Utility Maximisation in Incomplete Markets

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Preface

In these lectures we give a short introduction to the basic concepts of Mathematical Finance, focusing on the notion of "no arbitrage", and subsequently apply these notions to the problem of optimizing dynamically a portfolio in an incomplete financial market with respect to a given utility function U.

In the first part we mainly restrict ourselves to the situation where the underlying probability space $(\Omega, \mathcal{F}, \mathbf{P})$ is finite, in order to reduce the functional-analytic difficulties to simple linear algebra. In my opinion, this allows — at least as a first step — for a clearer picture of the Mathematical Finance issues.

We then treat the problem of utility maximisation and, in particluar, its duality theory for a general semi-martingale models of financial market. Here we are rather informal and concentrate mainly on explaining the basic ideas, e.g., the notion of the asymptotic elasticity of a utility function U.

These notes are largely based on the surveys [S 03] and [S 01a] and, in particular, on the notes taken by P. Guasoni during my Cattedra Galileiana lectures at Scuola Normale Superiore in Pisa [S 04a]. We also refer to the original papers [KS 99] and [S 01] for more detailed information on the topics of the present lectures.

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Chapter 1

Problem Setting

We consider a model of a security market which consists of $d+1$ assets. We denote by $S = ((S_t^i)_{1\leq t\leq T})_{0\leq i\leq d}$ the price process of the d stocks and suppose that the price of the asset S^0, called the "bond" or "cash account", is constant, i.e.,$S_t^0 \equiv 1$. The latter assumption does not restrict the generality of the model as we always may choose the bond as numéraire, i.e., we may express the values of the other assets in units of the "bond". In other words, $((S_t^i)_{0\leq t\leq T})_{1\leq i\leq d}$, is an $\mathbb{R}^d$-valued semi-martingale modeling the discounted price process of d risky assets.

The process S is assumed to be a semimartingale, based on and adapted to a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\leq t\leq T}, \mathbf{P})$ satisfying the usual conditions of saturatedness and right continuity. As usual in mathematical finance, we consider a finite horizon T, but we remark that our results can also be extended to the case of an infinite horizon.

In chapter 2 we shall consider the case of finite Ω, in which case the paths of S are constant except for jumps at a finite number of times. We then can write S as $(S_t)_{t=0}^T = (S_0, S_1, \ldots, S_T)$, for some $T \in \mathbb{N}$.

The assumption that the bond is constant is mainly chosen for notational convenience as it allows for a compact description of self-financing portfolios: a self-financing portfolio Π is defined as a pair (x, H), where the constant x is the initial value of the portfolio and $H = (H^i)_{1\leq i\leq d}$ is a predictable S-integrable process specifying the amount of each asset held in the portfolio. The value process $X = (X_t)_{0\leq t\leq T}$ of such a portfolio Π at time t is given by

$$X_t = X_0 + \int_0^t H_u dS_u, \quad 0 \leq t \leq T, \tag{1.1}$$

where $X_0 = x$ and the integral refers to stochastic integration in $\mathbb{R}^d$.

In order to rule out doubling strategies and similar schemes generating arbitrage-profits (by going deeply into the red) we follow Harrison and Pliska ([HP 81], see also

[DS 94]), calling a predictable, S-integrable process *admissible*, if there is a constant $C \in \mathbb{R}_+$ such that, almost surely, we have

$$(H \cdot S)_t := \int_0^t H_u dS_u \geq -C, \quad \text{for } 0 \leq t \leq T. \tag{1.2}$$

Let us illustrate these general concepts in the case of an $\mathbb{R}^d$-valued process $S = (S_t)_{t=0}^T$ in finite, discrete time $\{0, 1, \ldots, T\}$ adapted to the filtration $(\mathcal{F}_t)_{t=0}^T$. In this case each $\mathbb{R}^d$-valued process $(H_t)_{t=1}^T$, which is predictable (i.e. each H_t is $\mathcal{F}_{t-1}$-measurable), is S-integrable, and the stochastic integral reduces to a finite sum

$$\begin{aligned}
(H \cdot S)_t &= \int_0^t H_u dS_u && (1.3)\\
&= \sum_{u=1}^t H_u \Delta S_u && (1.4)\\
&= \sum_{u=1}^t H_u (S_u - S_{u-1}), && (1.5)
\end{aligned}$$

where $H_u \Delta S_u$ denotes the inner product of the vectors H_u and $\Delta S_u = S_u - S_{u-1}$ in $\mathbb{R}^d$, i.e.

$$H_u \Delta S_u = \sum_{j=1}^d H_u^j (S_u^j - S_u^{j-1}). \tag{1.6}$$

Of course, each such trading strategy H is admissible if the underlying probability space Ω is finite.

