Robust Duality in Parametric Convex Optimization *

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Revised Version: June 20, 2012

Abstract

Modelling of convex optimization in the face of data uncertainty often gives rise to families of parametric convex optimization problems. This motivates us to present, in this paper, a duality framework for a family of parametric convex optimization problems. By employing conjugate analysis, we present robust duality for the family of parametric problems by establishing strong duality between associated dual pair. We first show that robust duality holds whenever a constraint qualification holds. We then show that this constraint qualification is also necessary for robust duality in the sense that the constraint qualification holds if and only if robust duality holds for every linear perturbation of the objective function. As an application, we obtain a robust duality theorem for the best approximation problems with constraint data uncertainty under a strict feasibility condition.

Keywords. Parametric convex optimization, conjugate duality, strong duality, uncertain conical convex programs, best approximations.

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*The authors are grateful to the referees and the editor for their constructive comments and helpful suggestions which have contributed to the final preparation of the paper.

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1 Introduction

Consider the family of (primal) conical convex optimization problems of the form

$$\{(P_u)\}_{u\in\mathcal{U}}: \qquad \left\{ \min\nolimits_{x\in X}\{f(x) : x\in S,\ g_u(x)\in -C\}\right\}_{u\in\mathcal{U}},$$

where the cone-constraint depends on a parameter u, which belongs to a given set $\mathcal{U}$, $S\subseteq X$ is a nonempty convex set, $C\subseteq Y$ is a nonempty convex cone, $f: X\to \mathbb{R}\cup\{+\infty\}$ is a proper convex function and for each $u\in\mathcal{U}$, $g_u: X\to Y$ is a C-convex function, and X and Y are Hausdorff locally convex spaces. Parametric family of optimization problems of the form $\{(P_u)\}_{u\in\mathcal{U}}$ often arises in scientific modelling of real-world decision problems [1, 12, 17] and covers optimization problems in the face of data uncertainty [14, 15].

A *robust primal solution* of the family $\{(P_u)\}_{u\in\mathcal{U}}$ is obtained by solving the single problem:

$$(RP)\ \min_{x\in X}\{f(x) : x\in S,\ g_u(x)\in -C,\ \forall u\in\mathcal{U}\},$$

where the constraints are enforced for every parameter u in the prescribed set $\mathcal{U}$. By associating the Lagrangian dual for each parameter u, we form the family of dual optimization problems:

$$\left\{ \sup_{y^*\in C^*}\inf_{x\in S}\{f(x)+\langle y^*, g_u(x)\rangle\}\right\}_{u\in\mathcal{U}}.$$

A *robust dual solution* is obtained by solving the single dual problem:

$$(ODP)\ \sup_{(u,y^*)\in\mathcal{U}\times C^*}\inf_{x\in S}\{f(x)+\langle y^*, g_u(x)\rangle\},$$

where the supremum is taken over all $(u,y^*)\in\mathcal{U}\times C^*$. As usual, for an attained infimum (supremum) instead of inf (sup) we write min (max).

Robust duality for the family $\{(P_u)\}_{u\in\mathcal{U}}$ states that

$$\min_{x\in X}\{f(x) : x\in S,\ g_u(x)\in -C,\ \forall u\in\mathcal{U}\} = \max_{(u,y^*)\in\mathcal{U}\times C^*}\inf_{x\in S}\{f(x)+\langle y^*, g_u(x)\rangle\}$$

whenever (RP) attains its minimum. The significance of this robust duality is that the dual problem can be solved easily for some classes of robust convex problems. For instance, the dual of a robust best approximation problem with affine parameterized data uncertainty is a finite dimensional convex optimization, for details see [16]. For related results in the uncertainty free cases, see [20].

As an illustration, consider the simple uncertain least squares optimization problem

$$\min\{\tfrac{1}{2}(x_1^2+x_2^2) : a_1x_1+a_2x_2\le b\},$$

where $b\in\mathbb{R}$ and the constraint data $(a_1,a_2)\in\mathbb{R}^2$ is uncertain and it belongs to the uncertainty set $\mathcal{U}$. The effect of uncertain data can be captured by the parameterized problem

$$\{(P1_u)\}_{u\in\mathcal{U}} \qquad \left\{ \min\{\tfrac{1}{2}(x_1^2+x_2^2) : a_1(u)x_1+a_2(u)x_2\le b\}\right\}_{u\in\mathcal{U}}.$$

A solution of the problem

$$(RP1) \qquad \min\{\tfrac{1}{2}(x_1^2+x_2^2) : a_1(u)x_1 + a_2(u)x_2 \le b \ \forall u \in \mathcal{U}\}$$

gives us a primal robust solution with "the worst primal objective value". On the other hand, for each $u \in \mathcal{U}$, the corresponding Lagrangian dual problem of $(P1_u)$ is

$$\sup_{y^* \in \mathbb{R}_+} \inf_{x \in \mathbb{R}^2} \{\tfrac{1}{2}(x_1^2+x_2^2) + y^*(a_1(u)x_1 + a_2(u)x_2 - b)\}.$$

