Chapter 1
A comparison of some recent regularity conditions for Fenchel duality

Radu Ioan Boţ and Ernö Robert Csetnek

Abstract This article provides an overview on regularity conditions for Fenchel duality in convex optimization. Our attention is focused, on the one hand, on three generalized interior-point regularity conditions expressed by means of the quasi interior and of the quasi-relative interior and, on the other hand, on two closedness-type conditions that have been recently introduced in the literature. We discuss how they do relate to each other, but also to several other classical ones and illustrate these investigations by numerous examples.

Key words: convex optimization, Fenchel duality, quasi interior, quasi-relative interior, generalized interior-point regularity conditions, closedness-type regularity conditions
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1.1 Introduction

The primal problem we investigate in this section is an unrestricted optimization problem having as objective function the sum of two proper and convex functions defined on a separated locally convex space. To it we attach the *Fenchel dual* problem and further we concentrate ourselves on providing regularity conditions for *strong duality* for this primal-dual pair, which is the situation when the optimal objective values of the two problems coincide and the dual has an optimal solution. First of all, we bring into the discussion several conditions of this kind that one can

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find in the literature, where along the one which asks for the *continuity* of one of the two functions at a point from the intersection of the effective domains, we enumerate some classical generalized interior-point ones. Here we refer to the regularity conditions employing the *interior*, but also the *algebraic interior* (cf. [21]), the *intrinsic core* (cf. [17]) and the *strong-quasi relative interior* (cf. [1, 24]) of the difference of the domains of the two functions. The latter conditions guarantee strong duality if we suppose additionally that the two functions are lower semicontinuous and the space we work within is a Fréchet one. A general scheme containing the relations between these sufficient conditions is also furnished.

The central role in the paper is played by some regularity conditions for Fenchel duality recently introduced in the literature. First of all, we consider some regularity conditions expressed via the *quasi interior* and *quasi-relative interior* (cf. [7,8]), which presents the advantage that they do not ask for any topological assumption regarding the functions involved and work in general separated locally convex spaces. We consider three conditions of this kind, relate them to each other, but also to the classical ones mentioned above. By means of some examples we are able to underline their wider applicability, by providing optimization problems where these are fulfilled, while the consecrated ones fail.

The second class of recently introduced regularity conditions we discuss here is the one of the so-called *closedness-type regularity conditions*, which additionally ask for lower semicontinuity for the two functions, but work in general separated locally convex spaces, too. We discuss here two closedness-type conditions (cf. [9, 10]), we relate them to each other, to the classical interior-point ones, but also, more important, to the ones expressed via the quasi interior and quasi-relative interior. More precisely, we show that, unlike in finite-dimensional spaces, in the infinite-dimensional setting these two classes of regularity conditions for Fenchel duality are not comparable. In this way we give a negative answer to an open problem stated in [19, Remark 4.3].

The paper is organized as follows. In Section 1.2 we introduce some elements of convex analysis, whereby the accent is put on different generalized interiority notions. The notions quasi interior and quasi-relative interior are also introduced and some of their important properties are mentioned. The third section starts with the definition of the Fenchel dual problem, followed by a subsection dedicated to the classical interior-point regularity conditions. The second subsection of Section 1.3 deals with the new conditions expressed via the quasi interior and quasi-relative interior, while in the third one the closedness-type conditions are studied.

1.2 Preliminary notions and results

Consider X a (real) separated locally convex space and X^* its topological dual space. We denote by $w(X^*,X)$ the weak* topology on X^* induced by X. For a nonempty set $U \subseteq X$, we denote by $\operatorname{co}(U), \operatorname{cone}(U), \operatorname{coneco}(U), \operatorname{aff}(U), \operatorname{lin}(U), \operatorname{int}(U)$, $\operatorname{cl}(U)$, its *convex hull, conic hull, convex conic hull, affine hull, linear hull, interior* and *closure*, respectively. In case U is a linear subspace of X we denote by $U^{\perp}$ the *annihilator* of U. Let us mention the following property: if U is convex then

$$\operatorname{coneco}(U \cup \{0\}) = \operatorname{cone}(U). \tag{1.1}$$

If $U \subseteq \mathbb{R}^n$ ($n \in \mathbb{N}$) we denote by $\operatorname{ri}(U)$ the *relative interior* of U, that is the interior of U with respect to its affine hull. We denote by $\langle x^*, x\rangle$ the value of the linear continuous functional $x^* \in X^*$ at $x \in X$ and by $\ker x^*$ the *kernel* of x^*. The *indicator function* of U, $\delta_U : X \to \overline{\mathbb{R}}$, is defined as

$$\delta_U(x) = \begin{cases} 0, & \text{if } x \in U, \\ +\infty, & \text{otherwise,} \end{cases}$$

where $\overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$ is the extended real line. We make the following conventions: $(+\infty) + (-\infty) = +\infty$, $0 \cdot (+\infty) = +\infty$ and $0 \cdot (-\infty) = 0$. For a function $f : X \to \overline{\mathbb{R}}$ we denote by $\operatorname{dom} f = \{x \in X : f(x) < +\infty\}$ the *domain* of f and by $\operatorname{epi} f = \{(x,r) \in X \times \mathbb{R} : f(x) \le r\}$ its *epigraph*. Moreover, we denote by $\widehat{\operatorname{epi}}(f) = \{(x,r) \in X \times \mathbb{R} : (x,-r) \in \operatorname{epi} f\}$, the symmetric of $\operatorname{epi} f$ with respect to the x-axis. For a given real number α, $f - \alpha : X \to \overline{\mathbb{R}}$ is, as usual, the function defined by $(f-\alpha)(x) = f(x) - \alpha$ for all $x \in X$. We call f *proper* if $\operatorname{dom} f \neq \emptyset$ and $f(x) > -\infty$ for all $x \in X$. The *normal cone* of U at $x \in U$ is $N_U(x) = \{x^* \in X^* : \langle x^*, y - x\rangle \le 0 \ \forall y \in U\}$.

The *Fenchel-Moreau conjugate* of f is the function $f^* : X^* \to \overline{\mathbb{R}}$ defined by

$$f^*(x^*) = \sup_{x \in X} \{\langle x^*, x\rangle - f(x)\} \ \forall x^* \in X^*.$$

We have the so-called *Young-Fenchel inequality*

$$f^*(x^*) + f(x) \ge \langle x^*, x\rangle \ \forall x \in X \ \forall x^* \in X^*.$$

Having $f, g : X \to \overline{\mathbb{R}}$ two functions we denote by $f \square g : X \to \overline{\mathbb{R}}$ their *infimal convolution*, defined by $f \square g(x) = \inf_{u \in X}\{f(u) + g(x-u)\}$ for all $x \in X$. We say that the infimal convolution is *exact at* $x \in X$ if the infimum in its definition is attained. Moreover, $f \square g$ is said to be *exact* if it is exact at every $x \in X$.

Let us recall in the following the most important generalized interiority notions introduced in the literature. The set $U \subseteq X$ is supposed to be nonempty and convex. We have:

- $\operatorname{core}(U) := \{x \in U : \operatorname{cone}(U - x) = X\}$, the *algebraic interior* (the *core*) of U (cf. [21, 26]);
- $\operatorname{icr}(U) := \{x \in U : \operatorname{cone}(U - x) \text{ is a linear subspace of } X\}$, the *relative algebraic interior* (*intrinsic core*) of U (cf. [2, 18, 26]);
- $\operatorname{sqri}(U) := \{x \in U : \operatorname{cone}(U - x) \text{ is a closed linear subspace of } X\}$ the *strong quasi-relative interior* (*intrinsic relative algebraic interior*) of U (cf. [3, 26]).

We mention the following characterization of the strong quasi-relative interior (cf. [17, 26]): $x \in \operatorname{sqri}(U) \Leftrightarrow x \in \operatorname{icr}(U)$ and $\operatorname{aff}(U - x)$ is a closed linear subspace.

The *quasi-relative interior* of U is the set (cf. [4])

$$\operatorname{qri}(U) = \{x \in U : \operatorname{cl}\big(\operatorname{cone}(U - x)\big) \text{ is a linear subspace of } X\}.$$

The quasi-relative interior of a convex set is characterized by means of the normal cone as follows.

