# Duality for the multiobjective location model involving sets as existing facilities

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**Abstract**

In this paper we consider the optimization problem with a multiobjective composed convex function as objective function, namely, being a composite of a convex and componentwise increasing vector function with a convex vector function. By the conjugacy approach, we obtain a dual problem for it. The existence of weak and strong duality is proved.

Using this general result, we introduce the dual problem for the multiobjective location problem in a general normed space, in which the existing facilities are represented by sets of points.

The biobjective Weber-minimax problem, the multiobjective Weber problem and the multiobjective minimax problem with demand sets are studied as particular cases of this problem.

**Keywords**: multiobjective duality, location problems, optimality conditions, Weber problem, minimax problem

The goal of this paper is to construct, in a general normed space, a dual for a primal multiobjective problem, each component of the multiobjective function being a composite of a convex and componentwise increasing function with a convex vector function. In the past, optimization problems with the objective function being a composed convex function have been considered by different authors. We remind the works [9] and [10] where some results with regard to duality have been given. Recently, optimization problems of this type have found applications in goal programming problems [4] and average distance problems [14].

This article is based on the work of Boţ and Wanka [2], where the authors have examined the case of a single objective function, for which a geometrical characterization of the set of optimal solutions was treated in [13] by Nickel, Puerto and Rodriguez-Chia. Here we study the duality for a multiobjective problem $(P)$. For our original multiobjective problem Pareto-efficient and properly efficient solutions are considered. In order to do this, we consider first the linearly scalarized problem $(P_\lambda)$ and use a dual problem $(D_\lambda)$ to derive strong duality and optimality conditions, which later are used to obtain duality assertions for the original and dual multiobjective problem. This dual problem $(D_\lambda)$ results from a special perturbation of the primal problem, by applying the Fenchel-Rockafellar duality concept based on conjugacy and perturbation (cf. [5]). Therefore, we have the possibility to construct different dual problems to a primal optimization problem, by perturbing it in different ways (see, for instance [22] and [23]).

Among the large number of papers and books dealing with different approaches to multiobjective duality we mention as a representative selection the books [6], [8], [16] and the papers [3], [11], [12], [17], [20], [22], [24] and [25]. Beside presentations in the sense of approaches for general formulated problems there are a lot of contributions devoted to the duality for multiobjective programming problems of special type, as for example linear problems [7], location and approximation problems [18], [20], portfolio optimization problems [19], [21], etc. In this paper, applying the conjugacy approach and using an appropiate perturbation as the authors in [2], we construct a dual for $(P_\lambda)$ and we prove the existence of strong duality between them. The dual problem is given in terms of conjugate functions and has the advantage that its structure gives an idea how to formulate the multiobjective dual problem to the original problem. By means of strong duality for $(P_\lambda)$ and its dual, we derive some optimality conditions for the primal optimization problem and construct a dual for the multiobjective primal problem $(P)$.

Using the general result, we introduce the dual problem and study the weak and strong duality for the multiobjective location problem in a general normed space in which the existing facilities are represented by sets. Afterwards, as particular cases of this problem, the multiobjective Weber and minimax location problems are studied.

# 1 The optimization problem with a multiobjective composed convex function as objective function

Let $(X, \|\cdot\|)$ be a normed space, $g_j : X \to \mathbb{R},\ j = 1, ..., m,$ convex and continuous functions and $f = (f_1, \ldots, f_l)^T,\ f_i : \mathbb{R}^m \to \mathbb{R},\ i = 1, \ldots, l,$ convex and componentwise increasing functions, i.e. for $y = (y_1, \ldots, y_m),\ z = (z_1, \ldots, z_m) \in \mathbb{R}^m,$ such that $y_j \geq$

$z_j, j = 1, ..., m$, holds $f_i(y) \geq f_i(z)$ for $i = 1, \ldots, l$.

We consider the following multiobjective optimization problem

$$(P) \quad \text{v-}\min_{x \in X} f(g(x))$$

with $g(x) = (g_1(x), \ldots, g_m(x))$.

The problem $(P)$ is a multiobjective optimization problem in the form of a vector minimum problem and for such kind of problems different notions of solutions are known. We will use in our paper the so-called efficient and properly efficient solutions. Let us recall the two solution concepts.

**Definition 1.** *An element $\bar{x} \in X$ is said to be efficient (or Pareto - efficient) with respect to $(P)$ if from*

$$f(g(x)) \underset{\mathbb{R}^l_+}{\leqq} f(g(\bar{x})) \text{ for } x \in X, \text{ follows } f(g(x)) = f(g(\bar{x})).$$

*Remark 1.* Here we consider the partial ordering in $\mathbb{R}^l$ given by the cone $\mathbb{R}^l_+ = \{y = (y_1, \ldots, y_l)^T \in \mathbb{R}^l : y_i \geq 0, i = 1, \ldots, l\}$ by $y^1 \leqq_{\mathbb{R}^l_+} y^2$ iff $y^2 - y^1 \in \mathbb{R}^l_+$.

**Definition 2.** *An element $\bar{x} \in X$ is said to be properly efficient with respect to $(P)$ if there exists $\lambda = (\lambda_1, \ldots, \lambda_l)^T \in int\mathbb{R}^l_+$ (i.e. $\lambda_i > 0$, $i = 1, \ldots, l$), such that $\sum\limits_{i=1}^{l} \lambda_i f_i(g(\bar{x})) \leq \sum\limits_{i=1}^{l} \lambda_i f_i(g(x)), \forall\, x \in X$.*

## 2 Duality for the scalarized problem

In order to study the duality for the multiobjective problem $(P)$, first we will study the duality for the scalarized problem (cf. Definition 2).

$$(P_\lambda) \quad \inf_{x \in X} \sum_{i=1}^{l} \lambda_i f_i(g(x)),$$

where $\lambda = (\lambda_1, \ldots, \lambda_l)^T \in int\mathbb{R}^l_+$ is a fixed vector. For $(P_\lambda)$ we will derive a dual $(D_\lambda)$ by means of the conjugacy approach, which permits us to construct different dual problems to an original primal problem depending on the kind of perturbation.

We introduce the following perturbation function $\Psi : \underbrace{X \times \ldots \times X}_{m+1} \times \mathbb{R}^m \to \mathbb{R}$,

$$\Psi(x, q, d) = \sum_{i=1}^{l} \lambda_i f_i((g_1(x + q_1), \ldots, g_m(x + q_m)) + d),$$

with the perturbation variables $q = (q_1, \dots, q_m) \in X \times \dots \times X$ and $d = (d_1, \dots, d_m) \in \mathbb{R}^m$. A dual problem to $(P_\lambda)$, obtained by using the perturbation function $\Psi$, is then

$$(D_\lambda) \qquad \sup_{\substack{q^* = (q_1^*, \dots, q_m^*), \\ q_j^* \in X^*, j = 1, ..., m, \\ d^* \in \mathbb{R}^m}} \{-\Psi^*(0, q^*, d^*)\},$$

where $\Psi^* : \underbrace{X^* \times \dots \times X^*}_{m+1} \times \mathbb{R}^m \to \mathbb{R} \cup \{+\infty\}$ is the conjugate function of $\Psi$, $q_j^* \in X^*$, $j = 1, ..., m$, and $d^* = (d_1^*, \dots, d_m^*)^T \in \mathbb{R}^m$ are the dual variables. $X^*$ denotes the topological dual space to $X$. The conjugate function of $\Psi$ is by definition

$$\begin{aligned}
\Psi^*(x^*, q^*, d^*) &= \sup_{\substack{x \in X,\, d \in \mathbb{R}^m,\, q = (q_1, \dots, q_m), \\ q_j \in X,\, j = 1, \dots, m}} \left\{ \langle x^*, x \rangle + \sum_{j=1}^m \langle q_j^*, q_j \rangle + \langle d^*, d \rangle - \Psi(x, q, d) \right\} \\
&= \sup_{\substack{x \in X,\, d \in \mathbb{R}^m,\, q = (q_1, \dots, q_m), \\ q_j \in X,\, j = 1, \dots, m}} \left\{ \langle x^*, x \rangle + \sum_{j=1}^m \langle q_j^*, q_j \rangle + \langle d^*, d \rangle \right. \\
&\qquad \left. - \sum_{i=1}^l \lambda_i f_i((g_1(x + q_1), \dots, g_m(x + q_m)) + d) \right\},
\end{aligned}$$

where $\langle \cdot, \cdot \rangle$ denotes the bilinear pairing between $X^*$ and $X$ (for $\langle x^*, x \rangle$ and $\langle q_j^*, q_j \rangle$, $j = 1, ..., m$) and, respectively, the scalar product in $\mathbb{R}^m$ (for $\langle d^*, d \rangle$). To calculate this expression we introduce first the new variables $r_j$ instead of $q_j$ and then $t$ instead of $d$ by

$$r_j = x + q_j \in X, \; j = 1, ..., m,$$

and

$$t = (g_1(x + q_1), \dots, g_m(x + q_m)) + d \; \in \mathbb{R}^m.$$

This implies

$$\Psi^*(x^*, q^*, d^*) = \sup_{\substack{x\in X,\, t\in\mathbb{R}^m,\\ r_j\in X,\, j=1,\ldots,m}} \left\{ \langle x^*, x\rangle + \sum_{j=1}^{m} \langle q_j^*, r_j - x\rangle \ + \langle d^*, t\rangle - \langle d^*, (g_1(r_1), \ldots, g_m(r_m))\rangle \right.$$

$$\left. - \ \sum_{i=1}^{l} \lambda_i f_i(t) \right\} = \sup_{\substack{x\in X,\, t\in\mathbb{R}^m,\\ r_j\in X,\, j=1,\ldots,m}} \left\{ \ \langle x^*, x\rangle + \sum_{j=1}^{m} \langle q_j^*, r_j\rangle - \sum_{j=1}^{m} \langle q_j^*, x\rangle + \langle d^*, t\rangle \right.$$

$$\left. - \ \langle d^*, (\,g_1(r_1), \ldots, g_m(r_m))\rangle - \sum_{i=1}^{l} \lambda_i\, f_i(t) \right\} = \sup_{t\in\mathbb{R}^m} \left\{ \langle d^*, t\rangle - \sum_{i=1}^{l} \lambda_i f_i(t) \right\}$$

$$+ \ \sup_{\substack{r_j\in X,\\ j=1,\ldots,m}} \left\{ \sum_{j=1}^{m} \langle q_j^*, r_j\rangle - \langle d^*, (g_1(r_1), \ldots, g_m(r_m))\rangle \right\} + \sup_{x\in X} \left\langle x^* - \sum_{j=1}^{m} q_j^*, x \right\rangle$$

$$= \left( \sum_{i=1}^{l} \lambda_i f_i \right)^* (d^*) + \sum_{j=1}^{m} \sup_{r_j\in X} \left( \langle q_j^*, r_j\rangle - d_j^* g_j(r_j) \right) + \sup_{x\in X} \left\langle x^* - \sum_{j=1}^{m} q_j^*, x \right\rangle$$

$$= \left( \sum_{i=1}^{l} \lambda_i f_i \right)^* (d^*) + \sum_{j=1}^{m} (d_j^* g_j)^*(q_j^*) + \sup_{x\in X} \left\langle x^* - \sum_{j=1}^{m} q_j^*, x \right\rangle.$$

