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ORIGINAL PAPER

Error bound results for convex inequality systems via conjugate duality

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Abstract The aim of this paper is to implement some new techniques, based on conjugate duality in convex optimization, for proving the existence of global error bounds for convex inequality systems. First of all, we deal with systems described via one convex inequality and extend the achieved results, by making use of a celebrated scalarization function, to convex inequality systems expressed by means of a general vector function. We also propose a second approach for guaranteeing the existence of global error bounds of the latter, which meanwhile sharpens the classical result of Robinson.

Keywords Error bounds · Duality in convex programming · Conjugate functions

Mathematics Subject Classification (2000) 49N15 · 90C25 · 90C31

1 Introduction and preliminaries

Consider a real normed space $(X, \|\cdot\|)$ and a proper and convex function $f : X \to \overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$ such that $S = \{x \in X : f(x) \le 0\}$ is nonempty. We say that the *global error bound* holds for the inequality

$$f(x) \le 0, \quad x \in X, \tag{1}$$

Dedicated to Professor Marco A. López on the occasion of his 60th birthday.

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if there exists a constant $\alpha > 0$ (depending on the initial data) such that

$$d(x, S) \leq \alpha f(x)_+ \quad \forall x \in X, \tag{2}$$

where $\gamma_+ = \max\{\gamma, 0\}$ for $\gamma \in \overline{\mathbb{R}}$.

In this article we first implement some new techniques for proving the existence of a global error bound for (1) under classical assumptions by making use of the convex conjugate duality.

Then we consider for a further real normed space $(Y, \|\cdot\|)$, partially ordered by the nonempty convex closed cone $K \subseteq Y$, and a proper K-convex function $g : X \to Y^\bullet = Y \cup \{\infty_K\}$ such that $Q = \{x \in X : g(x) \in -K\}$ is nonempty, the inequality system

$$g(x) \leq_K 0, \quad x \in X, \tag{3}$$

for which we say that the *global error bound* holds if there exists a constant $\alpha > 0$ (depending on the initial data) such that

$$d(x, Q) \leq \alpha d\big(g(x), -K\big) \quad \forall x \in X. \tag{4}$$

The issue of the existence of error bounds for (3), a topic which has its roots in the paper of Hoffman (1952) (also see Lewis and Pang 1998; Klatte 1998; Pang 1997; Zălinescu 2001), was investigated in the seminal paper of Robinson (1975), and, by particularizing the statements given there, one can easily obtain corresponding assertions for the existence of global error bounds for (1) (for $Y = \mathbb{R}$, $K = \mathbb{R}_+$, and $Y^\bullet = \mathbb{R} \cup \{+\infty\}$). In Sect. 3 we go into the opposite direction, namely, when having the global error bound results for (1), we show how one can derive corresponding statements for (3) and make to this end use of an appropriate scalarization function. An alternative approach for guaranteeing the existence of a global error bound for (3), this time without assuming closedness for the cone K, is proposed in Sect. 4. The approach uses, as a starting point, a result due to Simons (2008), and it provides a sharpening of the result of Robinson (1975). We close the paper by deriving some conclusions and by proposing some topics for further research.

To make the paper self-consistent, we consider in the following some preliminary notion and results (see Boţ et al. 2009, 2010; Ekeland and Témam 1976; Hiriart-Urruty and Lemaréchal 1993; Rockafellar 1970; Zălinescu 2002). Having a real normed space $(X, \|\cdot\|)$, we denote by $(X^*, \|\cdot\|_*)$ its topological dual space. By $\langle x^*, x\rangle = x^*(x)$ we denote the value of the continuous linear functional $x^* \in X^*$ at $x \in X$, while $B(0,1) = \{x \in X : \|x\| \leq 1\}$ and $B_*(0,1) = \{x^* \in X^* : \|x^*\|_* \leq 1\}$ are the *closed unit balls* of X and X^*, respectively. Given a subset S of X, by $\operatorname{int} S$, $\operatorname{ri} S$, and $\operatorname{cl} S$ we denote its *interior*, *relative interior*, and *closure*, respectively. The function $\delta_S : X \to \overline{\mathbb{R}}$, defined by $\delta_S(x) = 0$ for $x \in S$ and $\delta_S(x) = +\infty$ otherwise, is the *indicator function* of S, while $\sigma_S : X^* \to \overline{\mathbb{R}}$, defined by $\sigma_S(x^*) = \sup_{x \in S}\langle x^*, x\rangle$, is the *support function* of S. Further, $d(\cdot, S) : X \to \overline{\mathbb{R}}$, $d(x, S) = \inf_{s \in S}\|x - s\|$, is the *distance function* of the set S, and it is always Lipschitz continuous, being convex when S is a convex set.

On $\overline{\mathbb{R}}$ we consider the following conventions: $(+\infty) - (+\infty) = +\infty$, $0(+\infty) = +\infty$, and $0(-\infty) = 0$. Having a function $f : X \to \overline{\mathbb{R}}$, we use the classical notation for its *domain* $\operatorname{dom} f = \{x \in X : f(x) < +\infty\}$, its *epigraph* $\operatorname{epi} f =$

$\{(x,r) \in X \times \mathbb{R} : f(x) \le r\}$, and its *lower level set at level* $r \in \mathbb{R}$, $L(f,r) = \{x \in X : f(x) \le r\}$. The *lower semicontinuous hull* of $f : X \to \overline{\mathbb{R}}$ is the function $\operatorname{cl} f : X \to \overline{\mathbb{R}}$ which has as epigraph $\operatorname{cl}(\operatorname{epi} f)$. We call f *proper* if $f(x) > -\infty$ for all $x \in X$ and $\operatorname{dom} f \neq \emptyset$. Further, by $f_+ : X \to \overline{\mathbb{R}}$ we denote the function defined by $f_+(x) = f(x)_+$ for $x \in X$. The *conjugate function* of f is $f^* : X^* \to \overline{\mathbb{R}}$ defined by $f^*(x^*) = \sup\{\langle x^*, x\rangle - f(x) : x \in X\}$, while the *biconjugate function* of f is $f^{**} : X^{**} \to \overline{\mathbb{R}}$ defined by $f^{**}(x^{**}) = \sup\{\langle x^{**}, x^*\rangle - f^*(x^*) : x^* \in X^*\}$. For all $x \in X$, one has $f^{**}(x) = \sup\{\langle x^*, x\rangle - f^*(x^*) : x^* \in X^*\}$. Regarding a function and its conjugate, we have the *Young–Fenchel inequality* $f^*(x^*) + f(x) \ge \langle x^*, x\rangle$ for all $x \in X$ and $x^* \in X^*$. When $S \subseteq X$, one obviously has $\delta_S^* = \sigma_S$.