A solution of the problem

$$(ODP1) \quad \sup_{(u,y^*) \in \mathcal{U} \times \mathbb{R}_+} \inf_{x \in \mathbb{R}^2} \{\tfrac{1}{2}(x_1^2+x_2^2) + y^*(a_1(u)x_1 + a_2(u)x_2 - b)\}$$

gives us the dual robust solution with "the best dual objective value." Thus, the robust strong duality for the family $\{(P1_u)\}_{u \in \mathcal{U}}$ reads:

$$\begin{aligned} & \min\{\tfrac{1}{2}(x_1^2+x_2^2) : a_1(u)x_1 + a_2(u)x_2 \le b \ \forall u \in \mathcal{U}\} \\ = & \max_{(u,y^*) \in \mathcal{U} \times \mathbb{R}_+} \inf_{x \in \mathbb{R}^2} \{\tfrac{1}{2}(x_1^2+x_2^2) + y^*(a_1(u)x_1 + a_2(u)x_2 - b)\} \end{aligned}$$

and states that "primal worst value" equals to the "dual best value" and both the primal worst value and the dual best values are attained.

The purpose of this paper is to establish a complete characterization of robust duality by employing the conjugate duality theory. We first prove robust duality using a new robust subdifferential condition. For related conditions we refer the reader to [14]. Then we establish that the subdifferential condition is in some sense the weakest condition guaranteeing robust duality for the given family $\{(P_u)\}_{u \in \mathcal{U}}$. As an application, we derive robust duality for a constrained best approximation problem in the face of constraint data uncertainty under a strict feasibility condition.

The outline of the paper is as follows. Section 2 presents some preliminaries of convex analysis, which aim to make the paper self-contained. Section 3 introduces a so-called general robust subdifferential condition, which further provides a basic constraint qualification condition for the existence of robust duality for the primal-dual pair $(RP)-(ODP)$. Section 4 presents robust duality for a best approximation model problem under data uncertainty.

2 Preliminaries: Conjugate Analysis

Consider two Hausdorff locally convex vector spaces X and Y and their topological dual spaces X^* and Y^*, respectively, endowed with the corresponding weak* topologies, and denote by $\langle x^*, x \rangle = x^*(x)$ the value at $x \in X$ of the continuous linear functional $x^* \in X^*$. Consider also the *projection function* $\mathrm{Pr}_X : X \times Y \to X$, defined by $\mathrm{Pr}_X(x,y) = x \ \forall (x,y) \in X \times Y$. Let $\overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$. Given a subset S of X, by $\mathrm{int}\, S$ we denote its *interior* and by $\delta_S : X \to \overline{\mathbb{R}}$,

$$\delta_S(x) = \begin{cases} 0, & \text{if } x \in S, \\ +\infty, & \text{otherwise,} \end{cases}$$

its *indicator function*.

Having a function $f : X \to \overline{\mathbb{R}}$ we use the classical notations for *domain* $\operatorname{dom} f := \{x \in X : f(x) < +\infty\}$, *epigraph* $\operatorname{epi} f := \{(x, r) \in X \times \mathbb{R} : f(x) \le r\}$ and *conjugate function* $f^* : X^* \to \overline{\mathbb{R}}$, $f^*(x^*) := \sup\{\langle x^*, x\rangle - f(x) : x \in X\}$. The conjugate function of the indicator function of a set $S \subseteq X$ is the so-called *support function* of S, $\sigma_S : X^* \to \overline{\mathbb{R}}$, $\sigma_S(x^*) = \sup_{x\in S}\langle x^*, x\rangle$.

For a function $f : X \to \overline{\mathbb{R}}$ and its conjugate one has the *Young-Fenchel's inequality* $f^*(x^*) + f(x) \ge \langle x^*, x\rangle$ for all $x \in X$ and $x^* \in X^*$. We call f *proper* if $f(x) > -\infty$ $\forall x \in X$ and $\operatorname{dom} f \ne \emptyset$. We say that f is *convex*, if $\operatorname{epi} f$ is a convex set and that f is *lower semicontinuous*, if $\operatorname{epi} f$ is closed.

For $f : X \to \overline{\mathbb{R}}$ proper, if $f(x) \in \mathbb{R}$ the *(convex) subdifferential* of f at x is $\partial f(x) := \{x^* \in X^* : f(y) - f(x) \ge \langle x^*, y - x\rangle\ \forall y \in X\}$, while if $f(x) = +\infty$ we take by convention $\partial f(x) := \emptyset$. The *normal cone* of a nonempty set $S \subseteq X$ at $x \in X$ is $N_S(x) := \partial\delta_S(x)$.

For a proper function $f : X \to \overline{\mathbb{R}}$ one can also prove that for $x \in X$ and $x^* \in X^*$

$$x^* \in \partial f(x) \Leftrightarrow f(x) + f^*(x^*) \le \langle x^*, x\rangle \Leftrightarrow f(x) + f^*(x^*) = \langle x^*, x\rangle.$$

Given two proper functions $f, g : X \to \overline{\mathbb{R}}$, the *infimal convolution* of f and g is defined by $f\Box g : X \to \overline{\mathbb{R}}$, $(f\Box g)(x) = \inf\{f(y) + g(x - y) : y \in X\}$. If there is an $y \in X$ such that $(f\Box g)(x) = f(y) + g(x - y)$ we say that the infimal convolution is *exact* at x.

