The Fenchel duality in set-valued vector optimization

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Contents

Introduction

Since many optimization problems encountered in economics and other fields involve set-valued mappings constraints and set-valued mappings objectives, set-valued optimization problems have received an increasing amount of attention in recent years(see Corley[3], Luc[10], Jahn and Rauh[10], Postolica[12] and the references cited here).

For the studying of the set-valued optimization problems it has been developed a very interesting duality theory. The first researches of the Lagrangian duality for vector optimization problems had been made by Corley[2], Dolecki and Malivert[4] and by Song in [15] and [16].

In the same time it has been developed a theory which represents a generalization of the Fenchel duality for set-valued mappings. This theory is based, like in the scalar optimization, on the very elegant concept of perturbed problems and conjugate mappings. On the other hand, it is interesting to observe the diversity of generalizations of the Fenchel duality theory, caused by the different definitions of the notion of efficiency. Postolica[13] has developed such a generalization for efficiency of set-valued mappings by using the concept of nuclear cone. Malivert[11] has considered Fenchel type duality for weak efficiency of set-valued mappings and recently, Song[18] has also considered Fenchel duality by using of a new concept of weak efficiency, appeared in the last decade.

In the first part of this work we present the Fenchel duality theory for set-valued mappings developed by Song[18] and give a generalization for a stability criterion. In the second part we formulate and solve the set-valued optimization problem with constraints and finally, by considering of a particular case, we rediscover some results from the scalar optimization.

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1 Preliminaries and weak supremum properties

Let Y be a real topological vector space partially ordered by a closed, convex and pointed cone S with a nonempty interior $intS$ in Y. We use the following notations

$$\begin{array}{lll} y \geqq y' & \text{iff} & y - y' \in S, \\ y \geq y' & \text{iff} & y - y' \in S \setminus \{0\}, \\ y > y' & \text{iff} & y - y' \in intS. \end{array}$$

We add two imaginary points $+\infty$ and $-\infty$ to Y and denote the extended space by $\overline{Y}$. Let consider, by convention, that Ø is a subset of $\overline{Y}$. We also consider that for any $y \in Y$, $-\infty < y < +\infty$. We can now extend the addition and the scalar multipilcation of Y to $\overline{Y}$ by using the following conventions

$$\begin{array}{lll} (\pm\infty) + y = y + (\pm\infty) = (\pm\infty), & \text{for all} & y \in Y \\ (\pm\infty) + (\pm\infty) = (\pm\infty) & & \\ \lambda(\pm\infty) = (\pm\infty), & \text{for} & \lambda > 0 \\ \lambda(\pm\infty) = (\mp\infty), & \text{for} & \lambda < 0. \end{array}$$

For a given set $Z \subset \overline{Y}$ we define $A(Z)$, the set of all points above Z, and $B(Z)$, the set of all points below Z, by

$$A(Z) = \{y \in \overline{Y} | \exists y' \in Z \quad s.t. \quad y > y'\}$$

and

$$B(Z) = \{y \in \overline{Y} | \exists y' \in Z \quad s.t. \quad y < y'\}$$

respectively. Clearly, $A(Z) \subset Y \cup \{+\infty\}$ and $B(Z) \subset Y \cup \{-\infty\}$. We define weak maximal points and weak supremal points as follows(see Postolica[13], Tanino[20], Dolecki and Malivert[4])

Definition 1.1
Given a set $Z \subset \overline{Y}$, a point $p \in \overline{Y}$ is said to be a **weak maximal** point of Z if $p \in Z$ and $p \notin B(Z)$, that is, if $p \in Z$ and there is no $y \in Z$ such that $p < y$. The set of all weak maximal points of Z is called the **weak maximum** of Z and is denoted by $WMaxZ$. The **weak minimum** of Z, $WMinZ$ is defined analogously.

Definition 1.2
Given a set $Z \subset \overline{Y}$, a point $p \in \overline{Y}$ is said to be a **weak supremal** point of Z if $p \notin B(Z)$ and $B(\{p\}) \subset B(Z)$, that is, if there is no $y \in Z$ such that $p < y$ and if $y' < p$ for $y' \in \overline{Y}$ implies that there exists a point $y \in Z$ such that $y' < y$. The set of all weak supremal points of Z is called the **weak supremum** of Z and is denoted by $WSupZ$. The **weak infimum** of Z, $WInfZ$ is defined analogously.

Remark 1.1
(i) $WMax\ \emptyset = \emptyset$ and $WSup\ \emptyset = \{-\infty\}$.
(ii) $-WMax(-Z) = WMin(Z)$ and $-WSup(-Z) = WInf(Z)$.
(iii) $\forall Z \subset \overline{Y}$, $-A(-Z) = B(Z)$.

Tanino[20] has proved the following properties of weak maximum and weak supremum in the case when $Y = R^p$. They are also valid in the general case, considered by us.

Proposition 1.1
$WMaxZ = Z \cap WSupZ$.

Propositon 1.2
(i) $WSupZ = \{-\infty\}$ if and only if $B(Z) = \emptyset$. This is the case when and only when $Z = \emptyset$ or $Z = \{-\infty\}$.
(ii) $WSupZ = \{+\infty\}$ if and only if $B(Z) = Y \cup \{-\infty\}$.
(iii)Except the above cases, $WSupZ \subset Y$.

The following definition of the closure of $B(Z)$ in $\overline{Y}$ has been given by Kawasaki[8] and is useful to characterize the set $WSupZ$.

Definition 1.3
For a set $Z \subset \overline{Y}$, let the **closure** of $B(Z)$ in $\overline{Y}$ be:

$$clB(Z) = \begin{cases} \{-\infty\}, & \text{if } B(Z) = \emptyset \\ \overline{Y}, & \text{if } B(Z) = Y \cup \{-\infty\} \\ cl[B(Z) \cap Y] \cup \{-\infty\}, & \text{otherwise.} \end{cases}$$

The symbol "cl" in the right-hand side means the usual closure in Y.

Proposition 1.3
If $Z \subset \overline{Y}$, then $B(clB(Z)) = B(Z)$.

Proof. If $B(Z) = \emptyset$ or $B(Z) = Y \cup \{-\infty\}$, then the result is obviously true. The point $-\infty$ is contained in both sets. Thus let $y \in B(Z)$ and $y \neq -\infty$. Then, there exists $y' \in Z \cap Y$ such that $y < y'$. Hence,

$$\alpha y + (1-\alpha)y' \in B(Z), \text{ for all } \alpha, 0 < \alpha < 1.$$

Taking the limit when $\alpha \to 0$, it follows that $y' \in clB(Z)$ and so $y \in B(cl(B(Z))$.

Conversely, suppose that $y \in B(clB(Z))$ and that $y \neq -\infty$. Then, there exists $y' \in clB(Z)$ such that $y' - y \in intS$. This implies that there exists $V \subset S$ such that $V + y$ is a neighbourhood of y' and then, we have, $(V + y) \cap B(Z) \neq \emptyset$. Therefore, it follows that $y \in B(Z)$.

Proposition 1.4
If $Z \subset \overline{Y}$, then $WSupZ = [clB(Z)] \setminus B(Z) = WMax[clB(Z)]$.

Proof. Using that for all $Z \subset \overline{Y}$, $WMaxZ = Z \setminus B(Z)$, it's obviously true that $WMax[clB(Z)] = [clB(Z)] \setminus B(clB(Z)) = [clB(Z)] \setminus B(Z)$.

Let $p \in [clB(Z)] \setminus B(Z)$. Since the other cases are trivial, we will consider the case when $p \in cl[B(Z) \cap Y]$. Let $y \in Y$ such that $y < p$ or equivalent, $p - y \in intS$. There exists, then, $V \subset S$ such that $V + y$ is a neighbourhood of p. From $(V + y) \cap B(Z) \neq \emptyset$ it follows that there exists $z \in Z$ such that $y < z$. Hence, $B(\{p\}) \subset B(Z)$ and so $p \in WSupZ$.

Conversely, suppose that $p \in WSupZ$. This means that $p \notin B(Z)$ and $B(\{p\}) \subset B(Z)$. Let consider an arbitrary $s \in intS$. Then,

$$p - \alpha s \in B(\{p\}), \ \forall \ \alpha > 0.$$

By taking the limit when $\alpha \to 0$, it follows that $p \in clB(Z)$. Hence, $p \in [clB(Z)] \setminus B(Z)$.

The last two propositions conduce us to the following corollary.

Corollary 1.1
If $Z \subset \overline{Y}$, then $WSupZ = WSup(B(Z)) = WSup(clB(Z))$.

Proposition 1.5
If $y \in Y$ and $d \in intS$, then there exists $\alpha_0 \geq 0$ such that $y + \alpha d \in intS$, $\forall \alpha \geq \alpha_0$.

Proof. If α_0 does not exist, then we can consider a sequence $\{\alpha_k\}$ such that $\alpha_k \geq 0$, $\forall k \in N$, $\alpha_k \to +\infty$ and $y + \alpha_k d \notin intS$. Since $intS$ is a cone, $\frac{y}{\alpha_k} + d \notin intS$. Using that $(intS)^c$ is a closed set and taking the limit when $k \to +\infty$, we have that $d \notin intS$ which is a contradiction.

Proposition 1.6
If $Z \subset \overline{Y}$, then $B(Z) = B(WSupZ)$.

