On some variants of Schu's lemma

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Abstract. In this note, we demonstrate that an incorrect statement has been propagated in multiple papers, stemming from the substitution of "lim" with "limsup" for a sequence in Lemma 1.3 of the paper [J. Schu: Weak and strong convergence to fixed points of asymptotically nonexpansive mappings, Bull. Austral. Math. Soc. 43 (1991), 153–159]. This occurred over a span of more than 20 years, with the earliest paper we identified using this incorrect statement dating back to 2002.

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In Schu's paper [1], the following result is stated without proof, with "[8]" referring to Zeidler's book, which we cite as [2]:

"LEMMA 1.3. (compare [8, p. 484]) *Let* $(E, \|\cdot\|)$ *be a uniformly convex Banach space,* $0 < b < c < 1$*,* $a \geq 0$*,* $(t_n) \subset [b, c]^{\mathbb{N}}$ *and* $(x_n), (y_n) \in E^{\mathbb{N}}$ *such that* $\limsup \|x_n\| \leq a$*,* $\limsup \|y_n\| \leq a$ *and* $\lim \|t_n x_n + (1 - t_n) y_n\| = a$*. Then* $\lim \|x_n - y_n\| = 0$."

Interestingly, Dotson had presented a similar (in fact equivalent) result 21 years earlier, but most of the literature we know of using this result attributes it to Schu. On page 68 of [3], Dotson states:

"The following lemma is an easy consequence of uniform convexity.

LEMMA 3. *Suppose E is a uniformly convex Banach space. Suppose* $0 < a < b < 1$*, and* $\{t_n\}$ *is a sequence in* $[a, b]$*. Suppose* $\{w_n\}$*,* $\{y_n\}$ *are sequences in E such that* $\|w_n\| \leq 1$*,* $\|y_n\| \leq 1$ *for all n. Define* $\{z_n\}$ *in E by* $z_n = (1 - t_n) w_n + t_n y_n$*. If* $\lim \|z_n\| = 1$*, then* $\lim \|w_n - y_n\| = 0$."

Note that [3, Lemma 3] is recalled in [4, page 376].

Taking $t_n := t \in (0, 1)$ for $n \in \mathbb{N}^*$ $(:= \{1, 2, ..., n, ...\})$, [1, Lemma 1.3] reduces to Problem 10.1 (c) from [2, p. 484]. Additionally, note that the version of [1, Lemma 1.3] where the interval $[b, c]$ is replaced by $[\varepsilon, 1 - \varepsilon]$ with $\varepsilon \in (0, 1)$, is attributed also to "(Zeidler (1986))" in [5, Lemma 2.1].

In [6], a variant of Schu's lemma, formulated with a **weaker hypothesis**, is introduced as follows:

"In 1991, Sahu [21] established an important property of UCBS, which can be stated as follows.

Lemma 2.2. [21] *Assume that* Ω *is a UCBS and* $\{z_\ell\}$ *be a sequence in* $(0,1)$ *for all* $\ell \geq 1$*. If* $\{a_\ell\}$ *and* $\{b_\ell\}$ *are in* Ω *such that* $\limsup_{\ell\to\infty} \|a_\ell\| \leq m$, $\limsup_{\ell\to\infty} \|b_\ell\| \leq m$ *and* $\lim_{\ell\to\infty} \|(1-z_\ell)a_\ell + z_\ell b_\ell\| = m$ *for some* $m \geq 0$*. Then* $\lim_{\ell\to\infty} \|a_\ell - b_\ell\| = 0$."

In fact, reference [21] from the preceding quoted text is none other than Schu's paper [1]. Additionally, note that Lemma 2.5 from [7], also attributed to [1], is equivalent to [6, Lemma 2.2].

Note that this variant of Schu's lemma [1, Lemma 1.3] is **false** for any non-trivial normed space as illustrated by the following example.