Passing again to the general setting of an $\mathbb{R}^d$-valued semi-martingale $S = (S_t)_{0 \leq t \leq T}$ we denote as in [KS 99] by $\mathcal{M}^e(S)$ (resp. $\mathcal{M}^a(S)$) the set of probability measures $\mathbf{Q}$ equivalent to $\mathbf{P}$ (resp. absolutely continuous with respect to $\mathbf{P}$) such that for each admissible integrand H, the process $H \cdot S$ is a local martingale under $\mathbf{Q}$.

We shall assume the following version of the no-arbitrage condition on S:

Assumption 1.1 *The set $\mathcal{M}^e(S)$ is not empty.*[1]

[1] If follows from [DS 94] and [DS 98] that Assumption 1.1 is equivalent to the condition of "no free lunch with vanishing risk". This property can also be equivalently characterised in terms of the existence of a measure $\mathbf{Q} \sim \mathbf{P}$ such that the process S itself (rather than the integrals $H \cdot S$ for admissible integrands) is "something like a martingale". The precise notion in the general semi-martingale setting is that S is a sigma-martingale under $\mathbf{Q}$ (see [DS 98]); in the case when S is locally bounded (resp. bounded) the term "sigma-martingale" may be replaced by the more

In these notes we shall mainly be interested in the case when $\mathcal{M}^e(S)$ is not reduced to a singleton, i.e., the case of an *incomplete* financial market.

After having specified the process S modeling the financial market we now define the function $U(x)$ modeling the utility of an agent's wealth x at the terminal time T.

We make the classical assumptions that $U : \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$ is *increasing on* $\mathbb{R}$, *continuous* on $\{U > -\infty\}$, *differentiable and strictly concave* on the interior of $\{U > -\infty\}$, and that marginal utility tends to zero when wealth tends to infinity, i.e.,

$$U'(\infty) := \lim_{x \to \infty} U'(x) = 0. \tag{1.7}$$

These assumptions make good sense economically and it is clear that the requirement (1.7) of marginal utility decreasing to zero, as x tends to infinity, is necessary, if one is aiming for a general existence theorem for optimal investment.

As regards the behavior of the (marginal) utility at the other end of the wealth scale we shall distinguish two cases.

Case 1 (negative wealth not allowed): in this setting we assume that U satifies the conditions $U(x) = -\infty$, for $x < 0$, while $U(x) > -\infty$, for $x > 0$, and the so-called *Inada conditions*

$$U'(0) := \lim_{x \searrow 0} U'(x) = \infty. \tag{1.8}$$

Case 2 (negative wealth allowed): in this case we assume that $U(x) > -\infty$, for all $x \in \mathbb{R}$, and that

$$U'(-\infty) := \lim_{x \searrow -\infty} U'(x) = \infty. \tag{1.9}$$

Typical examples for case 1 are

$$U(x) = \ln(x), \quad x > 0, \tag{1.10}$$

or

$$U(x) = \frac{x^\alpha}{\alpha}, \quad \alpha \in (-\infty, 1) \setminus \{0\}, \quad x > 0, \tag{1.11}$$

whereas a typical example for case 2 is

$$U(x) = -e^{-\gamma x}, \quad \gamma > 0, \qquad\qquad x \in \mathbb{R}. \tag{1.12}$$

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familiar term "local martingale" (resp. "martingale").

Readers who are not too enthusiastic about the rather subtle distinctions between martingales, local martingales and sigma-martingales may find some relief by noting that, in the case of finite Ω, or, more generally, for bounded processes S, these three notions coincide.

We again note that it is natural from economic considerations to require that the marginal utility tends to infinity when the wealth x tends to the infimum of its allowed values.