Proof. It is clear that $B(WSupZ) \subset B(Z)$. If $WSupZ = \{+\infty\}$ or $\{-\infty\}$, then the converse inclusion is obvious. Because $-\infty$ is contained in both sets we can choose an element $y' \in B(Z)$, $y' \neq -\infty$. This means that there exists $y \in Y \cap Z$ such that $y' < y$. Let take an arbitrary $d \in intS$. It follows that there exists $\alpha_0 \geq 0$ such that $y + \alpha d \notin clB(Z)$, $\forall \alpha > \alpha_0$, since otherwise $Y \subset clB(Z)$. Let now define a nonnegative number $\bar{\alpha}$ by

$$\bar{\alpha} = sup\{\alpha | \ y + \alpha d \in clB(Z)\}.$$

It's clear that $y + \bar{\alpha} d \in WSupZ = WMax[clB(Z)]$. Since $y' < y \leqq y + \bar{\alpha} d$ we have proved that $B(Z) \subset B(WSupZ)$.

Corollary 1.2
If $Z \subset \overline{Y}$, then $A(Z) = A(WInfZ)$.

For Proposition 1.4 and Proposition 1.6 results the next corollary.

Corollary 1.3
If $Z \subset \overline{Y}$, then $Z \subset clB(Z) = (WSupZ) \cup B(Z) = WSupZ \cup B(WSupZ)$.

Proposition 1.7
If $Z \subset \overline{Y}$, then $\overline{Y} = WSupZ \cup A(WSupZ) \cup B(WSupZ)$ and the three sets in the right-hand side are disjoint.

Proof. It is obvious that the three sets are disjoint. Since $WSupZ \cup B(WSupZ)$ $= clB(Z)$, from Corollary 1.3, we have to prove that $y \in A(WSupZ)$ if $y \notin clB(Z)$. When $WSupZ = \{-\infty\}$ or $\{+\infty\}$, the above statement is true. Since $\{+\infty\} \in A(WSupZ)$ we consider $y \neq +\infty$ such that $y \notin clB(Z)$ and we will prove that $y \in A(WSupZ)$. Let take an arbitrary $d \in intS$. By Proposition 1.5, $y - \alpha d \in B(Z)$ for a sufficiently large $\alpha > 0$. Let $\bar{\alpha} = inf\{\alpha > 0 | \; y - \alpha d \in B(Z)\}$ and $\bar{y} = y - \bar{\alpha} d$. We have to show that $\bar{y} \in$ WSupZ or equivalent, using that $\bar{y} \in clB(Z)$, we have to show that $\bar{y} \notin B(Z)$.
Supposing that $\bar{y} \in B(Z)$ it follows that $y - \alpha d \in B(Z)$ for some α smaller then $\bar{\alpha}$. This contradicts the definition of $\bar{\alpha}$. Therefore, $\bar{y} \notin B(Z)$ and so $y \in A(WSupZ)$.

Using the definition of $B(Z)$, the following results are obviuos.

Proposition 1.8
(i) $B(Z_1 + Z_2) = B(Z_1) + B(Z_2)$, for $Z_1, Z_2 \subset \overline{Y}$, where it is assumed that the sum $+\infty - \infty$ does not occur.
(ii)$B(\bigcup_{i \in I} Z_i) = \bigcup_{i \in I} B(Z_i)$, for $Z_i \in \overline{Y}$ $(i \in I)$.

Proposition 1.9
Let F_1 and F_2 be set-valued mappings from a space X to $\overline{Y}$. Then

$$WSup \bigcup_{x \in X} [F_1(x) + F_2(x)] = WSup \bigcup_{x \in X} [F_1(x) + WSupF_2(x)]$$

where it is assumed that the sum $+\infty - \infty$ does not occur.

Proof. By using Proposition 1.6, Proposition 1.8 and Corollary 1.1 we obtain

$$WSup \bigcup_{x \in X} [F_1(x) + F_2(x)] = WSupB(\bigcup_{x \in X} [F_1(x) + F_2(x)]) =$$

$$WSup \bigcup_{x \in X} [B(F_1(x)) + B(F_2(x))] = WSup \bigcup_{x \in X} [B(F_1(x)) + B(WSupF_2(x))] =$$

$$WSupB(\bigcup_{x \in X} [F_1(x) + WSupF_2(x)]) = WSup \bigcup_{x \in X} [F_1(x) + WSupF_2(x)].$$

From Proposition 1.9, we obtain the following corollaries, which will be very important for this work.

Corollary 1.4
For $Z_1, Z_2 \subset \overline{Y}$, $WSup(Z_1 + Z_2) = WSup(WSupZ_1 + WSupZ_2)$.

Corollary 1.5
If F is a set-valued mapping from X to $\overline{Y}$, then

$$WSup \bigcup_{x \in X} F(x) = WSup \bigcup_{x \in X} WSupF(x).$$

Corollary 1.6
For $Z_1, Z_2 \subset \overline{Y}$, $WSup(Z_1 \cup Z_2) = WSup(WSupZ_1 \cup WSupZ_2)$.

Corollary 1.7
For $Z \subset \overline{Y}$, $WSupZ = WSup(WSupZ)$.

Proposition 1.10
If $Z \subset \overline{Y}$, then $Z \cap Y \subset \overline{Z \cap Y + intS} \subset WInfZ \cup A(Z)$.

Proof. Let $z \in Z \cap Y$ and let V be a neighbourhood of z in Y. This implies that $V - z$ is a neighbourhood of 0 in Y. From $0 \in \overline{intS}$ it follows that $(V - z) \cap intS \neq \emptyset$ or, equivalent, that $V \cap (Z \cap Y + intS) \neq \emptyset$. This means that $z \in \overline{Z \cap Y + intS}$. To prove that $\overline{Z \cap Y + intS} \subset WInfZ \cup A(Z)$ we use Corollary 1.3 and Remark 1.1 . These give us the following relation

$$\overline{Z \cap Y + intS} \subset WInf(Z \cap Y + intS) \cup A(Z \cap Y + intS) \subset WInfZ \cup A(Z).$$

The last proposition of this chapter provides a characterization of the weak supremum of a set by scalarization under the convexity assumption. The proof of this proposition has been given by Sawaragi,Nakayama,Tanino[14].

Propositon 1.11
Let Y and Y^* be put in duality by the bilinear pairing $< \cdot, \cdot >$ and let

$$S^* = \{\mu \in Y^* | < \mu, s > \geq 0 \ \forall s \in S\}$$

be the dual cone of S. Let assume now that $< \mu, \pm\infty > = \pm\infty$ for any $\mu \in S^*$. Then

$$WSupZ \supset \bigcup_{\mu \in S^* \setminus \{0\}} \{\hat{y} \in clB(Z) | < \mu, \hat{y} > = sup_{y \in Z} < \mu, y >\}$$

for any arbitrary set $Z \subset \overline{Y}$, and the converse inclusion is also valid if $clB(Z)$ is a convex set.

Remark 1.2
Analogous results hold for weak minimum and weak supremum of a set.

2 The conjugate mapping and subdifferentiability of a set-valued mapping

In this chapter we define two new notions which will be very important for the development of this theory. Let X and Y be topological vector spaces and let $L(X,Y)$ be the space of all linear and continuos operators from X to Y. Let F be a set-valued mapping from X to $\overline{Y}$.

Definition 2.1
The set $domF = \{x \in X | F(x) \neq \emptyset,\ F(x) \neq \{+\infty\}\}$ is called the **effective domain** of F.

Remark 2.1
(i)For any set $A \subset X$, $F(A) = \bigcup_{x\in A} F(x)$.
(ii)We can admit that $F(x_0) = \emptyset$ for some $x_0 \in X$, if we adopt the convetion that for any set $Z \subset \overline{Y}$, $\emptyset + Z = \emptyset$ and $\lambda\emptyset = \emptyset$.

Definition 2.2
The set-valued mapping F^* from $L(X,Y)$ to $\overline{Y}$ defined by

$$F^*(T) = WSup \bigcup_{x\in X} [Tx - F(x)],\ \text{for } T \in L(X,Y)$$

is called the **conjugate mapping** of F. Moreover, the set-valued mapping F^{**} from X to $\overline{Y}$ defined by

$$F^{**}(x) = WSup \bigcup_{T\in L(X,Y)} [Tx - F^*(T)],\ \text{for } x \in X$$

is called the **biconjugate mapping** of F.

Remark 2.2
Let consider $X = Y = R^2$, $S = R^2_+$ and let $F : R^2 \to R^2$ be the vector-valued norm,

$$F(x) = |||x||| = \begin{pmatrix} |x_1| \\ |x_2| \end{pmatrix},\ x = (x_1, x_2) \in R^2.$$

Our aim is to calculate the conjugate mapping of F. By Definition 2.2, we have

$$F^*(T) = WSup_{x\in R^2} \left\{ Tx - \begin{pmatrix} |x_1| \\ |x_2| \end{pmatrix} \right\},\ \text{for } T \in L(R^2, R^2).$$

T can be then represented like a 2×2 matrix. Let $T = \begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix}$. We obtain

$$F^*(T) = WSup_{x\in R^2} \begin{pmatrix} t_{11}x_1 + t_{12}x_2 - |x_1| \\ t_{21}x_1 + t_{22}x_2 - |x_2| \end{pmatrix}.$$

If $t_{21} \neq 0$ or $t_{21} \neq 0$, then $F^*(T) = +\infty$. If $t_{12} = t_{21} = 0$, then

$$F^*(T) = WSup_{x \in R^2} \begin{pmatrix} t_{11}x_1 - |x_1| \\ t_{22}x_2 - |x_2| \end{pmatrix}.$$

It's easy to observe that if $|t_{11}| > 1$ or $|t_{22}| > 1$, then the weak supremum is also $+\infty$ and that otherwise the weak supremum is 0. In conclusion,

$$F^*(T) = \begin{cases} 0, & \text{if } T = \begin{pmatrix} t_{11} & 0 \\ 0 & t_{22} \end{pmatrix},\ |t_{11}| \leq 1, |t_{22}| \leq 1 \\ +\infty, & \text{otherwise.} \end{cases}$$

The concept of conjugate mapping of a set-valued mapping has been introduced by Postolica[13] basing on supremum and by Luc[9] basing on Pareto maximum.