Example 1. *Let* $(X, \|\cdot\|)$ *be a non-trivial normed vector space,* $x \in X \setminus \{0\}$, $x_n := x$ *and* $y_n := -x$ *for* $n \in \mathbb{N}^*$*; moreover, consider* $\{t_n\} \subset (0,1)$ *with* $t_n \to 0$*. Then* $\|x_n\| = \|y_n\| = \|x\|$*, and so* $\limsup_{n\to\infty} \|x_n\| = \limsup_{n\to\infty} \|y_n\| = \|x\| > 0$*; moreover,* $t_n x_n + (1-t_n)y_n = (2t_n - 1)x$*, and so*

$$\lim_{n\to\infty} \|t_n x_n + (1-t_n)y_n\| = \lim_{n\to\infty} |2t_n - 1| \cdot \|x\| = \|x\|.$$

It is obvious that $\lim_{n\to\infty} \|x_n - y_n\| = 2\|x\| \neq 0$*.*

In [8], Laowang and Panyanak introduced an extension of Schu's lemma to the setting of uniformly convex hyperbolic spaces, first announced in [9, Lemma 2.7] for CAT(0) spaces, as follows:

"The following result is a characterization of uniformly convex hyperbolic spaces which is an analog of Lemma 1.3 of Schu [25]. It can be applied to a CAT(0) space as well.

Lemma 2.9. *Let* (X, d, W) *be a uniformly convex hyperbolic space with modulus of convexity* η*, and let* $x \in X$*. Suppose that* η *increases with* r *(for a fixed* ε*) and suppose that* $\{t_n\}$ *is a sequence in* $[b, c]$ *for some* $b, c \in (0,1)$*, and* $\{x_n\}$*,* $\{y_n\}$ *are sequences in* X *such that* $\limsup_{n\to\infty} d(x_n, x) \leq r$, $\limsup_{n\to\infty} d(y_n, x) \leq r$*, and* $\lim_{n\to\infty} d((1-t_n)x_n) \oplus t_n y_n, x) = r$ *for some* $r \geq 0$*. Then* $\lim_{n\to\infty} d(x_n, y_n) = 0$*. (2.17)*"

The proof of [8, Lemma 2.9] is an adaptation to this context of that for Problem 10.1 (c) in Zeidler's book [2].

The formulation and the proof of [8, Lemma 2.9] are replicated in [10, Lemma 4.5], while, for $t_n := 1/2$ for $n \in \mathbb{N}^*$, it can be found as Lemma 2.2 in [11].

Lemma 2.9 from [8] is (practically) replicated also in [12, Lemma 2.5], where an *alternative proof* is provided. However, this proof contains the following *false assertion*:

"Since $\limsup_{n\to\infty} d(x_n, x) \leq r$ and $\limsup_{n\to\infty} d(y_n, x) \leq r$ we have: (i) $d(x_n, x) \leq r + \frac{1}{n}$; (ii) $d(y_n, x) \leq r + \frac{1}{n}$ for each $n \geq 1$".

Lemma 2.9 from [8] is also presented as Lemma 2.5 in [13], with a reference to [13, Lemma 2.3], and as Lemma 1.3 in [14].

Note that [8, Lemma 2.9] is stated and proved for X as a CAT(0) space in [15, Lemma 3.2] under the supplementary condition $0 < b(1-c) \leq \frac{1}{2}$. This condition appears in many papers from our bibliography. It is easy to observe that this condition is superfluous. Indeed, $b, c \in (0,1)$ implies that $0 < b(1-c)$, while the fact that (there exists) $t_n \in [b,c]$ implies that $b \leq c$; hence $0 < b \leq c < 1$. Because $b^2 + (b-1)^2 = 2b^2 - 2b + 1 > 0$ and $b > 0$, it follows that $\frac{1}{2b} > 1 - b \geq 1 - c$. Consequently, $b(1-c) < \frac{1}{2}$.