Setting $x^* = 0$, the dual problem of $(P_\lambda)$ takes the form

$$(D_\lambda) \qquad \sup_{\substack{q_j^*\in X^*,\, j=1,\ldots,m,\\ d^*\in\mathbb{R}^m}} \left\{ - \left( \sum_{i=1}^{l} \lambda_i f_i \right)^* (d^*) - \sum_{j=1}^{m} (d_j^* g_j)^*(q_j^*) + \inf_{x\in X} \left\langle \sum_{j=1}^{m} q_j^*, x \right\rangle \right\}.$$

In the objective function of $(D_\lambda)$, if $\sum_{j=1}^{m} q_j^* \neq 0$, then it holds $\inf_{x\in X} \left\langle \sum_{j=1}^{m} q_j^*, x \right\rangle = -\infty$. Thus, for the calculation of $(D_\lambda)$ only $\sum_{j=1}^{m} q_j^* = 0$ is relevant. On the other hand (cf. [15]),

$$\left(\sum_{i=1}^{l} \lambda_i f_i\right)^* (d^*) = \inf \left\{ \sum_{i=1}^{l} (\lambda_i f_i)^*(d^{i*}) \ : \ \sum_{i=1}^{l} d^{i*} = d^* \right\}.$$

Further, because $\lambda_i > 0$, we get $(\lambda_i f_i)^*(d^{i*}) = \lambda_i f_i^*(\frac{d^{i*}}{\lambda_i})$, for each $i = 1, ..., l$. We can make the substitutions $a^i = (a_1^i, ..., a_m^i) := \frac{1}{\lambda_i} d^{i*}$, $i = 1, \ldots, l$, and then the dual $(D_\lambda)$ becomes

$$(D_\lambda) \qquad \sup_{\substack{a^i\in\mathbb{R}^m,\ \sum_{i=1}^{l} \lambda_i a^i = d^*,\ d^*\in\mathbb{R}^m,\\ q_j^*\in X^*,\ \sum_{j=1}^{m} q_j^* = 0}} \left\{ - \sum_{i=1}^{l} \lambda_i f_i^*(a^i) - \sum_{j=1}^{m} (d_j^* g_j)^*(q_j^*) \right\}.$$

Let us point out that, by construction, between $(P_\lambda)$ and $(D_\lambda)$ weak duality holds (cf. [5]), i.e. $\inf(P_\lambda) \geq \sup(D_\lambda)$. But, we are interested in the existence of strong duality, i.e. $\inf(P_\lambda) = \sup(D_\lambda)$, or even $\inf(P_\lambda) = \max(D_\lambda)$, meaning the existence of the solution of the dual problem. This can be shown by proving that the problem $(P_\lambda)$ is stable (cf. [5]).

**Proposition 1.** *The function* $\Psi : \underbrace{X \times \ldots \times X}_{m+1} \times \mathbb{R}^m \to \mathbb{R}$,

$$\Psi(x, q, d) = \sum_{i=1}^{l} \lambda_i f_i((g_1(x + q_1), \ldots, g_m(x + q_m)) + d)$$

*is convex.*

The convexity of $\Psi$ follows from the convexity of the functions $f_i,\ i = 1, \ldots, l$, and $g_j,\ j = 1, ...m$, and the fact that $f_i,\ i = 1, \ldots, l$, are componentwise increasing functions.

**Proposition 2.** *If* $\inf(P_\lambda) > -\infty$, *then the dual problem has a solution and strong duality holds, i.e.*

$$\inf(P_\lambda) = \max(D_\lambda).$$

*Proof.* See Theorem 1 in [2]. □

To investigate later the multiobjective duality for $(P)$ we need the optimality conditions regarding to the scalar problem $(P_\lambda)$ and its dual $(D_\lambda)$. These are formulated in the following theorem.

**Theorem 1.**

*(1) Let* $\bar{x} \in X$ *be a solution to* $(P_\lambda)$. *Then there exists a tupel* $(\bar{a}, \bar{q}^*, \bar{d}^*)$, *such that the following optimality conditions are satisfied*

*(i)* $f_i(g(\bar{x})) + f_i^*(\bar{a}^i) - \langle \bar{a}^i, g(\bar{x}) \rangle = 0,\ i = 1, \ldots, l,$

*(ii)* $(\bar{d}_j^* g_j)^*(\bar{q}_j^*) + \bar{d}_j^* g_j(\bar{x}) - \langle \bar{q}_j^*, \bar{x} \rangle = 0,\ j = 1, \ldots, m,$

*(iii)* $\sum\limits_{j=1}^{m} \bar{q}_j^* = 0,$

*(iv)* $\sum\limits_{i=1}^{l} \lambda_i \bar{a}^i = \bar{d}^*.$

*(2) If* $\bar{x} \in X$ *and* $(\bar{a}, \bar{q}^*, \bar{d}^*)$ *satisfy (i)-(iv), then* $\bar{x}$ *is a solution to* $(P_\lambda)$, $(\bar{a}, \bar{q}^*, \bar{d}^*)$ *is a solution to* $(D_\lambda)$ *and strong duality holds, i.e.*

$$\sum_{i=1}^{l} \lambda_i f_i(g(\bar{x})) = -\sum_{i=1}^{l} \lambda_i f_i^*(\bar{a}^i) - \sum_{j=1}^{m} (\bar{d}_j^* g_j)^*(\bar{q}_j^*). \tag{1}$$

*Remark 2.* Obviously, the tupel $(\bar{a}, \bar{q}^*, \bar{d}^*)$ in the part (1) of Theorem 1 is a solution of $(D_\lambda)$ (cf. the proof).

*Proof.*

(1) By Proposition 2, it follows that there exist $\bar{a} = (\bar{a}^1, ..., \bar{a}^l)$, $\bar{a}^i \in \mathbb{R}^m$, $i = 1, ..., l$, $\bar{q}^* = (\bar{q}_1^*, ..., \bar{q}_m^*)$, $\bar{q}_j^* \in X^*$, $j = 1, ..., m$, and $\bar{d}^* \in \mathbb{R}^m$, such that $(\bar{a}, \bar{q}^*, \bar{d}^*)$ is a solution to $(D_\lambda)$ and $\inf(P_\lambda) = \max(D_\lambda)$. This means that $\sum\limits_{j=1}^{m} \bar{q}_j^* = 0$, $\sum\limits_{i=1}^{l} \lambda_i \bar{a}^i = \bar{d}^*$, i.e. $(iii)$ and $(iv)$ are true, and

$$\sum_{i=1}^{l} \lambda_i f_i(g(\bar{x})) = -\sum_{i=1}^{l} \lambda_i f_i^*(\bar{a}^i) - \sum_{j=1}^{m} (\bar{d}_j^* g_j)^*(\bar{q}_j^*).$$

This equality is equivalent to

$$\sum_{i=1}^{l} \lambda_i f_i(g(\bar{x})) + \sum_{i=1}^{l} \lambda_i f_i^*(\bar{a}^i) - \sum_{i=1}^{l} \lambda_i \langle \bar{a}^i, g(\bar{x}) \rangle + \sum_{j=1}^{m} (\bar{d}_j^* g_j)^*(\bar{q}_j^*)$$
$$+ \sum_{j=1}^{m} \bar{d}_j^* g_j(\bar{x}) - \sum_{j=1}^{m} \langle \bar{q}_j^*, \bar{x} \rangle + \sum_{i=1}^{l} \lambda_i \langle \bar{a}^i, g(\bar{x}) \rangle - \sum_{j=1}^{m} \bar{d}_j^* g_j(\bar{x}) = 0.$$

From here follows

$$\sum_{i=1}^{l} \lambda_i \{ f_i^*(\bar{a}^i) + f_i(g(\bar{x})) - \langle \bar{a}^i, g(\bar{x}) \rangle \} + \sum_{j=1}^{m} \{ (\bar{d}_j^* g_j)^*(\bar{q}_j^*) + \bar{d}_j^* g_j(\bar{x}) - \langle \bar{q}_j^*, \bar{x} \rangle \}$$
$$+ \sum_{i=1}^{l} \lambda_i \langle \bar{a}^i, g(\bar{x}) \rangle - \sum_{j=1}^{m} \bar{d}_j^* g_j(\bar{x}) = 0.$$

But $\langle \sum\limits_{i=1}^{m} \lambda_i \bar{a}^i, g(\bar{x}) \rangle - \langle \bar{d}^*, g(\bar{x}) \rangle = 0$, which implies that

$$\sum_{i=1}^{l} \lambda_i \{ f_i^*(\bar{a}^i) + f_i(g(\bar{x})) - \langle \bar{a}^i, g(\bar{x}) \rangle \} + \sum_{j=1}^{m} \{ (\bar{d}_j^* g_j)^*(\bar{q}_j^*) + \bar{d}_j^* g_j(\bar{x}) - \langle \bar{q}_j^*, \bar{x} \rangle \} = 0. \tag{2}$$

Because of the Young-Fenchel inequality which is expressing that for a function $f$ and its conjugate $f^*$, $f(x) + f^*(x^*) \geqq \langle x^*, x \rangle$ is fulfilled, obviously all terms of the sum in (2) are non-negative and therefore they must be even equal to zero. This gives the optimality conditions $(i)$ and $(ii)$.