Remark 2.3
If there exists some $x_0 \in X$ such that $-\infty \in F(x_0)$, then $F^* \equiv +\infty$. Conversely, if exists $T_0 \in L(X,Y)$ such that $F^*(T_0) = \{-\infty\}$, then $F \equiv \emptyset$ or $F \equiv +\infty$. We shall only consider the case when $domF \neq \emptyset$.

Remark 2.4
When f is single-valued from X to $\overline{Y}$, then its conjugate and biconjugate mappings can be defined by identifying f with the set-valued mapping $x \longrightarrow \{f(x)\}$.

We will present now few interesting properties of the conjugate and biconjugate mapping of a set-valued mapping.

Proposition 2.1
Let $x_0 \in X$. If we define a new set-valued mapping G from X to $\overline{Y}$ by $G(x) = F(x + x_0)$, $\forall x \in X$, then
(i) $G^*(T) = F^*(T) - Tx_0$, $\forall\ T \in L(X,Y)$
(ii)$G^{**}(x) = F^{**}(x + x_0)$, $\forall\ x \in X$.

Proposition 2.2
Let $y_0 \in Y$. Then,
(i)$(F + y_0)^*(T) = F^*(T) - y_0$, $\forall\ T \in L(X,Y)$
(ii)$(F + y_0)^{**}(x) = F^{**}(x) + y_0$, $\forall\ x \in X$.

Proposition 2.3
Let $WInfF$ be another set-valued mapping from X to $\overline{Y}$, defined by $(WInfF)(x) = WInfF(x)$. Then $F^*(T) = (WInfF)^*(T)$, for any $T \in L(X,Y)$ and $F^{**}(x) = (WInfF)^{**}(x)$, for any $x \in X$.

The proves of these three propositions are based on the definitions of the conjugate and biconjugate mapping and has been given by Tanino[21].

Proposition 2.4(Extension of Fenchel's Inequality)
For any $x_0 \in X$ and any $T \in L(X,Y)$,

$$[F(x_0) - Tx_0] \bigcap B(-F^*(T)) = \emptyset.$$

Proof. Since $F^*(T) = WSup \bigcup_{x\in X}[Tx - F(x)]$, it results from Corollary 1.2 that

$$Tx_0 - F(x_0) \subset \bigcup_{x\in X}[Tx - F(x)] \subset WSup \bigcup_{x\in X}[Tx - F(x)] \bigcup B(WSup \bigcup_{x\in X}[Tx - F(x)])$$

$$= F^*(T) \cup B(F^*(T)).$$

From Proposition 1.7 it follows that $[Tx_0 - F(x_0)] \bigcap A(F^*(T)) = \emptyset$, which is equivalent, by Remark 1.1(ii), with $[F(x_0) - Tx_0] \bigcap B(-F^*(T)) = \emptyset$.

Corollary 2.1
If $y \in F(0)$ and $y' \in -F^*(T)$ for $T \in L(X,Y)$, then $y \not< y'$.

Corollary 2.2
If $y_0 \in F(x_0)$ and $y'_0 \in F^{**}(x_0)$, then $y_0 \not< y'_0$. In other words, $F(x_0) \subset F^{**}(x_0) \bigcup A(F^{**}(x_0))$.

Proof. From Proposition 2.4 we have, $\forall\ T \in L(X,Y)$,

$$F(x_0) \bigcap B(Tx_0 - F^*(T)) = \emptyset.$$

However, by Proposition 1.6, it follows

$$B(\bigcup_{T\in L(X,Y)}[Tx_0 - F^*(T)]) = B(WSup \bigcup_{T\in L(X,Y)}[Tx_0 - F^*(T)]) = B(F^{**}(x_0))$$

and further, by Proposition 1.8(ii), it follows that $F(x_0) \bigcap B(F^{**}(x_0)) = \emptyset$. From the definition of the biconjugate mapping and from Proposition 1.7 follows that $F(x_0) \subset F^{**}(x_0) \bigcup A(F^{**}(x_0))$, which is the conclusion of the corollary.

Definition 2.3
Let $x_0 \in X$ and $y_0 \in F(x_0)$. An operator $T \in L(X,Y)$ is called **subgradient** of F at (x_0, y_0) if

$$Tx_0 - y_0 \in WMax \bigcup_{x\in X}[Tx - F(x)].$$

The set of all subgradients of F at (x_0, y_0) is called the **subdifferential** of F at (x_0, y_0) and is denoted by $\partial F(x_0, y_0)$. Moreover, let $\partial F(x_0) = \bigcup_{y\in F(x_0)} \partial F(x_0, y)$. F is said to be **subdifferentiable** at x_0 when $\partial F(x_0, y_0) \neq \emptyset,\ \forall\ y_0 \in F(x_0)$.

Remark 2.5
Postolica[12] observed that there exists a strong connection between the classical

subdifferential and the subdifferential defined for set-valued mappings. Indeed, if X is a real topological vector space and $\varphi : X \longrightarrow \overline{R}$ is a point-valued real function, then we can define the set-valued mapping $\widetilde{\varphi}$ from X to $\overline{R}$ by

$$\widetilde{\varphi}(x) = \begin{cases} \{t|\ t \geq \varphi(x)\}, & x \in dom\varphi \\ \emptyset, & x \in X \setminus dom\varphi, \end{cases}$$

where $dom\varphi = \{x \in X|\ \varphi(x) < +\infty\}$. If $x_0 \in dom\varphi$, then we shall prove that $\partial\varphi(x_0) = \partial\widetilde{\varphi}(x_0)$. We have

$$\partial\widetilde{\varphi}(x_0) = \{T \in L(X,R)|\ \exists y_0\ \geq \varphi(x_0)\ \text{s.t.}\ y_0 - y \leq Tx_0 - Tu,\ \forall u \in X,\ \forall y \geq \varphi(u)\},$$

by Definition 2.3, and

$$\partial\varphi(x_0) = \{T \in L(X,R)|\varphi(x_0) - \varphi(u) \leq Tx_0 - Tu,\ \forall\ u \in X\}.$$

Let $T \in \partial\widetilde{\varphi}(x_0)$. Then, there exists $y_0\ \geq 0$ such that $y_0 - y \leq Tx_0 - Tu,\ \forall\ u \in X$, $\forall\ y \geq \varphi(u)$. Therefore, for all $u \in X$, taking $y = \varphi(u)$, we obtain $\varphi(x_0) - \varphi(u) \leq Tx_0 - Tu$ and this means that $T \in \partial\varphi(x_0)$.
Conversely, let $T \in \partial\varphi(x_0)$. For all $u \in X$ we have $\varphi(x_0) - \varphi(u) \leq Tx_0 - Tu$. It follows that $\forall\ u \in X,\ \forall\ y \geq \varphi(u)$,

$$\varphi(x_0) - y \leq \varphi(x_0) - \varphi(u) \leq Tx_0 - Tu.$$

Thus we obtain the contrary inclusion $\partial\varphi(x_0) \subseteq \partial\widetilde{\varphi}(x_0)$ and so is the equality $\partial\varphi(x_0) = \partial\widetilde{\varphi}(x_0)$ proved.

Proposition 2.5
If F is a set valued mapping from X to $\overline{Y}$, then $y_0 \in F(x_0)$ is in $\mathrm{WMin}\bigcup_{x\in X} F(x)$ if and only if $0 \in \partial F(x_0, y_0)$.

The proof of Proposition 2.5 results from the definition of the subgradient.

Proposition 2.6
Let $x_0 \in X$ and $y_0 \in F(x_0)$. Then, $T \in \partial F(x_0, y_0)$ if and only if $Tx_0 - y_0 \in F^*(T)$.

Proof. From the definition of the subgradient, $T \in \partial F(x_0, y_0)$ if and only if $Tx_0 - y_0 \in WMax\bigcup_{x\in X}[Tx - F(x)]$. By Proposition 1.1, this is equivalent with

$$Tx_0 - y_0 \in \bigcup_{x\in X}[Tx - F(x)] \bigcap WSup \bigcup_{x\in X}[Tx - F(x)]$$

which is equivalent, from the definition of the conjugate mapping, with

$$Tx_0 - y_0 \in \bigcup_{x\in X}[Tx - F(x)] \bigcap F^*(T).$$

Hence, the proposition is obviously true.

Finally, we will show that the subdifferentiability of a set-valued mapping guarantees an inclusion relationship between the mapping and its biconjugate mapping (see Tanino[21]).