When f is a proper, convex, and lower semicontinuous function, according to the *Fenchel–Moreau theorem*, one has $f(x) = f^{**}(x)$ for all $x \in X$. Given the proper functions $f, g : X \to \overline{\mathbb{R}}$, their *infimal convolution* is the function $f \Box g : X \to \overline{\mathbb{R}}$, $(f \Box g)(x) = \inf\{f(x-y) + g(y) : y \in X\}$. One has $(f \Box g)^* = f^* + g^*$.

Having another real normed space $(Y, \|\cdot\|)$, we call a set $K \subseteq Y$ *cone* if for all $\lambda \ge 0$ and all $k \in K$, one has $\lambda k \in K$. For a given cone $K \subseteq Y$, we denote by $K^* = \{\lambda \in Y^* : \langle \lambda, k\rangle \ge 0\ \forall k \in K\}$ its *dual cone*. A nonempty convex cone $K \subseteq Y$ induces on Y a partial order "$\le_K$", defined by $y \le_K z \Leftrightarrow z - y \in K$ for $y, z \in Y$. To Y we attach a greatest element with respect to "$\le_K$", which does not belong to Y, denoted by ∞_K, and let $Y^\bullet := Y \cup \{\infty_K\}$. Then for any $y \in Y^\bullet$, one has $y \le_K \infty_K$, and we consider on $Y^\bullet$ the operations $y + \infty_K = \infty_K + y = \infty_K$ for all $y \in Y$ and $t \cdot \infty_K = \infty_K$ for all $t \ge 0$. By convention, for every set $S \subseteq Y$, $d(\infty_K, S) := +\infty$ and $\langle \lambda, \infty_K\rangle := +\infty$ for all $\lambda \in C^*$.

A function $h : Y^\bullet \to \overline{\mathbb{R}}$ is said to be *K-increasing (K-decreasing)*, whenever for all $y, z \in Y$ with $y \le_K z$, one has $h(y) \le h(z)$ ($h(y) \ge h(z)$). A vector function $g : X \to Y^\bullet$ is said to be *proper* whenever its *domain* $\operatorname{dom} g = \{x \in X : g(x) \in Y\}$ is nonempty. For $\lambda \in K^*$, we denote by $(\lambda g) : X \to \overline{\mathbb{R}}$ the function defined by $(\lambda g)(x) = \langle \lambda, g(x)\rangle$. The vector function g is called *K-convex* if for all $x, y \in X$ and all $t \in [0,1]$, one has $g(tx + (1-t)y) \le_K tg(x) + (1-t)g(y)$. For a K-convex function $g : X \to Y^\bullet$, the set $Q = \{x \in X : g(x) \in -K\}$ turns out to be convex.

2 Error bounds: the scalar case

We say that the *Slater qualification condition* holds for inequality (1) if

$$\text{there exists } x_0 \in X \text{ such that } f(x_0) < 0. \tag{5}$$

In this section we show that the fulfillment of the Slater qualification condition, combined with the boundedness of S, ensures the existence of global error bounds for (1). To this aim, we first give an equivalent characterization of the existence of error bounds by means of conjugate functions.

Lemma 1 *Suppose that $f : X \to \overline{\mathbb{R}}$ is a proper convex function such that $S = \{x \in X : f(x) \le 0\}$ is nonempty. Then the global error bound holds for inequality (1) with constant $\alpha > 0$ if and only if*

$$\min_{\lambda \in [0,1]} (\lambda f)^*(x^*) \le \sigma_S(x^*) \quad \forall x^* \in X^* \textit{ with } \|x^*\|_* \le 1/\alpha. \tag{6}$$

Proof The key observation is that inequality (2) is fulfilled with constant $\alpha > 0$ if and only if

$$\big(1/\alpha d(\cdot, S)\big)^* \geq (f_+)^*. \tag{7}$$

Indeed, the direct implication is trivial, while for the reverse one, we use the fact that $f_+(x) \geq (f_+)^{**}(x)$ for all $x \in X$ and that $d(\cdot, S)$ is real-valued, convex, and continuous, which allows us to apply for it the Fenchel–Moreau theorem. As $d(\cdot, S) = \|\cdot\| \Box \delta_S$, we have that, for all $x^* \in X^*$,

$$\big(1/\alpha d(\cdot, S)\big)^*(x^*) = 1/\alpha\big(\|\cdot\|^*(\alpha x^*) + \delta_S^*(\alpha x^*)\big) = 1/\alpha \delta_{B_*(0,1)}(\alpha x^*) + \sigma_S(x^*). \tag{8}$$

Further, for all $x^* \in X^*$, we have (cf. Simons 2008, Lemma 45.1, see also Boţ and Wanka 2008)

$$(f_+)^*(x^*) = \min_{\lambda \in [0,1]} (\lambda f)^*(x^*). \tag{9}$$

The result follows now from (7), (8), and (9). □

Remark 1 (a) When f is additionally lower semicontinuous, one can alternatively use Zălinescu (2001, Theorem 2.1) (or Cornejo et al. 1997, Theorem 5.1) for proving the statement in Lemma 1.