Proposition 1. *(cf. [4]) Let U be a nonempty convex subset of X and $x \in U$. Then $x \in \operatorname{qri}(U)$ if and only if $N_U(x)$ is a linear subspace of X^*.*

Next we consider another generalized interiority notion introduced in connection with a convex set, which is close to the quasi-relative interior. The *quasi interior* of U is the set

$$\operatorname{qi}(U) = \{x \in U : \operatorname{cl}\big(\operatorname{cone}(U - x)\big) = X\}.$$

It can be characterized as follows.

Proposition 2. *(cf. [8, Proposition 2.4]) Let U be a nonempty convex subset of X and $x \in U$. Then $x \in \operatorname{qi}(U)$ if and only if $N_U(x) = \{0\}$.*

Remark 1. The above characterization of the quasi interior of a convex set was given in [16], where the authors supposed that X is a reflexive Banach space. It is proved in [8, Proposition 2.4] that this property holds in a more general context, namely in separated locally convex spaces.

We have the following relations between the different generalized interiority notions considered above

$$\operatorname{int}(U) \subseteq \operatorname{core}(U) \subseteq \begin{array}{c} \operatorname{sqri}(U) \subseteq \operatorname{icr}(U) \\ \\ \operatorname{qi}(U) \end{array} \subseteq \operatorname{qri}(U) \subseteq U, \tag{1.2}$$

all the inclusions being in general strict. As one can also deduce from some of the examples which follows in this paper in general between $\operatorname{sqri}(U)$ and $\operatorname{icr}(U)$, on the one hand, and $\operatorname{qi}(U)$, on the other hand, no relation of inclusion can be provided. In case $\operatorname{int}(U) \neq \emptyset$ all the generalized interior-notions considered in (1.2) collapse into $\operatorname{int}(U)$ (cf. [4, Corollary 2.14]).

It follows from the definition of the quasi-relative interior that $\operatorname{qri}(\{x\}) = \{x\}$ for all $x \in X$. Moreover, if $\operatorname{qi}(U) \neq \emptyset$, then $\operatorname{qi}(U) = \operatorname{qri}(U)$. Although this property is given in [20] in the case of real normed spaces, it holds also in separated locally convex spaces, as it easily follows from the properties given above. For U, V two convex subsets of X such that $U \subseteq V$, we have $\operatorname{qi}(U) \subseteq \operatorname{qi}(V)$, a property which is no longer true for the quasi-relative interior (however this holds whenever $\operatorname{aff}(U) = \operatorname{aff}(V)$, see [13, Proposition 1.12]). If X if finite-dimensional then $\operatorname{qri}(U) = \operatorname{sqri}(U) = \operatorname{icr}(U) = \operatorname{ri}(U)$ (cf. [4, 17]) and $\operatorname{core}(U) = \operatorname{qi}(U) = \operatorname{int}(U)$ (cf. [20, 21]). We refer the reader to [2, 4, 17, 18, 20, 21, 23, 26] and to the references therein for more properties and examples regarding the above considered generalized interiority notions.

Example 1. Take an arbitrary $p \in [1,+\infty)$ and consider the real Banach space $\ell^p = \ell^p(\mathbb{N})$ of real sequences $(x_n)_{n\in\mathbb{N}}$ such that $\sum_{n=1}^{\infty}|x_n|^p < +\infty$, equipped with the norm $\|\cdot\| : \ell^p \to \mathbb{R}$, $\|x\| = \left(\sum_{n=1}^{\infty}|x_n|^p\right)^{1/p}$ for all $x = (x_n)_{n\in\mathbb{N}} \in \ell^p$. Then (cf. [4])

$$\mathrm{qri}(\ell^p_+) = \{(x_n)_{n\in\mathbb{N}} \in \ell^p : x_n > 0\ \forall n \in \mathbb{N}\},$$

where $\ell^p_+ = \{(x_n)_{n\in\mathbb{N}} \in \ell^p : x_n \geq 0\ \forall n \in \mathbb{N}\}$ is the positive cone of ℓ^p. Moreover, one can prove that

$$\mathrm{int}(\ell^p_+) = \mathrm{core}(\ell^p_+) = \mathrm{sqri}(\ell^p_+) = \mathrm{icr}(\ell^p_+) = \emptyset.$$

In the setting of separable Banach spaces every nonempty closed convex set has a nonempty quasi-relative interior (cf. [4, Theorem 2.19], see also [2, Theorem 2.8] and [26, Proposition 1.2.9]) and every nonempty convex set which is not contained in a hyperplane possesses a nonempty quasi interior (cf. [20]). This result may fail if the condition X is separable is removed, as the following example shows.

Example 2. For $p \in [1,+\infty)$ consider the real Banach space

$$\ell^p(\mathbb{R}) = \{s : \mathbb{R} \to \mathbb{R} \mid \sum_{r\in\mathbb{R}} |s(r)|^p < \infty\},$$

equipped with the norm $\|\cdot\| : \ell^p(\mathbb{R}) \to \mathbb{R}$, $\|s\| = \left(\sum_{r\in\mathbb{R}}|s(r)|^p\right)^{1/p}$ for all $s \in \ell^p(\mathbb{R})$, where

$$\sum_{r\in\mathbb{R}} |s(r)|^p = \sup_{F\subseteq\mathbb{R}, F \text{ finite}} \sum_{r\in F} |s(r)|^p.$$

Considering the positive cone $\ell^p_+(\mathbb{R}) = \{s \in \ell^p(\mathbb{R}) : s(r) \geq 0\ \forall r \in \mathbb{R}\}$, we have (cf. [4, Example 3.11(iii)], see also [5, Remark 2.20]) that $\mathrm{qri}\left(\ell^p_+(\mathbb{R})\right) = \emptyset$.

Let us mention some properties of the quasi-relative interior. For the proof of $(i)-(ii)$ we refer to [2, 4], while property (iii) was proved in [8, Proposition 2.5] (see also [7, Proposition 2.3]).

Proposition 3. *Consider U a nonempty convex subset of X. Then:*

(i) $t\,\mathrm{qri}(U) + (1-t)U \subseteq \mathrm{qri}(U)\ \forall t \in (0,1]$*; hence* $\mathrm{qri}(U)$ *is a convex set.*

If, additionally, $\mathrm{qri}(U) \neq \emptyset$ *then:*

(ii) $\mathrm{cl}\left(\mathrm{qri}(U)\right) = \mathrm{cl}(U)$*;*

(iii) $\mathrm{cl}\Big(\mathrm{cone}\left(\mathrm{qri}(U)\right)\Big) = \mathrm{cl}\left(\mathrm{cone}(U)\right)$.

The first part of the next lemma was proved in [8, Lemma 2.6] (see also [7, Lemma 2.1]).

Lemma 1. *Let U and V be nonempty convex subsets of X and $x \in X$. Then:*

(i) if $\mathrm{qri}(U) \cap V \neq \emptyset$ *and* $0 \in \mathrm{qi}(U-U)$*, then* $0 \in \mathrm{qi}(U-V)$*;*

(ii) $x \in \operatorname{qi}(U)$ *if and only if* $x \in \operatorname{qri}(U)$ *and* $0 \in \operatorname{qi}(U-U)$.

Proof. (ii) Suppose that $x \in \operatorname{qi}(U)$. Then $x \in \operatorname{qri}(U)$ and since $U - x \subseteq U - U$ and $0 \in \operatorname{qi}(U-x)$, the direct implication follows. The reverse one follows as a direct consequence of (i) by taking $V := \{x\}$. $\square$

Remark 2. Consider the setting of Example 1. By applying the previous result we get (since $\ell^p_+ - \ell^p_+ = \ell^p$) that

$$\operatorname{qi}(\ell^p_+) = \operatorname{qri}(\ell^p_+) = \{(x_n)_{n\in\mathbb{N}} \in \ell^p : x_n > 0\ \forall n \in \mathbb{N}\}.$$

The proof of the duality theorem presented in the next section is based on the following separation theorem.

Theorem 1. *(cf. [8, Theorem 2.7]) Let U be a nonempty convex subset of X and $x \in U$. If $x \notin \operatorname{qri}(U)$, then there exists $x^* \in X^*, x^* \neq 0$, such that*

$$\langle x^*, y\rangle \leq \langle x^*, x\rangle\ \forall y \in U.$$

Viceversa, if there exists $x^* \in X^*$, $x^* \neq 0$, such that

$$\langle x^*, y\rangle \leq \langle x^*, x\rangle\ \forall y \in U$$

and

$$0 \in \operatorname{qi}(U-U),$$

then $x \notin \operatorname{qri}(U)$.

Remark 3. (a) The above separation theorem is a generalization to separated locally convex spaces of a result stated in [15, 16] in the framework of real normed spaces (cf. [8, Remark 2.8]).