When $f : X \to \overline{\mathbb{R}}$ is a given function, then one has for each $\lambda > 0$

$$\operatorname{epi}(\lambda f)^* = \lambda \operatorname{epi} f^*.$$

When $f, g : X \to \overline{\mathbb{R}}$ are proper functions with $\operatorname{dom} f \cap \operatorname{dom} g \ne \emptyset$, then

$$\operatorname{epi}(f + g)^* = \operatorname{epi} f^* + \operatorname{epi} g^* \tag{1}$$

if and only if

$$(f + g)^*(x^*) = \min\{f^*(y^*) + g^*(x^* - y^*) : y^* \in X^*\}\ \forall x^* \in X^* \tag{2}$$

For sufficient conditions, both of interiority- and closedness-type, for (1) and (2) we refer the reader to [3, 10, 21]. The ones to which we make appeal throughout this paper, for f and g proper and convex functions with $\operatorname{dom} f \cap \operatorname{dom} g \ne \emptyset$, are, on the one hand,

$$f \text{ or } g \text{ is continuous at a point in } \operatorname{dom} f \cap \operatorname{dom} g$$

and, one the other hand (see also [8, 19]),

$$f, g \text{ are lower semicontinuous and } \operatorname{epi} f^* + \operatorname{epi} g^* \text{ is weak}^*\text{-closed.}$$

When $C \subseteq Y$ is a convex cone and $g : X \to Y$ is a given function, then the set $\operatorname{epi}_C g := \{(x, y) \in X \times Y : y \in g(x) + C\}$ is called *C-epigraph* of g. In analogy to the notions considered for scalar functions, we say that g is C-convex, if $\operatorname{epi}_C g$ is a convex set and that g is *C-epi closed*, if $\operatorname{epi}_C g$ is a closed set. The C-epi closedness is the weakest among the notions given in the literature that aim to extend the lower semicontinuity for scalar functions to vector functions (for more on this topic, see [3]).

We close the section by stating a weak duality result for the family of functions $\Phi_u : X \times Y \to \overline{\mathbb{R}}$, where u is taken in a given uncertainty set $\mathcal{U}$, which will prove to be useful later in the paper.

Proposition 1 *Let the family of functions $\Phi_u : X \times Y \to \overline{\mathbb{R}}$, for $u \in \mathcal{U}$, be given. For each $x^* \in X^*$ it holds*

$$\inf_{x\in X}\{\sup_{u\in\mathcal{U}}\{\Phi_u(x,0)+\langle x^*,x\rangle\}\} \geq \sup_{(u,y^*)\in\mathcal{U}\times Y^*}\{-\Phi_u^*(-x^*,y^*)\}.$$

Proof. Let $x^* \in X^*$, $x \in X$ and $y^* \in Y^*$ be fixed. For each $u \in \mathcal{U}$, by Young-Fenchel's inequality, we have

$$\Phi_u(x,0)+\langle x^*,x\rangle \geq -\Phi_u^*(-x^*,y^*),$$

which implies

$$\sup_{u\in\mathcal{U}}\{\Phi_u(x,0)+\langle x^*,x\rangle\} \geq \sup_{u\in\mathcal{U}}\{-\Phi_u^*(-x^*,y^*)\}.$$

From here the conclusion follows automatically. ■

3 A Constraint Qualification for Robust Duality

In this section we first introduce a constraint qualification ensuring robust duality as a particularization of a more general subdifferential condition. We also show that the constraint qualification is also a characterization of robust duality, in the sense that the constraint qualification holds if and only if robust duality holds for every linear perturbation of the objective function of (RP).

Throughout this section, we assume that $\operatorname{dom} f \cap A \neq \emptyset$ where $A := \{x \in S : g_u(x) \in -C, \ \ \forall u \in \mathcal{U}\}$. Note that, by definition, the following inclusion is always satisfied

$$\partial(f+\delta_A)(x) \supseteq \bigcup_{\substack{u\in\mathcal{U},y^*\in C^*,\\ (y^*g_u)(x)=0}} \partial(f+(y^*g_u)+\delta_S)(x), \ \forall x \in \operatorname{dom} f \cap A, \tag{3}$$

where $(y^*g_u)(x) := \langle y^*, g_u(x)\rangle$ for all $x \in X$.

We now introduce the *Robust Basic Subdifferential Condition* as follows:

$$(RBSC) \quad \partial(f+\delta_A)(x) = \bigcup_{\substack{u\in\mathcal{U},y^*\in C^*,\\ (y^*g_u)(x)=0}} \partial(f+(y^*g_u)+\delta_S)(x), \ \forall x \in \operatorname{dom} f \cap A,$$

where $C^* = \{y^* \in Y^* : \langle y^*, y\rangle \geq 0 \ \forall y \in C\}$ denotes the *dual cone* of C, and for all $y^* \in Y^*$, $(y^*g_u)(\cdot) := \langle y^*, g_u(\cdot)\rangle$.

We begin by first showing that the robust basic subdifferential condition (RBSC) is sufficient for robust duality.