(2) All the calculations and transformations done within part (1) may be carried out in the reverse direction starting from the conditions $(i)$, $(ii)$, $(iii)$ and $(iv)$. Thus the equality (1) results, which is the strong duality, and shows that $\bar{x}$ solves $(P_\lambda)$ and $(\bar{a}, \bar{q}^*, \bar{d}^*)$ solves $(D_\lambda)$. □

# 3 The multiobjective dual problem

A dual multiobjective optimization problem $(D)$ to $(P)$ is introduced by

$$(D) \quad \underset{(a,q,d,\lambda,t)\in Y}{\text{v-max}} \; h(a,q,d,\lambda,t),$$

with

$$h(a,q,d,\lambda,t) = \begin{pmatrix} h_1(a,q,d,\lambda,t) \\ h_2(a,q,d,\lambda,t) \\ \vdots \\ h_l(a,q,d,\lambda,t) \end{pmatrix},$$

and

$$h_i(a,q,d,\lambda,t) = -f_i^*(a^i) - \frac{1}{l\lambda_i}\sum_{j=1}^{m}(d_j g_j)^*(q_j) + t_i, \; i = 1,\dots,l.$$

The dual variables are

$$a = (a^1,\dots,a^l), \; a^i \in \mathbb{R}^m, \; i = 1,\dots,l, \; q = (q_1,\dots,q_m), \; q_j \in X^*, \; j = 1,\dots,m,$$

$$d = (d_1,\dots,d_m)^T \in \mathbb{R}^m, \; \lambda = (\lambda_1,\dots,\lambda_l)^T \in \mathbb{R}^l, \; t = (t_1,\dots,t_l)^T \in \mathbb{R}^l,$$

and the set of constraints is

$$Y = \left\{ (a,q,d,\lambda,t) : \lambda \in int\mathbb{R}^l_+, \sum_{i=1}^{l}\lambda_i a^i = d, \; \sum_{j=1}^{m} q_j = 0, \; \sum_{i=1}^{l}\lambda_i t_i = 0 \right\}.$$

*Remark 3.* For the sake of simplicity of the denotation of the dual variables we write here and in the following $q_j$ and $q$ instead of $q_j^*$ and $q^*$ and $d$ instead of $d^*$.

**Definition 3.** *An element $(\bar{a},\bar{q},\bar{d},\bar{\lambda},\bar{t}) \in Y$ is said to be efficient (or Pareto-efficient) to $(D)$ if from $h(a,q,d,\lambda,t) \underset{\mathbb{R}^l_+}{\geqq} h(\bar{a},\bar{q},\bar{d},\bar{\lambda},\bar{t})$ for $(a,q,d,\lambda,t) \in Y$ follows $h(a,q,d,\lambda,t) = h(\bar{a},\bar{q},\bar{d},\bar{\lambda},\bar{t})$.*

The following theorem states the weak duality assertion.

**Theorem 2.** *There is no $x \in X$ and no $(a,q,d,\lambda,t) \in Y$ fulfilling $h(a,q,d,\lambda,t) \underset{\mathbb{R}^l_+}{\geqq} f(g(x))$ and $h(a,q,d,\lambda,t) \neq f(g(x))$.*

*Proof.* Let us assume that there exist $x \in X$ and $(a,q,d,\lambda,t) \in Y$, such that $h(a,q,d,\lambda,t) \underset{\mathbb{R}^l_+}{\geqq} f(g(x))$ and $h(a,q,d,\lambda,t) \neq f(g(x))$, i.e. $h_i(a,q,d,\lambda,t) \geq f_i(g(x))$,

$\forall\, i = 1, \ldots, l$, and $h_k(a, q, d, \lambda, t) > f_k(g(x))$ for at least one $k \in \{1, \ldots, l\}$. This means that

$$\sum_{i=1}^{l} \lambda_i h_i(a, q, d, \lambda, t) > \sum_{i=1}^{l} \lambda_i f_i(g(x)). \tag{3}$$

On the other hand, we have

$$\sum_{i=1}^{l} \lambda_i h_i(a, q, d, \lambda, t) = -\sum_{i=1}^{l} \lambda_i f_i^*(a^i) - \sum_{i=1}^{l} \lambda_i \frac{1}{l\lambda_i} \sum_{j=1}^{m} (d_j g_j)^*(q_j) + \sum_{i=1}^{l} \lambda_i t_i$$

$$= -\sum_{i=1}^{l} \lambda_i f_i^*(a^i) - \sum_{j=1}^{m} (d_j g_j)^*(q_j).$$

By the Young-Fenchel inequality

$$-f_i^*(a^i) \leq f_i(g(x)) - \langle a^i, g(x) \rangle, \ i = 1, \ldots, l$$

and

$$-(d_j g_j)^*(q_j) \leq d_j g_j(x) - \langle q_j, x \rangle, \ j = 1, ..., m,$$

we obtain

$$\begin{aligned}
\sum_{i=1}^{l} \lambda_i h_i(a, q, d, \lambda, t) \leq\ & \sum_{i=1}^{l} \lambda_i f_i(g(x)) - \sum_{i=1}^{l} \lambda_i \langle a^i, g(x) \rangle + \sum_{j=1}^{m} d_j g_j(x) - \sum_{j=1}^{m} \langle q_j, x \rangle \\
=\ & \sum_{i=1}^{l} \lambda_i f_i(g(x)) + \left\langle d - \sum_{i=1}^{l} \lambda_i a^i, g(x) \right\rangle - \left\langle \sum_{j=1}^{m} q_j, x \right\rangle \\
=\ & \sum_{i=1}^{l} \lambda_i f_i(g(x)),
\end{aligned}$$

and, therefore,

$$\sum_{i=1}^{l} \lambda_i h_i(a, q, d, \lambda, t) \leq \sum_{i=1}^{l} \lambda_i f_i(g(x)).$$

But this inequality contradicts relation (3). $\square$

The following theorem expresses the so-called strong duality between the two multiobjective problems $(P)$ and $(D)$.

**Theorem 3.** *Let $\bar{x}$ be a properly efficient element to $(P)$. Then an efficient solution $(\bar{a}, \bar{q}, \bar{d}, \bar{\lambda}, \bar{t}) \in Y$ to $(D)$ exists and strong duality $f(g(\bar{x})) = h(\bar{a}, \bar{q}, \bar{d}, \bar{\lambda}, \bar{t})$ holds.*

*Proof.* Assume $\bar{x}$ to be properly efficient to $(P)$. From Definition 2 the existence of a corresponding vector $\tilde{\lambda} = (\tilde{\lambda}_1, \dots, \tilde{\lambda}_l)^T \in int\mathbb{R}^l_+$ follows such that $\bar{x}$ solves the scalar problem

$$(P_{\tilde{\lambda}}) \quad \inf_{x \in X} \sum_{i=1}^{l} \tilde{\lambda}_i f_i(g(x)).$$

Because of $\inf(P_{\tilde{\lambda}}) > -\infty$, Proposition 2 ensures the existence of a solution $(\tilde{a}, \tilde{q}, \tilde{d})$ of the dual $(D_{\tilde{\lambda}})$ of $(P_{\tilde{\lambda}})$. The optimality conditions $(i) - (iv)$ are satisfied because of Theorem 1. Now we construct by means of $\bar{x}$ and $(\tilde{a}, \tilde{q}, \tilde{d})$ the efficient solution $(\bar{a}, \bar{q}, \bar{d}, \bar{\lambda}, \bar{t})$ of $(D)$. We consider $\bar{\lambda} := \tilde{\lambda},\ \bar{a} := \tilde{a},\ \bar{q} := \tilde{q},\ \bar{d} := \tilde{d}$. It remains to introduce $\bar{t} = (\bar{t}_1, \dots, \bar{t}_l)^T$. Let for $i = 1, \dots, l$,

$$\bar{t}_i := \frac{1}{l\bar{\lambda}_i} \sum_{j=1}^{m} (\bar{d}_j g_j)^*(\bar{q}_j) + \langle \bar{a}^i, g(\bar{x}) \rangle \in \mathbb{R}.$$

By the optimality conditions $(i) - (iv)$, for this tupel $(\bar{a}, \bar{q}, \bar{d}, \bar{\lambda}, \bar{t})$ it holds

$$\bar{\lambda} \in int\mathbb{R}^l_+, \ \sum_{i=1}^{l} \bar{\lambda}_i \bar{a}^i = \bar{d}, \ \sum_{j=1}^{m} \bar{q}_j = 0$$

and

$$\begin{aligned}
\sum_{i=1}^{l} \bar{\lambda}_i \bar{t}_i &= \sum_{i=1}^{l} \bar{\lambda}_i \tfrac{1}{l\bar{\lambda}_i} \sum_{j=1}^{m} (\bar{d}_j g_j)^*(\bar{q}_j) + \sum_{i=1}^{l} \bar{\lambda}_i \langle \bar{a}^i, g(\bar{x}) \rangle \\
&= \sum_{j=1}^{m} (\bar{d}_j g_j)^*(\bar{q}_j) + \left\langle \sum_{i=1}^{l} \bar{\lambda}_i \bar{a}^i, g(\bar{x}) \right\rangle \\
&= \sum_{j=1}^{m} (\bar{d}_j g_j)^*(\bar{q}_j) + \sum_{j=1}^{m} \bar{d}_j g_j(\bar{x}) \\
&= \left\langle \sum_{j=1}^{m} \bar{q}_j, \bar{x} \right\rangle = 0.
\end{aligned}$$

This means that the element $(\bar{a}, \bar{q}, \bar{d}, \bar{\lambda}, \bar{t})$ is feasible to $(D)$. It remains to show that the values of the objective functions are equal, i.e. $f(g(\bar{x})) = h(\bar{a}, \bar{q}, \bar{d}, \bar{\lambda}, \bar{t})$. Therefore we will prove that $f_i(g(\bar{x})) = h_i(\bar{a}, \bar{q}, \bar{d}, \bar{\lambda}, \bar{t})$ for each $i = 1, \dots, l$. For this we use the relation $(i)$ from Theorem 1 and obtain the following equalities

$$\begin{aligned}
h_i(\bar{a}, \bar{q}, \bar{d}, \bar{\lambda}, \bar{t}) &= -f_i^*(\bar{a}^i) - \tfrac{1}{l\bar{\lambda}_i} \sum_{j=1}^{m} (\bar{d}_j g_j)^*(\bar{q}_j) + \bar{t}_i \\
&= -f_i^*(\bar{a}^i) - \tfrac{1}{l\bar{\lambda}_i} \sum_{j=1}^{m} (\bar{d}_j g_j)^*(\bar{q}_j) + \tfrac{1}{l\bar{\lambda}_i} \sum_{j=1}^{m} (\bar{d}_j g_j)^*(\bar{q}_j) + \langle \bar{a}^i, g(\bar{x}) \rangle \\
&= -f_i^*(\bar{a}^i) + \langle \bar{a}^i, g(\bar{x}) \rangle \\
&= f_i(g(\bar{x})), \quad i = 1, \dots, l.
\end{aligned}$$

In conclusion,

$$f(g(\bar{x})) = h(\bar{a}, \bar{q}, \bar{d}, \bar{\lambda}, \bar{t}).$$

□

# 4 The case of monotonic norms

In this section we particularize the multiobjective problem presented in the previous section. Therefore, let be $\Phi_i : \mathbb{R}^m \to \mathbb{R},\ i = 1, \ldots, l$, monotonic norms on $\mathbb{R}^m$, i.e. (cf. [1])

$$\forall\ u,\ v \in \mathbb{R}^m,\ |u_j| \leq |v_j|,\ j = 1, ..., m,\ \text{it holds } \Phi_i(u) \leq \Phi_i(v).$$

Let us introduce now the following multiobjective problem

$$(P_\Phi) \quad \underset{x\in X}{\text{v-min}} \begin{pmatrix} \Phi_1^+(g(x)) \\ \vdots \\ \Phi_l^+(g(x)) \end{pmatrix},$$

where $\Phi_i^+(t) := \Phi_i(t^+),\ i = 1, ..., l$, with $t^+ = (t_1^+, \ldots, t_m^+)$ and $t_j^+ = \max\{0, t_j\}$, $j = 1, ..., m$.