(b) The condition $x \in U$ in Theorem 1 is essential (see [16, Remark 2]). However, if x is an arbitrary element of X, an alternative separation theorem has been given by Cammaroto and Di Bella in [12, Theorem 2.1]. Let us mention that some strict separation theorems involving the quasi-relative interior can be found in [13].

1.3 Fenchel duality

Let us briefly recall some considerations regarding Fenchel duality. We deal in the following with the following optimization problem

$$(P_F)\ \inf_{x\in X}\{f(x)+g(x)\},$$

where X is a separated locally convex space and $f, g : X \to \overline{\mathbb{R}}$ are proper functions such that $\operatorname{dom} f \cap \operatorname{dom} g \neq \emptyset$.

The classical *Fenchel dual problem* to (P_F) has the following form

$$(D_F) \quad \sup_{y^* \in X^*} \{-f^*(-y^*) - g^*(y^*)\}.$$

We denote by $v(P_F)$ and $v(D_F)$ the optimal objective values of the primal and dual problems, respectively. Weak duality always holds, that is $v(P_F) \geq v(D_F)$. In order to guarantee strong duality, the situation when $v(P_F) = v(D_F)$ and (D_F) has an optimal solution, several regularity conditions were introduced in the literature.

1.3.1 Classical interior-point regularity conditions

In this subsection we deal with generalized interior-point regularity conditions, by enumerating the classical ones existing in the literature and by studying the relations between them. Let us start by recalling the most known conditions of this type:

(RC_1^F) | $\exists x' \in \operatorname{dom} f \cap \operatorname{dom} g$ such that f (or g) is continuous at x';

(RC_2^F) | X is a Fréchet space, f and g are lower semicontinuous and $0 \in \operatorname{int}(\operatorname{dom} f - \operatorname{dom} g)$;

(RC_3^F) | X is a Fréchet space, f and g are lower semicontinuous and $0 \in \operatorname{core}(\operatorname{dom} f - \operatorname{dom} g)$;

(RC_4^F) | X is a Fréchet space, f and g are lower semicontinuous, $\operatorname{aff}(\operatorname{dom} f - \operatorname{dom} g)$ is a closed linear subspace of X and $0 \in \operatorname{icr}(\operatorname{dom} f - \operatorname{dom} g)$

and

(RC_5^F) | X is a Fréchet space, f and g are lower semicontinuous and $0 \in \operatorname{sqri}(\operatorname{dom} f - \operatorname{dom} g)$.

The condition (RC_3^F) was considered by Rockafellar (cf. [21]), (RC_5^F) by Attouch and Brézis (cf. [1]) and Zălinescu (cf. [24]), while Gowda and Teboulle proved that (RC_4^F) and (RC_5^F) are equivalent (cf. [17]).

Theorem 2. *Let $f, g : X \to \overline{\mathbb{R}}$ be proper and convex functions. If one of the regularity conditions (RC_i^F), $i \in \{1,2,3,4,5\}$, is fulfilled, then $v(P_F) = v(D_F)$ and (D_F) has an optimal solution.*

Remark 4. In case X is a Fréchet space and f, g are proper, convex and lower semicontinuous functions we have the following relations between the above regularity conditions (see also [17, 25] and [26, Theorem 2.8.7])

$$(RC_1^F) \Rightarrow (RC_2^F) \Leftrightarrow (RC_3^F) \Rightarrow (RC_4^F) \Leftrightarrow (RC_5^F).$$

Let us notice that the regularity conditions (RC_2^F) and (RC_3^F) are equivalent. Indeed, assume that X is a Fréchet space, f, g are proper, convex and lower semicontinuous functions such that $\operatorname{dom} f \cap \operatorname{dom} g \neq \emptyset$ and consider the *infimal value function* $h : X \to \overline{\mathbb{R}}$, defined by $h(y) = \inf_{x \in X}\{f(x) + g(x-y)\}$ for all $y \in X$. The function h is convex and not necessarily lower semicontinuous, while one has that $\operatorname{dom} h = \operatorname{dom} f - \operatorname{dom} g$. Nevertheless, the function $(x,y) \mapsto f(x) + g(x-y)$ is ideally convex (being convex and lower semicontinuous), hence h is li-convex (cf. [26, Proposition 2.2.18]). Now by [26, Theorem 2.2.20] it follows that $\operatorname{core}(\operatorname{dom} h) = \operatorname{int}(\operatorname{dom} h)$, which has as consequence the equivalence of the regularity conditions (RC_2^F) and (RC_3^F). Let us mention that this fact has been noticed in the setting of Banach spaces by S. Simons in [22, Corollary 14.3].

1.3.2 Interior-point regularity conditions expressed via quasi interior and quasi-relative interior

Taking into account the relations that exist between the generalized interiority notions presented in Section 1.2 a natural question arises: is the condition $0 \in \operatorname{qri}(\operatorname{dom} f - \operatorname{dom} g)$ sufficient for strong duality? The following example (which can be found in [17]) shows that even if we impose a stronger condition, namely $0 \in \operatorname{qi}(\operatorname{dom} f - \operatorname{dom} g)$, the above question has a negative answer and this means that we need to look for additional assumptions in order to guarantee Fenchel duality.

Example 3. Consider the Hilbert space $X = \ell^2(\mathbb{N})$ and the sets

$$C = \{(x_n)_{n \in \mathbb{N}} \in \ell^2 : x_{2n-1} + x_{2n} = 0 \ \forall n \in \mathbb{N}\}$$

and

$$S = \{(x_n)_{n \in \mathbb{N}} \in \ell^2 : x_{2n} + x_{2n+1} = 0 \ \forall n \in \mathbb{N}\},$$

which are closed linear subspaces of ℓ^2 and satisfy $C \cap S = \{0\}$. Define the functions $f, g : \ell^2 \to \overline{\mathbb{R}}$ by $f = \delta_C$ and $g(x) = x_1 + \delta_S(x)$, respectively, for all $x = (x_n)_{n \in \mathbb{N}} \in \ell^2$. One can see that f and g are proper, convex and lower semicontinuous functions with $\operatorname{dom} f = C$ and $\operatorname{dom} g = S$. As $v(P_F) = 0$ and $v(D_F) = -\infty$ (cf. [17, Example 3.3]), there is a duality gap between the optimal objective values of the primal problem and its Fenchel dual problem. Moreover, $S - C$ is dense in ℓ^2 (cf. [17]), thus $\operatorname{cl}\big(\operatorname{cone}(\operatorname{dom} f - \operatorname{dom} g)\big) = \operatorname{cl}(C - S) = \ell^2$. The last relation implies $0 \in \operatorname{qi}(\operatorname{dom} f - \operatorname{dom} g)$, hence $0 \in \operatorname{qri}(\operatorname{dom} f - \operatorname{dom} g)$.

We notice that if $v(P_F) = -\infty$, by the weak duality result follows that for the primal-dual pair $(P_F) - (D_F)$ strong duality holds. This is the reason why we suppose in what follows that $v(P_F) \in \mathbb{R}$.

Consider now the following regularity conditions expressed by means of the quasi interior and quasi-relative interior:

$$(RC_6^F) \left| \begin{array}{l} \operatorname{dom} f \cap \operatorname{qri}(\operatorname{dom} g) \neq \emptyset, \ 0 \in \operatorname{qi}(\operatorname{dom} g - \operatorname{dom} g) \text{ and} \\ (0,0) \notin \operatorname{qri}\Big[\operatorname{co}\Big((\operatorname{epi} f - \widehat{\operatorname{epi}}(g - v(P_F))) \cup \{(0,0)\}\Big)\Big]; \end{array} \right.$$

$$(RC_7^F)\ \left|\ \begin{array}{l} 0 \in \operatorname{qi}(\operatorname{dom} f - \operatorname{dom} g) \text{ and} \\ (0,0) \notin \operatorname{qri}\Big[\operatorname{co}\Big((\operatorname{epi} f - \widehat{\operatorname{epi}}(g - v(P_F))) \cup \{(0,0)\}\Big)\Big] \end{array}\right.$$

and

$$(RC_8^F)\ \left|\ \begin{array}{l} 0 \in \operatorname{qi}\big[(\operatorname{dom} f - \operatorname{dom} g) - (\operatorname{dom} f - \operatorname{dom} g)\big],\ 0 \in \operatorname{qri}(\operatorname{dom} f - \operatorname{dom} g) \text{ and} \\ (0,0) \notin \operatorname{qri}\Big[\operatorname{co}\Big((\operatorname{epi} f - \widehat{\operatorname{epi}}(g - v(P_F))) \cup \{(0,0)\}\Big)\Big]. \end{array}\right.$$

Let us notice that these three regularity conditions were first introduced in [8]. We study in the following the relations between these conditions. We remark that

$$\operatorname{epi} f - \widehat{\operatorname{epi}}(g - v(P_F)) =$$

$$\{(x-y, f(x)+g(y)-v(P_F)+\varepsilon) : x \in \operatorname{dom} f, y \in \operatorname{dom} g, \varepsilon \geq 0\},$$

thus if the set $\operatorname{epi} f - \widehat{\operatorname{epi}}(g - v(P_F)$ is convex, then $\operatorname{dom} f - \operatorname{dom} g$ is convex, too.