Theorem 2 *Assume that* $\operatorname{dom} f \cap A \neq \emptyset$ *and that* $(RBSC)$ *is fulfilled. Then, whenever* f *attains its minimum over* A,

$$\min_{x\in A} f(x) = \max_{(u,y^*)\in\mathcal{U}\times C^*} \inf_{x\in S}\{f(x) + \langle y^*, g_u(x)\rangle\}.$$

Proof. Define $\Phi_u : X \times Y \to \overline{\mathbb{R}}$ by $\Phi_u(x, y) = f(x) + \delta_{\{z\in S: g_u(z)\in y-C\}}(x)$. Then Φ_u is a proper function. For all $(x^*, y^*) \in X^* \times Y^*$, it holds

$$\Phi_u^*(x^*, y^*) = \begin{cases} (f + ((-y^*)g_u) + \delta_S)^*(x^*), & \text{if } y^* \in -C^*, \\ +\infty, & \text{otherwise.} \end{cases}$$

Thus, the conclusion will follow if we show that

$$\min_{x\in X}\{\sup_{u\in\mathcal{U}} \Phi_u(x, 0)\} = \max_{(u,y^*)\in\mathcal{U}\times Y^*}\{-\Phi_u^*(0, y^*)\}.$$

We claim that, for each $x \in \operatorname{dom} f \cap A$,

$$\operatorname{Pr}_{X^*} \bigcup_{u\in\mathcal{U}} \partial\Phi_u(x, 0) = \bigcup_{u\in\mathcal{U}, y^*\in C^*, (y^*g_u)(x)=0} \partial(f + (y^*g_u) + \delta_S)(x). \tag{4}$$

Granting this, (RBSC) can be rewritten as

$$\partial\left(\sup_{u\in\mathcal{U}} \Phi_u(\cdot, 0)\right)(x) = \operatorname{Pr}_{X^*} \bigcup_{u\in\mathcal{U}} \partial\Phi_u(x, 0),\ \forall x \in \operatorname{dom}\left(\sup_{u\in\mathcal{U}} \Phi_u(\cdot, 0)\right).$$

Denote by $\bar{x}$ the minimum of $\sup_{u\in\mathcal{U}} \Phi_u(\cdot, 0)$ over X, which is obviously an element in $\operatorname{dom}\sup_{u\in\mathcal{U}} \Phi_u(\cdot, 0) = \operatorname{dom} f \cap A$. Then $0 \in \partial\left(\sup_{u\in\mathcal{U}} \Phi_u(\cdot, 0)\right)(\bar{x})$. Since $(RBSC)$ is fulfilled, there exist $\bar{y}^* \in Y^*$ and $\bar{u} \in \mathcal{U}$ such that $(0, \bar{y}^*) \in \partial\Phi_{\bar{u}}(\bar{x}, 0)$. Thus,

$$\sup_{u\in\mathcal{U}} \Phi_u(\bar{x}, 0) = f(\bar{x}) = \Phi_{\bar{u}}(\bar{x}, 0) = -\Phi_{\bar{u}}^*(0, \bar{y}^*),$$

which, by taking into consideration Proposition 1, implies that

$$\min_{x\in X}\{\sup_{u\in\mathcal{U}} \Phi_u(x, 0)\} = \sup_{u\in\mathcal{U}} \Phi_u(\bar{x}, 0) = \max_{(u,y^*)\in\mathcal{U}\times Y^*}\{-\Phi_u^*(0, y^*)\}.$$

To verify (4), let $x \in \operatorname{dom} f \cap A$. Choosing an arbitrary

$$x^* \in \bigcup_{u\in\mathcal{U}, y^*\in C^*, (y^*g_u)(x)=0} \partial(f + (y^*g_u) + \delta_S)(x),$$

there exist $\bar{u} \in \mathcal{U}$ and $\bar{y}^* \in C^*$ such that $(\bar{y}^*g_{\bar{u}})(x) = 0$ and $x^* \in \partial(f+(\bar{y}^*g_{\bar{u}})+\delta_S)(x)$. Hence,

$$(f + (\bar{y}^*g_{\bar{u}}) + \delta_S)^*(x^*) + (f + (\bar{y}^*g_{\bar{u}}) + \delta_S)(x) = \langle x^*, x\rangle.$$

This gives us that

$$\Phi_{\bar{u}}^*(x^*, -\bar{y}^*) + \Phi_{\bar{u}}(x, 0) = \langle x^*, x\rangle$$

or, equivalently, $(x^*, -\bar{y}^*) \in \partial\Phi_{\bar{u}}(x, 0)$. Thus, $x^* \in \Pr_{X^*} \bigcup_{u\in\mathcal{U}} \partial\Phi_u(x, 0)$.

To show the opposite inclusion, let $x^* \in \Pr_{X^*} \bigcup_{u\in\mathcal{U}} \partial\Phi_u(x, 0)$. Then, there exist $\bar{y}^* \in Y^*$ and $\bar{u} \in \mathcal{U}$ such that $\Phi^*_{\bar{u}}(x^*, \bar{y}^*) + \Phi_{\bar{u}}(x, 0) = \langle x^*, x\rangle$. Thus $\bar{y}^* \in -C^*$ and $(f + ((-\bar{y}^*)g_{\bar{u}}) + \delta_S)^*(x^*) + f(x) = \langle x^*, x\rangle$. By Young-Fenchel's inequality one has

$$0 \geq (f + ((-\bar{y}^*)g_{\bar{u}}) + \delta_S)^*(x^*) + f(x) + ((-\bar{y}^*)g_{\bar{u}})(x) + \delta_S(x) - \langle x^*, x\rangle \geq 0.$$

This shows us that $x^* \in \partial(f + ((-\bar{y}^*)g_{\bar{u}}) + \delta_S)(x)$ and, consequently,

$$x^* \in \bigcup_{u\in\mathcal{U}, y^*\in C^*, (y^*g_u)(x)=0} \partial(f + (y^*g_u) + \delta_S)(x).$$

■

In the following we show that $(RBSC)$ is in some sense the weakest condition for robust duality.

