \in \mathbb{R}$ we have

$$\begin{aligned} ((v_1^*, v_2^*)^T \Phi)^*(0, y^*) &= \sup_{\substack{(x,y)^T \in U, \\ x-z \leq 0, z \in \mathbb{R}}} \{y^* z - y(v_1^* + v_2^*)\} \\ &= \sup_{(x,y)^T \in U} \left\{ -y(v_1^* + v_2^*) + \sup_{z \geq x} y^* z \right\} \\ &= -(v_1^* + v_2^*) + \delta_{(-\infty, 0]}(y^*) > -3(v_1^* + v_2^*). \end{aligned}$$

Therefore, our assumption is false, i.e. $(3, 3)^T \notin h_2^{Gs}(\mathcal{B}_2^{Gs})$. Consequently, (DVG_2^S) and (DVG_3^S) do not coincide in this situation.

Remark 4. Replacing the inequalities from the feasible sets of the vector duals to (PVG) by equalities we obtain other vector duals to (PVG) which have smaller feasible sets. All the investigations done in this paper can be considered for those vector duals, too.

For these vector-maximization problems we consider efficient solutions, defined below for (DVG_1^S) and analogously for the others. An element $(\bar{s}, \bar{v}^*, \bar{y}^*, \bar{v}) \in \mathcal{B}_1^{Gs}$ is said to be an *efficient solution* to the vector optimization problem (DVG_1^S) if $(\bar{s}, \bar{v}^*, \bar{y}^*, \bar{v}) \in \text{dom } h_1^{Gs}$ and $h_1^{Gs}(\bar{s}, \bar{v}^*, \bar{y}^*, \bar{v}) \in \text{Max}(h_1^{Gs}(\text{dom } h_1^{Gs}), K)$. Let us show now that for the just introduced dual problems there is weak duality.

Theorem 1. *There are no $x \in X$ and $(s, v) \in \mathcal{B}_3^{Gs}$ such that $F(x) \leq_K h_3^{Gs}(s, v)$.*

Proof. Assume to the contrary that there exist $x \in X$ and $(s, v) \in \mathcal{B}_3^{Gs}$ fulfilling $F(x) \leq_K h_3^{Gs}(s, v)$. Then $x \in \text{dom } F$ and it follows $s(F(x)) < s(v)$ since $s \in \mathcal{S}$. But from the way the feasible set of the vector dual is defined we get $s(v) \leq \inf_{z \in X} s(F(z))$ and combining these two inequalities we reach a contradiction. \square

The weak duality statements for the other two vector duals can be obtained as consequences of Theorem 1, having in mind the inclusions from Remark 2 and Remark 3.

Theorem 2. *There are no $x \in X$ and $(s, v^*, y^*, v) \in \mathcal{B}_1^{Gs}$ such that $F(x) \leq_K h_1^{Gs}(s, v^*, y^*, v)$.*

Theorem 3. *There are no $x \in X$ and $(s, y^*, v) \in \mathcal{B}_2^{Gs}$ such that $F(x) \leq_K h_2^{Gs}(s, y^*, v)$.*

Next we turn our attention to strong duality for the vector duals introduced in this paper. Due to the way it is constructed, for (DVG_3^S) strong duality follows at once, without any additional assumption.

Theorem 4. *If $\bar{x} \in X$ is an \mathcal{S} -properly efficient solution to (PVG) , there exists an efficient solution $(\bar{s}, \bar{v}) \in \mathcal{B}_3^{Gs}$ to (DVG_3^S) such that $F(\bar{x}) = h_3^{Gs}(\bar{s}, \bar{v}) = \bar{v}$.*

Proof. As $\bar{x} \in X$ is a properly efficient solution to (PVG) , $F(\bar{x}) \in V$ and there exists a function $\bar{s} \in \mathcal{S}$ such that $\bar{s}(F(\bar{x})) \leq \bar{s}(F(x))$ for all $x \in X$. Thus $\bar{s}(F(\bar{x})) \leq \inf_{x \in X} \bar{s}(F(x))$. Consequently, $(\bar{s}, F(\bar{x})) \in \mathcal{B}_3^{Gs}$ and $F(\bar{x}) = h_3^{Gs}(\bar{s}, \bar{F}(\bar{x}))$. The efficiency of $(\bar{s}, F(\bar{x}))$ to (DVG_3^S) follows immediately via Theorem 1. \square

To obtain strong duality for the other two vector duals we assigned to (PVG) we need some additional hypotheses. Thus, we take the function F to be K -convex and we impose the fulfillment of a suitable regularity condition. One can introduce different regularity conditions for each of the two remaining vector duals, see for instance [2, 4, 5], but we consider here only a classical one involving continuity, namely

$$(RC^S) \quad \left| \begin{array}{l} \forall s \in \mathcal{S} \exists x' \in X \text{ such that } (x', 0) \in \text{dom } \Phi, \Phi(x', \cdot) \text{ is} \\ \text{continuous at } 0 \text{ and } s \text{ is continuous at } \Phi(x', 0). \end{array} \right.$$

This regularity condition guarantees, as can be seen in the proof of the next statement, on one hand that there is strong duality for the scalarized problem attached to (PVG) and, on the other hand, that the conjugates of s and Φ can be separated. Of course one can consider other regularity conditions, following for instance [2, 4, 5]. The strong duality statements for these two vector duals follow.

Theorem 5. *If F is a K -convex vector function, the regularity condition (RC^S) is fulfilled and $\bar{x} \in X$ is an \mathcal{S} -properly efficient solution to (PVG) , there exist $\bar{s} \in \mathcal{S}$, $\bar{v}^* \in K^*$, $\bar{y}^* \in Y^*$ and $\bar{v} \in V$ such that $(\bar{s}, \bar{v}^*, \bar{y}^*, \bar{v}) \in \mathcal{B}_1^{Gs}$ is an efficient solution to (DVG_1^S) , $(\bar{s}, \bar{y}^*, \bar{v}) \in \mathcal{B}_2^{Gs}$ is an efficient solution to (DVG_2^S) and $F(\bar{x}) = h_1^{Gs}(\bar{s}, \bar{v}^*, \bar{y}^*, \bar{v}) = h_2^{Gs}(\bar{s}, \bar{y}^*, \bar{v}) = \bar{v}$.*

Proof. As $\bar{x} \in X$ is an \mathcal{S} -properly efficient solution to (PVG) , $F(\bar{x}) \in V$ and there exists a function $\bar{s} \in \mathcal{S}$ such that $\bar{s}(F(\bar{x})) \leq \bar{s}(F(x))$ for all $x \in X$. Thus $\bar{s}(F(\bar{x})) \leq \inf_{x \in X} \bar{s}(F(x))$, consequently, $\bar{s}(F(\bar{x})) = \min_{x \in X} \bar{s}(F(x))$.