**Proposition 3.** *The functions $\Phi_i^+ : \mathbb{R}^m \to \mathbb{R}$, $i = 1, \ldots, l$, are convex and componentwise increasing.*

*Proof.* See Proposition 2 in [2]. $\square$

In order to study the duality for the problem $(P_\Phi)$ we will study, like in section 2, the duality for the scalarized problem

$$(P_{\Phi\lambda}) \quad \inf_{x\in X} \sum_{i=1}^{l} \lambda_i \Phi_i^+(g(x)),$$

where $\lambda = (\lambda_1, \ldots, \lambda_l)^T \in int\mathbb{R}^l_+$ is a fixed vector. By the approach described in section 2, a dual problem to $(P_{\Phi\lambda})$ is

$$(D_{\Phi\lambda}) \quad \sup_{\substack{a^i\in\mathbb{R}^m,\ \sum\limits_{i=1}^{l}\lambda_i a^i = d^*,\ d^*\in\mathbb{R}^m, \\ q_j^*\in X^*,\ \sum\limits_{j=1}^{m} q_j^* = 0}} \left\{ -\sum_{i=1}^{l} \lambda_i (\Phi_i^+)^*(a^i) - \sum_{j=1}^{m} (d_j^* g_j)^*(q_j^*) \right\}.$$

**Proposition 4.** *The conjugate functions $(\Phi_i^+)^* : \mathbb{R}^m \to \mathbb{R}\cup\{+\infty\}$, $i = 1, \ldots, l$, of $\Phi_i^+$ are*

$$(\Phi_i^+)^*(a^i) = \begin{cases} 0, & \text{if } a^i \underset{\mathbb{R}^m_+}{\geqq} 0 \text{ and } \Phi_i^0(a^i) \leq 1, \\ +\infty, & \text{otherwise,} \end{cases}$$

*where $\Phi_i^0$ is the dual norm of $\Phi_i$ in $\mathbb{R}^m$.*

*Proof.* See Proposition 3 in [2]. $\square$

By Proposition 4, the dual of $(P_{\Phi\lambda})$ has the following form

$$(D_{\Phi\lambda}) \quad \sup_{\substack{a^i\in\mathbb{R}^m_+,\, \Phi_i^0(a^i)\le 1,\, i=1,\dots,l,\\ \sum\limits_{i=1}^{l}\lambda_i a^i=d^*,\, d^*\in\mathbb{R}^m_+,\; q_j^*\in X^*,\, \sum\limits_{j=1}^{m} q_j^*=0}} \left\{-\sum_{j=1}^{m}(d_j^*g_j)^*(q_j^*)\right\}.$$

Let us consider now the set-valued variable $I \subseteq \{1,\dots,m\}$. In the objective function of $(D_{\Phi\lambda})$ let us separate the terms for which $d_j^* > 0$ (i.e. $j\in I$) from the terms for which $d_j^*=0$ (i.e. $j\notin I$). Then it follows

$$(D_{\Phi\lambda}) \quad \sup_{\substack{a^i\in\mathbb{R}^m_+,\, \Phi_i^0(a^i)\le 1,\, i=1,\dots,l,\\ \sum\limits_{i=1}^{l}\lambda_i a^i=d^*,\, I\subseteq\{1,\dots,m\},\, d_j^*>0,\,(j\in I),\\ d_j^*=0,\,(j\notin I),\, q_j^*\in X^*,\, \sum\limits_{j=1}^{m} q_j^*=0}} \left\{-\sum_{j\in I}(d_j^*g_j)^*(q_j^*)-\sum_{j\notin I}(d_j^*g_j)^*(q_j^*)\right\}.$$

Let us notice that, in the case $d_j^*>0$ $(j\in I)$ there must exist at least one $i\in\{1,\dots,l\}$, such that $a_j^i>0$.

Because of $0^*(q_j^*)=\sup\limits_{x\in X}\{\langle q_j^*,x\rangle-0\}=\sup\limits_{x\in X}\langle q_j^*,x\rangle=\begin{cases}0, & \text{if } q_j^*=0,\\ +\infty, & \text{otherwise}\end{cases}$, in order to have supremum in $(D_{\Phi\lambda})$, we must take $q_j^*=0$ for all $j\notin I$.

The problem $(D_{\Phi\lambda})$ can then be written as

$$(D_{\Phi\lambda}) \quad \sup_{\substack{a^i\in\mathbb{R}^m_+,\, \Phi_i^0(a^i)\le 1,\, i=1,\dots,l,\\ \sum\limits_{i=1}^{l}\lambda_i a^i=d^*,\, I\subseteq\{1,\dots,m\},\, d_j^*>0,\,(j\in I),\\ d_j^*=0,\,(j\notin I),\, q_j^*\in X^*,\, j\in J,\, \sum\limits_{j\in I} q_j^*=0}} \left\{-\sum_{j\in I}(d_j^*g_j)^*(q_j^*)\right\}.$$

For $d_j^*>0$, i.e. $j\in I$, we apply again the formula $(d_j^*g_j)^*(q_j^*)=d_j^*g_j^*(\frac{1}{d_j^*}q_j^*)$ (cf. [5]). Denoting $q_j^*:=\frac{q_j^*}{d_j^*}$ for $j\in I$ we get

$$(D_{\Phi\lambda}) \quad \sup_{(I,a,d^*,q^*)\in Y_{\Phi\lambda}} \left\{-\sum_{j\in I} d_j^*g_j^*(q_j^*)\right\},$$

with

$$\begin{aligned} Y_{\Phi\lambda}=\Big\{(I,a,d^*,q^*): {} & I\subseteq\{1,\dots,m\},\, a=(a^1,\dots,a^l),\, a^i\in\mathbb{R}^m_+,\, \Phi_i^0(a^i)\le 1,\, i=1,\dots,l,\\ & d^*=(d_1^*,\dots,d_m^*)\in\mathbb{R}^m,\; q^*=(q_1^*,\dots,q_m^*),\; q_j^*\in X^*,\; j=1,\dots,m,\\ & \sum_{i=1}^{l}\lambda_i a^i=d^*,\; d_j^*>0,\, j\in I,\; d_j^*=0,\, j\notin I,\, \sum_{j\in I} d_j^*q_j^*=0\Big\}. \end{aligned}$$

Now we can eliminate the variable $d^*$, observing that $\sum\limits_{i=1}^{l} \lambda_i a^i_j = d^*_j$, for $j = 1, ..., m$. Then the dual becomes (setting $q := q^*$)

$$(D_{\Phi\lambda}) \qquad \sup_{(I,\, a,\, q) \in Y_{\Phi\lambda}} \left\{ -\sum_{j \in I} \left( \sum_{i=1}^{l} \lambda_i a^i_j \right) g^*_j(q_j) \right\}$$

with

$$\begin{aligned} Y_{\Phi\lambda} = \Bigg\{ (I,\, a,\, q) : I \subseteq \{1, \ldots, m\}, a = (a^1, \ldots, a^l), a^i \in \mathbb{R}^m_+, \Phi^0_i(a^i) \le 1, \\ q = (q_1, \ldots, q_m),\, q_j \in X^*,\, j = 1,\, \ldots, m, \sum_{j \in I} \left( \sum_{i=1}^{l} \lambda_i a^i_j \right) q_j = 0, \\ \sum_{i=1}^{l} \lambda_i a^i_j > 0,\, j \in I,\; a^i_j = 0,\, j \notin I,\; i = 1, \ldots, l \Bigg\}. \end{aligned}$$

Because of the functions $\Phi^+_i$, $i = 1, ..., l$, are convex and componentwise increasing, it follows that $\sum\limits_{i=1}^{l} \lambda_i \Phi^+_i$ is also convex and componentwise increasing. One can notice that $\inf(P_{\Phi\lambda})$ is finite, being greater than or equal to zero. This observation, together with Proposition 2, permits us to formulate the following strong duality theorem for the problems $(P_{\Phi\lambda})$ and $(D_{\Phi\lambda})$.

**Theorem 4.** *The dual problem $(D_{\Phi\lambda})$ has a solution and strong duality holds, i.e.*

$$\inf(P_{\Phi\lambda}) = \max(D_{\Phi\lambda}).$$

Analogously to problem $(P_\lambda)$ we can derive now the optimality conditions for $(P_{\Phi\lambda})$.

**Theorem 5.**

*(1) Let $\bar{x} \in X$ be a solution to $(P_{\Phi\lambda})$. Then there exists $(\bar{I}, \bar{a}, \bar{q}) \in Y_{\Phi\lambda}$, solution to $(D_{\Phi\lambda})$, such that the following optimality conditions are satisfied*

*(i)* $\bar{I} \subseteq \{1, \ldots, m\}$, $\bar{a}^i \in \mathbb{R}^m_+$, $\sum\limits_{i=1}^{l} \lambda_i \bar{a}^i_j > 0$, $j \in \bar{I}$, $\bar{a}^i_j = 0$, $j \notin \bar{I}$, $i = 1, \ldots, l$,

*(ii)* $\Phi^0_i(\bar{a}^i) \le 1$, $i = 1, \ldots, l$, $\sum\limits_{j \in \bar{I}} \left( \sum\limits_{i=1}^{l} \lambda_i \bar{a}^i_j \right) \bar{q}_j = 0$,

*(iii)* $\Phi^+_i(g(\bar{x})) = \langle \bar{a}^i, g(\bar{x}) \rangle$, $i = 1, \ldots, l$,

*(iv)* $g_j(\bar{x}) + g^*_j(\bar{q}_j) = \langle \bar{q}_j, \bar{x} \rangle$, $j \in \bar{I}$.