Theorem 2.1
Let F be a set valued mapping from X to $\overline{Y}$. If F is subdifferentiable at x_0, then $F(x_0) \subset F^{**}(x_0)$. Moreover, if, in addition, $F(x_0) = WInfF(x_0)$, then $F(x_0) = F^{**}(x_0)$.

Proof. In view of Proposition 2.1 it suffices to prove the case when $x_0 = 0$. Let $y_0 \in F(0)$. Since F is subdifferentiable at 0, there exists $T_0 \in L(X, Y)$ such that $y_0 \in -F^*(T_0)$. Then, from Corollary 2.1, it follows that it doesn't exist any $T \in L(X, Y)$ and $y \in -F^*(T)$ such that $y_0 < y$. This is equivalent with

$$y_0 \in WMax \bigcup_{T \in L(X,Y)} [-F^*(T)] \subset WSup \bigcup_{T \in L(X,Y)} [-F^*(T)] = F^{**}(0).$$

Thus we have proved that $F(0) \subset F^{**}(0)$.
Next, we assume that $F(0) = WInfF(0)$ and take an arbitrary $y_0 \in F^{**}(0)$. By Proposition 1.7,

$$\overline{Y} = F(0) \cup A(F(0)) \cup B(F(0)).$$

In view of Corollary 2.2, $y_0 \notin A(F(0))$. If $y_0 \notin B(F(0))$, then it follows that $y_0 \in F(0)$ which ends our proof.
Let suppose that $y_0 \in B(F(0))$. Then, there exists $y' \in F(0)$ such that $y_0 < y'$. Since F is assumed to be subdifferentiable at 0, exists $T' \in L(X, Y)$ such that $y' \in -F^*(T')$. However, this implies that $y_0 \in B(-F^*(T'))$ and hence contradicts the assumption $y_0 \in F^{**}(0) = WSup \bigcup_{T \in L(X,Y)} [-F^*(T)]$.

3 The set-valued optimization problem

Let X,Y be real Hausdorff topological vector spaces and $S \subset Y$ be a closed, convex and pointed cone, with a nonempty interior in Y. Let F be a set-valued mapping from X to $Y \cup \{+\infty\}$ with $domF \neq \emptyset$ and let consider the vector optimization problem

(P) $\min\limits_{x \in X} F(x)$.

To solve this problem means to find the set $WInf(\mathrm{P}) = WInfF(X)$ or the set $WMin(\mathrm{P}) = WMinF(X)$.

In order to formulate the dual problem of (P), Song[18] and Tanino[21] had used the Fenchel-Rockafellar method and their results represent a generalization for the case of the point-valued real functions, treated by Ekeland and Temam[5].

Let us introduce a perturbation parameter $z \in Z$ and imbed the primal problem (P) into a family of vector optimization problems, where Z is another real Hausdorff topological vector space. Let Φ be a set-valued mapping from $X \times Z$ to $Y \cup \{+\infty\}$ such that

$$\Phi(x, 0) = F(x),\ \forall x \in X.$$

By using of the so-called perturbation mapping Φ, we can now formulate, for all $z \in Z$, a new optimization problem

($\mathrm{P_z}$) $\min\limits_{x \in X} \Phi(x, z)$.

Definition 3.1
The set-valued mapping W from Z to $\overline{Y}$ defined by

$$W(z) = WInf(P_z) = WInf \bigcup_{x \in X} \Phi(x, z)$$

is called the **value mapping** of the problem (P).

Remark 3.1
It is clear that $W(0) = WInf(\mathrm{P})$.

In accordance with Definition 2.2, the conjugate mapping of Φ is the set-valued mapping Φ^* from $L(X, Y) \times L(Z, Y)$ to $\overline{Y}$ defined by

$$\Phi^*(T, \Lambda) = WSup \bigcup_{(x,z) \in X \times Z} (Tx + \Lambda z - \Phi(x, z)),$$

for $T \in L(X,Y)$ and $\Lambda \in L(Z,Y)$. Therefore, by Remark 1.1(ii),

$$-\Phi^*(0,\Lambda) = -WSup \bigcup_{(x,z)\in X\times Z} (\Lambda z - \Phi(x,z)) = WInf \bigcup_{(x,z)\in X\times Z} (\Phi(x,z) - \Lambda z).$$

We define the dual problem to (P) as

(D) $\max_{\Lambda\in L(Z,Y)} -\Phi^*(0,\Lambda)$.

For the problems (P) and (D) we have the following duality results.

Proposition 3.1(weak duality)
For any $x \in X$ and $\Lambda \in L(Z,Y)$,

$$-\Phi^*(0,\Lambda) \cap A(\Phi(x,0)) = \emptyset$$

and hence

$$WSup(\mathrm{D}) \cap A(WInf(\mathrm{P})) = \emptyset.$$

Proof. Let suppose that there exist $y_1 \in \Phi(x,0)$ and $y_2 \in -\Phi^*(0,\Lambda)$ such that $y_1 < y_2$. As we proved, $-\Phi^*(0,\Lambda) = WInf \bigcup_{(x,z)\in X\times Z}(\Phi(x,z) - \Lambda z)$, which means that $y_2 \in WInf \bigcup_{(x,z)\in X\times Z}(\Phi(x,z) - \Lambda z)$. From

$$y_1 \in \Phi(x,0) \subset \bigcup_{(x,z)\in X\times Z} (\Phi(x,z) - \Lambda z)$$

we obtain the contradiction.

Corollary 3.1

$$WMin \bigcup_{x\in X} \Phi(x,0) \bigcap WMax \bigcup_{\Lambda\in L(Z,Y)} -\Phi^*(0,\Lambda) \neq \emptyset$$

if and only if there exist $x_0 \in domF$ and $\Lambda_0 \in L(Z,Y)$ such that

$$0 \in \Phi(x_0,0) + \Phi^*(0,\Lambda_0),$$

or equivalent,

$$(0,\Lambda_0) \in \partial\Phi(x_0,0).$$

Proof. If $y_0 \in WMin \bigcup_{x\in X} \Phi(x,0) \bigcap WMax \bigcup_{\Lambda\in L(Z,Y)} -\Phi^*(0,\Lambda)$,then there exist $x_0 \in domF$ and $\Lambda_0 \in L(Z,Y)$ such that $y_0 \in \Phi(x_0,0)$ and $-y_0 \in \Phi^*(0,\Lambda_0)$, which implies that $0 \in \Phi(x_0,0) + \Phi^*(0,\Lambda_0)$.
Conversely, let $y_0 \in \Phi(x_0,0) \cap (-\Phi^*(0,\Lambda_0))$. If $y_0 \notin WMin \bigcup_{x\in X} \Phi(x,0)$, then there exist $x_1 \in X$ and $y_1 \in \Phi(x_1,0)$ such that $y_0 > y_1$. This contradicts the result of Proposition 3.1. Analogous we can prove that $y_0 \in WMax \bigcup_{\Lambda\in L(Z,Y)} -\Phi^*(0,\Lambda)$.

Similar weak duality assertions has been given by Luc[9] for Pareto minimality.

Proposition 3.2
For all $\Lambda \in L(Z, Y)$,

$$W^*(\Lambda) = \Phi^*(0, \Lambda).$$

Proof. By de definition of the conjugate mapping we have,

$$W^*(\Lambda) = WSup \bigcup_{z \in Z} (\Lambda z - W(z)) = WSup \bigcup_{z \in Z} (\Lambda z - WInf \bigcup_{x \in X} \Phi(x, z)) =$$

$$WSup \bigcup_{z \in Z} WSup \bigcup_{x \in X} (\Lambda z - \Phi(x, z)).$$

By Corollary 1.5, we obtain

$$W^*(\Lambda) = WSup \bigcup_{(x,z) \in X \times Z} (\Lambda z - \Phi(x, z)) = \Phi^*(0, \Lambda).$$

Remark 3.2
Using Proposition 3.2 we can rewrite $WSup(\mathrm{D})$ as

$$WSup(\mathrm{D}) = WSup \bigcup_{\Lambda \in L(Z,Y)} [-W^*(\Lambda)] = W^{**}(0).$$

Since $WInf(\mathrm{P}) = W(0)$, the relationship between the primal problem $WInf(\mathrm{P})$ and the dual problem $WSup(\mathrm{D})$ is nothing else but the relationship between $W(0)$ and $W^{**}(0)$.

Definition 3.2
The primal problem (P) is said to be **stable** if the value mapping W is subdifferentiable at 0.

The following theorem has been proved by Song[17] and represents a sufficient condition for strong duality between (P) and (D).