Using now [5, Theorem 3.2.1], the fulfillment of (RC^S) yields the existence of $\bar{y}^* \in Y^*$ such that $\sup_{y^* \in Y^*} \{-(\bar{s} \circ \Phi)^*(0, y^*)\}$ is attained at \bar{y}^* and $\bar{s}(F(\bar{x})) = -(\bar{s} \circ \Phi)^*(0, \bar{y}^*)$. Taking $\bar{v} = F(\bar{x})$, it follows that $(\bar{s}, \bar{y}^*, \bar{v}) \in \mathcal{B}_2^{Gs}$ and $F(\bar{x}) = h_2^{Gs}(\bar{s}, \bar{y}^*, \bar{v}) = \bar{v}$.

On the other hand, the hypotheses yield (see Remark 2) also the existence of $\bar{v}^* \in K^*$ such that $(\bar{s} \circ \Phi)^*(0, \bar{y}^*) = \bar{s}^*(\bar{v}^*) + (\bar{v}^* \Phi)^*(0, \bar{y}^*)$, thus $h_1^{Gs}(\bar{s}, \bar{v}^*, \bar{y}^*, \bar{v}) = \bar{v}$, too, and $(\bar{s}, \bar{v}^*, \bar{y}^*, \bar{v}) \in \mathcal{B}_1^{Gs}$. The efficiency of $(\bar{s}, \bar{v}^*, \bar{y}^*, \bar{v}) \in \mathcal{B}_1^{Gs}$ to (DVG_1^S) follows immediately by Theorem 2, while the one of $(\bar{s}, \bar{y}^*, \bar{v}) \in \mathcal{B}_2^{Gs}$ to (DVG_2^S) is a consequence of Theorem 3. \square

Often, when $\text{int}(K) \neq \emptyset$, the scalarization functions considered in the literature are not strongly K -increasing, but strictly K -increasing. Following ideas from [3, 5], one can notice that such scalarization functions can be brought into the vector duality framework we treat in this paper by employing the cone $\widehat{K} = \text{int}(K) \cup \{0\}$. Note that a weakly efficient solution to (PVG) is actually an efficient solution to it when working with the cone \widehat{K} . It can also be verified that every strictly K -increasing on $F(\text{dom } F) + K$ function is actually strongly \widehat{K} -increasing on $F(\text{dom } F) + K$. Consider another set of *scalarization functions*

$$\mathcal{T} \subseteq \left\{ s : V \rightarrow \overline{\mathbb{R}} : F(\text{dom } F) + K \subseteq \text{dom } s \text{ and } s \text{ is proper, convex} \right. \\ \left. \text{and strictly } K\text{-increasing on } F(\text{dom } F) + K \right\}.$$

By convention, we extend any $s \in \mathcal{T}$ with the value $s(\infty_K) = +\infty$. We say that an element $\bar{x} \in X$ is a \mathcal{T} -properly efficient solution to (PVG) if $F(\bar{x}) \in \text{PMin}_{\mathcal{T}}(F(\text{dom } F), K)$.

With respect to the \mathcal{T} -properly efficient solutions of the primal problem (PVG) one can define three vector duals that are obtained from $(DVG_i^{\mathcal{S}})$, $i = \{1, 2, 3\}$, by replacing \mathcal{S} with \mathcal{T} , namely

$$(DVG_1^{\mathcal{T}}) \quad \text{WMax}_{(s, v^*, y^*, v) \in \mathcal{B}_1^{G\mathcal{T}}} h_1^{G\mathcal{T}}(s, v^*, y^*, v),$$

where

$$\mathcal{B}_1^{G\mathcal{T}} = \left\{ (s, v^*, y^*, v) \in \mathcal{T} \times K^* \times Y^* \times V : s(v) \leq -s^*(v^*) - (v^* \Phi)^*(0, y^*) \right\}$$

and

$$h_1^{G\mathcal{T}}(s, v^*, y^*, v) = v,$$

$$(DVG_2^{\mathcal{T}}) \quad \text{WMax}_{(s, y^*, v) \in \mathcal{B}_2^{G\mathcal{T}}} h_2^{G\mathcal{T}}(s, y^*, v),$$

where

$$\mathcal{B}_2^{G\mathcal{T}} = \left\{ (s, y^*, v) \in \mathcal{T} \times Y^* \times V : s(v) \leq -(s \circ \Phi)^*(0, y^*) \right\}$$

and

$$h_2^{G\mathcal{T}}(s, y^*, v) = v,$$

and, respectively,

$$(DVG_3^{\mathcal{T}}) \quad \text{WMax}_{(s, v) \in \mathcal{B}_3^{G\mathcal{T}}} h_3^{G\mathcal{T}}(s, v),$$

where

$$\mathcal{B}_3^{G\mathcal{T}} = \left\{ (s, v) \in \mathcal{T} \times V : s(v) \leq \inf_{x \in X} s(F(x)) \right\}$$

and

$$h_3^{G\mathcal{T}}(s, v) = v.$$

One can show that

$$h_1^{G\mathcal{T}}(\mathcal{B}_1^{G\mathcal{T}}) \subseteq h_2^{G\mathcal{T}}(\mathcal{B}_2^{G\mathcal{T}}) \subseteq h_3^{G\mathcal{T}}(\mathcal{B}_3^{G\mathcal{T}}). \quad (1)$$

To these problems we consider weakly efficient solutions, directly defined only for (DVG_1^T) , since for the other two vector duals they can be given analogously. An element $(\bar{s}, \bar{v}^*, \bar{y}^*, \bar{v}) \in \mathcal{B}_1^{G^T}$ is said to be a *weakly efficient solution* to the vector optimization problem (DVG_1^T) if $(\bar{s}, \bar{v}^*, \bar{y}^*, \bar{v}) \in \text{dom } h_1^{G^T}$ and $h_1^{G^T}(\bar{s}, \bar{v}^*, \bar{y}^*, \bar{v}) \in \text{WMax}(h_1^{G^T}(\text{dom } h_1^{G^T}), K)$.

The weak and strong duality statements concerning (PVG) and these vector duals follow as direct consequences of Theorems 1–5. Note that in this case the regularity condition (RC^S) can be weakened to

$$(RC) \mid \exists x' \in X \text{ such that } (x', 0) \in \text{dom } \Phi \text{ and } \Phi(x', \cdot) \text{ is continuous at } 0,$$

because the continuity assumptions for s are no longer necessary under the hypothesis $\text{int}(K) \neq \emptyset$, as it can be seen below in the proof of Theorem 7. Assuming also all over this paper that $\text{int}(K) \neq \emptyset$ would make (RC) (and its special cases) the only regularity condition considered, since the proof of Theorem 5 could be then modified analogously to the one of Theorem 7.

Theorem 6.

- (a) *There are no $x \in X$ and $(s, v^*, y^*, v) \in \mathcal{B}_1^{G^T}$ such that $F(x) <_K h_1^{G^T}(s, v^*, y^*, v)$.*
- (b) *There are no $x \in X$ and $(s, y^*, v) \in \mathcal{B}_2^{G^T}$ such that $F(x) <_K h_2^{G^T}(s, y^*, v)$.*
- (c) *There are no $x \in X$ and $(s, v) \in \mathcal{B}_3^{G^T}$ such that $F(x) <_K h_3^{G^T}(s, v)$.*

Theorem 7.