(b) For a fixed $x^* \in X^*$, consider the primal optimization problem

$$(P_{x^*}) \qquad \inf_{x \in S} \langle -x^*, x \rangle$$

and its *Lagrange dual problem*

$$(D_{x^*}) \qquad \sup_{\lambda \geq 0} \inf_{x \in X} \big\{\langle -x^*, x \rangle + \lambda f(x)\big\}.$$

Since weak duality always holds, that is, $v(P_{x^*}) \geq v(D_{x^*})$, where $v(P_{x^*}), v(D_{x^*})$ are the optimal objective values of (P_{x^*}) and (D_{x^*}), respectively, one can easily derive the inequality

$$\sigma_S(x^*) = -v(P_{x^*}) \leq -v(D_{x^*}) = \inf_{\lambda \geq 0} (\lambda f)^*(x^*).$$

Hence, (6) in Lemma 1 can be equivalently written as

$$\min_{\lambda \in [0,1]} (\lambda f)^*(x^*) = \sigma_S(x^*) \quad \forall x^* \in X^* \text{ with } \|x^*\|_* \leq 1/\alpha. \tag{10}$$

This means that the global error bound holds for (1) with constant $\alpha > 0$ if and only if for all $x^* \in 1/\alpha B_*(0, 1)$, one has $v(P_{x^*}) = v(D_{x^*})$ and the dual (D_{x^*}) has an optimal solution $\bar{\lambda}$ in the interval $[0, 1]$ (this can be seen as a *sharp strong duality* statement for the primal-dual pair $(P_{x^*}) - (D_{x^*})$).

(c) One can easily notice that when f is proper and convex and the Slater qualification condition for (1) is fulfilled, then for $(P_{x^*}) - (D_{x^*})$, strong duality holds for all

$x^* \in X^*$ (see, for instance, Boţ et al. 2009, 2010; Zălinescu 2002), which is nothing else than

$$\sigma_S(x^*) = -v(P_{x^*}) = -v(D_{x^*}) = \min_{\lambda \geq 0}(\lambda f)^*(x^*). \tag{11}$$

We come now to the first global error bound result for (1), for the proof of which we use conjugate duality techniques and also a useful characterization of the continuity of the conjugate of a function given by Rockafellar (1966).

Theorem 2 *Suppose that $f : X \to \overline{\mathbb{R}}$ is a proper, convex, and lower semicontinuous function such that the Slater qualification condition for* (1) *is fulfilled and $S = \{x \in X : f(x) \leq 0\}$ is a bounded set. Then for inequality* (1), *the global error bound holds.*

Proof We first show that for all $r \in \mathbb{R}$, the lower level set $L(f, r)$ is bounded. To this aim, we fix $r \in \mathbb{R}$. Let $x_0 \in X$ be such that $f(x_0) < 0$ and $M \geq 0$ fulfilling $\|x\| \leq M$ for all $x \in S$. There exists a sufficiently small $\lambda \in (0, 1)$ such that the inequality $f(x_0) + \lambda(r - f(x_0)) < 0$ is fulfilled. Take now an arbitrary element $x \in L(f, r)$. The function f being convex, we get

$$f\big(x_0 + \lambda(x - x_0)\big) \leq (1 - \lambda) f(x_0) + \lambda f(x) \leq f(x_0) + \lambda\big(r - f(x_0)\big) < 0;$$

hence, $\|x_0 + \lambda(x - x_0)\| \leq M$, which ensures that $\|x\| \leq 1/\lambda(\|x_0\| + M) + \|x_0\|$. Therefore, $L(f, r)$ is bounded. As $r \in \mathbb{R}$ was arbitrarily chosen, by Rockafellar (1966, Theorem 7A(a)) the conjugate f^* is finite and strongly continuous at 0. Moreover, the Slater qualification condition ensures that $f^*(0) \geq -f(x_0) > 0$.

Suppose in the following that the global error bound does not hold for inequality (1). Applying Lemma 1 and taking into consideration Remark 1(b) and (c), it follows that for all $\alpha > 0$, there exist $x_\alpha^* \in X^*$, $\|x_\alpha^*\|_* \leq 1/\alpha$, and $r_\alpha \in \mathbb{R}$ such that

$$\min_{\lambda \in [0,1]} (\lambda f)^*(x_\alpha^*) > r_\alpha > \min_{\lambda \geq 0}(\lambda f)^*(x_\alpha^*).$$

By taking $\alpha := n$ ($n \in \mathbb{N}$), we obtain the existence of sequences $x_n^* \in X^*$, $\|x_n^*\|_* \leq 1/n$, $r_n \in \mathbb{R}$, and $\lambda_n \in \mathbb{R}$, $\lambda_n > 1$ ($n \in \mathbb{N}$) such that

$$r_n > (\lambda_n f)^*(x_n^*) = \lambda_n f^*(1/\lambda_n x_n^*) \quad \forall n \in \mathbb{N} \tag{12}$$

and

$$r_n < (\lambda f)^*(x_n^*) = \lambda f^*(1/\lambda x_n^*) \quad \forall n \in \mathbb{N}\ \forall \lambda \in (0, 1]. \tag{13}$$

In the argumentation below we use that f^* is finite and continuous at 0, $f^*(0) > 0$, and $x_n^* \to 0$.

We consider two cases: the first one where the sequence λ_n is unbounded. Then there exists a subsequence λ_{n_k} ($k \in \mathbb{N}$) such that $\lambda_{n_k} \to \infty$ ($k \to \infty$). From (12) we get $r_{n_k} \to \infty$ ($k \to \infty$), which contradicts (13).

Suppose now that the sequence λ_n is bounded. There exists a convergent subsequence λ_{n_i} ($i \in \mathbb{N}$) such that $\lambda_{n_i} \to \overline{\lambda} \in [1, \infty)$ ($i \to \infty$). From (12) and (13) we obtain

$$\overline{\lambda} f^*(0) \le \liminf_{i\to\infty} r_{n_i} \le \limsup_{i\to\infty} r_{n_i} \le \lambda f^*(0) \quad \forall\lambda \in (0, 1],$$

which is, of course, impossible.