Theorem 3.1
If the problem (P) is stable, then

$$WInf(\mathrm{P}) = WSup(\mathrm{D}) = WMax(\mathrm{D}).$$

Proof. If the problem (P) is stable, then the value mapping W is subdifferentiable at 0 and, by Theorem 2.1, implies that $W(0) \subset W^{**}(0)$.
From the definition of W, Corollary 1.7 and Remark 1.1(ii),

$$WInfW(0) = WInf\ WInf \bigcup_{x \in X} \Phi(x, 0) = WInf \bigcup_{x \in X} \Phi(x, 0) = W(0)$$

and, by using of the second part of Theorem 2.1, we obtain that $W(0) = W^{**}(0)$. By Remark 3.2, this is equivalent to $WInf(\mathrm{P}) = WSup(\mathrm{D})$.
From $WMax(\mathrm{D}) \subset WSup(\mathrm{D}) = WInf(\mathrm{P})$ it results that it remains to prove that $WInf(\mathrm{P}) \subset WMax(\mathrm{D})$. Let $\bar{y} \in WInf(\mathrm{P}) = W(0)$. Since W is subdifferentiable at 0, there exists $\bar{\Lambda} \in L(Z, Y)$ such that $\bar{\Lambda} \in \partial W(0, \bar{y})$. Thus,

$$-\bar{y} \in WMax \bigcup_{z \in Z} (\bar{\Lambda} z - W(z)) \subset W^*(\bar{\Lambda}) = \Phi^*(0, \bar{\Lambda}).$$

If $\bar{y} \notin WMax(\mathrm{D})$, then there exist $\Lambda_0 \in L(Z, Y)$ and $y_0 \in -\Phi^*(0, \Lambda_0)$ such that $\bar{y} < y_0$. From the definition of the weak supremum, by using that $\bar{y} \in W(0) = WInf \bigcup_{x \in X} \Phi(x, 0)$ and that $\bar{y} < y_0$, it follows that there exist $x_1 \in X$ and $y_1 \in \Phi(x_1, 0)$ such that $y_1 < y_0$. This is a contradiction with Proposition 3.1. Thus, $WInf(\mathrm{P}) \subset WMax(\mathrm{D})$.

For the case of point-valued functions the first part of the Theorem 3.1 has been proved by Tanino[21]. Duality assesrtions for Pareto minimality under similar assumptions has also been obtained by Luc[9] and by Isac and Postolica[6].

4 Sufficient criteria for stability

In the following we shall present some sufficient criteria for stability, in the sense of weak minimality.

Definition 4.1
Let F be a set-valued mapping from X to $\overline{Y}$. The set

$$epiF = \{(x, y) \in X \times Y |\ y \in (F(x) + S) \cup A(F(x))\}$$

is called the **epigraph** of F.

Proposition 4.1
Let F be a set-valued mapping from X to $\overline{Y}$ and $x_0 \in X$. It holds:

$$\{y | y \in (F(x_0) + S) \cup A(F(x_0))\} = \begin{cases} \{y | y \in (F(x_0) + S)\}, & \text{if } -\infty \notin F(x_0) \\ Y, & \text{if } -\infty \in F(x_0). \end{cases}$$

The proof of Proposition 4.1 is evident. This last result is one of the reasons why we will consider, starting from now, the mapping F as a set-valued mapping from X to $Y \cup \{+\infty\}$. The epigraph of F will be then

$$epiF = \{(x, y) \in X \times Y |\ y \in F(x) + S\}.$$

Definition 4.2
A set-valued mapping F from X to $\overline{Y}$ is said to be **S-convex** if its epigraph is convex. If F is a set-valued mapping from X to $Y \cup \{+\infty\}$, then F is S-convex if and only if for all $t \in [0, 1]$ and all $x_1, x_2 \in X$,

$$tF(x_1) \cap Y + (1 - t)F(x_2) \cap Y \subset F(tx_1 + (1 - t)x_2) \cap Y + S.$$

Definition 4.2
A set-valued mapping F from X to $\overline{Y}$ is said to be **weakly S-upper bounded** on a set $A \subset X$ if there exists a point $b \in Y$ such that $(x, b) \in epiF$, for every $x \in A$.

Remark 4.1
If F is a set-valued mapping from X to $Y \cup \{+\infty\}$, then F is weakly S-upper bounded on a set $A \subset X$ if and only if $\exists\, b \in Y$ such that $F(x) \cap (b - S) \neq \emptyset$, for all $x \in A$.

At this point we will define the set-valued mapping Ψ from Z to $Y \cup \{+\infty\}$ as

$$\Psi(z) = \bigcup_{x \in X} \Phi(x, z) = \Phi(X, z).$$

Proposition 4.1
If Φ is S-convex, then also Ψ is S-convex.

Proof. Let consider $t \in [0,1]$ and $z_1, z_2 \in Z$. We have

$$t\Psi(z_1) \cap Y + (1-t)\Psi(z_2) \cap Y = \bigcup_{x\in X} t\Phi(x, z_1) \cap Y + \bigcup_{x\in X} (1-t)\Phi(x, z_2) \cap Y =$$

$$\bigcup_{(x,y)\in X\times X} [t\Phi(x, z_1)\cap Y+(1-t)\Phi(x, z_2)\cap Y \subset \bigcup_{(x,y)\in X\times X} \Phi(tx+(1-t)y, tz_1+(1-t)z_2)$$

$$\cap\, Y + S = \bigcup_{u\in X} t\Phi(u, tz_1 + (1-t)z_2) \cap Y + S = \Psi(tz_1 + (1-t)z_2) + S.$$

By Definition 4.2, we have that Ψ is S-convex.

Proposition 4.2
Let Ψ be a S-convex set-valued mapping from Z to $Y \cup \{+\infty\}$. Then, the value mapping W is a S-convex set-valued mapping from X to $\overline{Y}$.

Proof. Since W is defined from X to $\overline{Y}$, to prove that W is S-convex means to prove that $epiW$ is a convex set. Let $(z_1, y_1), (z_2, y_2) \in epiW$ and $t \in [0,1]$. From the definition of the epigraph, it follows that $y_i \in (W(z_i) + S) \cup A(W(z_i)), i = 1, 2$. From the definition of the weak infimum, for every $\varepsilon \in intS$, there exist $\bar{y}_i \in \Psi(z_i) \cap Y, i = 1, 2$, such that $y_i + \varepsilon > \bar{y}_i, i = 1, 2$.
Since Ψ is S-convex and $t \in [0,1]$,

$$t\bar{y}_1 + (1-t)\bar{y}_2 \in t\Psi(z_1) \cap Y + (1-t)\Psi(z_2) \cap Y \subset \Psi(tz_1 + (1-t)z_2) \cap Y + S.$$

Hence,

$$ty_1+(1-t)y_2+\varepsilon \in \Psi(tz_1+(1-t)z_2)\cap Y+intS+S \subset \Psi(tz_1+(1-t)z_2)\cap Y+intS.$$

This is equivalent with $ty_1 + (1-t)y_2 + \varepsilon \in A(\Psi(tz_1 + (1-t)z_2) \cap Y)$. From Corollary 1.2 and the definition of the value mapping we obtain that

$$ty_1 + (1-t)y_2 + \varepsilon \in A(W(tz_1 + (1-t)z_2)) \cap Y)$$

or equivalent,

$$ty_1 + (1-t)y_2 + \varepsilon \in W(tz_1 + (1-t)z_2) \cap Y + intS.$$

Since ε is arbitrary, by Proposition 1.10,

$$ty_1 + (1-t)y_2 \in \overline{W(tz_1 + (1-t)z_2) \cap Y + intS} \subset W(tz_1 + (1-t)z_2)$$

$$\cup\, A(W(tz_1 + (1-t)z_2)).$$

Thus, $(tz_1 + (1-t)z_2, ty_1 + (1-t)y_2) \in epiW$ and so W is S-convex.

We note that Ψ and W satisfy the following relation.

Proposition 4.3

$$epi\Psi \subset epiW \subset \overline{epi\Psi}.$$

Proof. Let $(z, y) \in epi\Psi$. Since Ψ is a set-valued mapping from Z to $Y \cup \{+\infty\}$, implies that $y \in \Psi(z) + S$. By Proposition 1.10 and Corollary 1.2, we have

$$y \in \Psi(z) + S \subset (W(z) + S) \cup A(W(z)).$$

This means that $epi\Psi \subset epiW$.
For the second inclusion, let $(z, y) \in epiW$. If $y \in A(W(z)$, then

$$y \in A(WInf\Psi(z)) = A(\Psi(z))$$

and then $(z, y) \in epi\Psi$. If $y \in W(z) \cap Y + S \setminus intS$, then there exists $s \in S \setminus intS$ such that $y - s \in W(z)$. Let $\{s_\alpha\}$ be a sequence in $intS$ such that s_α converges to s. This means that the sequence $\{y - s + s_\alpha\}$ converges to y and for all α, $y - s + s_\alpha \in A(W(z)) = A(WInf\Psi(z)) = A(\Psi(z))$. It follows that $\{(z, y - s + s_\alpha)\}$ is a sequence which belongs to the epigraph of Ψ and converges to (z, y). In conclusion, $(z, y) \in \overline{epi\Psi}$.

The next theorem gives an important stability criterion for the problem (P). This criterion has been formulated by Tanino[21] and extended by Song[18] for the more general case of the set-valued mappings.

Theorem 4.1
Suppose that Ψ is a S-convex set-valued mapping from $X \times Z$ to $Y \cup \{+\infty\}$ and that the value mapping W is weakly S-upper bounded on a neighbourhood of 0 in Z. Then the problem (P) is stable.