Theorem 3 *Assume that* $\operatorname{dom} f \cap A \neq \emptyset$. *Then the following statements are equivalent:*

(i) $(RBSC)$ *is fulfilled.*

(ii) *For each* $x^* \in X^*$ *such that* $f + \langle x^*, \cdot\rangle$ *attains its minimum over* A *it holds*

$$\min_{x\in A}\{f(x) + \langle x^*, x\rangle\} = \max_{(u,y^*)\in\mathcal{U}\times C^*} \inf_{x\in S}\{f(x) + \langle x^*, x\rangle + \langle y^*, g_u(x)\rangle\}.$$

Proof. "(i) ⇒ (ii)." Let $x^* \in X^*$ be such that $f + \langle x^*, \cdot\rangle$ attains its minimum over X. Then, ii holds by applying Theorem 2 with f replaced by $f + \langle x^*, \cdot\rangle$.
"(ii) ⇒ (i)." Let $\bar{x} \in \operatorname{dom} f \cap A$ be fixed. It follows from (3) that

$$\partial(f + \delta_A)(\bar{x}) \supseteq \bigcup_{\substack{u\in\mathcal{U}, y^*\in C^*,\\ (y^*g_u)(\bar{x})=0}} \partial(f + (y^*g_u) + \delta_S)(\bar{x}).$$

This means that we have to prove only the opposite inclusion. To see this, let $x^* \in \partial(f+\delta_A)(\bar{x})$. This is equivalent to the condition that $0 \in \partial(f+\delta_A+\langle -x^*, \cdot\rangle)(\bar{x})$. So, $\bar{x}$ is a minimum of the function $f + \langle -x^*, \cdot\rangle$ over A. Consequently, by (ii), there exist $\bar{y}^* \in C^*$ and $\bar{u} \in \mathcal{U}$ such that

$$f(\bar{x}) + \langle -x^*, \bar{x}\rangle = \min_{x\in A}\{f(x) + \langle -x^*, x\rangle\} = \inf_{x\in S}\{f(x) + \langle -x^*, x\rangle + \langle \bar{y}^*, g_{\bar{u}}(x)\rangle\}.$$

As $\bar{x} \in A$, in particular, we have $f(\bar{x}) + \langle -x^*, \bar{x}\rangle \leq f(\bar{x}) + \langle -x^*, \bar{x}\rangle + \langle \bar{y}^*, g_{\bar{u}}(\bar{x})\rangle$ which implies that $(\bar{y}^*g_{\bar{u}})(\bar{x}) \geq 0$. Note that $\bar{y}^* \in C^*$ and $g_{\bar{u}}(\bar{x}) \in -C$. It follows that $(\bar{y}^*g_{\bar{u}})(\bar{x}) = 0$. This shows that $\bar{x}$ is a minimum of the function $f + \langle -x^*, \cdot\rangle + (\bar{y}^*g_{\bar{u}}) + \delta_S$ over X, and so,

$$0 \in \partial\big(f + \langle -x^*, \cdot\rangle + (\bar{y}^*g_{\bar{u}}) + \delta_S\big)(\bar{x}).$$

Hence, $x^* \in \partial\big(f + (\bar{y}^*g_{\bar{u}}) + \delta_S\big)(\bar{x})$. ■

In the special case when $\mathcal{U}$ is a singleton, we obtain the following characterization for Lagrangian duality established in [4, 6].

Corollary 4 *Assume that* $\operatorname{dom} f \cap A \neq \emptyset$. *Then the following statements are equivalent:*

(i) *For each* $x^* \in X^*$ *one has*

$$\min_{x\in A}(f + \langle x^*, \cdot\rangle)(x) = \max_{y^*\in C^*} \inf_{x\in S}\{f(x) + \langle x^*, x\rangle + \langle y^*, g(x)\rangle\}.$$

(ii) *For each* $x \in \operatorname{dom} f \cap A$, *it holds*

$$\partial(f + \delta_A)(x) = \bigcup_{y^*\in C^*, (y^*g)(x)=0} \partial(f + (y^*g) + \delta_S)(x) \ \forall x \in \operatorname{dom} f \cap A.$$

Proof. The conclusion follows from Theorem 3 by letting $\mathcal{U} = \{u\}$. ■

When taking f identical to zero in $(RBSC)$, this turns into what we call to be the *Robust Basic Constraint Qualification*:

$$(RBCQ) \quad N_A(x) = \bigcup_{\substack{u\in\mathcal{U}, y^*\in C^*,\\ (y^*g_u)(x)=0}} \partial((y^*g_u) + \delta_S)(x) \ \forall x \in A.$$

The next theorem completely characterizes via $(RBCQ)$ the existence of robust duality.