- (a) *If $\bar{x} \in X$ is a \mathcal{T} -properly efficient solution to (PVG) , there exist $\bar{s} \in \mathcal{T}$ and $\bar{v} \in V$ such that $(\bar{s}, \bar{v}) \in \mathcal{B}_3^{G^T}$ is a weakly efficient solution to (DVG_3^T) and $F(\bar{x}) = h_3^{G^T}(\bar{s}, \bar{v}) = \bar{v}$.*
- (b) *If F is a K -convex vector function, the regularity condition (RC) is fulfilled and $\bar{x} \in X$ is a \mathcal{T} -properly efficient solution to (PVG) , there exist $\bar{s} \in \mathcal{T}$, $\bar{v}^* \in K^*$, $\bar{y}^* \in Y^*$ and $\bar{v} \in V$ such that $(\bar{s}, \bar{v}^*, \bar{y}^*, \bar{v}) \in \mathcal{B}_1^{G^T}$ is a weakly efficient solution to (DVG_1^T) , $(\bar{s}, \bar{v}^*, \bar{v}) \in \mathcal{B}_2^{G^T}$ is a weakly efficient solution to (DVG_2^T) and $F(\bar{x}) = h_1^{G^T}(\bar{s}, \bar{v}^*, \bar{y}^*, \bar{v}) = h_2^{G^T}(\bar{s}, \bar{y}^*, \bar{v}) = \bar{v}$.*

Proof. We prove only (b), since (a) can be shown analogously to the proof of Theorem 4. As $\bar{x} \in X$ is a \mathcal{T} -properly efficient solution to (PVG) , $F(\bar{x}) \in V$ and there exists a function $\bar{s} \in \mathcal{T}$ such that $\bar{s}(F(\bar{x})) \leq \bar{s}(F(x))$ for all $x \in X$. Thus $\bar{s}(F(\bar{x})) \leq \inf_{x \in X} \bar{s}(F(x))$, consequently, $\bar{s}(F(\bar{x})) = \min_{x \in X} \bar{s}(F(x))$. Note also that the optimization problem

$$\inf_{x \in X} \bar{s}(F(x)),$$

is actually nothing else than

$$\inf_{\substack{x \in X, y \in Y, \\ \Phi(x, 0) - y \in -K}} \bar{s}(y).$$

To the latter problem we attach its Lagrange dual

$$\sup_{v^* \in K^*} \inf_{\substack{x \in X, \\ y \in Y}} [\bar{s}(y) + \langle v^*, \Phi(x, 0) - y \rangle],$$

which can be rewritten as

$$\sup_{v^* \in K^*} \{ -\bar{s}^*(v^*) - ((v^* \Phi)(\cdot, 0))^*(0) \}.$$

Because (RC) holds, we obtain an $x' \in \text{dom } F$ such that $\Phi(x', 0) + \text{int}(K) \subseteq \text{dom } \bar{s}$ and also a $y' \in \text{dom } \bar{s}$ such that $\Phi(x', 0) - y' \in -\text{int}(K)$. Using now [5, Theorem 3.2.9], we obtain that for the primal-dual pair of scalar optimization problems introduced above there is strong duality, thus there exists $\bar{v}^* \in K^*$ such that $\bar{s}(F(\bar{x})) = -\bar{s}^*(\bar{v}^*) - ((\bar{v}^* \Phi)(\cdot, 0))^*(0)$. Applying now [5, Theorem 3.2.1], (RC) yields also the existence of $\bar{y}^* \in Y^*$ such that $((\bar{v}^* \Phi)(\cdot, 0))^*(0) = (\bar{v}^* \Phi)^*(0, \bar{y}^*)$.

Taking $\bar{v} = F(\bar{x})$, it follows that $(\bar{s}, \bar{v}^*, \bar{y}^*, \bar{v}) \in \mathcal{B}_1^{G^T}$ and $F(\bar{x}) = h_1^{G^T}(\bar{s}, \bar{v}^*, \bar{y}^*, \bar{v}) = \bar{v}$. Using (1) (see also Remark 2), it follows that $(\bar{s}, \bar{y}^*, \bar{v}) \in \mathcal{B}_2^{G^T}$ and $h_2^{G^T}(\bar{s}, \bar{y}^*, \bar{v}) = \bar{v}$.

The weak efficiency of $(\bar{s}, \bar{v}^*, \bar{y}^*, \bar{v}) \in \mathcal{B}_1^{G^T}$ to (DVG_1^T) follows immediately by Theorem 6(a), while the one of $(\bar{s}, \bar{y}^*, \bar{v}) \in \mathcal{B}_2^{G^T}$ to (DVG_2^T) is a consequence of Theorem 6(b). \square

3 Vector duality via several particular scalarizations

In the third section of this paper we consider several concrete scalarization functions, obtaining vector duals and corresponding duality statements by particularizing the set \mathcal{S} or \mathcal{T} , respectively. Namely, we present the linear scalarization, the set scalarization and the (semi)norm scalarization. Since all these scalarizations are made with continuous functions, taking into consideration Remark 2 and Theorem 4 it is clear that it makes no sense to deal in this section with all the vector duals we considered to (PVG) , thus we work only with (DVG_1^S) and (DVG_1^T) , respectively.

Before proceeding, let us mention that different other scalarizations were considered in the literature, from which we recall some here. From the scalarizations involving strongly K -increasing functions we mention the one using continuous sublinear functions from [28], the one with quadratic functions in finite dimensional spaces from [9] (see also [3]) and the one containing penalty functions from [37]. Regarding the scalarizations involving strictly K -increasing functions, we have the one with continuous sublinear functions from [23], the maximum-linear scalarization met in papers like [16, 25, 27] (see also [3, 5]) with its special case the weighted Tchebyshev scalarization for which we refer to [10, 19, 20, 24, 33], the scalarization with oriented distances from [39] and the bottleneck scalarization considered in [17].

3.1 Linear scalarization

The *linear scalarization* is the most famous and used scalarization method in the literature and it operates with strongly (and also strictly) K -increasing linear continuous functions. From the huge amount of works where it appears we mention here only [1, 5, 19, 26]. We deal first with the case of the strongly K -increasing linear functions. Take the set of scalarization functions

$$\mathcal{S}_l = \{s_{v^*} : V \rightarrow \mathbb{R} : v^* \in K^{*0}, s_{v^*}(v) = \langle v^*, v \rangle \forall v \in V\}$$

and recall that like in the general case every function $s_{v^*} \in \mathcal{S}_l$ is extended with the value $s_{v^*}(\infty_K) = +\infty$. Each $s_{v^*} \in \mathcal{S}_l$ is a linear continuous strongly K -increasing function and $\text{dom } s_{v^*} = V$.