Proof. If $W(0) = \{-\infty\}$, then $W^* \equiv \{+\infty\}$. From Proposition 2.6 results that W is subdifferentiable at 0. Hence we may assume that $W(0) \neq \{-\infty\}$.
By Proposition 4.2, we have that W is S-convex. Since W is weakly S-upper bounded on a neighbourhood of 0 in Z, we have $0 \in int(domW)$.
For the beginning we will prove that $W(z) \neq \{-\infty\}$, for all $z \in domW$. Indeed, suppose that, there exists $z_0 \in domW$ such that $W(z_0) = \{-\infty\}$. Since $0 \in int(domW)$, there exists $\varepsilon > 0$ such that $z_1 = -\varepsilon z_0 \in domW$. Since $W(z_0) = \{-\infty\}$, $(z_0, y) \in epiW$, for all $y \in Y$. Let $y_1 \in Y$, such that $(z_1, y_1) \in epiW$. Because of the S-convexity of W or equivalent, because of the convexity of $epiW$,

$$\left(\frac{1}{1+\varepsilon}z_1 + \frac{\varepsilon}{1+\varepsilon}z_0, \frac{1}{1+\varepsilon}y_1 + \frac{\varepsilon}{1+\varepsilon}y\right) \in epiW, \text{ for all } y \in Y.$$

Hence, by using that $z_1 = -\varepsilon z_0$, it results that $(0, y) \in epiW$, for all $y \in Y$ and hence $A(W(0)) = Y \cup \{+\infty\}$. By Corollary 1.2 and Corollary 1.7,

$$W(0) = WInfW(0) = \{-\infty\}.$$

But this is a contradiction.
Let now U be a neighbourhood of 0 in Z such that W is weakly S-upper bounded on U. There exists, then, $b \in Y$ such that $W(z) \cap (b - S) \neq \emptyset$, for all $z \in U$. Let consider an arbitrary $s_0 \in intS$. There exists, then, V, a neighbourhood of 0 in Y, such that $s_0 + V \subset S$. It follows that, for all $y \in b + s_0 + V$ and for all $z \in U$, $W(z) \cap (b - S) \neq \emptyset$. This means that $(0, b + s_0) \in int(epiW)$.
Let $\bar{y} \in W(0)$. Since $W(0) = WInfW(0)$ and $0 \in int(domW)$, then $\bar{y} \neq +\infty$ and $(0, \bar{y})$ is a boundary point of the convex set $epiW$ in $Z \times Y$. By a standard separation theorem, there exists $(z^*, y^*) \in Z^* \times Y^* \setminus \{(0, 0)\}$ such that

$$< z^*, 0 > + < y^*, \bar{y} > \leq < z^*, z > + < y^*, y >, \text{ for all } (z, y) \in epiW.$$

Since $(0, \bar{y} + s) \in epiW$ for all $s \in S$, we have $< y^*, s > \geq 0$, for all $s \in S$. This means that $y^* \in S^*$. Let assume that $y^* = 0$. Hence,

$$< z^*, z > \geq 0, \text{ for all } z \in domW.$$

From $0 \in int(domW)$ it follows that there exists V_1, a ballanced and absorbing neighbourhood of 0, such that $0 \in V_1 \subset domW$. It implies that for all $z \in V_1$, $< z^*, z >= 0$. Since V_1 is absorbing, $< z^*, z >= 0$, for all $z \in Z$ or equivalent, $z^* = 0$. This is a contradiction with $(z^*, y^*) \neq (0, 0)$. In conclusion, $y^* \neq 0$.
Hence, there exists $e \in intS$ such that $< y^*, e >= 1$. We can now define the function $T \in L(Z, Y)$ such that $Tz = - < z^*, z > e$, for all $z \in Z$. It's clear that $z^* = -y^*T$. Thus, we have

$$< y^*, \bar{y} - T0 > \leq < y^*, y - Tz >, \text{ for all } z \in Z, y \in W(z).$$

(For $y = +\infty$, $< y^*, +\infty >= +\infty \in \overline{R}$.) From Proposition 1.11, it follows that $\bar{y} - T0 \in WInf \bigcup_{z \in Z}[W(z) - Tz]$ and hence $T0 - \bar{y} \in W^*(T)$. By Proposition 2.6, $T \in \partial W(0, \bar{y})$ and thus W is subdifferentiable at 0.

The next theorem shows under what conditions for a set-valued mapping F from X to Y there exists at every point $x \in X$ a neighbourhood $U_x \subset X$ such that F should be weakly S-upper bounded on U_x, for all $x \in X$.

Theorem 4.2
Let F be a S-convex set-valued mapping from X to Y. Let assume that $F(x) \neq \emptyset$, for all $x \in X$. If there exist an $x_0 \in X$ and a neighbourhood $U_{x_0} \subset X$ of x_0 such that F is weakly S-upper bounded on U_{x_0}, then for every $x \in X$ there exists a neighbourhood $U_x \subset X$ of x such that F is weakly S-upper bounded on U_x, $\forall x \in X$.

Proof. For $x_0 \in X$, since F is weakly S-upper bounded on U_{x_0}, it follows that there exists $b_{x_0} \in Y$ such that $b_{x_0} \in F(x) + S$, for all $x \in U_{x_0}$.
Let $\tilde{x} \in X$ and $\lambda > 1$. It's clear that $\frac{\lambda-1}{\lambda}(U_{x_0} - x_0) + \tilde{x}$ is a neighbourhood of $\tilde{x}$ in X. Let $U_{\tilde{x}}$ be another neighbourhood of $\tilde{x}$ in X such that $U_{\tilde{x}} \subset \frac{\lambda-1}{\lambda}(U_{x_0} - x_0) + \tilde{x}$.

It follows that $\forall\ x \in U_{\tilde{x}}$, there exists $y \in U_{x_0} - x_0$ such that $x - \tilde{x} = \frac{\lambda-1}{\lambda} y$. By using that $F(x_0 + \lambda(\tilde{x} - x_0)) \neq \emptyset$, it results that $\exists\ b_{x_0+\lambda(\tilde{x}-x_0)} \in Y$ such that

$$b_{x_0+\lambda(\tilde{x}-x_0)} \in F(x_0 + \lambda(\tilde{x} - x_0)) + S.$$

Because F is a S-convex mapping, we have for all $y \in U_{x_0} - x_0$,

$$\frac{1}{\lambda} b_{x_0+\lambda(\tilde{x}-x_0)} + (1-\frac{1}{\lambda}) b_{x_0} \in \frac{1}{\lambda} F(x_0+\lambda(\tilde{x}-x_0)) \cap Y + S + (1-\frac{1}{\lambda}) F(x_0+y) \cap Y + S \subset$$

$$F(\frac{1}{\lambda}(x_0 + \lambda(\tilde{x} - x_0)) + (1 - \frac{1}{\lambda})(x_0 + y)) \cap Y + S = F(\frac{\lambda - 1}{\lambda} y + \tilde{x}) + S.$$

Then, for all $x \in U_{\tilde{x}}$,

$$\frac{1}{\lambda} b_{x_0+\lambda(\tilde{x}-x_0)} + (1 - \frac{1}{\lambda}) b_{x_0} \in F(x) + S$$

and this means that F is weakly S-upper bounded on $U_{\tilde{x}}$, which is a neighbourhood of $\tilde{x}$ in X.

Remark 4.2
A sufficient condition which assures that a set-valued mapping F is weakly S-upper bounded on a neighbourhood of a point x_0 has been given by Song[17]. A set-valued maping F from X to $\overline{Y}$ is said to be **S-Hausdorff lower continous** (see Aubin and Frankowska[1]) at $x_0 \in X$ if, for every neighborhood V of zero in Y, there exists a neighbourhood U of zero in X such that

$$F(x_0) \subset F(x) + V + S, \text{ for all } x \in (x_0 + U) \cap domF.$$

Song[17] has proved that if $intS \neq \emptyset$ and if a set-valued mapping F from X to $Y \cup \{+\infty\}$ is S-Hausdorff lower continous at $x_0 \in int(domF)$, then F is weakly S-upper bounded on some neighbourhood of x_0.

5 The set-valued optimization problem with constraints

Let U, W, $(X_i)_{i=\overline{1,n}}$, Y be real Hausdorff topological vector spaces, $S_i \in L(U, X_i)$, $i = \overline{1,n}$ be linear and continous mappings, $(F_i)_{i=\overline{1,n}}$ be S-convex set-valued mappings from X_i to $Y \cup \{+\infty\}$, $i = \overline{1,n}$, $x_i \in X_i$, $i = \overline{1,n}$ be fixed points and $\lambda_i > 0$, $i = \overline{1,n}$ be fixed positive constants. Let consider V, a convex and closed set in U, Q a closed, convex and pointed cone in W such that $intQ \neq \emptyset$ and G a Q-convex set-valued mapping from U to W.
In this chapter we will consider the following optimization problem

$$(\mathrm{P_c}) \quad \min_{\substack{u\in V \\ G(u)\cap(-Q)\neq\emptyset}} \sum_{i=1}^{n} \lambda_i F_i(x_i - S_i u).$$

Let $Z = X_1 \times \ldots \times X_n \times W$ be the perturbation space and let define the perturbation mapping Φ from $U \times Z$ to $Y \cup \{+\infty\}$, for all $(u, z_1, \ldots, z_n, \gamma) \in U \times Z = U \times X_1 \times \ldots \times X_n \times W$, as follows,

$$\Phi(u, z_1, \ldots, z_n, \gamma) = \begin{cases} \sum_{i=1}^{n} \lambda_i F_i(x_i - S_i u + z_i), & \text{if } u \in V,\ G(u) \cap (\gamma - Q) \neq \emptyset \\ \emptyset, & \text{otherwise.} \end{cases}$$