*(2) If $\bar{x} \in X$, $(\bar{I}, \bar{a}, \bar{q})$ is feasible to $(D_{\Phi\lambda})$ and (i)-(iv) are fulfilled, then $\bar{x}$ is a solution to $(P_{\Phi\lambda})$, $(\bar{I}, \bar{a}, \bar{q})$ is a solution to $(D_{\Phi\lambda})$ and strong duality holds, i.e.*

$$\sum_{i=1}^{l} \lambda_i \Phi_i^+(g(\bar{x})) = -\sum_{j\in\bar{I}} \left(\sum_{i=1}^{l} \lambda_i \bar{a}_j^i\right) g_j^*(\bar{q}_j).$$

*Proof.*

(1) By Theorem 4, it follows that there exists $(\bar{I}, \bar{a}, \bar{q}) \in Y_{\Phi\lambda}$, a solution to $(D_{\Phi\lambda})$, such that $(i)-(ii)$ are fulfilled and

$$\sum_{i=1}^{l} \lambda_i \Phi_i^+(g(\bar{x})) = -\sum_{j\in\bar{I}} \Big(\sum_{i=1}^{l} \lambda_i \bar{a}_j^i\Big) g_j^*(\bar{q}_j).$$

This is equivalent to

$$\sum_{i=1}^{l}\lambda_i \Phi_i^+(g(\bar{x}))+\sum_{i=1}^{l}\lambda_i(\Phi_i^+)^*(\bar{a}^i)-\sum_{i=1}^{l}\lambda_i\langle\bar{a}^i, g(\bar{x})\rangle+$$

$$\sum_{i=1}^{l} \lambda_i\langle\bar{a}^i, g(\bar{x})\rangle+\sum_{j\in\bar{I}} \left(\sum_{i=1}^{l} \lambda_i \bar{a}_j^i\right) g_j^*(\bar{q}_j)=0.$$

Hence,

$$\sum_{i=1}^{l} \lambda_i[\Phi_i^+(g(\bar{x})) + (\Phi_i^+)^*(\bar{a}^i) - \langle\bar{a}^i, g(\bar{x})\rangle]+$$

$$\sum_{i=1}^{l} \lambda_i\langle\bar{a}^i, g(\bar{x})\rangle + \sum_{j\in\bar{I}} \left(\sum_{i=1}^{l} \lambda_i \bar{a}_j^i\right) g_j^*(\bar{q}_j) = 0.$$

Because of $(i)$ and $(ii)$, it holds

$$\sum_{i=1}^{l} \lambda_i\langle\bar{a}^i, g(\bar{x})\rangle + \sum_{j\in\bar{I}} \left(\sum_{i=1}^{l} \lambda_i \bar{a}_j^i\right) g_j^*(\bar{q}_j) = \sum_{i=1}^{l} \lambda_i\langle\bar{a}^i, g(\bar{x})\rangle$$

$$+\sum_{j\in\bar{I}} \left(\sum_{i=1}^{l} \lambda_i \bar{a}_j^i\right) g_j^*(\bar{q}_j) - \left\langle \sum_{j\in\bar{I}} \Big(\sum_{i=1}^{l} \lambda_i \bar{a}_j^i\Big)\bar{q}_j, \bar{x} \right\rangle = \sum_{i=1}^{l} \lambda_i \left(\sum_{j\in\bar{I}} \bar{a}_j^i g_j(\bar{x})\right)$$

$$+\sum_{j\in\bar{I}} \left(\sum_{i=1}^{l} \lambda_i \bar{a}_j^i\right) \big(g_j^*(\bar{q}_j) - \langle\bar{q}_j, \bar{x}\rangle\big)$$

$$= \sum_{j \in \bar{I}} \left( \sum_{i=1}^{l} \lambda_i \bar{a}_j^i \right) \big( g_j(\bar{x}) + g_j^*(\bar{q}_j) - \langle \bar{q}_j, \bar{x} \rangle \big),$$

implying

$$\sum_{i=1}^{l} \lambda_i \big[ \Phi_i^+(g(\bar{x})) + (\Phi_i^+)^*(\bar{a}^i) - \langle \bar{a}^i, g(\bar{x}) \rangle \big] +$$

$$\sum_{j \in \bar{I}} \left( \sum_{i=1}^{l} \lambda_i \bar{a}_j^i \right) \big( g_j(\bar{x}) + g_j^*(\bar{q}_j) - \langle \bar{q}_j, \bar{x} \rangle \big) = 0.$$

In conclusion, using again the Young-Fenchel inequality, we obtain

$$\Phi_i^+(g(\bar{x})) + (\Phi_i^+)^*(\bar{a}^i) - \langle \bar{a}^i, g(\bar{x}) \rangle = 0, \ i = 1, \ldots, l,$$

and

$$g_j(\bar{x}) + g_j^*(\bar{q}_j) - \langle \bar{q}_j, \bar{x} \rangle = 0, \ j \in \bar{I},$$

and, so, the equalities in $(iii)$ and $(iv)$ must hold. $\square$

Further, like in the general case, we can construct a multiobjective dual problem to the primal problem $(P_\Phi)$

$$(D_\Phi) \qquad \underset{(I,a,q,\lambda,t) \in Y_\Phi}{\text{v-max}} \begin{pmatrix} h_1(I, a, q, \lambda, t) \\ \vdots \\ h_l(I, a, q, \lambda, t) \end{pmatrix},$$

with

$$h_k(I, a, q, \lambda, t) = -\frac{1}{l\lambda_k} \sum_{j \in I} \left( \sum_{i=1}^{l} \lambda_i a_j^i \right) g_j^*(q_j) + t_k, \ k = 1, \ldots, l,$$

the dual variables

$$a = (a^1, \ldots, a^l), \ a^i \in \mathbb{R}^m, \ i = 1, \ldots, l, \ q = (q_1, \ldots, q_m), \ q_j \in X^*, j = 1, \ldots, m,$$

$$\lambda = (\lambda_1, \ldots, \lambda_l)^T \in \mathbb{R}^l, \ t = (t_1, \ldots, t_l)^T \in \mathbb{R}^l,$$

and the set of constraints

$$Y_\Phi = \left\{ (I, a, q, \lambda, t) : I \subseteq \{1, ..., m\}, a^i \in \mathbb{R}_+^m, \Phi_i^0(a^i) \leq 1, \sum_{j \in I} \left( \sum_{i=1}^{l} \lambda_i a_j^i \right) q_j = 0, \right.$$
$$\left. \left( \sum_{i=1}^{l} \lambda_i a_j^i \right) > 0, j \in I, a_j^i = 0, j \notin I, i = 1, \ldots, l, \lambda \in int\mathbb{R}_+^l, \ \sum_{k=1}^{l} \lambda_k t_k = 0 \right\}.$$

Let us present now the weak and strong duality theorems for these problems.

**Theorem 6.** *There is no* $x \in X$ *and no* $(I,a,q,\lambda,t) \in Y_\Phi$, *such that* $\Phi_i^+(g(x)) \le h_i(I,a,q,\lambda,t)$, $i=1,\ldots,l$, *and* $\Phi_k^+(g(x)) < h_k(I,a,q,\lambda,t)$ *for at least one* $k \in \{1,\ldots,l\}$.

**Theorem 7.** *Let* $\bar{x}$ *be a properly efficient element to* $(P_\Phi)$). *Then an efficient solution* $(\bar{I},\bar{a},\bar{q},\bar{\lambda},\bar{t}) \in Y_\Phi$ *to* $(D_\Phi)$ *exists and strong duality holds, i.e.*

$$\Phi_k^+(g(\bar{x})) = -\frac{1}{l\bar{\lambda}_k}\sum_{j\in\bar{I}}\left(\sum_{i=1}^{l}\bar{\lambda}_i\bar{a}_j^i\right)g_j^*(\bar{q}_j)+\bar{t}_k,\ k=1,\ldots,l.$$

# 5 The multiobjective location model involving sets as existing facilities

Let $\mathcal{A} = \{A_1,\ldots,A_m\}$ be a family of convex sets in $X$ such that $\bigcap\limits_{j=1}^{m}\bar{A}_j = \emptyset$. We consider the same vector function $d : X \to \mathbb{R}^m$ as in [2], i.e.

$$d(x) := (d_1(x,A_1),\ldots,d_m(x,A_m)),$$

where

$$d_j(x,A_j) = \inf\{\gamma_j(x-y_j) : y_j \in A_j\},\ j=1,...,m,$$

and $\gamma_j$, $j=1,...,m$, are continuous norms on $X$. For $j=1,...,m$, we consider the functions $g_j : X \to \mathbb{R}$, $g_j(x) = d_j(x,A_j)$. This means that the functions $g_j$, $j = 1,\ldots,m$, are convex and continuous on $X$. The multiobjective location problem with sets as existing facilities is

$$(P_\Phi(\mathcal{A})) \quad \operatorname*{v-min}_{x\in X}\begin{pmatrix}\Phi_1(d(x))\\ \vdots \\ \Phi_l(d(x))\end{pmatrix},$$

with $\Phi_i : \mathbb{R}^m \to \mathbb{R}$, $i=1,...,l$, monotonic norms on $\mathbb{R}^m$.

Because of

$$\Phi_i^+(d(x)) = \Phi_i((d(x))^+) = \Phi_i(d(x)),\ \forall\, x\in X,\ i=1,...,l,$$

where $(d(x))^+ = ((d_1(x))^+,\ldots,(d_m(x))^+)$ with $(d_i(x))^+ = \max\{0,d_i(x)\}$, for $i = 1\ldots,m$, we can write $(P_\Phi(\mathcal{A}))$ in the equivalent form

$$(P_\Phi(\mathcal{A})) \quad \operatorname*{v-min}_{x\in X}\begin{pmatrix}\Phi_1^+(d(x))\\ \vdots \\ \Phi_l^+(d(x))\end{pmatrix}.$$

This problem is a particular case of the problem studied in section 4. In order to study the duality for this problem, we will study again, at first, the duality for the scalarized problem

$$(P_{\Phi\lambda}(\mathcal{A})) \quad \inf_{x\in X} \sum_{i=1}^{l} \lambda_i \Phi_i^+(d(x)),$$

with $\lambda = (\lambda_1, \dots, \lambda_l)^T \in int\mathbb{R}_+^l$ fixed.