Theorem 5 *Assume that* $\operatorname{dom} f \cap A \neq \emptyset$. *Then, the following statements are equivalent:*

(i) $(RBCQ)$ *is fulfilled.*

(ii) *For each* $x^* \in X^*$ *that attains its minimum over* A *it holds*

$$\min_{x\in A}\langle x^*, x\rangle = \max_{(u,y^*)\in\mathcal{U}\times C^*} \inf_{x\in S}\{\langle x^*, x\rangle + \langle y^*, g_u(x)\rangle\}.$$

(iii) *For each proper function* $f : X \to \overline{\mathbb{R}}$ *that attains its minimum over* A *and such that* $(f + \delta_A)^*(0) = (f^*\square\sigma_A)(0)$ *and the infimal convolution is exact at* 0 *it holds*

$$\min_{x\in A} f(x) = \max_{(u,y^*)\in\mathcal{U}\times C^*} \inf_{x\in S}\{f(x) + \langle y^*, g_u(x)\rangle\}.$$

Proof. The equivalence "(i) $\Leftrightarrow$ (ii)" follows as a direct consequence of Theorem 3.

"(iii) $\Rightarrow$ (ii)." Consider an $x^* \in X^*$ that attains its minimum over A and $f(x) := \langle x^*, x\rangle$. As f is continuous, (2) implies that $(f + \delta_A)^*(0) = (f^*\square\sigma_A)(0) = \sigma_A(-x^*)$ and the infimal convolution is exact at 0, thus

$$\min_{x\in A}\langle x^*, x\rangle = \max_{(u,y^*)\in\mathcal{U}\times C^*} \inf_{x\in S}\{\langle x^*, x\rangle + \langle y^*, g_u(x)\rangle\}.$$

"(i) $\Rightarrow$ (iii)." Consider a proper function $f : X \to \overline{\mathbb{R}}$ that attains its minimum over A at $\bar{x} \in A$ and such that $(f+\delta_A)^*(0) = (f^*\square\sigma_A)(0)$ and the infimal convolution is exact at 0. Then $f(\bar{x}) = -(f + \delta_A)^*(0)$ and there exists some $x^* \in X^*$ fulfilling $(f + \delta_A)^*(0) = f^*(x^*) + \sigma_A(-x^*)$. Obviously, $\operatorname{dom} f \cap A \neq \emptyset$. Thus $f(\bar{x}) \in \mathbb{R}$. By using Young-Fenchel's inequality we get

$$0 = f(\bar{x}) + f^*(x^*) - \langle x^*, \bar{x}\rangle + \delta_A(\bar{x}) + \sigma_A(-x^*) - \langle -x^*, \bar{x}\rangle \geq 0,$$

which implies that $f(\bar{x}) + f^*(x^*) = \langle x^*, \bar{x}\rangle$ or, equivalently, $x^* \in \partial f(\bar{x})$ and also that $\delta_A(\bar{x}) + \sigma_A(-x^*) = \langle -x^*, \bar{x}\rangle$ or, equivalently, $-x^* \in \partial_A(\bar{x}) = N_A(\bar{x})$.

According to $(RBCQ)$, there exist $\bar{u} \in \mathcal{U}$ and $\bar{y}^* \in C^*$ such that $(\bar{y}^* g_{\bar{u}})(\bar{x}) = 0$ and $-x^* \in \partial((\bar{y}^* g_{\bar{u}}) + \delta_S)(\bar{x})$. Consequently, $((\bar{y}^* g_{\bar{u}}) + \delta_S)^*(-x^*) = \langle -x^*, \bar{x}\rangle = -f^*(x^*) - f(\bar{x})$. This yields

$$f(\bar{x}) = -f^*(x^*) - ((\bar{y}^* g_{\bar{u}}) + \delta_S)^*(-x^*) \leq -(f + (\bar{y}^* g_{\bar{u}}) + \delta_S)^*(0)$$

$$= \inf_{x \in S}\{f(x) + \langle \bar{y}^*, g_{\bar{u}}(x)\rangle\} \leq \sup_{(u,y^*) \in \mathcal{U} \times C^*} \inf_{x \in S}\{f(x) + \langle y^*, g_u(x)\rangle\} \leq \inf_{x \in A} f(x) = f(\bar{x}),$$

where the first inequality holds as $h_1^*(x^*) + h_2^*(-x^*) \geq (h_1 + h_2)^*(0)$ for any proper functions h_1, h_2. This completes the proof. ■

Remark 1 The assumption made on the proper function $f : X \to \overline{\mathbb{R}}$ in Theorem 5 (iii), namely, that $(f + \delta_A)^*(0) = (f^* \Box \sigma_A)(0)$ and the infimal convolution is exact at 0, is nothing else than asking for the existence of strong duality for the optimization problem

$$\inf_{x \in X}\{f(x) + \delta_A(x)\}$$

and its *Fenchel dual*

$$\sup_{x^* \in X^*}\{-f^*(x^*) - \sigma_A(-x^*)\}.$$

For f proper and convex and A a convex set (as already seen, S convex and g_u C-convex for all $u \in \mathcal{U}$ guarantees this), this is the case whenever f is continuous at some element in $\operatorname{dom} f \cap A$ (see, for instance, [21]). Additionally, when f is lower semicontinuous and A is closed (as already seen, S closed and g_u C-epi closed for all $u \in \mathcal{U}$ guarantees this), this is the case when $\operatorname{epi} f^* + \operatorname{epi} \sigma_A$ is weak*-closed (see [8]), but also when $f^* \Box \sigma_A$ is lower semicontinuous and exact at zero (see [7]).

In the light of the above remarks, one can notice that Theorem 5 extends [6, Theorem 5] (see also [5, Theorem 10] for a similar result) to parameterized constrained optimization problems.