Thus, our statement that the global error bound does not hold for inequality (1) is false, and the proof is complete. □

We show in the following that in the above theorem the lower semicontinuity of the function f can be dropped. To this aim, we work with the lower semicontinuous hull of f. For other considerations concerning the relation between the existence of global error bounds for (1) and the existence of global error bounds for a similar inequality, where f is replaced by $\mathrm{cl}\, f$, we refer to Hu and Wang (2010).

Theorem 3 *Suppose that $f : X \to \overline{\mathbb{R}}$ is a proper convex function such that the Slater qualification condition for* (1) *is fulfilled and $S = \{x \in X : f(x) \le 0\}$ is a bounded set. Then for inequality* (1), *the global error bound holds.*

Proof Let $x_0 \in X$ be such that $f(x_0) < 0$. We show first that the Slater qualification condition guarantees the equality

$$\mathrm{cl}\, S = \big\{x \in X : \mathrm{cl}\, f(x) \le 0\big\}. \tag{14}$$

Since the inclusion "⊆" is obvious, we prove only the reverse one. Let $x \in X$ be such that $\mathrm{cl}\, f(x) \le 0$. This means that $(x, 0) \in \mathrm{epi}(\mathrm{cl}\, f) = \mathrm{cl}(\mathrm{epi}\, f)$. Hence, there exist sequences $x_n \in X$, $r_n \in \mathbb{R}$ $(n \in \mathbb{N})$ such that $f(x_n) \le r_n$ for all $n \in \mathbb{N}$ and $(x_n, r_n) \to (x, 0)$ $(n \to \infty)$. We can suppose without losing the generality that $r_n \le 1/n^2$ for all $n \in \mathbb{N}$. Since $f(x_0) < 0$, there exists $n_0 \in \mathbb{N}$ such that $f(x_0) + (1 - 1/n)1/n < 0$ for all $n \ge n_0$. Define the sequence $y_n := (1/n)x_0 + (1 - 1/n)x_n$ $(n \in \mathbb{N})$. The convexity of the function f ensures

$$f(y_n) \le (1/n) f(x_0) + (1 - 1/n) f(x_n) \le (1/n) f(x_0) + (1 - 1/n)1/n^2 < 0 \quad \forall n \ge n_0;$$

hence, $y_n \in S$ for all $n \ge n_0$. Since $y_n \to x$ $(n \to \infty)$, we conclude that (14) holds.

We prove that the global error bound holds for the inequality

$$\mathrm{cl}\, f(x) \le 0, \quad x \in X, \tag{15}$$

that is, there exists a constant $\alpha > 0$ such that

$$d\big(x, \big\{y \in X : \mathrm{cl}\, f(y) \le 0\big\}\big) \le \alpha\big[\mathrm{cl}\, f(x)\big]_+ \quad \forall x \in X. \tag{16}$$

We consider to this aim two cases; the first one where $\mathrm{cl}\, f$ is not proper. Due to Ekeland and Témam (1976, Proposition 2.4) we get $\mathrm{cl}\, f(x) = -\infty$ for $x \in \mathrm{dom}(\mathrm{cl}\, f)$ and $\mathrm{cl}\, f(x) = +\infty$ for $x \notin \mathrm{dom}(\mathrm{cl}\, f)$. Then $\{y \in X : \mathrm{cl}\, f(y) \le 0\} = \mathrm{dom}(\mathrm{cl}\, f)$, and thus, (16) holds for arbitrary $\alpha > 0$.

In case the function $\mathrm{cl}\, f$ is proper, relation (14) and the fact that $\mathrm{cl}\, f(x_0) \le f(x_0) < 0$ guarantee that Theorem 2 can be applied for the function $\mathrm{cl}\, f$, and thus, the global error bounds holds for inequality (15). This means that there exists $\alpha > 0$ such that (16) holds. Since for all $x \in X$, one has $f(x) \ge \mathrm{cl}\, f(x)$ and (cf. (14)) $d(x, \{y \in X : \mathrm{cl}\, f(y) \le 0\}) = d(x, \mathrm{cl}\, S) = d(x, S)$, the global error bound holds for inequality (1) with the same constant $\alpha > 0$. □

In general the Slater qualification condition is not enough in order to guarantee the existence of global error bounds (see Lewis and Pang 1998, Example 2). However, for particular convex inequality systems, one can renounce to the boundedness condition. We prove in the following that in the above theorem, the assumption that the lower level set of f at level 0 is bounded can be removed in case $f : \mathbb{R}^m \to \mathbb{R}$ ($m \in \mathbb{N}$) is a convex quadratic function. Here we consider $\mathbb{R}^m$ to be endowed with an arbitrary norm.

Theorem 4 *Let A be an $m \times m$ symmetric positive semidefinite matrix ($m \in \mathbb{N}$), $b \in \mathbb{R}^m$, $c \in \mathbb{R}$, and for the function $f : \mathbb{R}^m \to \mathbb{R}$, defined by $f(x) = 1/2\langle x, Ax\rangle + \langle b, x\rangle - c$, let us assume that the Slater qualification condition for (1) is fulfilled. Then for inequality (1), the global error bound holds.*

Proof As in the proof of Theorem 2, we suppose that the global error bound does not hold for inequality (1); hence, there exist sequences $x_n^* \in \mathbb{R}^m$, $\|x_n^*\| \le 1/n$, $r_n \in \mathbb{R}$, $\lambda_n \in \mathbb{R}$, $\lambda_n > 1$ ($n \in \mathbb{N}$) such that

$$r_n > (\lambda_n f)^*(x_n^*) = \lambda_n f^*(1/\lambda_n x_n^*) \quad \forall n \in \mathbb{N} \tag{17}$$

and

$$r_n < (\lambda f)^*(x_n^*) = \lambda f^*(1/\lambda x_n^*) \quad \forall n \in \mathbb{N}\ \forall \lambda \in (0, 1]. \tag{18}$$