Furthermore, an extension of Schu's lemma to the setting of modular function spaces was introduced in [16] (see also [17, Lemma 4.2]):

"Lemma 3.2. *Let* $\rho \in \Re$ *be (UUC1) and let* $\{t_k\} \subset (0,1)$ *be bounded away from* 0 *and* 1. *If there exists* $R > 0$ *such that* $\limsup_{n\to\infty} \rho(f_n) \leq R$, $\limsup_{n\to\infty} \rho(g_n) \leq R$, $\lim_{n\to\infty} \rho(t_n f_n + (1-t_n) g_n) = R$, *then* $\lim_{n\to\infty} \rho(f_n - g_n) = 0$."

In the following we present a proof for [1, Lemma 1.3]:

Proof (of [1, Lemma 1.3]). Since $\limsup \|x_n\| \leq a$ and $\limsup \|y_n\| \leq a$, there exists $R > 0$ such that $\|x_n\|, \|y_n\| \leq R$ for every $n \in \mathbb{N}^*$. $(E, \|\cdot\|)$ being uniformly convex, by [18, Theorem 4.1 (ii)], $\|\cdot\|^2$ is a uniformly convex function on $B := \{x \in E \mid \|x\| \leq R\}$. This means that there exists a strictly increasing function $\rho : \mathbb{R}_+ \to \mathbb{R}_+$, with $\rho(0) = 0$, such that

$$\|\lambda x + (1-\lambda) y\|^2 \leq \lambda \|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\rho(\|x-y\|)$$

for all $x, y \in B$ and $\lambda \in [0,1]$. Setting $\gamma := \min\{b, 1-c\}$ (> 0), one has $\lambda(1-\lambda) \geq \gamma^2$, and so

$$\gamma^2 \rho(\|x_n - y_n\|) \leq t_n \|x_n\|^2 + (1-t_n)\|y_n\|^2 - \|t_n x_n + (1-t_n) y_n\|^2 \quad (1)$$

for all $n \in \mathbb{N}$. Assume that the conclusion does not hold. Then there exist a strictly increasing sequence $(n_k) \subset \mathbb{N}^*$ and $\varepsilon > 0$ such that $t_{n_k} \to t \in [b,c]$ as $k \to \infty$, and $\|x_{n_k} - y_{n_k}\| \geq \varepsilon$ for every $k \in \mathbb{N}^*$. Replacing n by n_k in (1), using that $\rho(\|x_{n_k} - y_{n_k}\|) \geq \rho(\varepsilon)$ for every $k \in \mathbb{N}^*$, and passing to lim sup in (1) as $k \to \infty$, one gets the contradiction $\gamma^2 \rho(\varepsilon) \leq ta^2 + (1-t)a^2 - a^2 = 0$. □

In [19], Kim, Kiuchi and Takahashi introduced a variant of Schu's lemma in uniformly convex Banach spaces, assuming a **weaker hypothesis**, with "[9]" referring to [1]:

"Lemma 2.4 ([9]). *Let* E *be a uniformly convex Banach space, let* $0 < b \leq t_n \leq c < 1$ *for all* $n \in N$, *and let* $\{x_n\}$ *and* $\{y_n\}$ *be sequences of* E *such that* $\overline{\lim}\|x_n\| \leq a$, $\overline{\lim}\|y_n\| \leq a$, *and* $\overline{\lim}\|t_n x_n + (1-t_n) y_n\| = a$ *for some* $a \geq 0$. *Then, it holds that* $\lim_{n\to\infty} \|x_n - y_n\| = 0$."