For later reference we summarize our assumptions on the utility function:

Assumption 1.2 (Usual Regularity Conditions) *A utility function $U : \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$ satisfies the* usual regularity conditions *if it is increasing on $\mathbb{R}$, continuous on $\{U > -\infty\}$, differentiable and strictly concave on the interior of $\{U > -\infty\}$, and satisfies*

$$U'(\infty) := \lim_{x \to \infty} U'(x) = 0. \tag{1.13}$$

Denoting by $\mathrm{dom}(U)$ *the interior of $\{U > -\infty\}$, we assume that we have one of the two following cases.*

Case 1: $\mathrm{dom}(U) =]0, \infty[$ *in which case U satisfies the condition*

$$U'(0) := \lim_{x \searrow 0} U'(x) = \infty. \tag{1.14}$$

Case 2: $\mathrm{dom}(U) = \mathbb{R}$ *in which case U satisfies*

$$U'(-\infty) := \lim_{x \searrow -\infty} U'(x) = \infty. \tag{1.15}$$

We now can give a precise meaning to the problem of maximizing the expected utility of terminal wealth. Define the value function

$$u(x) := \sup_{H \in \mathcal{H}} \mathbf{E}\left[U(x + (H \cdot S)_T)\right], \quad x \in \mathrm{dom}(U), \tag{1.16}$$

where H ranges through the family $\mathcal{H}$ of admissible S-integrable trading strategies. To exclude trivial cases we shall assume that the value function u is not degenerate:

Assumption 1.3

$$u(x) < \sup_{\xi} U(\xi), \quad \textit{for some} \quad x \in \mathrm{dom}(U). \tag{1.17}$$

Since u is clearly increasing, and $U(y) \leq U(x) + U'(x)(y - x)$ for any $y > x$, this assumption implies that

$$u(x) < \sup_{\xi} U(\xi), \quad \text{for all} \quad x \in \mathrm{dom}(U). \tag{1.18}$$

Under appropriate hypotheses (e.g., when Ω is finite) Assumptions 1.1 and 1.2 already imply Assumption 1.3.

Chapter 2

Models on Finite Probability Spaces

In order to reduce the technical difficulties of the theory of utility maximization to a minimum, we assume throughout this chapter that the probability space Ω will be finite, say, $\Omega = \{\omega_1, \omega_2, \ldots, \omega_N\}$. This assumption implies that all the differences among the spaces $L^\infty(\Omega, \mathcal{F}, \mathbf{P})$, $L^1(\Omega, \mathcal{F}, \mathbf{P})$ and $L^0(\Omega, \mathcal{F}, \mathbf{P})$ disappear, as all these spaces are simply isomorphic to $\mathbb{R}^N$. Hence all the functional analysis reduces to simple linear algebra in the setting of the present chapter.

Nevertheless we shall write $L^\infty(\Omega, \mathcal{F}, \mathbf{P})$, $L^1(\Omega, \mathcal{F}, \mathbf{P})$ etc. below (knowing very well that these spaces are isomorphic in the present setting) to indicate, what we shall encounter in the setting of the general theory.

Definition 2.1 *A model of a* finite financial market *is an $\mathbb{R}^{d+1}$-valued stochastic process $S = (S)_{t=0}^T = (S_t^0, S_t^1, \ldots, S_t^d)_{t=0}^T$, based on and adapted to the filtered stochastic base $(\Omega, \mathcal{F}, (\mathcal{F})_{t=0}^T, \mathbf{P})$. Without loss of generality we assume that $\mathcal{F}_0$ is trivial, that $\mathcal{F}_T = \mathcal{F}$ is the power set of Ω, and that $\mathbf{P}[\omega_n] > 0$, for all $1 \leq n \leq N$. We assume that the zero coordinate S^0, which we call the cash account, satisfies $S_t^0 \equiv 1$, for $t = 0, 1, \ldots, T$. The letter ΔS_t denotes the increment $S_t - S_{t-1}$.*

Definition 2.2 *$\mathcal{H}$ denotes the set of* trading strategies *for the financial market S. An element $H \in \mathcal{H}$ is an $\mathbb{R}^d$-valued process $(H_t)_{t=1}^T = (H_t^1, H_t^2, \ldots, H_t^d)_{t=1}^T$ which is predictable, i.e. each H_t is $\mathcal{F}_{t-1}$-measurable.*

We then define the stochastic integral $(H \cdot S)$ as the $\mathbb{R}$-valued process $((H \cdot S)_t)_{t=0}^T$ given by

$$(H \cdot S)_t = \sum_{k=1}^{t} (H_k, \Delta S_k), \quad t = 0, \ldots, T, \tag{2.1}$$

where $(\,.\,,\,.\,)$ denotes the inner product in $\mathbb{R}^d$.