The Slater condition ensures that $f^*(0) > 0$. Moreover, as $\mathrm{dom}\, f^* = b + A(\mathbb{R}^m)$ (cf. Hiriart-Urruty and Lemaréchal 1993, Chap. X, Example 1.1.4), from (17) we get $(1/\lambda_n)x_n^* \in b + A(\mathbb{R}^m)$ for all $n \in \mathbb{N}$. Since $(1/\lambda_n)x_n^* \to 0$ ($n \to +\infty$), we obtain

$$0 \in \mathrm{cl}\big(b + A\big(\mathbb{R}^m\big)\big) = b + A\big(\mathbb{R}^m\big) = \mathrm{dom}\, f^*.$$

This implies that $\mathrm{dom}\, f^* = b + A(\mathbb{R}^m) = A(\mathbb{R}^m)$ and $f^*(0) \in \mathbb{R}$. Thus, $\mathrm{ri}\,\mathrm{dom}\, f^* = \mathrm{ri}\, A(\mathbb{R}^m) = A(\mathbb{R}^m) = \mathrm{dom}\, f^*$ (cf. Rockafellar 1970, Theorem 6.6). From $(1/\lambda_n)x_n^* \in b + A(\mathbb{R}^m) = A(\mathbb{R}^m)$ we derive $x_n^* \in A(\mathbb{R}^m)$, and so $(1/\lambda)x_n^* \in A(\mathbb{R}^m)$ for all $n \in \mathbb{N}$ and all $\lambda \in (0, 1]$. Using the fact that f^* is continuous relative to $\mathrm{ri}\,\mathrm{dom}\, f^* = A(\mathbb{R}^m)$ (cf. Rockafellar 1970, Theorem 10.1), the proof can be continued in the lines of the second part of the proof of Theorem 2. □

Remark 2 Luo and Luo (1994) proved, by using some results from the linear algebra, the existence of global error bound results also for inequality systems of k ($k \in \mathbb{N}$) convex quadratic functions. At this moment we are not aware of how the techniques used in the proof of Theorem 4 can be extended to this more general situation.

3 Error bounds: from the scalar to the vector case

In this section we consider a further real normed space $(Y, \|\cdot\|)$, partially ordered by a *convex closed cone* $K \subseteq Y$ having a *nonempty interior*, and a proper vector function $g : X \to Y^\bullet$ such that $Q = \{x \in X : g(x) \in -K\}$ is nonempty. We will provide some existence results for the global error bound of the inequality system (3), which we deduce from the scalar case investigated above.

To this aim, we make use of the *oriented distance function*, which is a special scalarization function introduced by Hiriart-Urruty (1979a, 1979b). For $A \subseteq Y$, this function is defined by

$$\Delta_A : Y \to \overline{\mathbb{R}}, \Delta_A(y) = d(y, A) - d(y, Y \setminus A) \quad \text{for all } y \in Y.$$

Let us recall in the following the properties of the oriented distance function which we will use throughout this section (see Zaffaroni 2003, Proposition 3.2). Suppose that A is a nonempty, convex, and closed set such that $A \neq Y$. Then Δ_A is real-valued, convex, while

$$\left\{y \in Y : \Delta_A(y) \leq 0\right\} = A \quad \text{and} \quad \operatorname{int} A \subseteq \left\{y \in Y : \Delta_A(y) < 0\right\}.$$

If, additionally, A is a cone, then Δ_A is K-decreasing.

We say that the *Slater qualification condition* holds for inequality (3) if

$$\text{there exists } x_0 \in X \text{ such that } g(x_0) \in -\operatorname{int} K. \tag{19}$$

The following result was first proved by Robinson (1975) (in case the function g is defined on a nonempty convex subset of X). We give here an alternative proof for it, which relies on Theorem 3 and makes use of the oriented distance function.

Theorem 5 *Suppose that $g : X \to Y^\bullet$ is a proper K-convex function such that the Slater qualification condition for (3) is fulfilled and $Q = \{x \in X : g(x) \in -K\}$ is a bounded set. Then, for inequality system (3), the global error bound holds.*

Proof When $K = Y$, then (4) holds for an arbitrary $\alpha > 0$. Assume in the following that $K \neq Y$.

Consider the function $f : X \to \overline{\mathbb{R}}$ defined by $f(x) = \Delta_{-K}(g(x))$ for all $x \in X$. The properties of the oriented distance function guarantee that f is a proper convex function. Moreover,

$$\left\{x \in X : f(x) \leq 0\right\} = \left\{x \in X : \Delta_{-K}\big(g(x)\big) \leq 0\right\} = \left\{x \in X : g(x) \in -K\right\} = Q, \tag{20}$$

while the Slater qualification condition guarantees that $f(x_0) = \Delta_{-K}(g(x_0)) < 0$. This means that all the hypotheses of Theorem 3 are verified, and hence there exists $\alpha > 0$ such that

$$d(x, Q) = d\big(x, \left\{y \in X : f(y) \leq 0\right\}\big) \leq \alpha f(x)_+ \quad \forall x =\in X. \tag{21}$$

We close the proof by showing that the global error bound holds for the inequality system (3) with the same constant α. Take an arbitrary $x \in X$. If $g(x) = \infty_K$, then

(4) is obviously fulfilled. Further assume that $g(x) \in Y$. If $x \in Q$, that is, $g(x) \in -K$, then obviously $d(x, Q) = 0 = \alpha d(g(x), -K)$. If $x \notin Q$, that is, $g(x) \in Y \setminus (-K)$, then $f(x) > 0$ (cf. (20)). From (21) we get $d(x, Q) \le \alpha f(x)$, and the conclusion follows, since in this case $f(x) = d(g(x), -K)$. □

4 Sharpening the error bound result of Robinson

In this section we give an alternative proof for the existence of global error bounds for (3), succeeding meanwhile to sharpen the statement of Robinson (1975) concerning the bound $\alpha > 0$. We work in the same setting as in the previous section, excepting the fact that the cone K is no further assumed to be closed.

We start as in the scalar case with an equivalent characterization of the existence of error bounds by means of conjugate functions.