No proof is given for [19, Lemma 2.4]. However, Agarwal, O'Regan and Sahu formulated the following result in their book [20], for which a proof is also given:

"Theorem 2.3.13 *Let* X *be a uniformly convex Banach space and let* $\{t_n\}$ *be a sequence of real numbers in* $(0,1)$ *bounded away from* 0 *and* 1. *Let* $\{x_n\}$

and $\{y_n\}$ *be two sequences in* X *such that*

$$\limsup_{n\to\infty}\|x_n\| \le a, \quad \limsup_{n\to\infty}\|y_n\| \le a \quad \text{and} \limsup_{n\to\infty}\|t_n x_n + (1-t_n)y_n\| = a$$

for some $a \ge 0$*. Then* $\lim \|x_n - y_n\| = 0$*.*"

Replacing a by r in [20, Theorem 2.3.13] (above) and the pair (α, β) from its proof by (p, q) one gets the statement of [21, Theorem 2.7]; compare $R \in (a, a+1)$ in its proof with $r \in (a, a+1)$ from the proof of [20, Theorem 2.3.13]. Notice that [20] is mentioned in the bibliography of [21], but is not cited in this context.

Note that the (equivalent) variants [19, Lemma 2.4] and [20, Theorem 2.3.13] of Schu's lemma [1, Lemma 1.3] are **false** for any non-trivial normed space, as illustrated by the following example.

Example 2. *Let* $(X, \|\cdot\|)$ *be a non-trivial normed vector space,* $x \in X$ *with* $\|x\| = 1$ $(=: a)$*,* $x_n := x$*,* $y_{2n} := 0$*,* $y_{2n-1} := x$*,* $t_n := \lambda \in (0,1)$ *for* $n \in \mathbb{N}^*$*. Then* $\limsup_{n\to\infty}\|x_n\| = 1$, $\limsup_{n\to\infty}\|y_n\| = 1$, $t_n x_n + (1-t_n)y_n = \lambda x$ *for even* n*,* $t_n x_n + (1-t_n)y_n = x$ *for odd* n*, and so* $\limsup_{n\to\infty}\|t_n x_n + (1-t_n)y_n\| = \limsup_{n\to\infty} 1 = 1$*. It is obvious that* $\lim_{n\to\infty}\|x_n - y_n\|$ *does not exist, and so it is different from* 0*.*

A quite strange result, with the same conclusion as that of [1, Lemma 1.3], is the following one from [22] in which the reference [23] is Nakajo and Takahashi's paper [23]:

"Lemma 2.2. ([23]) *Assume that* X *is a uniformly convex Banach space and* $\{s_n\}$ *is sequence in* $[\delta, 1-\delta]$ *for* $\delta \in (0,1)$*. Assume that sequences* $\{x_n\}$ *and* $\{y_n\}$ *in* X *are such that* $\liminf_{n\to\infty}\|x_n\| \le c$*,* $\liminf_{n\to\infty}\|y_n\| \le c$*, and* $\liminf_{n\to\infty}\|s_n x_n + (1-s_n)y_n\| = c$ *for some* $c \ge 0$*. Then* $\liminf_{n\to\infty}\|x_n - y_n\| = 0$."

Note that this variant of Schu's lemma [1, Lemma 1.3] is **false** for any non-trivial normed space as illustrated by the following example.

Example 3. *Let* $(X, \|\cdot\|)$ *be a non-trivial normed vector space,* $x \in X$ *with* $\|x\| = 1$ $(=: c)$ *and take* $x_{2n-1} := y_{2n} := \frac{1}{2}x$*,* $x_{2n} := y_{2n-1} := \frac{3}{2}x$*,* $s_n := \frac{1}{2}$ *for* $n \in \mathbb{N}^*$*; then* $s_n x_n + (1-s_n)y_n = x$ *and* $x_n - y_n \in \{x, -x\}$ *for* $n \in \mathbb{N}^*$*, whence* $\liminf_{n\to\infty}\|x_n\| = \liminf_{n\to\infty}\|y_n\| = \frac{1}{2} \le c$*,* $\lim_{n\to\infty}\|s_n x_n + (1-s_n)y_n\| = \|x\| = c$ *and* $\lim_{n\to\infty}\|x_n - y_n\| = c > 0$*.*