Proposition 4. *Let $f,g : X \to \overline{\mathbb{R}}$ be proper functions such that $v(P_F) \in \mathbb{R}$ and $\operatorname{epi} f - \widehat{\operatorname{epi}}(g - v(P_F))$ is a convex subset of $X \times \mathbb{R}$ (the latter is the case if for instance f and g are convex functions). The following statements are true:*

(i) $(RC_7^F) \Leftrightarrow (RC_8^F)$*; if, moreover, f and g are convex, then* $(RC_6^F) \Rightarrow (RC_7^F) \Leftrightarrow (RC_8^F)$*;*

(ii) if (P_F) has an optimal solution, then $(0,0) \notin \operatorname{qri}\Big[\operatorname{co}\Big((\operatorname{epi} f - \widehat{\operatorname{epi}}(g - v(P_F))) \cup \{(0,0)\}\Big)\Big]$ *can be equivalently written as* $(0,0) \notin \operatorname{qri}\Big(\operatorname{epi} f - \widehat{\operatorname{epi}}(g - v(P_F))\Big)$*;*

(iii) if $0 \in \operatorname{qi}\big[(\operatorname{dom} f - \operatorname{dom} g) - (\operatorname{dom} f - \operatorname{dom} g)\big]$*, then* $(0,0) \notin \operatorname{qri}\Big[\operatorname{co}\Big((\operatorname{epi} f - \widehat{\operatorname{epi}}(g - v(P_F))) \cup \{(0,0)\}\Big)\Big]$ *is equivalent to* $(0,0) \notin \operatorname{qi}\Big[\operatorname{co}\Big((\operatorname{epi} f - \widehat{\operatorname{epi}}(g - v(P_F))) \cup \{(0,0)\}\Big)\Big]$.

Proof. (i) That (RC_7^F) is equivalent to (RC_8^F) is a direct consequence of Lemma 1(ii). Let us suppose that f and g are convex and (RC_6^F) is fulfilled. By applying Lemma 1(i) with $U := \operatorname{dom} g$ and $V := \operatorname{dom} f$ we get $0 \in \operatorname{qi}(\operatorname{dom} g - \operatorname{dom} f)$ or, equivalently, $0 \in \operatorname{qi}(\operatorname{dom} f - \operatorname{dom} g)$. This means that (RC_7^F) holds.

(ii) One can prove that the primal problem (P_F) has an optimal solution if and only if $(0,0) \in \operatorname{epi} f - \widehat{\operatorname{epi}}(g - v(P_F)))$ and the conclusion follows.

(iii) See [8, Remark 3.4 (a)]. □

Remark 5. (a) The condition $0 \in \operatorname{qi}(\operatorname{dom} f - \operatorname{dom} g)$ implies relation $0 \in \operatorname{qi}\big[(\operatorname{dom} f - \operatorname{dom} g) - (\operatorname{dom} f - \operatorname{dom} g)\big]$ in Proposition 4(iii). This is a direct consequence of the inclusion $\operatorname{dom} f - \operatorname{dom} g \subseteq (\operatorname{dom} f - \operatorname{dom} g) - (\operatorname{dom} f - \operatorname{dom} g)$.

(b) We have the following implication

$$(0,0) \in \operatorname{qi}\Big[\operatorname{co}\Big(\big(\operatorname{epi} f - \widehat{\operatorname{epi}}(g - v(P_F))\big) \cup \{(0,0)\}\Big)\Big] \Rightarrow 0 \in \operatorname{qi}(\operatorname{dom} f - \operatorname{dom} g).$$

Indeed, suppose that $(0,0) \in \mathrm{qi}\left[\mathrm{co}\left(\left(\mathrm{epi} f - \widehat{\mathrm{epi}}(g - v(P_F))\right) \cup \{(0,0)\}\right)\right]$. Then $\mathrm{cl}\left[\mathrm{coneco}\left(\left(\mathrm{epi} f - \widehat{\mathrm{epi}}(g - v(P_F))\right) \cup \{(0,0)\}\right)\right] = X \times \mathbb{R}$, hence (cf. (1.1))

$$\mathrm{cl}\left[\mathrm{cone}\left(\mathrm{epi} f - \widehat{\mathrm{epi}}(g - v(P_F))\right)\right] = X \times \mathbb{R}.$$

As the inclusion

$$\mathrm{cl}\left[\mathrm{cone}\left(\mathrm{epi} f - \widehat{\mathrm{epi}}(g - v(P_F))\right)\right] \subseteq \mathrm{cl}\left(\mathrm{cone}(\mathrm{dom} f - \mathrm{dom} g)\right) \times \mathbb{R}$$

trivially holds, we have $\mathrm{cl}\left(\mathrm{cone}(\mathrm{dom} f - \mathrm{dom} g)\right) = X$, that is $0 \in \mathrm{qi}(\mathrm{dom} f - \mathrm{dom} g)$. Hence the following implication is true

$$0 \notin \mathrm{qi}(\mathrm{dom} f - \mathrm{dom} g) \Rightarrow (0,0) \notin \mathrm{qi}\left[\mathrm{co}\left(\left(\mathrm{epi} f - \widehat{\mathrm{epi}}(g - v(P_F))\right) \cup \{(0,0)\}\right)\right].$$

Nevertheless, in the regularity conditions given above one cannot substitute the condition $(0,0) \notin \mathrm{qi}\left[\mathrm{co}\left(\left(\mathrm{epi} f - \widehat{\mathrm{epi}}(g - v(P_F))\right) \cup \{(0,0)\}\right)\right]$ by the stronger, but more handleable one $0 \notin \mathrm{qi}(\mathrm{dom} f - \mathrm{dom} g)$, since in all the regularity conditions (RC_i^F), $i \in \{6,7,8\}$, the other hypotheses imply $0 \in \mathrm{qi}(\mathrm{dom} f - \mathrm{dom} g)$ (cf. Proposition 4(i)).

We give now the following strong duality result concerning the primal-dual pair $(P_F) - (D_F)$. It was first stated in [8] under convexity assumptions for the functions involved.

Theorem 3. *Let $f, g : X \to \overline{\mathbb{R}}$ be proper functions such that $v(P_F) \in \mathbb{R}$ and $\mathrm{epi} f - \widehat{\mathrm{epi}}(g - v(P_F))$ is a convex subset of $X \times \mathbb{R}$ (the latter is the case if, for instance, f and g are convex functions). Suppose that either f and g are convex and (RC_6^F) is fulfilled, or one of the regularity conditions (RC_i^F), $i \in \{7,8\}$, holds. Then $v(P_F) = v(D_F)$ and (D_F) has an optimal solution.*

Proof. One has to use the techniques employed in the proof of [8, Theorem 3.5]. $\square$

When the condition $(0,0) \notin \mathrm{qri}\left[\mathrm{co}\left((\mathrm{epi} f - \widehat{\mathrm{epi}}(g - v(P_F))) \cup \{(0,0)\}\right)\right]$ is removed, the duality result given above may fail. In the setting of Example 3, strong duality does not hold. Moreover, it has been proved in [8, Example 3.12(b)] that $(0,0) \in \mathrm{qri}\left[\mathrm{co}\left((\mathrm{epi} f - \widehat{\mathrm{epi}}(g - v(P_F))) \cup \{(0,0)\}\right)\right]$.

The following example (given in [8, Example 3.13]) justifies the study of the regularity conditions expressed by means of the quasi interior and quasi-relative interior.