Lemma 6 *Suppose that* $g : X \to Y^\bullet$ *is a proper* K*-convex function such that* $Q = \{x \in X : g(x) \in -K\}$ *is nonempty. Then the global error bound holds for inequality* (3) *with constant* $\alpha > 0$ *if and only if*

$$\min_{\substack{\lambda \in K^* \\ \|\lambda\|_* \le 1}} (\lambda g)^*(x^*) \le \sigma_Q(x^*) \quad \forall x^* \in X^* \text{ with } \|x^*\|_* \le 1/\alpha. \tag{22}$$

Proof Relation (4) is equivalent to

$$\big(1/\alpha d(\cdot, Q)\big)^* \ge f^*, \tag{23}$$

where $f : X \to \overline{\mathbb{R}}$, $f = d(\cdot, -K) \circ g$. One can easily show that the function $d(\cdot, -K)$ is K-increasing, and hence f is proper and convex. Moreover, since $d(\cdot, -K)$ is continuous, we can apply (Boţ et al. 2009, Theorem 3.5.2(a)) in order to compute the conjugate of f. For all $x^* \in X^*$, we get

$$f^*(x^*) = \min_{\lambda \in K^*} \big[\big(d(\cdot, -K)\big)^*(\lambda) + (\lambda g)^*(x^*)\big]. \tag{24}$$

Since $d(\cdot, -K) = \| \cdot \| \Box \delta_{-K}$, we get, for all $\lambda \in K^*$,

$$\big(d(\cdot, -K)\big)^*(\lambda) = \big(\| \cdot \|\big)^*(\lambda) + \sigma_{-K}(\lambda),$$

which is equal to 0 for $\|\lambda\|_* \le 1$ and $+\infty$ otherwise. Hence,

$$f^*(x^*) = \min_{\substack{\lambda \in K^* \\ \|\lambda\|_* \le 1}} (\lambda g)^*(x^*) \quad \forall x^* \in X^*. \tag{25}$$

As

$$\big(1/\alpha d(\cdot, Q)\big)^* = 1/\alpha\big(\| \cdot \|^*(\alpha x^*) + \delta_Q^*(\alpha x^*)\big) = 1/\alpha \delta_{B_*(0,1)}(\alpha x^*) + \sigma_Q(x^*),$$

the result follows from (23) and (25). □

Remark 3 (a) One can notice that the equivalence in the above lemma remains true even if K fails to have a nonempty interior.

(b) For a fixed $x^* \in X^*$, consider the primal optimization problem

$$\left(P_{x^*}^v\right) \qquad \inf_{x\in Q} \langle -x^*, x\rangle$$

and its *Lagrange dual problem*

$$\left(D_{x^*}^v\right) \qquad \sup_{\lambda\in K^*} \inf_{x\in X} \left\{\langle -x^*, x\rangle + (\lambda g)(x)\right\}.$$

Since weak duality always holds, that is, $v(P_{x^*}^v) \geq v(D_{x^*}^v)$, where $v(P_{x^*}^v)$ and $v(D_{x^*}^v)$ are the optimal objective values of $(P_{x^*}^v)$ and $(D_{x^*}^v)$, respectively, one can easily derive the inequality

$$\sigma_Q(x^*) = -v\left(P_{x^*}^v\right) \leq -v\left(D_{x^*}^v\right) = \inf_{\lambda\in K^*} (\lambda g)^*(x^*).$$

Hence, (22) in Lemma 6 can be equivalently written as

$$\min_{\substack{\lambda\in K^*\\ \|\lambda\|_*\leq 1}} (\lambda g)^*(x^*) = \sigma_Q(x^*) \quad \forall x^* \in X^* \text{ with } \|x^*\|_* \leq 1/\alpha. \tag{26}$$

This means that the global error bound holds for (3) with constant $\alpha > 0$ if and only if for all $x^* \in 1/\alpha B_*(0, 1)$, one has $v(P_{x^*}^v) = v(D_{x^*}^v)$, and the dual $(D_{x^*}^v)$ has an optimal solution $\bar{\lambda}$ in the set $K^* \cap B_*(0, 1)$ (this can be seen as a *sharp strong duality* statement for the primal-dual pair $(P_{x^*}^v)$–$(D_{x^*}^v)$).

(c) One can easily notice that when g is proper K-convex and the Slater qualification condition for (3) is fulfilled, then for $(P_{x^*}^v)$–$(D_{x^*}^v)$, strong duality holds for all $x^* \in X^*$ (see, for instance, Boţ et al. 2009, 2010; Zălinescu 2002), which is nothing else than

$$\sigma_Q(x^*) = -v\left(P_{x^*}^v\right) = -v\left(D_{x^*}^v\right) = \min_{\lambda\in K^*} (\lambda g)^*(x^*). \tag{27}$$

Recall that, when having an $x \in X$ such that $g(x) \in Y \setminus (-K)$, under the supplementary assumptions that K is a closed, that the Slater qualification condition for (3) is fulfilled at $x_0 \in X$, and that Q is bounded, Robinson proved that (4) is fulfilled for $\alpha = \operatorname{diam} Q/\delta$, where $\delta > 0$ is such that $\delta B(0, 1) \subseteq g(x_0) + K$, and $\operatorname{diam} Q := \sup\{\|y - z\| : y, z \in Q\}$ is the *diameter* of the set Q.

Indeed, according to Robinson (1975), when $x \in X$ is such that $g(x) \in Y \setminus (-K)$, then for $\rho := d(g(x), -K) > 0$ and $\lambda := \rho/(\rho + \delta) \in (0, 1)$, one has $(1 - \lambda)x + \lambda x_0 \in Q$, which means that the set Q does not reduce to a singleton, that is, $\operatorname{diam} Q > 0$.