Below is a list of statements equivalent to [19, Lemma 2.4], given obviously without proofs, and used in the proofs of other results in the corresponding works. In most cases, [1, Lemma 1.3] is cited as the reference for this result; exceptions will be noted explicitly. For the interval $[b, c]$ for some $0 < b \le c < 1$ often the supplementary condition(s) $b < c$ and/or $0 < b(1-c) \le \frac{1}{2}$ are required, or this is replaced by $[\varepsilon, 1-\varepsilon]$ with $\varepsilon \in (0,1)$:

[24, Lemma 2.3]; [25, Lemma 2.1]; [26, Lemma 2.3]; [27, Lemma 2.4]; [28, Lemma 2.1]; [29, Lemma 5]; [30, Lemma 2.2]; [31, Lemma 2.1]; [32, Lemma 2.2]; [33, Lemma 2.1] (attributed to [34]); [35, Lemma 2.1]; [36, Lemma 2.2] in which $0 \le p < t_n \le q < 1$; [37, Lemma 2.2] in which $0 \le p < t_n \le q < 1$; [38, Lemma 2.2] in which $0 \le p < t_n \le q < 1$; [39, Lemma 2.2]; [40, Lemma 2.3]; [41, Theorem 4.3.1]; [42, Lemma 2.2]; [43, Lemma 2.3]; [44, Lemma 2.2] in which $0 \le p < t_n \le q < 1$ (attributed to [45]); [46, Lemma 2.6]; [47, Lemma 2.2]; [48, Lemma 2.1]; [49, Lemma 2.4]; [50, Lemma 2.2]; [51, Lemma 2]; [52, Lemma 2.3]; [53, Lemma 2.2]; [54, Lemma 6]; [55, Lemma 1.3]; [56, Lemma 2.9]; [57, Lemma 2.2]; [58, Lemma 3.2]; [59, Lemma 1.9] ([20]); [60, Lemma 2.2]; [61, Lemma 1.4] ([45]); [62, Lemma 2.2]; [63, Lemma 12]; [64, Lemma 4.2]; [65, Lemma 2.2]; [66, Lemma 2.1]; [67, Lemma 2.2]; [68, Lemma 2.4]; [69, Lemma 2.4]; [70, Lemma 2.10]; [71, Lemma 2.2] (attributed to [2]); [72, Lemma 2.2]; [73, Lemma 1.4]; [74, Lemma 2.3]; [75, Lemma 2.2]; [76, Lemma 2.7]; [77, Lemma 2.4]; [78, Lemma 2]; [79, Lemma 2.4]; [80, Lemma 2.4]; [81, Lemma 2.1]; [82, Lemma 2.6]; [83, Lemma 3]; [84, Lemma 1.4]; [85, Lemma 2.5]; [86, Lemma 1.7] in which $0 < g_n < 1$; [87, Lemma 2.1]; [88, Lemma 1.4]; [89, Lemma 1]; [90, Lemma 4.6] in which $0 \le t_n \le 1$; [91, Lemma 2.6]; [92, Lemma 5]; [93, Lemma 2.4]; [94, Lemma 