Note that the \mathcal{S}_l -properly efficient solutions to (PVG) are actually the classical properly efficient solutions in the sense of linear scalarization to it. Noticing that for all $k^* \in K^*$ one has $s_{v^*}^*(k^*) = \delta_{\{v^*\}}(k^*)$, the dual vector problem $(DVG_1^{\mathcal{S}_l})$ becomes

$$(DVG_1^{\mathcal{S}_l}) \quad \text{Max}_{(v^*, y^*, v) \in \mathcal{B}_1^{G_{\mathcal{S}_l}}} h_1^{G_{\mathcal{S}_l}}(v^*, y^*, v),$$

where

$$\mathcal{B}_1^{G_{\mathcal{S}_l}} = \{(v^*, y^*, v) \in K^{*0} \times Y^* \times V : \langle v^*, v \rangle \leq -(v^* \Phi)^*(0, y^*)\}$$

and

$$h_1^{G_{\mathcal{S}_l}}(v^*, y^*, v) = v.$$

Note that this is actually the vector dual to (PVG) considered in [14] and [5, Section 4.3]. The weak and strong duality statements for (PVG) and $(DVG_1^{\mathcal{S}_l})$ follow, with the remark that due to the continuity of the scalarization function the regularity condition we consider is (RC) .

Theorem 8.

- (a) *There are no $x \in X$ and $(v^*, y^*, v) \in \mathcal{B}_1^{G_{\mathcal{S}_l}}$ such that $F(x) \leq_K h_1^{G_{\mathcal{S}_l}}(v^*, y^*, v)$.*
- (b) *If F is a K -convex vector function, the regularity condition (RC) is fulfilled and $\bar{x} \in X$ is an \mathcal{S}_l -properly efficient solution to (PVG) , there exist $\bar{v}^* \in K^{*0}$, $\bar{y}^* \in Y^*$ and $\bar{v} \in V$ such that $(\bar{v}^*, \bar{y}^*, \bar{v}) \in \mathcal{B}_1^{G_{\mathcal{S}_l}}$ is an efficient solution to $(DVG_1^{\mathcal{S}_l})$ and $F(\bar{x}) = h_1^{G_{\mathcal{S}_l}}(\bar{v}^*, \bar{y}^*, \bar{v}) = \bar{v}$.*

On the other hand, in the case $\text{int}(K) \neq \emptyset$ one can take as set of scalarization functions also

$$\mathcal{T}_l = \{s_{v^*} : V \rightarrow \mathbb{R} : v^* \in K^* \setminus \{0\}, s_{v^*}(v) = \langle v^*, v \rangle \forall v \in V\}$$

with every function $s_{v^*} \in \mathcal{T}_l$ extended with the value $s_{v^*}(\infty_K) = +\infty$. Each $s_{v^*} \in \mathcal{T}_l$ is a linear continuous strictly K -increasing function and $\text{dom } s_{v^*} = V$.

Note that the \mathcal{T}_l -properly efficient solutions to (PVG) are actually the weakly efficient solutions to it. The dual vector problem $(DVG_1^{\mathcal{T}_l})$ becomes

in this case

$$(DVG_1^{T_l}) \quad \text{WMax}_{(v^*, y^*, v) \in \mathcal{B}_1^{G_{T_l}}} h_1^{G_{T_l}}(v^*, y^*, v),$$

where

$$\mathcal{B}_1^{G_{T_l}} = \left\{ (v^*, y^*, v) \in (K^* \setminus \{0\}) \times Y^* \times V : \langle v^*, v \rangle \leq -(v^* \Phi)^*(0, y^*) \right\}$$

and

$$h_1^{G_{T_l}}(v^*, y^*, v) = v.$$

Note that this is actually the vector dual to (PVG) considered in [5, Subsection 4.3.4]. The weak and strong duality statements for (PVG) and $(DVG_1^{T_l})$ follow.

Theorem 9.

- (a) *There are no $x \in X$ and $(v^*, y^*, v) \in \mathcal{B}_1^{G_{T_l}}$ such that $F(x) <_K h_1^{G_{T_l}}(v^*, y^*, v)$.*
- (b) *If F is a K -convex vector function, the regularity condition (RC) is fulfilled and $\bar{x} \in X$ is a weakly efficient solution to (PVG) , there exist $\bar{v}^* \in K^* \setminus \{0\}$, $\bar{y}^* \in Y^*$ and $\bar{v} \in V$ such that $(\bar{v}^*, \bar{y}^*, \bar{v}) \in \mathcal{B}_1^{G_{T_l}}$ is a weakly efficient solution to $(DVG_1^{T_l})$ and $F(\bar{x}) = h_1^{G_{T_l}}(\bar{v}^*, \bar{y}^*, \bar{v}) = \bar{v}$.*

3.2 Set scalarization

As *set scalarizations* we understand the scalarization approaches for which the scalarization functions are defined by means of some given sets. We consider here a quite general scalarization function inspired by the one used in works like [13, 31, 32, 35]. In this subsection $\text{int}(K)$ is taken nonempty.

Consider a fixed nonempty convex set $E \subseteq V$ which satisfies $\text{cl}(E) + \text{int}(K) \subseteq \text{int}(E)$. For all $\mu \in \text{int}(K)$ we define the scalarization function $s_\mu : V \rightarrow \mathbb{R}$ by

$$s_\mu(v) = \inf \{ t \in \mathbb{R} : v \in t\mu - \text{cl}(E) \},$$

extended with the value $s_\mu(\infty_K) = +\infty$. According to [13, 35], for $\mu \in \text{int}(K)$ the function s_μ is convex, continuous and strictly K -increasing. The set of scalarization functions is then

$$\mathcal{T}_s = \{ s_\mu : V \rightarrow \mathbb{R} : \mu \in \text{int}(K) \}.$$

An element $\bar{x} \in X$ is a \mathcal{T}_s -properly efficient solution to (PVG) if there exists $\mu \in \text{int}(K)$ such that $s_\mu(F(\bar{x})) \leq s_\mu(F(x))$ for all $x \in X$. Since the conjugate function of s_μ , $\mu \in \text{int}(K)$, is (cf. [3, 5])

$$s_\mu^* : V^* \rightarrow \overline{\mathbb{R}}, \quad s_\mu^*(v^*) = \begin{cases} \sigma_{-\text{cl}(E)}(v^*), & \text{if } \langle v^*, \mu \rangle = 1, \\ +\infty, & \text{otherwise,} \end{cases}$$

the dual vector problem attached to (PVG) via the set scalarization turns out to be

$$(DVG_1^{T_s}) \quad \text{WMax}_{(\mu, v^*, y^*, v) \in \mathcal{B}_1^{G_{T_s}}} h_1^{G_{T_s}}(\mu, v^*, y^*, v),$$

where

$$\mathcal{B}_1^{G_{T_s}} = \left\{ (\mu, v^*, y^*, v) \in \text{int}(K) \times (K^* \setminus \{0\}) \times Y^* \times V : \langle v^*, \mu \rangle = 1, \right. \\ \left. s_\mu(v) \leq -\sigma_{-\text{cl}(E)}(v^*) - (v^* \Phi)^*(0, y^*) \right\}$$

and

$$h_1^{G_{T_s}}(\mu, v^*, y^*, v) = v.$$

The weak and strong duality statements for (PVG) and $(DVG_1^{T_i})$ follow.