First of all, we want to notice that, whenever g is proper and K-convex, for guaranteeing that $\operatorname{diam} Q$ is a positive real number, along the Slater qualification condition at x_0 and boundedness for Q (which automatically imply that $\operatorname{diam} Q \in [0, +\infty)$), it is enough to assume that $\operatorname{dom} g \setminus \{x_0\} \neq \emptyset$. Indeed, let $x_1 \in \operatorname{dom} g \setminus \{x_0\}$. If $x_1 \in Q$, then the conclusion is obvious. Assume that $x_1 \notin Q$. Since $g(x_0) \in -\operatorname{int} K$, there exists $t \in (0, 1)$ such that $(1 - t)g(x_0) + tg(x_1) = g(x_0) + t(g(x_1) - g(x_0)) \in -\operatorname{int} K$. Thus,

$g((1-t)x_0 + tx_1) \in -K - \operatorname{int} K = -\operatorname{int} K$, which means that $(1-t)x_0 + tx_1 \in Q$ and so $\operatorname{diam} Q > 0$.

In the proof of the following statement we use a *sharp Lagrange multiplier* result due to Simons.

Theorem 7 *Suppose that $g : X \to Y^\bullet$ is a proper K-convex function such that the Slater qualification condition for (3) is fulfilled at $x_0 \in X$, i.e., $g(x_0) \in -\operatorname{int} K$, and $Q = \{x \in X : g(x) \in -K\}$ is a bounded set. If $\operatorname{dom} g \setminus \{x_0\} \neq \emptyset$, then for the inequality system (3), the global error bound holds with*

$$\alpha = \frac{\operatorname{diam} Q}{d(g(x_0), Y \setminus (-K))}.$$

Otherwise, every $\alpha > 0$ is a global error bound for (3).

Proof We consider here only the situation where $\operatorname{dom} g \setminus \{x_0\} \neq \emptyset$, as in the other one the conclusion follows automatically.

Taking into account Remark 3(b), it is enough to show that for all $x^* \in X^*$ with $\|x^*\|_* \le 1/\alpha$, strong duality holds for the primal-dual pair $(P^v_{x^*})$–$(D^v_{x^*})$ and that $(D^v_{x^*})$ has an optimal solution $\lambda \in K^*$ with $\|\lambda\|_* \le 1$.

Take an arbitrary $x^* \in X^*$ with $\|x^*\|_* \le 1/\alpha$. Since the Slater qualification condition is fulfilled, we can apply (Simons 2008, Theorem 6.6). It follows that strong duality holds for the pair $(P^v_{x^*})$–$(D^v_{x^*})$ and $(D^v_{x^*})$ has an optimal solution $\lambda \in K^*$ with

$$\begin{aligned}\|\lambda\|_* &\le \inf_{\substack{x\in X\\ g(x)\in -\operatorname{int} K}} \frac{\langle -x^*, x\rangle - \inf_{u\in Q}\langle -x^*, u\rangle}{d(g(x), Y\setminus -K)}\\ &\le \frac{\sup_{u\in Q}\langle x^*, u\rangle - \langle x^*, x_0\rangle}{d(g(x_0), Y\setminus -K)} = \frac{\sup_{u\in Q}\langle x^*, u - x_0\rangle}{d(g(x_0), Y\setminus -K)} \le \frac{(1/\alpha)\operatorname{diam} Q}{d(g(x_0), Y\setminus -K)} = 1,\end{aligned}$$

and the proof is complete. □

Remark 4 For $x_0 \in X$ with $g(x_0) \in -\operatorname{int} K$, we proved that for (3), the global error bound holds with $\alpha_{BC} := \operatorname{diam} Q/d(g(x_0), Y \setminus (-K))$, while Robinson (1975) obtained, as a bound for the same inequality system, $\alpha_R(\delta) := \operatorname{diam} Q/\delta$, where $\delta > 0$ is such that $\delta B(0,1) \subseteq g(x_0) + K$. In the following we prove that

$$\alpha_{BC} = \inf\bigl\{\alpha_R(\delta) : \delta > 0, \delta B(0,1) \subseteq g(x_0) + K\bigr\},$$

which actually means proving that

$$d\bigl(g(x_0), Y \setminus (-K)\bigr) = \sup\bigl\{\delta > 0 : \delta B(0,1) \subseteq g(x_0) + K\bigr\}. \tag{28}$$

Take first an arbitrary $\delta > 0$ such that $\delta B(0,1) \subseteq g(x_0) + K$. Then $d(g(x_0), Y \setminus (-K)) \ge \delta$. Indeed, if there exists $y_0 \in Y \setminus (-K)$ such that $\|g(x_0) - y_0\| < \delta$, then $g(x_0) - y_0 \in \delta B(0,1) \subseteq g(x_0) + K$, and hence $y_0 \in -K$, which is a contradiction. Thus,

$$d\bigl(g(x_0), Y \setminus (-K)\bigr) \ge \delta \quad \text{for all } \delta > 0 \text{ with } \delta B(0,1) \subseteq g(x_0) + K.$$

Take now an arbitrary $\delta > 0$ such that $\delta < d(g(x_0), Y \setminus (-K))$. Then one has that $\delta B(0,1) \subseteq g(x_0) + K$. Indeed, if there exists $y_0 \in Y \setminus (-K)$ such that $\|y_0 - g(x_0)\| \leq \delta$, then $\|y_0 - g(x_0)\| < d(g(x_0), Y \setminus -K) \leq \|y_0 - g(x_0)\|$, which is a contradiction. Thus (28) holds, and α_{BC} proves to be the infimum over the family of bounds proposed by Robinson (1975).

Remark 5 One of the anonymous reviewers brought into attention, in connection with the sharp error bound provided in Theorem 7 (Zheng 2003). There several error bound theorems for convex inclusions are proved. By an appropriate particularization of Zheng (2003, Corollary 3.4) one can get for inequality system (3) the same error bound as obtained by us in Theorem 7 via the sharp Lagrange multiplier. In its current form, Zheng (2003, Corollary 3.4) asks for the multifunction F, involved in its formulation, to have closed values, which would mean for the particularization to our setting that one needs to impose closedness for the cone K. The mentioned result is a direct consequence of Zheng (2003, Theorem 3.4), where the same topological assumption for F was considered. A careful reading of the proof of Zheng (2003, Theorem 3.4) makes clear that this result remains valid even in the absence of this hypothesis, the fact which allows the application of Zheng (2003, Corollary 3.4) in the general setting from Theorem 7, namely even if K fails to be closed.