Then the dual of $(P_{\Phi\lambda}(\mathcal{A}))$ is

$$(D_{\Phi\lambda}(\mathcal{A})) \quad \sup_{(I,\,a,\,q)\in Y_{\Phi\lambda}(\mathcal{A})} \left\{ -\sum_{j\in I} \left( \sum_{i=1}^{l} \lambda_i a_j^i \right) d_j^*(q_j) \right\},$$

where $d_j^*(q_j)$ is the conjugate function to $d_j(x, A_j), j = 1, ..., m$, and

$$\begin{aligned} Y_{\Phi\lambda}(\mathcal{A}) = \Big\{(I, a, q)\colon\ & I \subseteq \{1, \dots, m\}, a = (a^1, \dots, a^l), a^i \in \mathbb{R}_+^m, \Phi_i^0(a^i) \le 1, \\ & q = (q_1, \dots, q_m), q_j \in X^*, \left( \sum_{i=1}^{l} \lambda_i a_j^i \right) > 0, j \in I, \\ & a_j^i = 0, j \notin I, i = 1, ..., l, \sum_{j\in I} \left( \sum_{i=1}^{l} \lambda_i a_j^i \right) q_j = 0 \Big\}. \end{aligned}$$

Using the Theorems 4 and 5, we can present for $(P_{\Phi\lambda}(\mathcal{A}))$ and $(D_{\Phi\lambda}(\mathcal{A}))$ the strong duality theorem and the optimality conditions.

**Theorem 8.** *The dual problem $(D_{\Phi\lambda}(\mathcal{A}))$ has a solution and strong duality holds, i.e.*

$$\inf(P_{\Phi\lambda}(\mathcal{A})) = \max(D_{\Phi\lambda}(\mathcal{A})).$$

**Theorem 9.**

*(1) Let $\bar{x} \in X$ be a solution to $(P_{\Phi\lambda}(\mathcal{A}))$. Then there exists $(\bar{I}, \bar{a}, \bar{q}) \in Y_{\Phi\lambda}(\mathcal{A})$, solution to $(D_{\Phi\lambda}(\mathcal{A}))$, such that the following optimality conditions are satisfied*

*(i)* $\bar{I} \subseteq \{1, \dots, m\}, \bar{I} \neq \emptyset, \left( \sum_{i=1}^{l} \lambda_i \bar{a}_j^i \right) > 0, j \in \bar{I}, \bar{a}_j^i = 0,\ j \notin \bar{I}, i = 1, \dots, l,$

*(ii)* $\bar{a}^i \in \mathbb{R}_+^m,\ \Phi_i^0(\bar{a}^i) = 1,\ i = 1, \dots, l,\ \sum_{j\in\bar{I}} \left( \sum_{i=1}^{l} \lambda_i \bar{a}_j^i \right) \bar{q}_j = 0,$

*(iii)* $\Phi_i(d(\bar{x})) = \langle \bar{a}^i, d(\bar{x}) \rangle,\ i = 1, \dots, l,$

*(iv)* $\bar{x} \in \partial d_j^*(\bar{q}_j),\ j \in \bar{I}.$

(2) *If $\bar{x} \in X$, $(\bar{I}, \bar{a}, \bar{q})$ is feasible to $(D_{\Phi\lambda}(\mathcal{A}))$ and (i)-(iv) are fulfilled, then $\bar{x}$ is a solution to $(P_{\Phi\lambda}(\mathcal{A}))$, $(\bar{I}, \bar{a}, \bar{q})$ is a solution to $(D_{\Phi\lambda}(\mathcal{A}))$ and strong duality holds, i.e.*

$$\sum_{i=1}^{l} \lambda_i \Phi_i^+(d(\bar{x})) = -\sum_{j\in\bar{I}} \left(\sum_{i=1}^{l} \lambda_i \bar{a}_j^i\right) d_j^*(\bar{q}_j).$$

*Proof.*

(1) By Theorem 5 follows that there exists $(\bar{I}, \bar{a}, \bar{q}) \in Y_{\Phi\lambda}(\mathcal{A})$, solution to $(D_{\Phi\lambda}(\mathcal{A}))$, such that

$$\begin{array}{ll} (i') & \bar{I} \subseteq \{1, \dots, m\}, \bar{I} \neq \emptyset, \left(\sum\limits_{i=1}^{l} \lambda_i \bar{a}_j^i\right) > 0, j \in \bar{I}, \bar{a}_j^i = 0, j \notin \bar{I}, i = 1, \dots, l, \\ (ii') & \bar{a}^i \in \mathbb{R}_+^m,\ \Phi_i^0(\bar{a}^i) \leq 1,\ i = 1, \dots, l,\ \sum\limits_{j\in\bar{I}} \left(\sum\limits_{i=1}^{l} \lambda_i \bar{a}_j^i\right) \bar{q}_j = 0, \\ (iii') & \Phi_i^+(d(\bar{x})) = \langle \bar{a}^i, d(\bar{x})\rangle,\ i = 1, \dots, l, \\ (iv') & d_j(\bar{x}, A_j) + d_j^*(\bar{q}_j) = \langle \bar{q}_j, \bar{x}\rangle,\ j \in \bar{I}. \end{array}$$

We will prove that $(\bar{I}, \bar{a}, \bar{q})$ verifies the relations $(i) - (iv)$. If $\bar{I}$ would be empty, then it would follow by $(i')$ that $\bar{a}_j^i = 0, j = 1, \dots, m, i = 1, \dots, l$. From $(iii')$ it holds then $\Phi_i((d(\bar{x}))^+) = \Phi_i^+(d(\bar{x})) = 0$ which actually means that $d(\bar{x}) = (d(\bar{x}))^+ = 0$, i.e.

$$g_j(\bar{x}) = d_j(\bar{x}, A_j) = 0,\ j = 1, ..., m.$$

But, this would imply that $\bar{x} \in \bigcap\limits_{j=1}^{m} \bar{A}_j$. This is a contradiction to the hypothesis $\bigcap\limits_{j=1}^{m} \bar{A}_j = \emptyset$. By this, the relation $(i)$ is proved.

From $(iii')$, we have that

$$\Phi_i^+(d(\bar{x})) = \Phi_i(d(\bar{x})) = \langle \bar{a}^i, d(\bar{x})\rangle,\ i = 1, \dots, l,$$

and, so, $(iii)$ is also proved. From $(iv')$, we have that $\bar{q}_j \in \partial d_j(\bar{x}, A_j)$ for $j \in \bar{I}$ (cf. [5]). On the other hand, $d_j$ being a convex and continuous function, verifies (cf. [5])

$$\bar{q}_j \in \partial d_j(\bar{x}, A_j) \Leftrightarrow \bar{x} \in \partial d_j^*(\bar{q}_j),\ j \in \bar{I},$$

which proves $(iv)$.

Now, it remains us to show that $\Phi_i^0(\bar{a}^i) = 1,\ i = 1, ..., l$. By the definition of the dual norm, we have

$$\Phi_i^0(\bar{a}^i) = \sup_{\substack{\Phi_i(v)\leq 1 \\ v\in\mathbb{R}^m}} \{|\langle \bar{a}^i, v\rangle|\},\ i = 1, ..., l.$$

Because of $\bigcap\limits_{j=1}^{m} \bar{A}_j = \emptyset$, it holds $\Phi_i(d(\bar{x})) > 0$, for $i = 1, ..., l$. Let be $\bar{v}_i = \frac{1}{\Phi_i(d(\bar{x}))} d(\bar{x}) \in \mathbb{R}^m$. We have $\Phi_i(\bar{v}_i) = 1,\ i = 1, ..., l$, and then, by $(iii)$,

$$\Phi_i^0(\bar{a}^i) = \Phi_i(\bar{v}_i)\Phi_i^0(\bar{a}^i) \geq \langle \bar{a}^i, \bar{v}_i \rangle = \frac{\sum\limits_{k=1}^{m} \bar{a}_k^i d_k(\bar{x}, A_k)}{\Phi_i(d(\bar{x}))} = \frac{\langle \bar{a}^i, d(\bar{x}) \rangle}{\Phi_i(d(\bar{x}))} = 1.$$

In conclusion, by $(ii')$, $\Phi_i^0(\bar{a}^i) = 1,\ i = 1, \dots, l$. □

As a multiobjective dual problem of the primal problem $(P_\Phi(\mathcal{A}))$ we can introduce

$$(D_\Phi(\mathcal{A})) \qquad \underset{(I,a,q,\lambda,t) \in Y_\Phi(\mathcal{A})}{\text{v-max}} \begin{pmatrix} h_1^d(I, a, q, \lambda, t) \\ \vdots \\ h_l^d(I, a, q, \lambda, t) \end{pmatrix},$$

with

$$h_k^d(I, a, q, \lambda, t) = -\frac{1}{l\lambda_k} \sum_{j \in I} \left( \sum_{i=1}^{l} \lambda_i a_j^i \right) d_j^*(q_j) + t_k,\ k = 1, \dots, l,$$

the dual variables

$$I \subseteq \{1, \dots, m\},\ a = (a^1, \dots, a^l),\ a^i \in \mathbb{R}^m,\ q = (q_1, \dots, q_m),\ q_j \in X^*,$$
$$\lambda = (\lambda_1, \dots, \lambda_l)^T \in \mathbb{R}^l,\ t = (t_1, \dots, t_l)^T \in \mathbb{R}^l,$$

and the set of constraints

$$\begin{aligned} Y_\Phi(\mathcal{A}) = \Big\{ (I, a, q, \lambda, t) : & I \subseteq \{1, ..., m\},\ a^i \in \mathbb{R}_+^m,\ \Phi_i^0(a^i) \leq 1,\ i = 1, \dots, l, \\ & q_j \in X^*,\ j = 1, \dots, m,\ \left( \sum_{i=1}^{l} \lambda_i a_j^i \right) > 0,\ j \in I,\ a_j^i = 0,\ j \notin I, \\ & i = 1, \dots, l, \sum_{j \in I} \left( \sum_{i=1}^{l} \lambda_i a_j^i \right) q_j = 0, \lambda \in int\mathbb{R}_+^l, \sum_{k=1}^{l} \lambda_k t_k = 0 \Big\}. \end{aligned}$$

The following theorems state the weak and strong duality assertions applying Theorem 6 and Theorem 7.