Theorem 10.

- (a) *There are no $x \in X$ and $(\mu, v^*, y^*, v) \in \mathcal{B}_1^{G_{T_s}}$ such that $F(x) <_K h_1^{G_{S_s}}(\mu, v^*, y^*, v)$.*
- (b) *If F is a K -convex vector function, the regularity condition (RC) is fulfilled and $\bar{x} \in X$ is a weakly efficient solution to (PVG) , there exist $\bar{\mu} \in \text{int}(K)$, $\bar{v}^* \in K^* \setminus \{0\}$, $\bar{y}^* \in Y^*$ and $\bar{v} \in V$ such that $(\bar{\mu}, \bar{v}^*, \bar{y}^*, \bar{v}) \in \mathcal{B}_1^{G_{T_s}}$ is a weakly efficient solution to $(DVG_1^{T_s})$ and $F(\bar{x}) = h_1^{G_{T_s}}(\bar{\mu}, \bar{v}^*, \bar{y}^*, \bar{v}) = \bar{v}$.*

In the literature there are some interesting special cases of the set scalarization, from which we mention here the *scalarization with conical sets*, where $E = K$ and, since K is a convex cone, the condition $\text{cl}(E) + \text{int}(K) \subseteq \text{int}(K)$ is automatically fulfilled as equality, mentioned in papers like [28, 30], the *scalarization with sets generated by norms* for which we refer to [32, 38], having as a subcase the situation when oblique norms are employed and the solutions of (PVG) are then \mathcal{S} -properly efficient (see [29, 32]), and, finally, the *scalarization with polyhedral sets* treated in [36]. Note also that in [35] can be found a deeper analysis of an approach for embedding older classical scalarization functions into the set scalarization concept and that the set scalarization with its special instances was employed into vector duality in [3, 5].

3.3 (Semi)Norm scalarization

The *(semi)norm scalarization* has its roots in the fact that in some circumstances some (semi)norms on V turn out to be strongly K -increasing functions, as noted in different works from which we recall here only [19, 21, 22, 29, 37]. The scalarization functions we investigate in the following are based on strongly K -increasing gauges. This kind of scalarization functions has been used in [34] for location problems and in [7] for goal programming, but also papers like [8, 18, 26, 40] can be mentioned here since they contain different scalarizations involving (semi)norms.

Assume first that there exists $b \in V$ such that $\Phi(\text{dom } \Phi) \subseteq b + K$. We consider $E \subseteq V$ a convex set such that $0 \in \text{int}(E)$ and its *gauge (Minkowski function)* $\gamma_E : X \rightarrow \mathbb{R}$, defined by $\gamma_E(x) = \inf\{\lambda \geq 0 : x \in \lambda E\}$, is strongly K -increasing on K . Since $0 \in \text{int}(E)$ it yields that $\gamma_E(v) \in \mathbb{R}$ for all $v \in V$.

For every $a \in b - K$ define the scalarization function $s_a : V \rightarrow \overline{\mathbb{R}}$ by

$$s_a(v) = \begin{cases} \gamma_E(v - a), & \text{if } v \in b + K, \\ +\infty, & \text{otherwise,} \end{cases}$$

extended with the value $s_a(\infty_K) = +\infty$. All these functions are convex, continuous, because $0 \in \text{int}(E)$, and strongly K -increasing on $b + K$ and one also has $F(\text{dom } F) \subseteq b + K = \text{dom } s_a$ for all $a \in b - K$.

Considering the following family of scalarization functions

$$\mathcal{S}_g = \{s_a : V \rightarrow \overline{\mathbb{R}} : a \in b - K\},$$

we say that an element $\bar{x} \in X$ is an \mathcal{S}_g -properly efficient solution to (PVG) if there exists $a \in b - K$ such that $s_a(F(\bar{x})) \leq s_a(F(x))$ for all $x \in X$. Since from [3, 5] we know that

$$(s_a)^*(v^*) = \langle v^*, a \rangle + \min_{\substack{w^* \in -K^*, \\ \sigma_E(v^* - w^*) \leq 1}} \langle w^*, b - a \rangle \quad \forall v^* \in V^*,$$

the dual vector problem to (PVG) with respect to this scalarization is

$$(DVG_1^{\mathcal{S}_g}) \quad \text{Max}_{(a, v^*, y^*, w^*, v) \in \mathcal{B}_1^{G\mathcal{S}_g}} h_1^{G\mathcal{S}_g}(a, v^*, y^*, w^*, v),$$

where

$$\mathcal{B}_1^{G\mathcal{S}_g} = \left\{ (a, v^*, y^*, w^*, v) \in (b - K) \times K^{*0} \times Y^* \times (-K^*) \times (b + K) : \right. \\ \left. \sigma_E(v^* - w^*) \leq 1, \gamma_E(v - a) \leq \langle w^*, a - b \rangle - \langle v^*, a \rangle - (v^* \Phi)^*(0, y^*) \right\}$$

and

$$h_1^{G\mathcal{S}_g}(a, v^*, y^*, w^*, v) = v.$$

The weak and strong duality statements for (PVG) and $(DVG_1^{\mathcal{S}_g})$ follow, with the remark that due to the continuity of the scalarization function the regularity condition we consider is (RC) .

Theorem 11.

- (a) There are no $x \in X$ and $(a, v^*, y^*, w^*, v) \in \mathcal{B}_1^{G\mathcal{S}_g}$ such that $F(x) \leq_K h_1^{G\mathcal{S}_g}(a, v^*, y^*, w^*, v)$.
- (b) If F is a K -convex vector function, the regularity condition (RC) is fulfilled and $\bar{x} \in X$ is an \mathcal{S}_g -properly efficient solution to (PVG) , there exist $\bar{a} \in b - K$, $\bar{v}^* \in K^{*0}$, $\bar{y}^* \in Y^*$, $\bar{w}^* \in -K^*$ and $\bar{v} \in b + K$ such that $(\bar{a}, \bar{v}^*, \bar{y}^*, \bar{w}^*, \bar{v}) \in \mathcal{B}_1^{G\mathcal{S}_g}$ is an efficient solution to $(DVG_1^{\mathcal{S}_g})$ and $F(\bar{x}) = h_1^{G\mathcal{S}_g}(\bar{a}, \bar{v}^*, \bar{y}^*, \bar{w}^*, \bar{v}) = \bar{v}$.

Note that the duality approach described in this subsection can be considered also in the particular case when γ_E is a norm with the unit ball E .