Example 4. Consider the real Hilbert space $\ell^2 = \ell^2(\mathbb{N})$. We define the functions $f, g : \ell^2 \to \overline{\mathbb{R}}$ by

$$f(x) = \begin{cases} \|x\|, \text{ if } x \in x^0 - \ell^2_+, \\ +\infty, \text{ otherwise} \end{cases}$$

and

$$g(x) = \begin{cases} \langle c, x\rangle, \text{ if } x \in \ell^2_+, \\ +\infty, \quad \text{otherwise}, \end{cases}$$

respectively, where $x^0, c \in \ell^2_+$ are arbitrarily chosen such that $x^0_n > 0$ for all $n \in \mathbb{N}$. Note that

$$v(P_F) = \inf_{x \in \ell^2_+ \cap (x^0 - \ell^2_+)} \{\|x\| + \langle c, x\rangle\} = 0$$

and the infimum is attained at $x = 0$. We have $\operatorname{dom} f = x^0 - \ell^2_+ = \{(x_n)_{n\in\mathbb{N}} \in \ell^2 : x_n \le x^0_n \ \forall n \in \mathbb{N}\}$ and $\operatorname{dom} g = \ell^2_+$. By using Example 1 we get

$$\operatorname{dom} f \cap \operatorname{qri}(\operatorname{dom} g) = \{(x_n)_{n\in\mathbb{N}} \in \ell^2 : 0 < x_n \le x^0_n \ \forall n \in \mathbb{N}\} \neq \emptyset.$$

Also, $\operatorname{cl}\big(\operatorname{cone}(\operatorname{dom} g - \operatorname{dom} g)\big) = \ell^2$ and so $0 \in \operatorname{qi}(\operatorname{dom} g - \operatorname{dom} g)$. Further,

$$\operatorname{epi} f - \widehat{\operatorname{epi}}(g - v(P_F)) = \{(x - y, \|x\| + \langle c, y\rangle + \varepsilon) : x \in x^0 - \ell^2_+, y \in \ell^2_+, \varepsilon \ge 0\}.$$

In the following we prove that $(0,0) \notin \operatorname{qri}\Big(\operatorname{epi} f - \widehat{\operatorname{epi}}(g - v(P_F))\Big)$. Assuming the contrary, one would have that the set $\operatorname{cl}\Big[\operatorname{cone}\Big(\operatorname{epi} f - \widehat{\operatorname{epi}}(g - v(P_F))\Big)\Big]$ is a linear subspace of $\ell^2 \times \mathbb{R}$. Since $(0,1) \in \operatorname{cl}\Big[\operatorname{cone}\Big(\operatorname{epi} f - \widehat{\operatorname{epi}}(g - v(P_F))\Big)\Big]$ (take $x = y = 0$ and $\varepsilon = 1$), $(0,-1)$ must belong to this set, too. On the other hand, one can easily see that for all (x,r) belonging to $\operatorname{cl}\Big[\operatorname{cone}\Big(\operatorname{epi} f - \widehat{\operatorname{epi}}(g - v(P_F))\Big)\Big]$ it holds $r \ge 0$. This leads to the desired contradiction.

Hence the regularity condition (RC^F_6) is fulfilled, thus strong duality holds (cf. Theorem 3). On the other hand, ℓ^2 is a Fréchet space (being a Hilbert space), the functions f and g are proper, convex and lower semicontinuous and, as $\operatorname{sqri}(\operatorname{dom} f - \operatorname{dom} g) = \operatorname{sqri}(x^0 - \ell^2_+) = \emptyset$, none of the conditions (RC^F_i), $i \in \{1,2,3,4,5\}$, listed at the beginning of this section, can be applied for this optimization problem.

As for all $x^* \in \ell^2$ it holds $g^*(x^*) = \delta_{c - \ell^2_+}(x^*)$ and (cf. [26, Theorem 2.8.7])

$$f^*(-x^*) = \inf_{x_1^* + x_2^* = -x^*} \{\|\cdot\|^*(x_1^*) + \delta^*_{x^0 - l^2_+}(x_2^*)\} = \inf_{\substack{x_1^* + x_2^* = -x^*, \\ \|x_1^*\| \le 1, x_2^* \in \ell^2_+}} \langle x_2^*, x^0\rangle,$$

the optimal objective value of the Fenchel dual problem is

$$v(D_F) = \sup_{\substack{x_2^* \in \ell^2_+ - c - x_1^*, \\ \|x_1^*\| \le 1, x_2^* \in \ell^2_+}} \langle -x_2^*, x^0\rangle = \sup_{x_2^* \in \ell^2_+} \langle -x_2^*, x^0\rangle = 0$$

and $x_2^* = 0$ is the optimal solution of the dual.

The following example (see also [14, Example 2.5]) underlines the fact that in general the regularity condition (RC^F_7) (and automatically also (RC^F_8)) is weaker than (RC^F_6) (see also Example 9 below).

Example 5. Consider the real Hilbert space $\ell^2(\mathbb{R})$ and the functions $f,g:\ell^2(\mathbb{R})\to\overline{\mathbb{R}}$ defined for all $s\in\ell^2(\mathbb{R})$ by

$$f(s)=\begin{cases} s(1), \text{ if } s\in\ell^2_+(\mathbb{R}),\\ +\infty, \text{ otherwise}\end{cases}$$

and

$$g(s)=\begin{cases} s(2), \text{ if } s\in\ell^2_+(\mathbb{R}),\\ +\infty, \text{ otherwise,}\end{cases}$$

respectively. The optimal objective value of the primal problem is

$$v(P_F)=\inf_{s\in\ell^2_+(\mathbb{R})}\{s(1)+s(2)\}=0$$

and $s=0$ is an optimal solution (let us notice that (P_F) has infinitely many optimal solutions). We have $\operatorname{qri}(\operatorname{dom} g)=\operatorname{qri}(\ell^2_+(\mathbb{R}))=\emptyset$ (cf. Example 2), hence the condition (RC_6^F) fails. In the following we show that (RC_7^F) is fulfilled. One can prove that $\operatorname{dom} f-\operatorname{dom} g=\ell^2_+(\mathbb{R})-\ell^2_+(\mathbb{R})=\ell^2(\mathbb{R})$, thus $0\in\operatorname{qi}(\operatorname{dom} f-\operatorname{dom} g)$. Like in the previous example, we have

$$\operatorname{epi} f-\widehat{\operatorname{epi}}(g-v(P_F))=\{(s-s',s(1)+s'(2)+\varepsilon):s,s'\in\ell^2_+(\mathbb{R}),\varepsilon\geq 0\}$$

and with the same technique one can show that $(0,0)\notin\operatorname{qri}\Big(\operatorname{epi} f-\widehat{\operatorname{epi}}(g-v(P_F))\Big)$, hence the condition (RC_7^F) holds.

Let us take a look at the formulation of the dual problem. To this end we have to calculate the conjugates of f and g. Let us recall that the scalar product on $\ell^2(\mathbb{R})$, $\langle\cdot,\cdot\rangle:\ell^2(\mathbb{R})\times\ell^2(\mathbb{R})\to\mathbb{R}$ is defined by $\langle s,s'\rangle=\sup_{F\subseteq\mathbb{R},F\text{ finite}}\sum_{r\in F}s(r)s'(r)$, for $s,s'\in\ell^2(\mathbb{R})$ and that the dual space $\big(\ell^2(\mathbb{R})\big)^*$ is identified with $\ell^2(\mathbb{R})$. For an arbitrary $u\in\ell^2(\mathbb{R})$ we have

$$\begin{aligned}f^*(u)&=\sup_{s\in\ell^2_+(\mathbb{R})}\{\langle u,s\rangle-s(1)\}=\sup_{s\in\ell^2_+(\mathbb{R})}\left\{\sup_{F\subseteq\mathbb{R},F\text{ finite}}\sum_{r\in F}u(r)s(r)-s(1)\right\}\\&=\sup_{F\subseteq\mathbb{R},F\text{ finite}}\left\{\sup_{s\in\ell^2_+(\mathbb{R})}\left\{\sum_{r\in F}u(r)s(r)-s(1)\right\}\right\}.\end{aligned}$$

If there exists $r\in\mathbb{R}\setminus\{1\}$ with $u(r)>0$ or if $u(1)>1$, then one has $f^*(u)=+\infty$. Assuming the contrary, for every finite subset F of $\mathbb{R}$, independently from the fact that 1 belongs to F or not, it holds $\sup_{s\in\ell^2_+(\mathbb{R})}\{\sum_{r\in F}u(r)s(r)-s(1)\}=0$. Consequently,

$$f^*(u)=\begin{cases}0, & \text{if } u(r)\leq 0\ \forall r\in\mathbb{R}\setminus\{1\} \text{ and } u(1)\leq 1,\\ +\infty, & \text{otherwise.}\end{cases}$$

Similarly one can provide a formula for g^* and in this way we obtain that $v(D_F)=0$ and that $u=0$ is an optimal solution of the dual ((D_F) has actually infinitely many optimal solutions).