Remark 2 It should also be noted that, if S is a convex and closed set and g_u is a C-convex and C-epi closed function for each $u \in \mathcal{U}$ such that $A \neq \emptyset$, then the following closed convex condition used in [18]

$$\bigcup_{u \in \mathcal{U}, y^* \in C^*} \operatorname{epi}((y^* g_u) + \delta_S)^* \text{ is weak}^*\text{- closed and convex.} \tag{5}$$

is a sufficient condition for $(RBCQ)$.

4 Best Approximation under data Uncertainty

In this section, we apply our robust duality theory to best approximation problem under data uncertainty. Consider the uncertainty sets $\mathcal{U}_i \subseteq L^2[0,1] \times \mathbb{R}$, $i = 1, ..., m$, and the best approximation problem

$$\inf_{x \in L^2[0,1]} \left\{ \frac{1}{2} \int_0^1 x(t)^2 dt : \int_0^1 a_i(t)x(t)dt \leq \beta_i, i = 1, ..., m \right\}, \tag{6}$$

where the input data (a_i, β_i) is uncertain and $(a_i, \beta_i) \in \mathcal{U}_i$ for $i = 1, ..., m$. In the uncertainty free case, this best approximation model problem aims at finding a solution of minimum norm in $L^2[0,1]$ for a finite linear inequality system which have extensively studied (for example see [9, 11, 13, 20]). By denoting with $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ the inner product and the norm on $L^2[0,1]$, respectively, we consider the robust dual pair,

$$(RP) \qquad \inf_{x \in L^2[0,1]} \left\{ \frac{1}{2} \|x\|^2 : \langle a_i, x \rangle \leq \beta_i, \ \forall (a_i, \beta_i) \in \mathcal{U}_i, i = 1, ..., m \right\}$$

and

$$(ODP) \qquad \sup_{\substack{(a_i,\beta_i) \in \mathcal{U}_i, \lambda_i \geq 0, \\ i=1,...,m}} \inf_{x \in L^2[0,1]} \left\{ \frac{1}{2} \|x\|^2 + \left\langle \sum_{i=1}^m \lambda_i a_i, x \right\rangle - \sum_{i=1}^m \lambda_i \beta_i \right\}$$

or, equivalently (since $\left(\frac{1}{2}\| \cdot \|^2\right)^* = \frac{1}{2}\| \cdot \|^2$),

$$(ODP) \qquad \sup_{\substack{(a_i,\beta_i) \in \mathcal{U}_i, \lambda_i \geq 0, \\ i=1,...,m}} \left\{ -\frac{1}{2} \left\| \sum_{i=1}^m \lambda_i a_i \right\|^2 - \sum_{i=1}^m \lambda_i \beta_i \right\}.$$

Denote the feasible set of the problem (RP) by

$$A := \{x \in L^2[0,1] : \langle a_i, x \rangle \leq \beta_i, \ \forall (a_i, \beta_i) \in \mathcal{U}_i, i = 1, ..., m\}.$$

Next we furnish some simple sufficient conditions under which the Robust Basic Constraint Qualification is valid.

Theorem 6 *Assume that the sets $\mathcal{U}_i, i = 1, ..., m$, are convex and compact and that the following Slater-type condition is fulfilled:*

$$\exists x' \in L^2[0,1] \text{ such that } \langle a_i, x' \rangle < \beta_i \ \forall (a_i, \beta_i) \in \mathcal{U}_i, i = 1, ..., m.$$

Then $(RBCQ)$ holds.

Proof. In order to prove that $(RBCQ)$ is valid, according to Remark 2, it is enough to prove that the set

$$\bigcup_{\substack{(a_i,\beta_i) \in \mathcal{U}_i, \lambda_i \geq 0, \\ i=1,...,m}} \operatorname{epi} \left(\sum_{i=1}^m \lambda_i (\langle a_i, \cdot \rangle - \beta_i) \right)^* = \bigcup_{\substack{(a_i,\beta_i) \in \mathcal{U}_i, \lambda_i \geq 0, \\ i=1,...,m}} \sum_{i=1}^m \operatorname{epi} \left(\langle \lambda_i a_i, \cdot \rangle - \lambda_i \beta_i \right)^*$$

$$= \bigcup_{\substack{(a_i,\beta_i)\in\mathcal{U}_i,\lambda_i\geq 0,\\ i=1,...,m}} \sum_{i=1}^{m} \left(\{\lambda_i a_i\}\times[\lambda_i\beta_i,+\infty)\right) = \sum_{i=1}^{m} \bigcup_{\lambda_i\geq 0} \bigcup_{(a_i,\beta_i)\in\mathcal{U}_i} \{\lambda_i a_i\}\times[\lambda_i\beta_i,+\infty)$$

is (weak-) closed and convex.

To this end, consider for $i = 1,...,m$, the function $g_i : L^2[0,1] \to \mathbb{R}$, $g_i(x) = \sup_{(a_i,\beta_i)\in\mathcal{U}_i}\{\langle a_i, x\rangle - \beta_i\}$. The full domain of g_i is a consequence of the compactness of $\mathcal{U}_i$ and, since g_i is convex and lower semicontinuous, it is actually continuous, for $i = 1,...,m$ (see [**?**, Theorem 2.2.20]). Thus the feasible set of the problem (RP) can be written as $A := \{x \in L^2[0,1] : g_i(x) \leq 0, i = 1,...,m\}$.