Definition 2.3 *We call the subspace K of $L^0(\Omega, \mathcal{F}, \mathbf{P})$ defined by*

$$K = \{(H \cdot S)_T : H \in \mathcal{H}\} \tag{2.2}$$

the set of contingent claims attainable at price 0*.*

The economic interpretation is the following: the random variables $f = (H \cdot S)_T$, for some $H \in \mathcal{H}$, are precisely those contingent claims, i.e., the pay-off functions at time T depending on $\omega \in \Omega$ in an $\mathcal{F}_T$-measurable way, that an economic agent may replicate with zero initial investment, by pursuing some predictable trading strategy H.

For $a \in \mathbb{R}$, we call the *set of contingent claims attainable at price a* the affine space K_a obtained by shifting K by the constant function $a\mathbf{1}$, in other words the random variables of the form $a + (H \cdot S)_T$, for some trading strategy H. Again the economic interpretation is that these are precisely the contingent claims that an economic agent may replicate with an initial investment of a by pursuing some predictable trading strategy H.

Definition 2.4 *We call the convex cone C in $L^\infty(\Omega, \mathcal{F}, \mathbf{P})$ defined by*

$$C \;=\; \{g \in L^\infty(\Omega, \mathcal{F}, \mathbf{P}) \text{ s.t. there is } f \in K, f \geq g\}\,. \tag{2.3}$$

the set of contingent claims super-replicable at price 0*.*

Economically speaking, a contingent claim $g \in L^\infty(\Omega, \mathcal{F}, \mathbf{P})$ is *super-replicable at price* 0, if we can achieve it with zero net investment, subsequently pursuing some predictable trading strategy H — thus arriving at some contingent claim f — and then, possibly, "throwing away money" to arrive at g. This operation of "throwing away money" may seem awkward at this stage, but we shall see later that the set C plays an important role in the development of the theory. Observe that C is a convex cone containing the negative orthant $L^\infty_-(\Omega, \mathcal{F}, \mathbf{P})$. Again we may define C_a as the *contingent claims super-replicable at price a* if we shift C by the constant function $a\mathbf{1}$.

Definition 2.5 *A financial market S satifies the* no-arbitrage condition *(NA) if*

$$K \cap L^0_+(\Omega, \mathcal{F}, \mathbf{P}) = \{0\} \tag{2.4}$$

or, equivalently,

$$C \cap L^\infty_+(\Omega, \mathcal{F}, \mathbf{P}) = \{0\} \tag{2.5}$$

where 0 *denotes the function identically equal to zero.*

In other words we now have formalized the concept of an arbitrage possibility: it consists of the existence of a trading strategy H such that — starting from an initial investment zero — the resulting contingent claim $f = (H \cdot S)_T$ is non-negative and not identically equal to zero. If a financial market does not allow for arbitrage we say it satisfies the *no-arbitrage condition (NA)*.

Definition 2.6 *A probability measure* $\mathbf{Q}$ *on* $(\Omega, \mathcal{F})$ *is called an* equivalent martingale measure *for* S*, if* $\mathbf{Q} \sim \mathbf{P}$ *and* S *is a martingale under* $\mathbf{Q}$*.*

We denote by $\mathcal{M}^e(S)$ the set of equivalent martingale probability measures and by $\mathcal{M}^a(S)$ the set of all (not necessarily equivalent) martingale probability measures. The letter a stands for "absolutely continuous with respect to $\mathbf{P}$" which in the present setting (finite Ω and $\mathbf{P}$ having full support) automatically holds true, but which will be of relevance for general probability spaces $(\Omega, \mathcal{F}, \mathbf{P})$ later. We shall often identify a measure $\mathbf{Q}$ on $(\Omega, \mathcal{F})$ with its Radon-Nikodym derivative $\frac{d\mathbf{Q}}{d\mathbf{P}} \in L^1(\Omega, \mathcal{F}, \mathbf{P})$.