Let us compare in the following the regularity conditions expressed by means of the quasi interior and quasi relative interior with the classical ones form the literature, mentioned at the beginning of the section. To this end, we need an auxiliary result.

Proposition 5. *Suppose that for the primal-dual pair $(P_F)-(D_F)$ strong duality holds. Then* $(0,0)\notin \operatorname{qi}\Big[\operatorname{co}\Big((\operatorname{epi} f-\widehat{\operatorname{epi}}(g-v(P_F)))\cup\{(0,0)\}\Big)\Big]$.

Proof. By the assumptions we made, there exists $x^*\in X^*$ such that $v(P_F)=-f^*(-x^*)-g^*(x^*)=\inf_{x\in X}\{\langle x^*,x\rangle+f(x)\}+\inf_{x\in X}\{\langle -x^*,x\rangle+g(x)\}$, hence

$$v(P_F)\le \langle x^*,x\rangle+f(x)+\langle -x^*,y\rangle+g(y)\ \forall (x,y)\in X\times Y,$$

that is

$$\langle -x^*,x-y\rangle-(f(x)+g(y)-v(P_F))\le 0\ \forall (x,y)\in \operatorname{dom} f\times \operatorname{dom} g.$$

We obtain

$$\langle (-x^*,-1),(z,r)\rangle\le 0\ \forall (z,r)\in \operatorname{epi} f-\widehat{\operatorname{epi}}(g-v(P_F)),$$

hence

$$\langle (-x^*,-1),(z,r)\rangle\le 0\ \forall (z,r)\in \operatorname{co}\Big((\operatorname{epi} f-\widehat{\operatorname{epi}}(g-v(P_F)))\cup\{(0,0)\}\Big).$$

The last relation ensures $(-x^*,-1)\in N_{\left[\operatorname{co}\left((\operatorname{epi} f-\widehat{\operatorname{epi}}(g-v(P_F)))\cup\{(0,0)\}\right)\right]}(0,0)$ and Proposition 2 implies that $(0,0)\notin \operatorname{qi}\Big[\operatorname{co}\Big((\operatorname{epi} f-\widehat{\operatorname{epi}}(g-v(P_F)))\cup\{(0,0)\}\Big)\Big]$. □

A comparison of the above regularity conditions is provided in the following.

Proposition 6. *Suppose that X is a Fréchet space and $f,g:X\to\overline{\mathbb{R}}$ are proper, convex and lower semicontinuous functions. The following relations hold*

$$(RC_1^F)\Rightarrow(RC_2^F)\Leftrightarrow(RC_3^F)\Rightarrow(RC_7^F)\Leftrightarrow(RC_8^F).$$

Proof. In view of Remark 4 and Proposition 4(i) we have to prove only the implication $(RC_3^F)\Rightarrow(RC_7^F)$. Let us suppose that (RC_3^F) is fulfilled. We apply (1.2) and obtain $0\in\operatorname{qi}\big(\operatorname{dom} f-\operatorname{dom} g\big)$. Moreover, the regularity condition (RC_3^F) ensures strong duality for the pair $(P_F)-(D_F)$ (cf. Theorem 2), hence $(0,0)\notin \operatorname{qi}\Big[\operatorname{co}\Big((\operatorname{epi} f-\widehat{\operatorname{epi}}(g-v(P_F)))\cup\{(0,0)\}\Big)\Big]$ (cf. Proposition 5). Applying Proposition 4(iii) (see also Remark 5(a)) we get that the condition (RC_7^F) holds and the proof is complete. □

Remark 6. One can notice that the implications

$$(RC_1^F) \Rightarrow (RC_7^F) \Leftrightarrow (RC_8^F)$$

hold in the framework of separated locally convex spaces and for $f,g : X \to \overline{\mathbb{R}}$ proper and convex functions (nor completeness for the space neither lower semicontinuity for the functions is needed here).

Next we show that, in general, the conditions (RC_i^F), $i \in \{4,5\}$, cannot be compared with (RC_i^F), $i \in \{6,7,8\}$. Example 4 provides a situation for which (RC_i^F), $i \in \{6,7,8\}$, are fulfilled, unlike (RC_i^F), $i \in \{4,5\}$. In the following example the conditions (RC_i^F), $i \in \{4,5\}$, are fulfilled, while (RC_i^F), $i \in \{6,7,8\}$, fail.

Example 6. Consider $(X, \|\cdot\|)$ a nonzero real Banach space, $x_0^* \in X^* \setminus \{0\}$ and the functions $f,g : X \to \overline{\mathbb{R}}$ defined by $f = \delta_{\ker x_0^*}$ and $g = \|\cdot\| + \delta_{\ker x_0^*}$, respectively. The optimal objective value of the primal problem is

$$v(P_F) = \inf_{x \in \ker x_0^*} \|x\| = 0$$

and $\bar{x} = 0$ is the unique optimal solution of (P_F). The functions f and g are proper, convex and lower semicontinuous. Further, $\operatorname{dom} f - \operatorname{dom} g = \ker x_0^*$, which is a closed linear subspace of X, hence (RC_i^F), $i \in \{4,5\}$, are fulfilled. Moreover, $\operatorname{dom} g - \operatorname{dom} g = \operatorname{dom} f - \operatorname{dom} g = \ker x_0^*$ and it holds $\operatorname{cl}(\ker x_0^*) = \ker x_0^* \neq X$. Thus $0 \notin \operatorname{qi}(\operatorname{dom} g - \operatorname{dom} g)$ and $0 \notin \operatorname{qi}(\operatorname{dom} f - \operatorname{dom} g)$ and this means that all the three regularity conditions (RC_i^F), $i \in \{6,7,8\}$, fail.

The conjugate functions of f and g are $f^* = \delta_{(\ker x_0^*)^\perp} = \delta_{\mathbb{R}x_0^*}$ and, respectively, $g^* = \delta_{B_*(0,1)} \Box \delta_{\mathbb{R}x_0^*} = \delta_{B_*(0,1)+\mathbb{R}x_0^*}$ (cf. [26, Theorem 2.8.7]), where $B_*(0,1)$ is the closed unit ball of the dual space X^*. Hence $v(D_F) = 0$ and the set of optimal solutions of (D_F) coincides with $\mathbb{R}x_0^*$. Finally, let us notice that instead of $\ker x_0^*$ one can consider any closed linear subspace S of X such that $S \neq X$.

1.3.3 Closedness-type regularity conditions

Besides the generalized interior-point regularity conditions, there exist in the literature so-called *closedness-type regularity conditions* for conjugate duality. In the following we will recall two sufficient conditions of this type for Fenchel duality and we will relate them to the ones investigate in the previous subsection. Let these two conditions be:

$$(RC_9^F) \left| \begin{array}{l} f \text{ and } g \text{ are lower semicontinuous and} \\ \operatorname{epi} f^* + \operatorname{epi} g^* \text{ is closed in } (X^*, w(X^*,X)) \times \mathbb{R} \end{array} \right.$$

and

$$(RC_{10}^F) \left| \begin{array}{l} f \text{ and } g \text{ are lower semicontinuous, } f^* \Box g^* \text{ is } w(X^*,X)\text{-lower} \\ \text{semicontinuous on } X^* \text{ and exact at } 0. \end{array} \right.$$

The condition (RC_9^F) has been first considered by Burachik and Jeyakumar in Banach spaces (cf. [10]) and by Boţ and Wanka in separated locally convex spaces (cf. [9]), while the second one, (RC_{10}^F), has been introduced in [9]. We have the following duality results (cf. [9]).