By definition, its conjugate mapping will be

$$\Phi^*(T, T_1, \ldots, T_n, \Gamma) = WSup \bigcup_{\substack{u\in U \\ (z,\gamma)\in Z}} \left[Tu + \sum_{i=1}^{n} T_i z_i + \Gamma\gamma - \Phi(u, z, \gamma) \right],$$

for $z = (z_1, \ldots, z_n) \in X_1 \times \ldots \times X_n$, $\gamma \in W$, $T \in L(U, Y)$, $T_i \in L(X_i, Y)$, $i = \overline{1,n}$ and $\Gamma \in L(W, Y)$. From Remark 2.1(ii), follows that

$$\Phi^*(T, T_1, \ldots, T_n, \Gamma) = WSup \bigcup_{\substack{u\in V \\ (z,\gamma)\in Z \\ G(u)\cap(\gamma-Q)\neq\emptyset}} \left[Tu + \sum_{i=1}^{n} T_i x_i + \Gamma\gamma - \right.$$

$$\left. - \sum_{i=1}^{n} \lambda_i F_i(x_i - S_i u + z_i) \right] = WSup \bigcup_{\substack{u\in V \\ (z1,\ldots,z_n)\in X_1\times\ldots X_n}} \bigcup_{\gamma\in G(u)+Q} \left[Tu + \sum_{i=1}^{n} T_i z_i + \right.$$

$$\left. + \Gamma\gamma - \sum_{i=1}^{n} \lambda_i F_i(x_i - S_i u + z_i) \right].$$

For all $i = \overline{1,n}$, we make the substitution $y_i = x_i - S_i u + z_i$ and then we obtain

$$\Phi^*(T, T_1, \ldots, T_n, \Gamma) = WSup \bigcup_{\substack{u\in V \\ (y1,\ldots,y_n)\in X_1\times\ldots X_n}} \bigcup_{\gamma\in G(u)+Q} \left[\sum_{i=1}^{n} (T_i y_i - \lambda_i F_i(y_i)) + Tu + \right.$$

$$+\Gamma\gamma-\sum_{i=1}^{n}T_ix_i+\sum_{i=1}^{n}T_iS_iu\Bigg] = WSup\bigcup_{u\in V}\bigcup_{\gamma\in G(u)+Q}\Bigg[\sum_{i=1}^{n}\bigcup_{y_i\in X_i}(T_iy_i-\lambda_iF_i(y_i))+$$

$$+Tu+\Gamma\gamma-\sum_{i=1}^{n}T_ix_i+\sum_{i=1}^{n}T_iS_iu\Bigg].$$

Further, we have

$$\Phi^*(0,T_1,\ldots,T_n,\Gamma) = WSup\bigcup_{u\in V}\Bigg[\sum_{i=1}^{n}\bigcup_{y_i\in X_i}(T_iy_i-\lambda_iF_i(y_i))+\Gamma(G(u)+Q)-$$

$$-\sum_{i=1}^{n}T_ix_i+\sum_{i=1}^{n}T_iS_iu\Bigg] = WSup\Bigg[\sum_{i=1}^{n}\bigcup_{y_i\in X_i}(T_iy_i-\lambda_iF_i(y_i))+\Gamma(Q)-$$

$$-\sum_{i=1}^{n}T_ix_i+(\sum_{i=1}^{n}T_iS_i+\Gamma\circ G)(V)\Bigg].$$

We denoted by $\Gamma\circ G$ the set-valued mapping from U to Y, defined by $\Gamma\circ G(x)=\{\Gamma(y)|\ y\in G(x)\}$, for all $x\in U$.

Remark 5.1
If F is a set-valued mapping from X to Y and G is a point-valued mapping from X to Y, let define the sum of F and G, denoted by $F+G$, as the set-valued mapping from X to Y such that

$$(F+G)(x)=F(x)+G(x)=\{y+G(x)|y\in F(x)\},\ \text{for all } x\in X.$$

From Corollary 1.4, it follows

$$\Phi^*(0,T_1,\ldots,T_n,\Gamma) = WSup\left\{\sum_{i=1}^{n}\lambda_iWSup\bigcup_{y_i\in X_i}[\frac{T_i}{\lambda_i}y_i-F_i(y_i)]+WSup\Gamma(Q)-\right.$$

$$\left.-\sum_{i=1}^{n}T_ix_i+WSup[(\sum_{i=1}^{n}T_iS_i+\Gamma\circ G)(V)]\right\}.$$

By the definition of the conjugate mapping, it results

$$\Phi^*(0,T_1,\ldots,T_n,\Gamma) = WSup\left\{\sum_{i=1}^{n}\lambda_iF_i^*(\frac{T_i}{\lambda_i})+WSup\Gamma(Q)-\sum_{i=1}^{n}T_ix_i+\right.$$

$$\left.+WSup[(\sum_{i=1}^{n}T_iS_i+\Gamma\circ G)(V)]\right\}.$$

The dual problem of ($\mathrm{P_c}$) will be

$$(\mathrm{D_c}) \quad \max_{\substack{T_i \in L(X_i,Y),\ i=\overline{1,n} \\ \Gamma \in L(W,Y)}} -\Phi^*(0, T_1, \ldots, T_n, \Gamma).$$

To solve this problem means to find the set

$$WSup \bigcup_{\substack{T_i \in L(X_i,Y),\ i=\overline{1,n} \\ \Gamma \in L(W,Y)}} -\Phi^*(0, T_1, \ldots, T_n, \Gamma).$$

The dual problem can be then written

$$(\mathrm{D_c})\ WSup \bigcup_{\substack{T_i \in L(X_i,Y),\ i=\overline{1,n} \\ \Gamma \in L(W,Y)}} WInf \left\{ -\sum_{i=1}^{n} \lambda_i F_i^*(\frac{T_i}{\lambda_i}) - WSup\Gamma(Q) + \sum_{i=1}^{n} T_i x_i - \right.$$

$$\left. -WSup[(\sum_{i=1}^{n} T_i S_i + \Gamma \circ G)(V)] \right\}$$

or equivalent,

$$(\mathrm{D_c})\ WSup \bigcup_{\substack{T_i \in L(X_i,Y),\ i=\overline{1,n} \\ \Gamma \in L(W,Y)}} WInf \left\{ \sum_{i=1}^{n} \lambda_i \left[T_i x_i - F_i^*(T_i) \right] - WSup\Gamma(Q) - \right.$$

$$\left. -WSup \left[(\sum_{i=1}^{n} \lambda_i T_i S_i + \Gamma \circ G)(V) \right] \right\}.$$

We want to present in the second part of this chapter a necessary condition for the existence of the strong duality between the problems ($\mathrm{P_c}$) and ($\mathrm{D_c}$).

Proposition 5.1
The set-valued mapping Ψ from $Z = X_1 \times \ldots \times X_n \times W$ to $Y \cup \{+\infty\}$, $\Psi(z,\gamma) = \bigcup_{u \in V} \Phi(u,z,\gamma) = \Phi(V,z,\gamma)$ is S-convex. This will imply, by Proposition 4.2, that the set-valued mapping W from Z to $\overline{Y}$, $W(z,\gamma) = WInf\Psi(z,\gamma)$ is also S-convex.

Proof. To prove that Ψ is S-convex means to prove that

$$t\Psi(z^1,\gamma^1) \cap Y + (1-t)\Psi(z^2,\gamma^2) \cap Y \subset \Psi(tz^1 + (1-t)z^2, t\gamma^1 + (1-t)\gamma^2) \cap Y + S,$$

for all $(z^1,\gamma^1) = (z_1^1, \ldots, z_n^1, \gamma^1), (z^2,\gamma^2) = (z_1^2, \ldots, z_n^2, \gamma^2) \in Z = X_1 \times \ldots \times X_n \times W$ and for all $t \in [0,1]$.
We have

$$t\Psi(z^1,\gamma^1) \cap Y + (1-t)\Psi(z^2,\gamma^2) \cap Y = t \bigcup_{\substack{u \in V \\ G(u) \cap (\gamma^1 - Q) \neq \emptyset}} \sum_{i=1}^{n} \lambda_i F_i(x_i - S_i u + z_i^1) \cap Y +$$

$$+(1-t)\bigcup_{\substack{u\in V\\ G(u)\cap(\gamma^2-Q)\neq\emptyset}}\sum_{i=1}^{n}\lambda_i F_i(x_i-S_iu+z_i^2)\cap Y=$$

$$=\bigcup_{\substack{(u,v)\in V\times V\\ G(u)\cap(\gamma^1-Q)\neq\emptyset\\ G(v)\cap(\gamma^2-Q)\neq\emptyset}}\sum_{i=1}^{n}\lambda_i\left[tF_i(x_i-S_iu+z_i^1)\cap Y+(1-t)F_i(x_i-S_iu+z_i^2)\cap Y\right]\subset$$

$$\subset\bigcup_{\substack{(u,v)\in V\times V\\ G(u)\cap(\gamma^1-Q)\neq\emptyset\\ G(v)\cap(\gamma^2-Q)\neq\emptyset}}\sum_{i=1}^{n}\lambda_i F_i\left(x_i+tz_i^1+(1-t)z_i^2-S_i(tu+(1-t)v)\right)\cap Y+S\subset$$

$$\subset\bigcup_{\substack{w\in V\\ G(w)\cap(t\gamma^1+(1-t)\gamma^2-Q)\neq\emptyset}}\sum_{i=1}^{n}\lambda_i F_i\left(x_i+tz_i^1+(1-t)z_i^2-S_iw\right)\cap Y+S=$$

$$=\Psi\left(tz^1+(1-t)z^2,t\gamma^1+(1-t)\gamma^2)\right)\cap Y+S.$$

The main theorem of this chapter represents a stability criterion for the problem ($\mathrm{P_c}$).