**Theorem 10.** *There is no $x \in X$ and no $(I, a, q, \lambda, t) \in Y_\Phi(\mathcal{A})$, such that $\Phi_i(d(x)) \leq h_i^d(I, a, q, \lambda, t),\ i = 1, \dots, l,$ and $\Phi_k(d(x)) < h_k^d(I, a, q, \lambda, t)$ for at least one $k \in \{1, \dots, l\}$.*

**Theorem 11.** *Let $\bar{x}$ be a properly efficient element to $(P_\Phi(\mathcal{A}))$. Then there exists an efficient solution $(\bar{I}, \bar{a}, \bar{q}, \bar{\lambda}, \bar{t}) \in Y_\Phi(\mathcal{A})$ to $(D_\Phi(\mathcal{A}))$ and strong duality*

$$\Phi_k(d(\bar{x})) = -\frac{1}{l\bar{\lambda}_k} \sum_{j \in \bar{I}} \left( \sum_{i=1}^{l} \bar{\lambda}_i \bar{a}_j^i \right) d_j^*(\bar{q}_j) + \bar{t}_k,\ k = 1, ..., l,$$

*holds.*

# 6 The biobjective Weber-minimax problem with infimal distances

In this section, for the same data set $\mathcal{A} = \{A_1, \ldots, A_m\}$ as in the previous one, we consider a multiobjective minimization problem with a two-dimensional objective function, its first component being given by the Weber location problem and the second one by the minimax location problem with infimal distances. Thus, the primal problem is

$$(P_{WM}(\mathcal{A})) \quad \underset{x \in X}{\text{v-min}} \begin{pmatrix} \sum\limits_{j=1}^{m} w_j d_j(x, A_j) \\ \max\limits_{j=1,\ldots,m} w_j d_j(x, A_j) \end{pmatrix},$$

where $d_j(x, A_j) = \inf\limits_{y_j \in A_j} \gamma_j(x - y_j)$, $j = 1, ..., m$, and $w_j > 0$, $j = 1, \ldots, m$, are positive weights. Let be, for $j = 1, \ldots, m$, the continuous norms $\gamma_j' : X \to \mathbb{R}$, $\gamma_j' = w_j \gamma_j$ and the corresponding distance functions $d_j'(\cdot, A_j) : X \to \mathbb{R}$, $d_j'(x, A_j) = \inf\limits_{y_j \in A_j} \gamma_j'(x - y_j) = w_j d_j(x, A_j)$. This means that the primal problem $(P_{WM}(\mathcal{A}))$, as a special case of $(P_\Phi(\mathcal{A}))$ in section 5 with $\Phi_1 = l_1$ and $\Phi_2 = l_2$, becomes

$$(P_{WM}(\mathcal{A})) \quad \underset{x \in X}{\text{v-min}} \begin{pmatrix} l_1(d'(x)) \\ l_\infty(d'(x)) \end{pmatrix},$$

with $d'(x) = (d_1'(x, A_1), ..., d_m'(x, A_m))$ and the norms $l_1, l_\infty : \mathbb{R}^m \to \mathbb{R}$, $l_1(z) = \sum\limits_{j=1}^{m} |z_j|, l_\infty(z) = \max\limits_{j=1,\ldots,m} |z_j|$, for $z \in \mathbb{R}^m$. We remark that $l_1^0(z) = l_\infty(z)$ and $l_\infty^0(z) = l_1(z)$. Obviously, $l_1$ and $l_\infty$ are monotonic norms.

Taking into consideration the form of $D_\Phi(\mathcal{A})$ in section 5, observing $d_j'^*(q_j) = (w_j d_j)^*(q_j) = w_j d_j^*(\frac{1}{w_j} q_j)$, and, denoting by $q_j := \frac{1}{w_j} q_j$, we construct the multiobjective dual to the primal problem $(P_{WM}(\mathcal{A}))$. This becomes

$$(D_{WM}(\mathcal{A})) \quad \underset{(I, a, q, \lambda, t) \in \mathcal{Y}_{WM}(\mathcal{A})}{\text{v-max}} \begin{pmatrix} h_1(I, a, q, \lambda, t) \\ h_2(I, a, q, \lambda, t) \end{pmatrix},$$

with

$$\begin{aligned} h_1(I, a, q, \lambda, t) = &\ -\tfrac{1}{2\lambda_1} \sum_{j \in I} \left( \sum_{i=1}^{2} \lambda_i a_j^i \right) w_j d_j^*(q_j) + t_1, \\ h_2(I, a, q, \lambda, t) = &\ -\tfrac{1}{2\lambda_2} \sum_{j \in I} \left( \sum_{i=1}^{2} \lambda_i a_j^i \right) w_j d_j^*(q_j) + t_2, \end{aligned}$$

the dual variables

$$I \subseteq \{1, \ldots, m\},\ a = (a^1, a^2),\ a^1,\ a^2 \in \mathbb{R}^m,\ q = (q_1, \ldots, q_m),\ q_j \in X^*,$$

$$\lambda = (\lambda_1, \lambda_2)^T \in \mathbb{R}^2,\ t = (t_1, t_2)^T \in \mathbb{R}^2,$$

and the set of constraints

$$Y_{WM}(\mathcal{A})=\Big\{(I,a,q,\lambda,t): I\subseteq\{1,...,m\}, a^1,a^2\in\mathbb{R}^m_+, q_j\in X^*, j=1,\dots,m,$$
$$\max_{j=1,\dots,m}|a_j^1|\leq 1, \sum_{i=1}^m |a_j^2|\leq 1, \left(\sum_{i=1}^2 \lambda_i a_j^i\right)>0, j\in I, a_j^i=0, j\notin I, i=1,2,$$
$$\sum_{j\in I}\left(\sum_{i=1}^2 \lambda_i a_j^i\right) w_j q_j=0, \lambda\in int\mathbb{R}^2_+, \sum_{k=1}^2 \lambda_k t_k=0\Big\},$$

Let us give also for this problem the weak and strong duality theorems.

**Theorem 12.** *There is no $x\in X$ and no $(I,a,q,\lambda,t)\in Y_{WM}(\mathcal{A})$ such that*

$$\sum_{j=1}^m w_j d_j(x,A_j)\leq h_1(I,a,q,\lambda,t), \max_{j=1,\dots,m} w_j d_j(x,A_j)\leq h_2(I,a,q,\lambda,t)$$

*and*

$$\sum_{j=1}^m w_j d_j(x,A_j)<h_1(I,a,q,\lambda,t) \text{ or } \max_{j=1,\dots,m} w_j d_j(x,A_j)<h_2(I,a,q,\lambda,t).$$

**Theorem 13.** *Let $\bar{x}$ be a properly efficient element to $(P_{WM}(\mathcal{A}))$. Then there exists an efficient solution $(\bar{I},\bar{a},\bar{q},\bar{\lambda},\bar{t})\in Y_{WM}(\mathcal{A})$ to $(D_{WM}(\mathcal{A}))$ and the strong duality holds, i.e.*

$$\sum_{j=1}^m w_j d_j(\bar{x},A_j)=-\frac{1}{2\bar{\lambda}_1}\sum_{j\in\bar{I}}\left(\sum_{i=1}^2 \bar{\lambda}_i\bar{a}_j^i\right) w_j d_j^*(\bar{q}_j)+\bar{t}_1$$

*and*

$$\max_{j=1,\dots,m} w_j d_j(\bar{x},A_j)=-\frac{1}{2\bar{\lambda}_2}\sum_{j\in\bar{I}}\left(\sum_{i=1}^2 \bar{\lambda}_i\bar{a}_j^i\right) w_j d_j^*(\bar{q}_j)+\bar{t}_2.$$

# 7 The multiobjective Weber problem with infimal distances

We consider, as another application of the multiobjective duality results in section 5, the multiobjective Weber problem with infimal distances for the data $\mathcal{A}$

$$(P_W(\mathcal{A}))\quad \text{v-}\min_{x\in X}\begin{pmatrix}\sum_{j=1}^m w_j^1 d_j(x,A_j)\\ \vdots \\ \sum_{j=1}^m w_j^l d_j(x,A_j)\end{pmatrix},$$

where $d_j(x, A_j) = \inf\limits_{y_j \in A_j} \gamma_j(x - y_j),\ j = 1, ..., m$, and $w_j^i$, $j = 1, \ldots, m$, $i = 1, ..., l$, are positive weights. Again, the norms $\gamma_j$, $j = 1, \ldots, m$, are assumed to be continuous. Considering the norms $\Phi_i^W : \mathbb{R}^m \to \mathbb{R},\ i = 1, ..., l$, defined by

$$\Phi_i^W(x) := \sum_{j=1}^{m} w_j^i |x_j|,$$

we have

$$\Phi_i^W(d(x)) = \sum_{j=1}^{m} w_j^i d_j(x, A_j).$$

We notice that $\Phi_i^W, i = 1, ..., l$, are monotonic norms, with the dual norm $(\Phi_i^W)^0(x) = \max\limits_{j=1,\ldots,m} \frac{|x_j|}{w_j^i}$.
So, the primal problem $(P_W(\mathcal{A}))$ becomes

$$(P_W(\mathcal{A})) \qquad \operatorname*{v-min}_{x \in X} \begin{pmatrix} \Phi_1^W(d(x)) \\ \vdots \\ \Phi_l^W(d(x)) \end{pmatrix}.$$