3]; [95, Lemma 2.1]; [96, Lemma 2.1]; [97, Lemma 2.2]; [98, Lemma 3]; [99, Lemma 3]; [100, Lemma 2.2]; [101, Lemma 2.8]; [102, Lemma 2]; [103, Lemma 2.2]; [104, Lemma 3]; [105, Lemma 2]; [106, Lemma 2]; [107, Lemma 4]; [108, Lemma 2.2]; [109, Lemma 3]; [110, Lemma 4.2.4]; [111, Lemma 2.16]; [112, Lemma 2.14]; [113, Lemma 3]; [114, Lemma 1.2] ([4]); [115, Lemma 2.5] ([116]); [117, Lemma 2.10]; [118, Lemma 5]; [119, Lemma 2.7]; [120, Lemma 4]; [121, Lemma 2.4]; [122, Lemma 2.3]; [123, Lemma 1] in which $0 < \rho_n < 1$; [124, Lemma 2.7]; [125, Lemma 3]; [126, Lemmas 1.3, 2.4]; [127, Lemma 2.11]; [128, Lemma 2.3]; [129, Lemma 2] (attributed to [68]); [130, Lemma 2.5]; [131, Lemma 6]; [132, Lemma 12]; [133, Lemma (1.6)]; [134, Lemma 2]; [135, Lemma 1]; [136, Lemma 2.6]; [137, Lemma 2.6]; [138, Lemma 2.4]; [139, Lemma 2.2]; [140, Lemma 2.10]; [141, Lemma 1.4] in which $t_n \in [0, 1]$ ([20]); [142, Lemma 7]; [143, Lemma 2.1]; [144, Lemma 2.2]; [145, Lemma 3]; [146, Lemma 1]; [147, Lemma 1]; [148, Lemma 1]; [149, Lemma 1.6] in which $0 < s_n < 1$; [150, Lemma 2.3]; [151, Theorem 2.6] ([2] – compare with [21]); [152, Lemma 2.5]; [153, Lemma 2]; [154, Lemma 2.3]; [155, Lemma 2]; [156, Lemma 5]; [157, Lemma 5]; [158, Lemma 1]; [159, Lemma 4]; [160, Lemma 1.3] in which $0 < q \le u_n < 1$ ([59]); [161, Lemma (1.2)] ([20]); [162, Lemma 2.1]; [163, Lemma 2.12]; [164, Lemma 2.1]; [165, Lemma 1.3]; [166, Lemma 1]; [167, Lemma 2.4]; [168, Lemma 2.3]; [169, Lemma 2.1]; [170, Lemma 2.2]; [171, Lemma 2.11] in which $0 < \zeta_t < 1$; [172, Lemma 2.4]; [173, Lemma 2.1]; [174, Lemma 2.2]; [175, Lemma 2.4]; [176, Lemma 2.3]; [177, Lemma 1]; [178, Lemma 2.1] in which $0 < t_s < 1$; [179, Lemma 1]; [180, Lemma 4.2]; [181, Lemma 1]; [182, Lemma 1]; [183, Lemma 2.5]; [184, Lemma 2.1]; [185, Lemma 3]; [186, Lemma 2.6]; [187, Lemma 1]; [188, Lemma 2.5]; [189, Lemma 2.5]; [190, Lemma 3]; [191, Lemma 2.3].