Theorem 5.1
If there exists $u_0\in V$ such that $G(u_0)\cap -intQ\neq\emptyset$ and there exists U_i, a neighbourhood of $x_i-S_iu_0$, such that F_i is weakly S-upper bounded on U_i, $i=\overline{1,n}$, then the problem ($\mathrm{P_c}$) is stable.

Proof. From the definition of the weakly S-upper boundeness it follows that there exists $b_i\in Y$ such that $b_i\in F_i(z_i')+S$, for all $z_i'\in U_i$, $i=\overline{1,n}$. Let $V_i=U_i-(x_i-S_iu_0)$, $i=\overline{1,n}$. This means that V_i is a neighbourhood of 0 in X_i, $i=\overline{1,n}$. For all $z_i\in V_i$ we have

$$b_i\in F_i(x_i-S_iu_0+z_i)+S,\ i=\overline{1,n}$$

and further,

$$\sum_{i=1}^{n}\lambda_ib_i\in\sum_{i=1}^{n}\lambda_iF_i(x_i-S_iu_0+z_i)+S,$$

for all $z=(z_1,\ldots,z_n)\in V_1\times\ldots V_n$.
Let y_0 be an element from the intersection $G(u_0)\cap -intQ$. This is equivalent with $y_0\in G(u_0)$ and $-y_0\in intQ$. There exists, then, U_{y_0} a neighbourhood of $-y_0$ in W such that $U_{y_0}\subset Q$. Let $V_{y_0}=U_{y_0}+y_0$. V_{y_0} is a neighbourhood of 0 in W and for all $\gamma\in V_{y_0}$ we have $\gamma\in U_{y_0}+y_0\subset G(u_0)+Q$ or equivalent,

$$G(u_0)\cap(\gamma-Q)\neq\emptyset,\ \text{for all}\ \gamma\in V_{y_0}.$$

We obtained, finally, $\tilde{V} = V_1 \times \ldots V_n \times V_{y_0}$, a neighbourhood of 0 in $Z = X_1 \times \ldots \times X_n \times W$ such that for all $(z_1, \ldots, z_n, \gamma) \in V_1 \times \ldots V_n \times V_{y_0}$,

$$\begin{array}{l} G(u_0) \cap (\gamma - Q) \neq \emptyset \\ \sum_{i=1}^{n} \lambda_i F_i(x_i - S_i u_0 + z_i) \cap (\sum_{i=1}^{n} \lambda_i b_i - S) \neq \emptyset \\ u_0 \in V \end{array}$$

or equivalent,

$$\Phi(u_0, z, \gamma) \cap (\sum_{i=1}^{n} \lambda_i b_i - S) \neq \emptyset,$$

for all $(z, \gamma) = (z_1, \ldots, z_n, \gamma) \in \tilde{V} = V_1 \times \ldots V_n \times V_{y_0}$. This means that

$$\Phi(V, z, \gamma) \cap (\sum_{i=1}^{n} \lambda_i b_i - S) \neq \emptyset,$$

for all $(z, \gamma) = (z_1, \ldots, z_n, \gamma) \in \tilde{V} = V_1 \times \ldots V_n \times V_{y_0}$. From Corollary 1.3 and Remark 1.1, we have

$$\Phi(V, z, \gamma) \subset WInf\Phi(V, z, \gamma) \cup A(WInf\Phi(V, z, \gamma))$$

or equivalent,

$$\Phi(V, z, \gamma) \subset W(z, \gamma) \cup A(W(z, \gamma)),$$

for all $(z, \gamma) \in \tilde{V}$. Let assume, at first, that $W(z, \gamma) \cap (\sum_{i=1}^{n} \lambda_i b_i - S) \neq \emptyset$. Then it's clear that

$$\sum_{i=1}^{n} \lambda_i b_i \in W(z, \gamma) + S.$$

In the other case, if $A(W(z, \gamma)) \cap (\sum_{i=1}^{n} \lambda_i b_i - S) \neq \emptyset$, then there exists $y \in A(W(z, \gamma))$ such that $\sum_{i=1}^{n} \lambda_i b_i \in y + S$. By the definition of $A(W(z, \gamma))$ it follows that there exists $y' \in W(z, \gamma)$ such that $y \in y' + intS$. This implies that

$$\sum_{i=1}^{n} \lambda_i b_i \in y' + intS + S \subset W(z, \gamma) + S.$$

In conclusion, $(\sum_{i=1}^{n} \lambda_i b_i - S) \cap W(z, \gamma) \neq \emptyset$, for all $(z, \gamma) \in \tilde{V}$ and this means thata W is weakly S-upper bounded on a neighbourhood of 0 in Z. By Theorem 4.1, it follows that the problem $(\mathrm{P_c})$ is stable.

For the case when F_i are set-valued mappings from X_i to Y, $i = \overline{1, n}$, by using Theorem 4.2, we can formulate another stability criterion for the problem $(\mathrm{P_c})$.

Theorem 5.2
Let assume that there exists $u_0 \in V$ such that $G(u_0) \cap -intQ \neq \emptyset$. Let $(F_i)_{i=\overline{1,n}}$ be set-valued mappings from X_i to Y. If $F_i(z_i) \neq \emptyset$, for all $z_i \in X_i,\ i = \overline{1,n}$ and there exist $x_i^0 \in X_i$ and U_i^0 a neighbourhood of x_i^0 in X_i such that F_i is weakly S-upper bounded on $U_i^0,\ i = \overline{1,n}$, then the problem $(\mathrm{P_c})$ is stable.

Remark 5.2
Some particular cases of $(\mathrm{P_c})$ had been studied by Song[18] and by Tanino and Sawaragi[19]. For F, a set-valued mapping from U to R^n, and V a subset of U, the last two authors had also considered in their common paper the following optimization problem

$$(\tilde{\mathrm{P}}_{\mathrm{c}}) WMin \bigcup_{u \in V} F(u).$$

For $(\tilde{\mathrm{P}}_{\mathrm{c}})$ the strong duality theorems are true just under compactness assertions for the set V.

Remark 5.3
Finally, we will consider for the problem $(\mathrm{P_c})$ a particular case and our aim is to find its dual problem.
Let $(X_i)_{i=\overline{1,n}}, U$ and W be Hausdorff topological vector spaces, $F_i : X_i \to R$, $i = \overline{1,n}$ let be point-valued real convex functions, $A \in L(U,W)$ a linear and continous operator and $f \in W$. Further, let $V \subset U$ be a convex and closed set, Q be a convex, closed and pointed cone in W with nonempty interior, $S_i \in L(U, X_i)$, $i = \overline{1,n}$ be linear and continous operators, $x_i \in X_i,\ i = \overline{1,n}$ be fixed points and $\lambda_i > 0,\ i = \overline{1,n}$ be fixed positive constants. Let consider the following problem

$$(\mathrm{P_{rc}}) \quad \min_{\substack{u \in V \\ Au+f \in -Q}} \sum_{i=1}^{n} \lambda_i F_i(x_i - S_i u).$$

Let now define the set-valued mappings $\tilde{F}_i$ from X_i to R, $\tilde{F}_i(x) = \{F_i(x)\},\ i = \overline{1,n}$ and the set-valued mapping from U to W, $\tilde{G}(u) = \{Au + f\}$. Under this assumptions the problem $(\mathrm{P_{rc}})$ is equivalent with

$$(\tilde{\mathrm{P}}_{\mathrm{rc}}) \quad \min_{\substack{u \in V \\ \tilde{G}(u) \cap (-Q) \neq \emptyset}} \sum_{i=1}^{n} \lambda_i \tilde{F}_i(x_i - S_i u).$$

It is clear that $\tilde{F}_i$ are R_+-convex, $i = \overline{1,n}$ and that $\tilde{G}$ is Q-convex. The dual problem of $(\tilde{\mathrm{P}}_{\mathrm{rc}})$ will be

$$(\tilde{\mathrm{D}}_{\mathrm{rc}})\ WSup \bigcup_{\substack{T_i \in L(X_i, R),\ i=\overline{1,n} \\ \Gamma \in W*}} WInf \left\{ \sum_{i=1}^{n} \lambda_i \left[T_i x_i - \tilde{F}_i^*(T_i) \right] - WSup\Gamma(Q) - \right.$$

$$- < \Gamma, f > - WSup \bigcup_{u \in V} < \sum_{i=1}^{n} \lambda_i T_i S_i + \Gamma \circ A, u > \Bigg\}.$$

Using that

$$WSup\Gamma(Q) = \begin{cases} 0, & \text{if } \Gamma \leqq_{Q^*} 0 \Longleftrightarrow \Gamma \in -Q^* \\ +\infty, & \text{otherwise} \end{cases}$$

we obtain for the dual of $(\mathrm{P_{rc}})$ the following form

$$(\mathrm{D_{rc}}) \quad \sup_{\substack{T_i \in X_i^*,\ i=\overline{1,n} \\ \Gamma \in -Q^*}} \Bigg\{ \sum_{i=1}^{n} \lambda_i \left[T_i x_i - F_i^*(T_i) \right] -$$

$$- < \Gamma, f > - \sup_{u \in V} < \sum_{i=1}^{n} \lambda_i S_i^* T_i + A^* \Gamma, u > \Bigg\}.$$

The same dual problem for $(\mathrm{P_{rc}})$ was obtained by Wanka and Bot[22].

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