Remark 5. If V is a Hilbert space, then the norm of V is strongly K -increasing on K if and only if $K \subseteq K^*$ (cf. [19]). This is the case if, for instance, $V = \mathbb{R}^k$ and K is the nonnegative orthant in \mathbb{R}^k . Not only the Euclidean norm is strongly \mathbb{R}_+^k -increasing on \mathbb{R}_+^k , but also the oblique norms (cf. [29, 32]) are strongly \mathbb{R}_+^k -increasing on \mathbb{R}_+^k . Other conditions which ensure that a norm is strongly K -increasing on a given set have been investigated in [18, 19, 37].

4 Vector duality for particular instances of (PVG)

This section is dedicated to the implementation of the vector duality approach introduced in Section 2 for two large classes of vector optimization problems, that can be obtained as special cases of (PVG) . For carefully chosen perturbation functions we obtain vector duals for these particular primal vector problems via the general scalarization approach considered in this paper. In some places we rediscover duals already known in the literature, pointing this out where is the case.

4.1 Vector duality for unconstrained vector optimization problems

Let $f : X \rightarrow V^\bullet$ and $g : Y \rightarrow V^\bullet$ be given proper vector functions and $A : X \rightarrow Y$ a linear continuous mapping such that $\text{dom } f \cap A^{-1}(\text{dom } g) \neq \emptyset$. The primal unconstrained vector optimization problem we consider is

$$(PVA) \quad \text{Min}_{x \in X} [f(x) + g(Ax)].$$

Since (PVA) is a special case of (PVG) obtained by taking $F = f + g \circ A$, we use the approach developed in the second section in order to deal with it via duality. More precisely, for a convenient choice of the vector perturbation function Φ we obtain vector duals to (PVA) which are special cases of (DVG_1^S) and (DVG_1^T) , respectively.

In order to attach dual vector problems to (PVA) , consider the vector perturbation function

$$\Phi^A : X \times Y \rightarrow V^\bullet, \quad \Phi^A(x, y) = f(x) + g(Ax + y).$$

For $v^* \in K^*$ and $y^* \in Y^*$ one has $(v^* \Phi^A)^*(0, y^*) = (v^* f)^*(-A^* y^*) + (v^* g)^*(y^*)$. Now we are ready to formulate the vector duals to (PVA) that are special cases of (DVG_1^S) and (DVG_1^T) , namely

$$(DVA_1^S) \quad \text{Max}_{(s, v^*, y^*, v) \in \mathcal{B}_1^{AS}} h_1^{AS}(s, v^*, y^*, v),$$

where

$$\mathcal{B}_1^{AS} = \left\{ (s, v^*, y^*, v) \in \mathcal{S} \times K^* \times Y^* \times V : \right. \\ \left. s(v) \leq -s^*(v^*) - (v^* f)^*(-A^* y^*) - (v^* g)^*(y^*) \right\}$$

and

$$h_1^{AS}(s, v^*, y^*, v) = v,$$

and, when $\text{int}(K) \neq \emptyset$,

$$(DVA_1^T) \quad \text{WMax}_{(s, v^*, y^*, v) \in \mathcal{B}_1^{AT}} h_1^{AT}(s, v^*, y^*, v),$$

where

$$\mathcal{B}_1^{AT} = \left\{ (s, v^*, y^*, v) \in \mathcal{T} \times K^* \times Y^* \times V : \right. \\ \left. s(v) \leq -s^*(v^*) - (v^* f)^*(-A^* y^*) - (v^* g)^*(y^*) \right\}$$

and

$$h_1^{AT}(s, v^*, y^*, v) = v.$$

Due to the fact that in the case $V = \mathbb{R}$ these duals turn out to coincide with the classical Fenchel dual problem to the scalar optimization problem (PVA) we say that these two vector duals are the *Fenchel vector duals* to (PVA). The weak and strong duality statements for these two vector duals to (PVA) follow as special instances of Theorem 2, Theorem 5 and Theorem 7, with the regularity conditions from the general case becoming

$$(RCA^S) \quad \left| \begin{array}{l} \forall s \in \mathcal{S} \exists x' \in \text{dom } f \cap A^{-1}(\text{dom } g) \text{ such that } g \text{ is continuous} \\ \text{at } Ax' \text{ and } s \text{ is continuous at } f(x') + g(Ax'), \end{array} \right.$$

and, respectively,

$$(RCA) \quad \left| \exists x' \in \text{dom } f \cap A^{-1}(\text{dom } g) \text{ such that } g \text{ is continuous at } Ax' \right.$$

Theorem 12.

- (a) *There are no $x \in X$ and $(s, v^*, y^*, v) \in \mathcal{B}_1^{AS}$ such that $f(x) + g(Ax) \leq_K h_1^{AS}(s, v^*, y^*, v)$.*
- (b) *If f and g are K -convex vector functions, the regularity condition (RCA^S) is fulfilled and $\bar{x} \in X$ is an \mathcal{S} -properly efficient solution to (PVA), there exist $\bar{s} \in \mathcal{S}$, $\bar{v}^* \in K^*$, $\bar{y}^* \in Y^*$ and $\bar{v} \in V$ such that $(\bar{s}, \bar{v}^*, \bar{y}^*, \bar{v}) \in \mathcal{B}_1^{AS}$ is an efficient solution to (DVA_1^S) and $f(\bar{x}) + g(A\bar{x}) = h_1^{AS}(\bar{s}, \bar{v}^*, \bar{y}^*, \bar{v}) = \bar{v}$.*

Theorem 13. *Let $\text{int}(K) \neq \emptyset$.*

- (a) *There are no $x \in X$ and $(s, v^*, y^*, v) \in \mathcal{B}_1^{AT}$ such that $f(x) + g(Ax) <_K h_1^{AT}(s, v^*, y^*, v)$.*
- (b) *If f and g are K -convex vector functions, the regularity condition (RCA) is fulfilled and $\bar{x} \in X$ is a \mathcal{T} -properly efficient solution to (PVA), there exist $\bar{s} \in \mathcal{T}$, $\bar{v}^* \in K^*$, $\bar{y}^* \in Y^*$ and $\bar{v} \in V$ such that $(\bar{s}, \bar{v}^*, \bar{y}^*, \bar{v}) \in \mathcal{B}_1^{AT}$ is a weakly efficient solution to (DVA_1^T) and $f(\bar{x}) + g(A\bar{x}) = h_1^{AT}(\bar{s}, \bar{v}^*, \bar{y}^*, \bar{v}) = \bar{v}$.*

Special instances of the dual vector problems considered in this subsection can be found in [5, Section 4.1], where the linear scalarization is considered.

4.2 Vector duality for constrained vector optimization problems

Consider the nonempty convex set $S \subseteq X$ and the proper vector functions $f : X \rightarrow V^\bullet$ and $g : X \rightarrow Y^\bullet$ fulfilling $\text{dom } f \cap S \cap g^{-1}(C) \neq \emptyset$. Let the primal vector optimization problem with geometric and cone constraints

$$(PVC) \quad \underset{x \in \mathcal{A}}{\text{Min}} f(x),$$

where

$$\mathcal{A} = \{x \in S : g(x) \in -C\}.$$

Since (PVC) is a special case of (PVG) obtained by taking

$$F : X \rightarrow V^\bullet, F(x) = \begin{cases} f(x), & \text{if } x \in \mathcal{A}, \\ \infty_K, & \text{otherwise,} \end{cases}$$

we use the approach developed in the Section 2 in order to deal with it via duality. More precisely, for convenient choices of the vector perturbation function Φ we obtain vector duals to (PVC) which are special cases of (DVG_1^S) and (DVG_1^T) .