5 Conclusion and further research

We have shown that the theory of conjugate duality can be successfully applied in order to get existence results concerning global error bounds for convex inequality systems. We investigated in the first part the scalar case, and then we have proposed a bridge between the scalar and the vector case via the *oriented distance function* introduced by Hiriart-Urruty. In the last section we computed, by means of a Lagrange multiplier result due to Simons, a bound which sharpens the ones given by Robinson in the context of error bounds for convex inequality systems defined by vector functions.

An interesting future research topic in this area could be to find out if the conjugate duality techniques used in this paper can be implemented in case of error bounds defined by multifunctions. More precisely, having a multifunction $\Gamma : X \rightrightarrows Y$, where X, Y are real normed spaces, we say that Γ has a *global error bound at* $x_0 \in \operatorname{dom} \Gamma$, provided that there exists $\alpha > 0$ such that

$$d\big(y, \Gamma(x_0)\big) \leq \alpha d\big(x_0, \Gamma^{-1}(y)\big) \quad \forall y \in Y.$$

This is a generalization of the notions considered in this paper. We refer to Li and Singer (1998), Zălinescu (2003), Klatte (1998) for conditions which guarantee the existence of error bounds in this context.

Another direction, which could be of interest, is to analyze if instead of the Slater qualification condition, which requires the nonemptiness of the interior of the cone K, some weaker conditions could be considered, in order to guarantee the existence of error bounds. Recall that there exist generalizations of the classical interior, like the *algebraic interior*, the *strong quasi-relative interior*, and the *quasi-relative interior*, which play an important role in the formulation of regularity conditions ensuring strong duality in convex optimization; see Boţ et al. (2009, 2010), Csetnek (2010), Zălinescu (2002). With this respect, let us mention that a first weakening of the Slater qualification condition comes via Zheng (2003, Corollary 3.4) from which one can deduce that the conclusion of Theorem 7 remains true even if one assumes instead that there exists $x_0 \in X$ such that

$$d\big(g(x_0), Y \setminus -K\big) B(0,1) \cap \operatorname{aff}\big(g(X) + K\big) \subseteq g(x_0) + K,$$

where aff stands for the *affine hull* of a set.

Finally, it could be challenging to see if the technique used in the proof of Theorem 4 can be generalized to k convex quadratic functions (with the corresponding global error bound notion in the vector case). We know that a similar result remains valid in this case, too (cf. Luo and Luo 1994, Theorem 3.1).

References

Boţ RI (2010) Conjugate duality in convex optimization. Lecture notes in economics and mathematical systems, vol 637. Springer, Berlin

Boţ RI, Wanka G (2008) The conjugate of the pointwise maximum of two convex functions revisited. J Glob Optim 41(4):625–632

Boţ RI, Grad S-M, Wanka G (2009) Duality in vector optimization. Springer, Berlin

Cornejo O, Jourani A, Zălinescu C (1997) Conditioning and upper-Lipschitz inverse subdifferentials in nonsmooth optimization problems. J Optim Theory Appl 95(1):127–148

Csetnek ER (2010) Overcoming the failure of the classical generalized interior-point regularity conditions in convex optimization. Applications of the duality theory to enlargements of maximal monotone operators. Logos, Berlin

Ekeland I, Témam R (1976) Convex analysis and variational problems. North-Holland, Amsterdam

Hiriart-Urruty J-B (1979a) New concepts in nondifferentiable programming. Bull Soc Math Fr, Mém 60:57–85

Hiriart-Urruty J-B (1979b) Tangent cones, generalized gradients and mathematical programming in Banach spaces. Math Oper Res 4(1):79–97

Hiriart-Urruty J-B, Lemaréchal C (1993) Convex analysis and minimization algorithms I, II. Springer, Berlin

Hoffman AJ (1952) On approximate solutions of systems of linear inequalities. J Res Natl Bur Stand 49:263–265

Hu H, Wang Q (2010) Local and global error bounds for proper function. Pac J Optim 6(1):177–186

Klatte D (1998) Hoffman's error bound for systems of convex inequalities. In: Fiacco AV (ed) Mathematical programming with data perturbations. Dekker, New York, pp 185–199

Lewis AS, Pang J-S (1998) Error bounds for convex inequality systems. In: Generalized convexity, generalized monotonicity: recent results, Luminy, 1996. Nonconvex optimization and its applications, vol 27. Kluwer Academic, Dordrecht, pp 75–110

Li W, Singer I (1998) Global error bounds for convex multifunctions and applications. Math Oper Res 23(2):443–462

Luo X-D, Luo Z-Q (1994) Extension of Hoffman's error bound to polynomial systems. SIAM J Optim 4(2):383–392

Pang J-S (1997) Error bounds in mathematical programming. Math Program, Ser A 79(1–3):299–332

Robinson SM (1975) An application of error bounds for convex programming in a linear space. SIAM J Control Optim 13:271–273

Rockafellar RT (1966) Level sets and continuity of conjugate convex functions. Trans Am Math Soc 123(1):43–63

Rockafellar RT (1970) Convex analysis. Princeton University Press, Princeton

Simons S (2008) From Hahn–Banach to monotonicity. Springer, Berlin

Zaffaroni A (2003) Degrees of efficiency and degrees of minimality. SIAM J Control Optim 42(3):1071–1086

Zălinescu C (2001) Weak sharp minima, well-behaving functions and global error bounds for convex inequalities in Banach spaces. In: Proceedings of the 12th Baikal International conference on optimization methods and their applications, Irkutsk, Russia, pp 272–284

Zălinescu C (2002) Convex analysis in general vector spaces. World Scientific, Singapore

Zălinescu C (2003) A nonlinear extension of Hoffman's error bound for linear inequalities. Math Oper Res 28(3):524–532

Zheng XY (2003) Error bounds for set inclusions. Sci China Ser A 46:750–763