Lemma 2.7 *For a probability measure* $\mathbf{Q}$ *on* $(\Omega, \mathcal{F})$ *the following are equivalent:*

(i) $\mathbf{Q} \in \mathcal{M}^a(S)$*,*

(ii) $\mathbf{E_Q}[f] = 0$*, for all* $f \in K$*,*

(iii) $\mathbf{E_Q}[g] \leq 0$*, for all* $g \in C$*.*

Proof The equivalences are rather trivial, as (ii) is tantamount to the very definition of S being a martingale under $\mathbf{Q}$, and the equivalence of (ii) and (iii) is straightforward. ■

After having fixed these formalities we may formulate and prove the central result of the theory of pricing and hedging by no-arbitrage, sometimes called the "fundamental theorem of asset pricing", which in its present form (i.e., finite Ω) is due to Harrison and Pliska [HP 81].

Theorem 2.8 (Fundamental Theorem of Asset Pricing) *For a financial market* S *modeled on a finite stochastic base* $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t=0}^T, \mathbf{P})$ *the following are equivalent:*

(i) S *satisfies (NA).*

(ii) $\mathcal{M}^e(S) \neq \emptyset$*.*

Proof (ii) $\Rightarrow$ (i): This is the obvious implication. If there is some $\mathbf{Q} \in \mathcal{M}^e(S)$ then by lemma 2.7 we have that

$$\mathbf{E}_{\mathbf{Q}}[g] \leq 0, \quad \text{for } g \in C. \tag{2.6}$$

On the other hand, if there were $g \in C \cap L^\infty_+$, $g \neq 0$, then, using the assumption that $\mathbf{Q}$ is equivalent to $\mathbf{P}$, we would have

$$\mathbf{E}_{\mathbf{Q}}[g] > 0, \tag{2.7}$$

a contradiction.

(i) $\Rightarrow$ (ii) This implication is the important message of the theorem which will allow us to link the no-arbitrage arguments with martingale theory. We give a functional analytic existence proof, which will be generalizable — in spirit — to more general situations.

By assumption the space K intersects L^∞_+ only at 0. We want to separate the disjoint convex sets $L^\infty_+ \backslash \{0\}$ and K by a hyperplane induced by a linear functional $\mathbf{Q} \in L^1(\Omega, \mathcal{F}, \mathbf{P})$ which is *strictly positive* on $L^\infty_+ \backslash \{0\}$. Unfortunately this is a situation, where the usual versions of the separation theorem (i.e., the Hahn-Banach Theorem) do not apply (even in finite dimensions!). Indeed, one usually assumes that one of the convex sets is compact in order to obtain a strict separation.

One way to overcome this difficulty (in finite dimension) is to consider the convex hull of the unit vectors $(\mathbf{1}_{\{\omega_n\}})_{n=1}^N$ in $L^\infty(\Omega, \mathcal{F}, \mathbf{P})$ i.e.

$$P := \left\{ \sum_{n=1}^N \mu_n \mathbf{1}_{\{\omega_n\}} : \mu_n \geq 0, \sum_{n=1}^N \mu_n = 1 \right\}. \tag{2.8}$$

This is a convex, compact subset of $L^\infty_+(\Omega, \mathcal{F}, \mathbf{P})$ and, by the *(NA)* assumption, disjoint from K. Hence we may strictly separate the sets P and K by a linear functional $\mathbf{Q} \in L^\infty(\Omega, \mathcal{F}, \mathbf{P})^* = L^1(\Omega, \mathcal{F}, \mathbf{P})$, i.e., find $\alpha < \beta$ such that

$$\begin{array}{rcccl} \mathbf{E}_{\mathbf{Q}}[f] & = & \langle \mathbf{Q}, f \rangle & \leq & \alpha \quad \text{for} \quad f \in K, \\ \mathbf{E}_{\mathbf{Q}}[h] & = & \langle \mathbf{Q}, h \rangle & \geq & \beta \quad \text{for} \quad h \in P. \end{array} \tag{2.9}$$