Consider first the Lagrange type vector perturbation function

$$\Phi^{CL} : X \times Y \rightarrow V^\bullet, \Phi^{CL}(x, y) = \begin{cases} f(x), & \text{if } x \in S, g(x) \in y - C, \\ \infty_K, & \text{otherwise.} \end{cases}$$

For $v^* \in K^*$ and $y^* \in Y^*$ we have $(v^* \Phi^{CL})^*(0, y^*) = ((v^* f) - (y^* g) + \delta_S)^*(0) + \delta_{-C^*}(y^*)$, so the *Lagrange vector duals* to (PVC) are (note the change of sign of y^*)

$$(DVCL_1^S) \quad \underset{(s, v^*, y^*, v) \in \mathcal{B}_1^{CLS}}{\text{Max}} h_1^{CLS}(s, v^*, y^*, v),$$

where

$$\mathcal{B}_1^{CLS} = \{(s, v^*, y^*, v) \in \mathcal{S} \times K^* \times C^* \times V : s(v) \leq -s^*(v^*) - ((v^* f) + (y^* g) + \delta_S)^*(0)\}$$

and

$$h_1^{CLS}(s, v^*, y^*, v) = v,$$

and, when $\text{int}(K) \neq \emptyset$,

$$(DVCL_1^T) \quad \underset{(s, v^*, y^*, v) \in \mathcal{B}_1^{CLT}}{\text{WMax}} h_1^{CLT}(s, v^*, y^*, v),$$

where

$$\mathcal{B}_1^{CLT} = \{(s, v^*, y^*, v) \in \mathcal{T} \times K^* \times C^* \times V : s(v) \leq -s^*(v^*) - ((v^* f) + (y^* g) + \delta_S)^*(0)\}$$

and

$$h_1^{CLT}(s, v^*, y^*, v) = v.$$

The weak and strong duality statements follow as special instances of Theorem 2, Theorem 5 and Theorem 7, with the regularity conditions becoming

$$(RCCL^{\mathcal{S}}) \quad \left| \begin{array}{l} \forall s \in \mathcal{S} \exists x' \in \text{dom } f \cap S \text{ such that } g(x') \in -\text{int}(C) \\ \text{and } s \text{ is continuous at } f(x'), \end{array} \right.$$

and, respectively,

$$(RCCL) \quad \left| \quad \exists x' \in \text{dom } f \cap S \text{ such that } g(x') \in -\text{int}(C), \right.$$

which is the classical *Slater constraint qualification* extended to the vector case.

Theorem 14.

- (a) *There are no $x \in X$ and $(s, v^*, y^*, v) \in \mathcal{B}_1^{CL\mathcal{S}}$ such that $f(x) \leq_K h_1^{CL\mathcal{S}}(s, v^*, y^*, v)$.*
- (b) *If f is a K -convex vector function, g is a C -convex vector function, the regularity condition $(RCCL^{\mathcal{S}})$ is fulfilled and $\bar{x} \in X$ is an \mathcal{S} -properly efficient solution to (PVC) , there exist $\bar{s} \in \mathcal{S}$, $\bar{v}^* \in K^*$, $\bar{y}^* \in C^*$ and $\bar{v} \in V$ such that $(\bar{s}, \bar{v}^*, \bar{y}^*, \bar{v}) \in \mathcal{B}_1^{CL\mathcal{S}}$ is an efficient solution to $(DVCL_1^{\mathcal{S}})$ and $f(\bar{x}) = h_1^{CL\mathcal{S}}(\bar{s}, \bar{v}^*, \bar{y}^*, \bar{v}) = \bar{v}$.*

Theorem 15. *Let $\text{int}(K) \neq \emptyset$.*

- (a) *There are no $x \in X$ and $(s, v^*, y^*, v) \in \mathcal{B}_1^{CL\mathcal{T}}$ such that $f(x) <_K h_1^{CL\mathcal{T}}(s, v^*, y^*, v)$.*
- (b) *If f is a K -convex vector function, g is a C -convex vector function, the regularity condition $(RCCL)$ is fulfilled and $\bar{x} \in X$ is a \mathcal{T} -properly efficient solution to (PVC) , there exist $\bar{s} \in \mathcal{T}$, $\bar{v}^* \in K^*$, $\bar{y}^* \in C^*$ and $\bar{v} \in V$ such that $(\bar{s}, \bar{v}^*, \bar{y}^*, \bar{v}) \in \mathcal{B}_1^{CL\mathcal{T}}$ is a weakly efficient solution to $(DVCL_1^{\mathcal{T}})$ and $f(\bar{x}) = h_1^{CL\mathcal{T}}(\bar{s}, \bar{v}^*, \bar{y}^*, \bar{v}) = \bar{v}$.*

When (PVG) is particularized to (PVC) and Φ to Φ^{CL} , one can find in the literature special instances of all three vector duals with respect to \mathcal{S} -properly efficient solutions proposed in this article. For instance, the vector dual obtained from $(DVG_3^{\mathcal{S}})$ by means of linear scalarization was considered in [5], while in [12] one can find the vector dual which is a special case of $(DVG_2^{\mathcal{S}})$, but with the scalarization functions taken moreover continuous. In [15], in the same framework, a vector dual similar to $(DVG_2^{\mathcal{S}})$ is considered, but with the inequality from the constraints replaced by equality. In [5, 18, 19] it is treated the vector dual obtained in this framework from $(DVG_1^{\mathcal{S}})$ by using the linear scalarization. Regarding the vector duals with respect to \mathcal{T} -properly efficient solutions, the one obtained from $(DVG_3^{\mathcal{T}})$ by means of linear scalarization was considered in [5], in [31] the special case of $(DVG_2^{\mathcal{S}})$ in this framework is mentioned, but with the scalarization functions taken also continuous, while $(DVG_1^{\mathcal{T}})$ via the linear scalarization can be found in [5, 19].