Theorem 4. *Let $f,g : X \to \overline{\mathbb{R}}$ be proper and convex functions such that* $\operatorname{dom} f \cap \operatorname{dom} g \neq \emptyset$*. If (RC_9^F) is fulfilled, then*

$$(f+g)^*(x^*) = \min\{f^*(x^*-y^*)+g^*(y^*) : y^* \in X^*\} \ \forall x^* \in X^*. \tag{1.3}$$

Theorem 5. *Let $f,g : X \to \overline{\mathbb{R}}$ be proper and convex functions such that* $\operatorname{dom} f \cap \operatorname{dom} g \neq \emptyset$*. If (RC_{10}^F) is fulfilled, then $v(P_F) = v(D_F)$ and (D_F) has an optimal solution.*

Remark 7. (a) Let us notice that condition (1.3) is referred in the literature as *stable strong duality* (see [6, 11, 22] for more details) and obviously guarantees strong duality for $(P_F)-(D_F)$. When $f,g : X \to \overline{\mathbb{R}}$ are proper, convex and lower semicontinuous functions with $\operatorname{dom} f \cap \operatorname{dom} g \neq \emptyset$ one has in fact that (RC_9^F) is fulfilled if and only if (1.3) holds (cf. [9, Theorem 3.2]).

(b) If f,g are proper, convex and lower semicontinuous such that $\operatorname{dom} f \cap \operatorname{dom} g \neq \emptyset$, then $(RC_9^F) \Rightarrow (RC_{10}^F)$ (cf. [9, Section 4]). Moreover, there are examples showing that in general (RC_{10}^F) is weaker than (RC_9^F) (see [9]). Finally, let us mention that (under the same hypotheses) $f^*\square g^*$ is a $w(X^*,X)$-lower semicontinuous function on X^* if and only if $(f+g)^* = f^*\square g^*$. This is a direct consequence of the equality $(f+g)^* = \operatorname{cl}(f^*\square g^*)$, where the closure is considered with respect to the weak* topology on X^* (cf. [9, Thorem 2.1]).

(c) In case X is a Fréchet space and f,g are proper, convex and lower semicontinuous functions we have the following relations between the regularity conditions considered for the primal-dual pair $(P_F)-(D_F)$ (cf. [9], see also [17] and [26, Theorem 2.8.7])

$$(RC_1^F) \Rightarrow (RC_2^F) \Leftrightarrow (RC_3^F) \Rightarrow (RC_4^F) \Leftrightarrow (RC_5^F) \Rightarrow (RC_9^F) \Rightarrow (RC_{10}^F).$$

We refer to [6,9,10,22] for several examples showing that in general the implications above are strict. The implication $(RC_1^F) \Rightarrow (RC_9^F) \Rightarrow (RC_{10}^F)$ holds in the general setting of separated locally convex spaces (in the hypotheses that f,g are proper, convex and lower semicontinuous).

We observe that if X is a finite-dimensional space and f,g are proper, convex and lower semicontinuous, then $(RC_6^F) \Rightarrow (RC_7^F) \Leftrightarrow (RC_8^F) \Rightarrow (RC_9^F) \Rightarrow (RC_{10}^F)$. However, in the infinite-dimensional setting this is no longer true. In the following two examples the conditions (RC_9^F) and (RC_{10}^F) are fulfilled, unlike (RC_i^F), $i \in \{6,7,8\}$ (we refer to [6, 9, 10, 19, 22] for examples in the finite-dimensional setting).

Example 7. Consider the same setting as in Example 6. We know that (RC_5^F) is fulfilled, hence also (RC_9^F) and (RC_{10}^F) (cf. Remark 7(c)). This is not surprising, since $\operatorname{epi} f^* + \operatorname{epi} g^* = (B_*(0,1)+\mathbb{R}x_0^*) \times [0,\infty)$, which is closed in $(X^*, w(X^*,X)) \times \mathbb{R}$ (by the Banach-Alaoglu Theorem the unit ball $B_*(0,1)$ is compact in $(X^*, w(X^*,X))$). As shown in Example 6, none of the regularity conditions (RC_i^F), $i \in \{6,7,8\}$, is fulfilled.

Example 8. Consider the real Hilbert space $\ell^2(\mathbb{R})$ and the functions $f,g : \ell^2(\mathbb{R}) \to \overline{\mathbb{R}}$ defined by $f = \delta_{\ell^2_+(\mathbb{R})}$ and $g = \delta_{-\ell^2_+(\mathbb{R})}$, respectively. We have $\operatorname{qri}(\operatorname{dom} f - \operatorname{dom} g) = \operatorname{qri}\big(\ell^2_+(\mathbb{R})\big) = \emptyset$ (cf. Example 2), hence all the generalized interior-point regularity conditions (RC_i^F), $i \in \{1,2,3,4,5,6,7,8\}$, fail (see also Proposition 4(i)). The conjugate functions of f and g are $f^* = \delta_{-\ell^2_+(\mathbb{R})}$ and $g^* = \delta_{\ell^2_+(\mathbb{R})}$, respectively, hence $\operatorname{epi} f^* + \operatorname{epi} g^* = \ell^2(\mathbb{R}) \times [0,\infty)$, that is the condition (RC_9^F) holds (hence also (RC_{10}^F), cf. Remark 7(b)). One can see that $v(P_F) = v(D_F) = 0$ and $y^* = 0$ is an optimal solution of the dual problem.

The next issue we investigate concerns the relation between the generalized interior-point conditions (RC_i^F), $i \in \{6,7,8\}$ and the closedness-type ones (RC_9^F) and (RC_{10}^F). In the last two examples the conditions (RC_9^F) and (RC_{10}^F) are fulfilled, while (RC_i^F), $i \in \{6,7,8\}$, fail. In the following we provide an example for which (RC_7^F) is fulfilled, unlike (RC_i^F), $i \in \{9,10\}$. In this way we give a negative answer to an open problem stated in [19, Remark 4.3], concomitantly proving that in general (RC_7^F) (and automatically also (RC_8^F)) and (RC_9^F) are not comparable.

Example 9. (see also [14, Example 2.7]) Like in Example 3, consider the real Hilbert space $X = \ell^2(\mathbb{N})$ and the sets

$$C = \{(x_n)_{n\in\mathbb{N}} \in \ell^2 : x_{2n-1} + x_{2n} = 0 \ \forall n \in \mathbb{N}\}$$

and

$$S = \{(x_n)_{n\in\mathbb{N}} \in \ell^2 : x_{2n} + x_{2n+1} = 0 \ \forall n \in \mathbb{N}\},$$

which are closed linear subspaces of ℓ^2 and satisfy $C \cap S = \{0\}$. Define the functions $f,g : \ell^2 \to \overline{\mathbb{R}}$ by $f = \delta_C$ and $g = \delta_S$, respectively, which are proper, convex and lower semicontinuous. The optimal objective value of the primal problem is $v(P_F) = 0$ and $\bar{x} = 0$ is the unique optimal solution of $v(P_F)$. Moreover, $S - C$ is dense in ℓ^2 (cf. [17, Example 3.3]), thus $\operatorname{cl}\big(\operatorname{cone}(\operatorname{dom} f - \operatorname{dom} g)\big) = \operatorname{cl}(C - S) = \ell^2$. This implies $0 \in \operatorname{qi}(\operatorname{dom} f - \operatorname{dom} g)$. Further, one has

$$\operatorname{epi} f - \widehat{\operatorname{epi}}(g - v(P_F)) = \{(x-y,\varepsilon) : x \in C, y \in S, \varepsilon \geq 0\} = (C - S) \times [0,+\infty)$$

and $\operatorname{cl}\Big[\operatorname{cone}\Big(\operatorname{epi} f - \widehat{\operatorname{epi}}(g - v(P_F))\Big)\Big] = \ell^2 \times [0,+\infty)$, which is not a linear subspace of $\ell^2 \times \mathbb{R}$, hence $(0,0) \notin \operatorname{qri}\Big(\operatorname{epi} f - \widehat{\operatorname{epi}}(g - v(P_F))\Big)$. All together, we get that the condition (RC_7^F) is fulfilled, hence strong duality holds (cf. Theorem 3). One can prove that $f^* = \delta_{C^\perp}$ and $g^* = \delta_{S^\perp}$, where

$$C^\perp = \{(x_n)_{n\in\mathbb{N}} \in \ell^2 : x_{2n-1} = x_{2n} \ \forall n \in \mathbb{N}\}$$

and

$$S^{\perp} = \{(x_n)_{n\in\mathbb{N}} \in \ell^2 : x_1 = 0, x_{2n} = x_{2n+1}\ \forall n \in \mathbb{N}\}.$$

Further, $v(D_F) = 0$ and the set of optimal solutions of the dual problem is exactly $C^{\perp} \cap S^{\perp} = \{0\}$.