As K is a linear space, we have $\alpha \geq 0$ and may, in fact, replace α by 0. Hence $\beta > 0$. Therefore $\langle \mathbf{Q}, \mathbf{1} \rangle > 0$, and we may normalize $\mathbf{Q}$ such that $\langle \mathbf{Q}, \mathbf{1} \rangle = 1$. As $\mathbf{Q}$ is strictly positive on each $\mathbf{1}_{\{\omega_n\}}$, we therefore have found a probability measure $\mathbf{Q}$ on $(\Omega, \mathcal{F})$ equivalent to $\mathbf{P}$ such that condition (ii) of lemma 2.7 holds true. In other words, we found an equivalent martingale measure $\mathbf{Q}$ for the process S. ■

Corollary 2.9 *Let S satisfy (NA) and $f \in L^\infty(\Omega, \mathcal{F}, \mathbf{P})$ be an attainable contingent claim so that*

$$f = a + (H \cdot S)_T, \tag{2.10}$$

for some $a \in \mathbb{R}$ and some trading strategy H.

Then the constant a and the process $(H \cdot S)$ are uniquely determined by (2.10) *and satisfy, for every $\mathbf{Q} \in \mathcal{M}^e(S)$,*

$$a = \mathbf{E}_{\mathbf{Q}}[f], \quad \textit{and} \quad a + (H \cdot S)_t = \mathbf{E}_{\mathbf{Q}}[f|\mathcal{F}_t] \quad \textit{for} \quad 0 \leq t \leq T. \tag{2.11}$$

Proof As regards the uniqueness of the constant $a \in \mathbb{R}$, suppose that there are two representations $f = a^1 + (H^1 \cdot S)_T$ and $f = a^2 + (H^2 \cdot S)_T$ with $a^1 \neq a^2$. Assuming w.l.o.g. that $a^1 > a^2$ we find an obvious arbitrage possibility: we have $a^1 - a^2 = ((H^1 - H^2) \cdot S)_T$, i.e. the trading strategy $H^1 - H^2$ produces a strictly positive result at time T, a contradiction to *(NA)*.

As regards the uniqueness or the process $H \cdot S$ we simply apply a conditional version of the previous argument: assume that $f = a + (H^1 \cdot S)_T$ and $f = a + (H^2 \cdot S)_T$ such that the processes $H^1 \cdot S$ amd $H^2 \cdot S$ are not identical. Then there is $0 < t < T$ such that $(H^1 \cdot S)_t \neq (H^2 \cdot S)_t$; w.l.g. $A := \{(H^1 \cdot S)_t > (H^2 \cdot S)_t\}$ is a non-empty event, which clearly is in $\mathcal{F}_t$. Hence, using the fact hat $(H^1 \cdot S)_T = (H^2 \cdot S)_T$, the trading strategy $H := (H^2 - H^1)\chi_A \cdot \chi_{]t,T]}$ is a predictable process producing an arbitrage, as $(H \cdot S)_T = 0$ outside A, while $(H \cdot S)_T = (H^1 \cdot S)_t - (H^2 \cdot S)_t > 0$ on A, which again contradicts *(NA)*.

Finally, the equations in (2.11) result from the fact that, for every predictable process H and every $\mathbf{Q} \in \mathcal{M}^a(S)$, the process $H \cdot S$ is a $\mathbf{Q}$-martingale. Noting that, for a measure $\mathbf{Q} \sim \mathbf{P}$, the conditional expectation $\mathbf{E}_{\mathbf{Q}}[f|\mathcal{F}_t]$ is $\mathbf{P}$-a.s. well-defined we thus obtain (2.11) for each $\mathbf{Q} \in \mathcal{M}^e(S)$. ■

Denote by $\mathrm{cone}(\mathcal{M}^e(S))$ and $\mathrm{cone}(\mathcal{M}^a(S))$ the cones generated by the convex sets $\mathcal{M}^e(S)$ and $\mathcal{M}^a(S)$ respectively. The subsequent result clarifies the polar relation between these cones and the cone C. Recall (see, e.g., [S 66]) that, for a pair (E, E') of vector spaces in separating duality via the scalar product $\langle ., . \rangle$, the polar C^0 of a set C in E is defined as

$$C^0 = \{g \in E' : \langle f, g \rangle \leq 1, \text{ for all } f \in C\}. \tag{2.12}$$

In the case when C is closed under multiplication with positive scalars (e.g., if C is a convex cone) the polar C^0 may equivalently be defined by

$$C^0 = \{g \in E' : \langle f, g \rangle \leq 0, \text{ for all } f \in C\}. \tag{2.13}$$

The *bipolar theorem* (see, e.g., [S 66]) states that the bipolar $C^{00} := (C^0)^0$ of a set C in E is the $\sigma(E, E')$-closed convex hull of C.