A second vector perturbation function that can be considered for (PVC) is the Fenchel type vector perturbation function

$$\Phi^{CF} : X \times X \rightarrow V^{\bullet}, \quad \Phi^{CF}(x, y) = \begin{cases} f(x + y), & \text{if } x \in \mathcal{A}, \\ \infty_K, & \text{otherwise.} \end{cases}$$

Using it the following *Fenchel vector duals* obtained as special cases of (DVG_1^S) and (DVG_1^T) can be attached to (PVC)

$$(DVC F_1^S) \quad \text{Max}_{(s, v^*, y^*, v) \in \mathcal{B}_1^{CF_S}} h_1^{CF_S}(s, v^*, y^*, v),$$

where

$$\mathcal{B}_1^{CF_S} = \left\{ (s, v^*, y^*, v) \in \mathcal{S} \times K^* \times Y^* \times V : s(v) \leq -s^*(v^*) - (v^* f)^*(-y^*) - \sigma_{\mathcal{A}}(y^*) \right\}$$

and

$$h_1^{CF_S}(s, v^*, y^*, v) = v,$$

and, when $\text{int}(K) \neq \emptyset$,

$$(DVC F_1^T) \quad \text{WMax}_{(s, v^*, y^*, v) \in \mathcal{B}_1^{CF_T}} h_1^{CF_T}(s, v^*, y^*, v),$$

where

$$\mathcal{B}_1^{CF_T} = \left\{ (s, v^*, y^*, v) \in \mathcal{T} \times K^* \times Y^* \times V : s(v) \leq -s^*(v^*) - (v^* f)^*(-y^*) - \sigma_{\mathcal{A}}(y^*) \right\}$$

and

$$h_1^{CF_T}(s, v^*, y^*, v) = v.$$

The way we named these vector duals comes from the fact that rewriting (PVC) in the form of (PVA) (where g is taken to be $\delta_{\mathcal{A}}$ and A the identity operator), one can derive $(DVC F_1^S)$ and $(DVC F_1^T)$ directly from the Fenchel vector duals considered in the previous subsection. Consequently, in this case we do not give again the weak and strong duality statements, since they can be obtained directly from both the general case and the unconstrained case.

The last vector perturbation function we consider in this section is the Fenchel-Lagrange type vector perturbation function $\Phi^{CFL} : X \times X \times Y \rightarrow V^\bullet$,

$$\Phi^{CFL}(x, z, y) = \begin{cases} f(x+z), & \text{if } x \in S, g(x) \in y - C, \\ \infty_K, & \text{otherwise.} \end{cases}$$

For $v^* \in K^*$, $z^* \in X^*$ and $y^* \in Y^*$ one has $(v^* \Phi^{CFL})^*(0, z^*, y^*) = (v^* f)^*(z^*) + (-(y^* g) + \delta_S)^*(-z^*) + \delta_{-C^*}(y^*)$. Consequently, the *Fenchel-Lagrange vector duals* to (PVC) obtained, by making use of the vector perturbation function Φ^{CFL} , from the vector duals introduced in Section 2 are

$$(DVC FL_1^S) \quad \text{Max}_{(s, v^*, z^*, y^*, v) \in \mathcal{B}_1^{CFL_S}} h_1^{CFL_S}(s, v^*, z^*, y^*, v),$$

where

$$\mathcal{B}_1^{CFL_S} = \left\{ (s, v^*, z^*, y^*, v) \in \mathcal{S} \times K^* \times X^* \times C^* \times V : s(v) \leq -s^*(v^*) - (v^* f)^*(z^*) - ((y^* g) + \delta_S)^*(-z^*) \right\}$$

and

$$h_1^{CFL_S}(s, v^*, z^*, y^*, v) = v,$$

and, when $\text{int}(K) \neq \emptyset$,

$$(DVCFL_1^T) \quad \text{Max}_{(s, v^*, z^*, y^*, v) \in \mathcal{B}_1^{CFLT}} h_1^{CFLT}(s, v^*, z^*, y^*, v),$$

where

$$\mathcal{B}_1^{CFLT} = \left\{ (s, v^*, z^*, y^*, v) \in \mathcal{T} \times K^* \times X^* \times C^* \times V : \right. \\ \left. s(v) \leq -s^*(v^*) - (v^* f)^*(z^*) - ((y^* g) + \delta_S)^*(-z^*) \right\}$$

and

$$h_1^{CFLT}(s, v^*, z^*, y^*, v) = v.$$

These are nothing but the vector duals introduced via the general scalarization in [3] in the finitely dimensional case and then extended to infinite dimensions in [5, Section 4.4]. In both these works the scalarization functions are then particularized, like in Section 3. Before giving the weak and strong duality statements for these vector duals, we consider the regularity conditions

$$(RCCFL^S) \quad \left| \begin{array}{l} \forall s \in \mathcal{S} \exists x' \in \text{dom } f \cap S \text{ such that } f \text{ is continuous at } x', \\ g(x') \in -\text{int}(C) \text{ and } s \text{ is continuous at } f(x'), \end{array} \right.$$

and, respectively,

$$(RCCFL) \quad \left| \begin{array}{l} \exists x' \in \text{dom } f \cap S \text{ such that } f \text{ is continuous at } x' \text{ and} \\ g(x') \in -\text{int}(C). \end{array} \right.$$

Theorem 16.

- (a) *There are no $x \in X$ and $(s, v^*, z^*, y^*, v) \in \mathcal{B}_1^{CFLS}$ such that $f(x) \leq_K h_1^{CFLS}(s, v^*, z^*, y^*, v)$.*
- (b) *If f is a K -convex vector function, g is a C -convex vector function, the regularity condition $(RCCFL^S)$ is fulfilled and $\bar{x} \in X$ is an \mathcal{S} -properly efficient solution to (PVC) , there exist $\bar{s} \in \mathcal{S}$, $\bar{v}^* \in K^*$, $\bar{z}^* \in X^*$, $\bar{y}^* \in C^*$ and $\bar{v} \in V$ such that $(\bar{s}, \bar{v}^*, \bar{z}^*, \bar{y}^*, \bar{v}) \in \mathcal{B}_1^{CFLS}$ is an efficient solution to $(DVCFL_1^S)$ and $f(\bar{x}) = h_1^{CFLS}(\bar{s}, \bar{v}^*, \bar{z}^*, \bar{y}^*, \bar{v}) = \bar{v}$.*

Theorem 17. *Let $\text{int}(K) \neq \emptyset$.*

- (a) *There are no $x \in X$ and $(s, v^*, z^*, y^*, v) \in \mathcal{B}_1^{CFLT}$ such that $f(x) <_K h_1^{CFLT}(s, v^*, z^*, y^*, v)$.*
- (b) *If f is a K -convex vector function, g is a C -convex vector function, the regularity condition $(RCCFL^T)$ is fulfilled and $\bar{x} \in X$ is a \mathcal{T} -properly efficient solution to (PVC) , there exist $\bar{s} \in \mathcal{T}$, $\bar{v}^* \in K^*$, $\bar{z}^* \in X^*$, $\bar{y}^* \in C^*$ and $\bar{v} \in V$ such that $(\bar{s}, \bar{v}^*, \bar{z}^*, \bar{y}^*, \bar{v}) \in \mathcal{B}_1^{CFLT}$ is a weakly efficient solution to $(DVCFL_1^T)$ and $f(\bar{x}) = h_1^{CFLT}(\bar{s}, \bar{v}^*, \bar{z}^*, \bar{y}^*, \bar{v}) = \bar{v}$.*

Obviously, one can particularize the scalarization function in each situation treated in this section, using the scalarization functions dealt in Section 3 with or others from the literature.

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