Due to section 5, a multiobjective dual problem to $(P_W(\mathcal{A}))$ is

$$(DP_W(\mathcal{A})) \qquad \operatorname*{v-max}_{(I,a,q,\lambda,t) \in Y_W(\mathcal{A})} \begin{pmatrix} h_1^W(I, a, q, \lambda, t) \\ \vdots \\ h_l^W(I, a, q, \lambda, t) \end{pmatrix},$$

with

$$h_k^W(I, a, q, \lambda, t) = -\frac{1}{l\lambda_k} \sum_{j \in I} \left( \sum_{i=1}^{l} \lambda_i a_j^i \right) d_j^*(q_j) + t_k,\ k = 1, ..., l,$$

the dual variables

$$I \subseteq \{1, \ldots, m\},\ a = (a^1, ..., a^l),\ a^i \in \mathbb{R}^m,\ q = (q_1, ..., q_m),\ q_j \in X^*,$$

$$\lambda = (\lambda_1, ..., \lambda_l)^T \in \mathbb{R}^l,\ t = (t_1, ..., t_l)^T \in \mathbb{R}^l,$$

and the set of constraints

$$\begin{aligned} Y_W(\mathcal{A}) = \Big\{ & (I, a, q, \lambda, t) : I \subseteq \{1, ..., m\},\ a^i \in \mathbb{R}_+^m,\ q_j \in X^*, , j = 1, \ldots, m, \\ & \lambda \in int\mathbb{R}_+^l,\ \max_{j=1,\ldots,m} \frac{a_j^i}{w_j^i} \leq 1,\ \left( \sum_{i=1}^{l} \lambda_i a_j^i \right) > 0, j \in I, a_j^i = 0, j \notin I, i = 1, \ldots, l, \\ & \sum_{j \in I} \left( \sum_{i=1}^{l} \lambda_i a_j^i \right) q_j = 0, \sum_{k=1}^{l} \lambda_k t_k = 0 \Big\}, \end{aligned}$$

which can be written equivalently as (setting $a_j^i := \frac{a_j^i}{w_j^i}$)

$$(DP_W(\mathcal{A})) \qquad \underset{(I,a,q,\lambda,t)\in Y_W(\mathcal{A})}{\text{v-max}} \begin{pmatrix} h_1^W(I,a,q,\lambda,t) \\ \vdots \\ h_l^W(I,a,q,\lambda,t) \end{pmatrix},$$

with

$$h_k^W(I,a,q,\lambda,t) = -\frac{1}{l\lambda_k}\sum_{j\in I}\left(\sum_{i=1}^{l}\lambda_i w_j^i a_j^i\right) d_j^*(q_j) + t_k,\ k = 1,...,l,$$

the dual variables

$$I \subseteq \{1,\dots,m\},\ a = (a^1,...,a^l),\ a^i \in \mathbb{R}^m,\ q = (q_1,...,q_m),\ q_j \in X^*,$$

$$\lambda = (\lambda_1,...,\lambda_l)^T \in \mathbb{R}^l,\ t = (t_1,...,t_l)^T \in \mathbb{R}^l,$$

and the set of constraints

$$\begin{aligned} Y_W(\mathcal{A}) = \Big\{ &(I,a,q,\lambda,t) : I \subseteq \{1,...,m\}, a^i \in \mathbb{R}^m_+,\ q_j \in X^*,\ j = 1,\dots,m, \\ &\lambda \in int\mathbb{R}^l_+,\ \max_{j=1,\dots,m} a_j^i \le 1,\ \left(\sum_{i=1}^{l}\lambda_i w_j^i a_j^i\right) > 0, j \in I, a_j^i = 0, j \notin I, i = 1,\dots,l, \\ &\sum_{j\in I}\left(\sum_{i=1}^{l}\lambda_i w_j^i a_j^i\right) q_j = 0, \sum_{k=1}^{l}\lambda_k t_k = 0\Big\}. \end{aligned}$$

Using the Theorems 10 and 11 we can formulate the following duality results.

**Theorem 14.** *There is no $x \in X$ and no $(I,a,q,\lambda,t)$ in $Y_W(\mathcal{A})$ such that $\sum_{j=1}^{m} w_j^i d_j(x,A_j) \le h_i^W(I,a,q,\lambda,t),\ i = 1,...,l,$ and $\sum_{j=1}^{m} w_j^k d_j(x,A_j) < h_k^W(I,a,q,\lambda,t)$ for at least one $k \in \{1,...,l\}$.*

**Theorem 15.** *Let $\bar{x}$ be a properly efficient element to $(P_W(\mathcal{A}))$. Then there exists an efficient solution $(\bar{I},\bar{a},\bar{q},\bar{\lambda},\bar{t}) \in Y_W(\mathcal{A})$ to $(DP_W(\mathcal{A}))$ and strong duality, i.e.*

$$\sum_{j=1}^{m} w_j^k d_j(\bar{x},A_j) = -\frac{1}{l\bar{\lambda}_k}\sum_{j\in\bar{I}}\left(\sum_{i=1}^{l}\bar{\lambda}_i w_j^i \bar{a}_j^i\right) d_j^*(\bar{q}_j) + \bar{t}_k,\ k = 1,...,l,$$

*holds.*

# 8 The multiobjective minimax location problem with infimal distances

The last optimization problem we are going to consider in this paper is the multiobjective minimax location problem with infimal distances for the data $\mathcal{A}$

$$(P_M(\mathcal{A})) \quad \operatorname*{v-min}_{x\in X} \begin{pmatrix} \max\limits_{j=1,\dots,m} w_j^1 d_j(x,A_j) \\ \vdots \\ \max\limits_{j=1,\dots,m} w_j^l d_j(x,A_j) \end{pmatrix},$$

where $d_j(x,A_j) = \inf\limits_{y_j\in A_j} \gamma_j(x-y_j),\ j=1,...,m$, and $w_j^i$, $j=1,\dots,m$, $i=1,...,l$, are positive weights. Considering the norms $\Phi_i^M : \mathbb{R}^m \to \mathbb{R},\ i=1,...,l$, defined by

$$\Phi_i^M(x) = \max_{j=1,\dots,m} w_j^i |x_j|,$$

we have that

$$\Phi_i^M(d(x)) = \max_{j=1,\dots,m} w_j^i d_j(x,A_j).$$

We notice that $\Phi_i^M, i = 1,...,l$, are monotonic norms, with the dual norm $(\Phi_i^M)^0(x) = \sum\limits_{j=1}^m \frac{|x_j|}{w_j^i}$.

Thus, the primal problem $(P_M(\mathcal{A}))$ becomes

$$(P_M(\mathcal{A})) \quad \operatorname*{v-min}_{x\in X} \begin{pmatrix} \Phi_1^M(d(x)) \\ \vdots \\ \Phi_l^M(d(x)) \end{pmatrix}.$$

Its multiobjective dual problem is (cf. section 5)

$$(DP_M(\mathcal{A})) \quad \operatorname*{v-max}_{(I,a,q,\lambda,t)\in Y_M(\mathcal{A})} \begin{pmatrix} h_1^M(I,a,q,\lambda,t) \\ \vdots \\ h_l^M(I,a,q,\lambda,t) \end{pmatrix},$$

with

$$h_k^M(I,a,q,\lambda,t) = -\frac{1}{l\lambda_k} \sum_{j\in I} \left( \sum_{i=1}^l \lambda_i a_j^i \right) d_j^*(q_j) + t_k,\ k=1,...,l,$$

the dual variables

$$I \subseteq \{1,\dots,m\},\ a=(a^1,...,a^l),\ a^i\in\mathbb{R}^m,\ q=(q_1,...,q_m),\ q_j\in X^*,$$

$$\lambda = (\lambda_1,...,\lambda_l)^T \in \mathbb{R}^l,\ t=(t_1,...,t_l)^T\in\mathbb{R}^l,$$

and the set of constraints

$$Y_M(\mathcal{A}) = \Big\{(I,a,q,\lambda,t) : I \subseteq \{1,...,m\}, a^i \in \mathbb{R}^m_+,\ q_j \in X^*,\ j=1,\dots,m,$$
$$\sum_{j=1}^m \frac{a^i_j}{w^i_j} \leq 1, \left(\sum_{i=1}^l \lambda_i a^i_j\right) > 0, j \in I,\ a^i_j = 0,\ j \notin I,\ i=1,\dots,l,$$
$$\sum_{j\in I}(\sum_{i=1}^l \lambda_i a^i_j) q_j = 0, \lambda \in int\mathbb{R}^l_+, \sum_{k=1}^l \lambda_k t_k = 0\Big\},$$

which can be written equivalently as (setting $a^i_j := \frac{a^i_j}{w^i_j}$)

$$(DP_M(\mathcal{A})) \qquad \underset{(I,a,q,\lambda,t)\in Y_M(\mathcal{A})}{\text{v-max}} \begin{pmatrix} h^M_1(I,a,q,\lambda,t) \\ \vdots \\ h^M_l(I,a,q,\lambda,t) \end{pmatrix},$$

with

$$h^M_k(I,a,q,\lambda,t) = -\frac{1}{l\lambda_k}\sum_{j\in I}\left(\sum_{i=1}^l \lambda_i w^i_j a^i_j\right) d^*_j(q_j) + t_k,\ \ k=1,...,l,$$

the dual variables

$$I \subseteq \{1,\dots,m\},\ a=(a^1,...,a^l),\ a^i \in \mathbb{R}^m,\ \ q=(q_1,...,q_m),\ q_j \in X^*,$$
$$\lambda = (\lambda_1,...,\lambda_l)^T, \in \mathbb{R}^l,\ \ t=(t_1,...,t_l)^T \in \mathbb{R}^l,$$

and the set of constraints

$$Y_M(\mathcal{A}) = \Big\{(I,a,q,\lambda,t) : I \subseteq \{1,...,m\}, a^i \in \mathbb{R}^m_+,\ q_j \in X^*,\ j=1,\dots,m,$$
$$\sum_{j=1}^m a^i_j \leq 1, \left(\sum_{i=1}^l \lambda_i w^i_j a^i_j\right) > 0, j \in I,\ a^i_j = 0,\ j \notin I,\ i=1,\dots,l,$$
$$\sum_{j\in I}\left(\sum_{i=1}^l \lambda_i w^i_j a^i_j\right) q_j = 0,\ \lambda \in int\mathbb{R}^l_+, \sum_{k=1}^l \lambda_k t_k = 0\Big\}.$$

*Remark 4.* We emphazise the interesting observation that both dual problems $(DP_W(\mathcal{A}))$ and $(DP_M(\mathcal{A}))$ differ only in the constraints $\max\limits_{i=1,...,m} a^i_j \leq 1$ and $\sum\limits_{i=1}^m a^i_j \leq 1$, respectively.

The corresponding duality results for $(DP_M(\mathcal{A}))$ are the following.

**Theorem 16.** *There is no $x \in X$ and no $(I,a,q,\lambda,t) \in Y_M(\mathcal{A})$ such that $\max\limits_{j=1,...,m} w^i_j d_j(x,A_j) \leq h^M_i(I,a,q,\lambda,t)$, $i=1,...,l$, and $\max\limits_{j=1,...,m} w^k_j d_j(x,A_j) < h^M_k(I,a,q,\lambda,t)$ for at least one $k \in \{1,...,l\}$.*

**Theorem 17.** *Let $\bar{x}$ be a properly efficient element to $(P_M(\mathcal{A}))$. Then there exists an efficient solution $(\bar{I}, \bar{a}, \bar{q}, \bar{\lambda}, \bar{t}) \in Y_M(\mathcal{A})$ to $(DP_M(\mathcal{A}))$ and strong duality, i.e.*

$$\max_{j=1,\ldots,m} w_j^k d_j(\bar{x}, A_j) = -\frac{1}{l\bar{\lambda}_k} \sum_{j \in \bar{I}} \left( \sum_{i=1}^{l} \bar{\lambda}_i w_j^i \bar{a}_j^i \right) d_j^*(\bar{q}_j) + \bar{t}_k, k = 1, ..., l,$$

*holds.*

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