After these general considerations we pass to the concrete setting of the cone $C \subseteq L^\infty(\Omega, \mathcal{F}, \mathbf{P})$ of contingent claims super-replicable at price 0. Note that in our finite-dimensional setting this convex cone is closed as it is the algebraic sum

of the closed linear space K (a linear space in $\mathbb{R}^N$ is always closed) and the closed polyhedral cone $L^\infty_-(\Omega, \mathcal{F}, \mathbf{P})$ (the verification, that the algebraic sum of a space and a polyhedral cone in $\mathbb{R}^N$ is closed, is an easy, but not completely trivial exercise). Hence we deduce from the bipolar theorem, that C equals its bipolar C^{00}.

Proposition 2.10 *Suppose that S satisfies (NA). Then the polar of C is equal to cone$(\mathcal{M}^a(S))$ and $\mathcal{M}^e(S)$ is dense in $\mathcal{M}^a(S)$. Hence the following assertions are equivalent for an element $g \in L^\infty(\Omega, \mathcal{F}, \mathbf{P})$*

(i) $g \in C$,

(ii) $\mathbf{E_Q}[g] \leq 0$, *for all* $g \in \mathcal{M}^a(S)$,

(iii) $\mathbf{E_Q}[g] \leq 0$, *for all* $g \in \mathcal{M}^e(S)$,

Proof The fact that the polar C^0 and cone$(\mathcal{M}^a(S))$ coincide, follows from lemma 2.7 and the observation that $C \supseteq L^\infty_-(\Omega, \mathcal{F}, \mathbf{P})$ implies $C^0 \subseteq L^\infty_+(\Omega, \mathcal{F}, \mathbf{P})$. Hence the equivalence of (i) and (ii) follows from the bipolar theorem.

As regards the density of $\mathcal{M}^e(S)$ in $\mathcal{M}^a(S)$ we first deduce from theorem 2.8 that there is at least one $\mathbf{Q}^* \in \mathcal{M}^e(S)$. For any $\mathbf{Q} \in \mathcal{M}^a(S)$ and $0 < \mu \leq 1$ we have that $\mu\mathbf{Q}^* + (1-\mu)\mathbf{Q} \in \mathcal{M}^e(S)$, which clearly implies the density of $\mathcal{M}^e(S)$ in $\mathcal{M}^a(S)$. The equivalence of (ii) and (iii) now is obvious. ■

The subsequent theorem tells us precisely what the principle of no arbitrage can tell us about the possible prices for a contingent claim f. It goes back to the work of D. Kreps [K 81] and was subsequently extended by several authors.

For given $f \in L^\infty(\Omega, \mathcal{F}, \mathbf{P})$, we call $a \in \mathbb{R}$ an *arbitrage-free price*, if in addition to the financial market S, the introduction of the contingent claim, which pays the random amount f at time $t = T$ and can be bought or sold at price a at time $t = 0$, does not create an arbitrage possibility. Mathematically speaking, this can be formalized as follows. Let $C^{f,a}$ denote the cone spanned by C and the linear space spanned by $f - a$; then a is an arbitrage-free price for f if $C^{f,a} \cap L^\infty_+(\Omega, \mathcal{F}, \mathbf{P}) = \{0\}$.

Theorem 2.11 (Pricing by No-Arbitrage) *Assume that S satisfies (NA) and let $f \in L^\infty(\Omega, \mathcal{F}, \mathbf{P})$. Define*

$$\overline{\pi}(f) = \sup\{\mathbf{E_Q}[f] : \mathbf{Q} \in \mathcal{M}^e(S)\}, \tag{2.14}$$

$$\underline{\pi}(f) = \inf\{\mathbf{E_Q}[f] : \mathbf{Q} \in \mathcal{M}^e(S)\}. \tag{2.15}$$

Either $\underline{\pi}(f) = \overline{\pi}(f)$, in which case f is attainable at price $\pi(f) := \underline{\pi}(f) = \overline{\pi}(f)$, i.e. $f = \pi(f) + (H \cdot S)_T$ for some $H \in \mathcal{H}$; therefore $\pi(f)$ is the unique arbitrage-free price for f.