Next, we calculate the conjugate function of $g_i, i = 1,...,m$. For $i = 1,....m$ and $x^* \in L^2[0,1]$ it holds

$$\begin{aligned} g_i^*(x^*) &= \sup_{x\in L^2[0,1]} \left\{ \langle x^*, x\rangle - \sup_{(a_i,\beta_i)\in\mathcal{U}_i} \{\langle a_i, x\rangle - \beta_i\} \right\} \\ &= \sup_{x\in L^2[0,1]} \min_{(a_i,\beta_i)\in\mathcal{U}_i} \{\langle x^* - a_i, x\rangle + \beta_i\} \\ &= -\inf_{x\in L^2[0,1]} \max_{(a_i,\beta_i)\in\mathcal{U}_i} \{-\langle x^* - a_i, x\rangle - \beta_i\}. \end{aligned}$$

Due to a minimax theorem (see, for instance, [21, Theorem 2.10.2]), we have

$$\begin{aligned} g_i^*(x^*) &= -\max_{(a_i,\beta_i)\in\mathcal{U}_i} \inf_{x\in L^2[0,1]} \{-\langle x^* - a_i, x\rangle - \beta_i\} \\ &= \min_{(a_i,\beta_i)\in\mathcal{U}_i} \sup_{x\in L^2[0,1]} \{\langle x^* - a_i, x\rangle + \beta_i\} \end{aligned}$$

which means that $g_i^*(x^*) = +\infty$, if $x^* \notin \mathrm{Pr}_{L^2[0,1]}\mathcal{U}_i$, being equal to $\min\{\beta_i : (x^*, \beta_i) \in \mathcal{U}_i\}$, otherwise. Consequently.

$$\operatorname{epi} g_i^* = \bigcup_{(a_i,\beta_i)\in\mathcal{U}_i} \{a_i\}\times[\beta_i,+\infty) = \mathcal{U}_i + (\{0\}\times\mathbb{R}_+), i = 1,...,m,$$

and, hence,

$$\bigcup_{\substack{(a_i,\beta_i)\in\mathcal{U}_i,\lambda_i\geq 0,\\ i=1,...,m}} \operatorname{epi}\left(\sum_{i=1}^{m} \lambda_i(\langle a_i,\cdot\rangle - \beta_i)\right)^* = \sum_{i=1}^{m} \bigcup_{\lambda_i\geq 0} \operatorname{epi}(\lambda_i g_i)^*$$

$$= \bigcup_{\substack{\lambda_i\geq 0,\\ i=1,...,m}} \sum_{i=1}^{m} \operatorname{epi}(\lambda_i g_i)^* = \bigcup_{\substack{\lambda_i\geq 0,\\ i=1,...,m}} \operatorname{epi}\left(\sum_{i=1}^{m} \lambda_i g_i\right)^*,$$

where the last equality follows from (1) by taking into account that the functions g_i are continuous, for $i = 1,...,m$. Due to the Slater-type condition and using again the compactness of the uncertainty sets, we have that

$$g_i(x') < 0 \text{ for all } i = 1,...,m,$$

fact which, via [3, Theorem 8.2, Theorem 8.3 and Remark 8.4], implies that

$$\bigcup_{\substack{(a_i,\beta_i)\in\mathcal{U}_i,\lambda_i\geq 0,\\ i=1,\ldots,m}} \operatorname{epi}\left(\sum_{i=1}^m \lambda_i(\langle a_i,\cdot\rangle - \beta_i)\right)^* = \bigcup_{\substack{\lambda_i\geq 0,\\ i=1,\ldots,m}} \operatorname{epi}\left(\sum_{i=1}^m \lambda_i g_i\right)^* = \operatorname{epi}\sigma_A$$

and furnishes, finally, the (weak-) closeness and the convexity of

$$\bigcup_{\substack{(a_i,\beta_i)\in\mathcal{U}_i,\lambda_i\geq 0,\\ i=1,\ldots,m}} \operatorname{epi}\left(\sum_{i=1}^m \lambda_i(\langle a_i,\cdot\rangle - \beta_i)\right)^*.$$

Consequently, as pointed out in Remark 2, $(RBCQ)$ holds. ■

This means that, under the hypotheses of Theorem 6, the statement (iii) in Theorem 5 is true. Taking further into consideration that the objective function of (RP) is continuous and coercive, we see that (RP) attains its minimum over A (see, for instance [2, Proposition 11.14]). Hence,

$$\min_{x\in L^2[0,1]}\left\{\frac{1}{2}\|x\|^2 : \langle a_i,x\rangle \leq \beta_i \ \forall (a_i,\beta_i)\in\mathcal{U}_i, i=1,\ldots,m\right\}$$

$$= \max_{\substack{(a_i,\beta_i)\in\mathcal{U}_i,\lambda_i\geq 0,\\ i=1,\ldots,m}}\left\{-\frac{1}{2}\left\|\sum_{i=1}^m \lambda_i a_i\right\|^2 - \sum_{i=1}^m \lambda_i\beta_i\right\}.$$

Acknowledgements. R.I. Boţ would like to thank Prof. V. Jeyakumar and Dr. G.Y. Li for their hospitality during his stay in September 2011 at the School of Mathematics and Statistics of the University of New South Wales in Sydney.

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