In the setting of CAT(0) spaces, Thakur, Thakur and Postolache formulated the following result in [192], which assumes a **weaker hypothesis** than that in [8, Lemma 2.9], recalled above:

"Lemma 2.3 ([25], Lemma 4.5) *Let X be a $CAT(0)$ space, $x \in X$ be a given point and $\{t_n\}$ be a sequence in $[b, c]$ with $b, c \in (0, 1)$ and $0 < b(1-c) \leq \frac{1}{2}$. Let $\{x_n\}$ and $\{y_n\}$ be any sequences in X such that $\limsup_{n\to\infty} d(x_n, x) \leq r$, $\limsup_{n\to\infty} d(y_n, x) \leq r$ and $\limsup_{n\to\infty} d((1-t_n)x_n \oplus t_n y_n, x) = r$, for some $r \geq 0$. Then $\lim_{n\to\infty} d(x_n, y_n) = 0$.*"

The reference "[25]" mentioned in [192, Lemma 2.3] is our reference [10]. Of course, no proof is given for this "stronger" version of [8, Lemma 2.9] in [192]. Since any Hilbert space is CAT(0), Example 2 shows that [192, Lemma 2.3] is **false**.

Below is a list of statements equivalent to [192, Lemma 2.3] used in the proofs of other results in the corresponding works. For each paper, we add the version of [8, Lemma 2.9] that is cited, or just the work if a particular result is not mentioned:

[193, Lemma 2.2] ([12, 194]); [195, Lemma 2.2] with $\limsup_{n\to\infty} d((1-t_n)x_n \oplus t_n y_n, x) \leq r$ ([196, Lemma 2.9], but there is not a statement called "Lemma 2.9" in [196]); [197, Lemma 2.5] ([11]); [198, Lemma 1.12] ([10, Lemma 4.5]); [199, Lemma 1.11] ([14]); [200, Lemma 2] ([10, Lemma 4.5]); [201, Lemma 2.11] ([8]); [202, Lemma 2.5] ([10, Lemma 4.5]); [203, Lemma 2.2] ([10, Lemma 4.5]); [204, Lemma 2.7] ([8, Lemma 2.9]); [205, Lemma 2.12] ([206, Lemma 1.3], which does not exists); [207, Lemma 2.6] ([199]); [208, Lemma 2.11] ([12]); [209, Lemma 2.5] ([9]); [210, Lemma 2.5] ([8]); [211, Lemma 3] ([12]); [212, Lemma 5] ([8]); [213, Lemma 2.2] ([10, Lemma 4.5]); [214, Lemma 2] ([8]); [215, Lemma 1.8] ([12]); [216, Lemma 1.4] ([12]); [217, Lemma 5.6] with $\limsup_{n\to\infty} d(t_n x_n \oplus (1-t_n)y_n, x) \leq r$ ([8]); [218, Lemma 2.14] ([196]); [219, Lemma 2.2] ([199]); [220, Lemma 2.2] ([12]); [221, Lemma 2.1] ([12]); [222, Lemma 3.8] ([12]); [223, Lemma 3] ([8]); [224, Lemma 2.10] ([8]); [225, Lemma 2.4] ([8]); [226, Lemma 12] ([1]); [227, Lemma 2.10] ([9]); [228, Lemma 2.3] with $\limsup_{n\to\infty} d(W(x_n, y_n, t_n), x) \leq c$ ([12]); [229, Lemma 2.8] ([12]); [230, Lemma 5] ([1]).

In the setting of modular function spaces, Bejenaru and Postolache formulated the following result in [231], which assumes a **weaker hypothesis** than that in [16, Lemma 3.2], recalled above:

"Lemma 1. *Suppose that ρ satisfies property (UUC1) and let $\{\alpha_l\} \subset [a, b]$, where $0 < a < b < 1$. If there exists a positive real number r such that $\limsup_{l\to\infty} \rho(\alpha_l x_l + (1-\alpha_l)y_l) = r$, $\limsup_{l\to\infty} \rho(x_l) \leq r$ and $\limsup_{l\to\infty} \rho(y_l) \leq r$, then $\lim_{l\to\infty} \rho(x_l - y_l) = 0$ ([8], cf. [5]).*"

The references "[8]" and "[5]" mentioned in [231, Lemma 1] are our references [232] and [16], respectively, where [232, Lemma 2.7] is nothing else than [16, Lemma 3.2]. Of course, no proof is given for this "stronger" version of [16, Lemma 3.2] in [231]. Since any L^p space with $p \in [1, \infty)$ is a modular space, Example 2 demonstrates that [231, Lemma 1] is **false**.

Below is a list of statements equivalent to [231, Lemma 1] used in the proofs of other results in the corresponding works. For each paper, we add the version of [16, Lemma 3.2] that is cited, or just the work if a particular result is not mentioned:

[233, Lemma 2.6] (attributed to "[8], cf. [4,21]", that is [234], [17] and [16], respectively, although there is not a result equivalent to [231, Lemma 1] in [234]); [235, Lemma 1] ([17]); [236, Lemma 2.6] ([17]).

The list of works with incorrect variations of Schu's lemma mentioned in this article is by no means complete.

We would like to emphasize that we have only partially verified the correctness of the results in the aforementioned works that rely on various variants of Schu's lemma, such as [6, Lemma 2.2], [19, Lemma 2.4], [20, Theorem 2.3.13], [195, Lemma 2.2] or [231, Lemma 1]. While some of these works directly apply an incorrect variant of Schu's lemma in their proofs, others, though they strangely cite a false variant, do verify that the third condition is met by "lim".

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