We show that (RC_{10}^F) is not fulfilled (hence (RC_9^F) fails too, cf. Remark 7(b)). Let us consider the element $e^1 \in \ell^2$, defined by $e_1^1 = 1$ and $e_k^1 = 0$ for all $k \in \mathbb{N}\setminus\{1\}$. We compute $(f+g)^*(e^1) = \sup_{x\in\ell^2}\{\langle e^1, x\rangle - f(x) - g(x)\} = 0$ and $(f^*\square g^*)(e^1) = \delta_{C^{\perp}+S^{\perp}}(e^1)$. If we suppose that $e^1 \in C^{\perp}+S^{\perp}$, then we would have $(e^1+S^{\perp})\cap C^{\perp} \neq \emptyset$. However, it has been proved in [17, Example 3.3] that $(e^1+S^{\perp})\cap C^{\perp} = \emptyset$. This shows that $(f^*\square g^*)(e^1) = +\infty > 0 = (f+g)^*(e^1)$. Via Remark 7(b) it follows that the condition (RC_{10}^F) is not fulfilled and, consequently, (RC_i^F), $i \in \{1,2,3,4,5,9\}$, fail, too (cf. Remark 7(c)), unlike condition (RC_7^F). Concerning (RC_6^F), one can see that this condition is not fulfilled, since $0 \in \operatorname{qi}(\operatorname{dom} g - \operatorname{dom} g)$ does not hold.

In the next example the conditions (RC_i^F), $i \in \{6,7,8\}$, are fulfilled and (RC_9^F) fails.

Example 10. The example we consider in the following is inspired by [22, Example 11.3]. Consider X an arbitrary Banach space, C a convex and closed subset of X and x_0 an extreme point of C which is not a support point of C. Taking for instance $X = \ell^2$, $1 < p < 2$ and $C := \left\{x \in \ell^2 : \sum_{n=1}^{\infty} |x_n|^p \leq 1\right\}$ one can find extreme points in C that are not support points (see [22]). Consider the functions $f, g : X \to \overline{\mathbb{R}}$ defined as $f = \delta_{x_0 - C}$ and $g = \delta_{C - x_0}$, respectively. They are both proper, convex and lower semicontinuous and fulfill, as x_0 is an extreme point of C, $f + g = \delta_{\{0\}}$. Thus $v(P_F) = 0$ and $\bar{x} = 0$ is the unique optimal solution of (P_F). We show that, different to the previous example, (RC_6^F) is fulfilled and this will guarantee that both (RC_7^F) and (RC_8^F) are valid, too (cf. Proposition 4 (i)). To this end we notice first that $x_0 \in \operatorname{qi}(C)$. Assuming the contrary, one would have that there exists $x^* \in X^*\setminus\{0\}$ such that $\langle x^*, x_0\rangle = \sup_{x\in C}\langle x^*, x\rangle$ (cf. Proposition 2), contradicting the hypothesis that x_0 is not a support point of C. This means that $x_0 \in \operatorname{qri}(C)$, too, and so $0 \in \operatorname{dom} f \cap \operatorname{qri}(\operatorname{dom} g)$. Further, since it holds $\operatorname{cl}(\operatorname{cone}(C - x_0)) \subseteq \operatorname{cl}(\operatorname{cone}(C - C))$, we have $\operatorname{cl}(\operatorname{cone}(C - C)) = X$ and from here $0 \in \operatorname{qi}(C - C) = \operatorname{qi}(\operatorname{dom} g - \operatorname{dom} g)$. Noticing that

$$\operatorname{epi} f - \widehat{\operatorname{epi}}(g - v(P_F)) = \{(x - y, \varepsilon) : x, y \in C, \varepsilon \geq 0\} = (C - C)\times[0, +\infty),$$

it follows that $\operatorname{cl}\left[\operatorname{cone}\left(\operatorname{epi} f - \widehat{\operatorname{epi}}(g - v(P_F))\right)\right] = X \times [0, +\infty)$, which is not a linear subspace of $X \times \mathbb{R}$. Thus $(0,0) \notin \operatorname{qri}\left(\operatorname{epi} f - \widehat{\operatorname{epi}}(g - v(P_F))\right)$ and this has as consequence the fact that (RC_6^F) is fulfilled. Hence strong duality holds (cf. Theorem 3), $v(D_F) = 0$ and 0 is an optimal solution of the dual problem.

We show that (RC_9^F) is not fulfilled. Assuming the contrary, one would have that the equality in (1.3) holds for all $x^* \in X^*$. On the other hand, in [22, Example 11.3] it is proven that this is the case only when $x^* = 0$ and this provides the desired contradiction.

Remark 8. Consider the following optimization problem

$$(P_F^A)\ \inf_{x\in X}\{f(x)+(g\circ A)(x)\},$$

where X and Y are separated locally convex spaces with topological dual spaces X^* and Y^*, respectively, $A:X\to Y$ is a linear continuous mapping, $f:X\to\overline{\mathbb{R}}$ and $g:Y\to\overline{\mathbb{R}}$ are proper functions such that $A(\operatorname{dom} f)\cap\operatorname{dom} g\neq\emptyset$. The Fenchel dual problem to (P_F^A) is

$$(D_F^A)\ \sup_{y^*\in Y^*}\{-f^*(-A^*y^*)-g^*(y^*)\},$$

where $A^*:Y^*\to X^*$ is the *adjoint operator*, defined by $\langle A^*y^*,x\rangle=\langle y^*,Ax\rangle$ for all $y^*\in Y^*$ and $x\in X$. We denote by $v(P_F^A)$ and $v(D_F^A)$ the optimal objective values of the primal and the dual problem, respectively, and suppose that $v(P_F^A)\in\mathbb{R}$. We consider the set

$$A\times\mathrm{id}_{\mathbb{R}}(\operatorname{epi} f)=\{(Ax,r)\in Y\times\mathbb{R}:f(x)\le r\}.$$

By using the approach presented in the previous section one can provide similar discussions regarding strong duality for the primal-dual pair $(P_F^A)-(D_F^A)$. To this end, we introduce the following functions: $F,G:X\times Y\to\overline{\mathbb{R}}$, $F(x,y)=f(x)+\delta_{\{u\in X:Au=y\}}(x)$ and $G(x,y)=g(y)$ for all $(x,y)\in X\times Y$. The functions F and G are proper and their domains fulfill the relation

$$\operatorname{dom} F-\operatorname{dom} G=X\times\big(A(\operatorname{dom} f)-\operatorname{dom} g\big).$$

Since $\operatorname{epi} F=\{(x,Ax,r):f(x)\le r\}$ and $\widehat{\operatorname{epi}}(G-v(P_F^A))=\{(x,y,r):r\le -G(x,y)+v(P_F^A)\}=X\times\widehat{\operatorname{epi}}(g-v(P_F^A))$, we obtain

$$\operatorname{epi} F-\widehat{\operatorname{epi}}(G-v(P_F^A))=X\times\Big(A\times\mathrm{id}_{\mathbb{R}}(\operatorname{epi} f)-\widehat{\operatorname{epi}}(g-v(P_F^A))\Big).$$

Moreover,

$$\inf_{(x,y)\in X\times Y}\{F(x,y)+G(x,y)\}=\inf_{x\in X}\{f(x)+(g\circ A)(x)\}=v(P_F^A).$$

On the other hand, for all $(x^*,y^*)\in X^*\times Y^*$ we have $F^*(x^*,y^*)=f^*(x^*+A^*y^*)$ and

$$G^*(x^*,y^*)=\begin{cases} g^*(y^*), & \text{if } x^*=0,\\ +\infty, & \text{otherwise.}\end{cases}$$

Therefore

$$\sup_{\substack{x^*\in X^*\\ y^*\in Y^*}}\{-F^*(-x^*,-y^*)-G^*(x^*,y^*)\}=\sup_{y^*\in Y^*}\{-f^*(-A^*y^*)-g^*(y^*)\}=v(D_F^A).$$

For more details concerning this approach we refer to [8, 14].

We remark that Borwein and Lewis gave in [4] some regularity conditions by means of the quasi-relative interior, in order to guarantee strong duality for (P_F^A) and (D_F^A). However, they considered a more restrictive case, namely when the codomain of the operator A is finite-dimensional. Here we have considered the more general case, when both spaces X and Y are infinite-dimensional.

Finally, let us notice that several regularity conditions by means of the quasi interior and quasi-relative interior were introduced in the literature in order to guarantee strong duality between a primal optimization problem with geometric and cone constraints and its Lagrange dual problem. However, they have either contradictory assumptions, like in [12], or superfluous conditions, like in [16]. For a detailed argumentation of these considerations and also for correct alternative strong duality results in the case of Lagrange duality we refer to [7, 8].

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