Or $\underline{\pi}(f) < \overline{\pi}(f)$, in which case $\{\mathbf{E_Q}[f] : \mathbf{Q} \in \mathcal{M}^e(S)\}$ equals the open interval $]\underline{\pi}(f), \overline{\pi}(f)[$, which in turn equals the set of arbitrage-free prices for the contingent claim f.

Proof First observe that the set $\{\mathbf{E}_{\mathbf{Q}}[f] : \mathbf{Q} \in \mathcal{M}^e(S)\}$ forms a bounded non-empty interval in $\mathbb{R}$, which we denote by I.

We claim that a number a is in I, iff a is an arbitrage-free price for f. Indeed, supposing that $a \in I$ we may find $\mathbf{Q} \in \mathcal{M}^e(S)$ s.t. $\mathbf{E}_{\mathbf{Q}}[f-a] = 0$ and therefore $C^{f,a} \cap L^\infty_+(\Omega, \mathcal{F}, \mathbf{P}) = \{0\}$.

Conversely suppose that $C^{f,a} \cap L^\infty_+(\Omega, \mathcal{F}, \mathbf{P}) = \{0\}$. Note that $C^{f,a}$ is a closed convex cone (it is the albegraic sum of the linear space $\mathrm{span}(K, f-a)$ and the closed, polyhedral cone $L^\infty_-(\Omega, \mathcal{F}, \mathbf{P})$). Hence by the same argument as in the proof of theorem 2.8 there exists a probability measure $\mathbf{Q} \sim \mathbf{P}$ such that $\mathbf{Q}|_{C^{f,a}} \leq 0$. This implies that $\mathbf{E}_{\mathbf{Q}}[f-a] = 0$, i.e., $a \in I$.

Now we deal with the boundary case: suppose that a equals the right boundary of I, i.e., $a = \overline{\pi}(f) \in I$, and consider the contingent claim $f - \overline{\pi}(f)$; by definition we have $\mathbf{E}_{\mathbf{Q}}[f - \overline{\pi}(f)] \leq 0$, for all $\mathbf{Q} \in \mathcal{M}^e(S)$, and therefore by proposition 2.10, that $f - \overline{\pi}(f) \in C$. We may find $g \in K$ such that $g \geq f - \overline{\pi}(f)$. If the sup in (2.14) is attained, i.e., if there is $\mathbf{Q}^* \in \mathcal{M}^e(S)$ such that $\mathbf{E}_{\mathbf{Q}^*}[f] = \overline{\pi}(f)$, then we have $0 = \mathbf{E}_{\mathbf{Q}^*}[g] \geq \mathbf{E}_{\mathbf{Q}^*}[f - \overline{\pi}(f)] = 0$ which in view of $\mathbf{Q}^* \sim \mathbf{P}$ implies that $f - \overline{\pi}(f) \equiv g$; in other words f is attainable at price $\overline{\pi}(f)$. This in turn implies that $\mathbf{E}_{\mathbf{Q}}[f] = \overline{\pi}(f)$, for all $\mathbf{Q} \in \mathcal{M}^e(S)$, and therefore I is reduced to the singleton $\{\overline{\pi}(f)\}$.

Hence, if $\underline{\pi}(f) < \overline{\pi}(f)$, $\overline{\pi}(f)$ connot belong to the interval I, which is therefore open on the right hand side. Passing from f to $-f$, we obtain the analogous result for the left hand side of I, which therefore equals $I =]\underline{\pi}(f), \overline{\pi}(f)[$. ■

Corollary 2.12 (complete financial markets) *For a financial market S satisfying the no-arbitrage condition (NA) the following are equivalent:*

(i) $\mathcal{M}^e(S)$ consists of a single element $\mathbf{Q}$.

(ii) Each $f \in L^\infty(\Omega, \mathcal{F}, \mathbf{P})$ may be represented as

$$f = a + (H \cdot S)_T, \quad \text{for some } a \in \mathbb{R}, \text{ and } H \in \mathcal{H}. \tag{2.16}$$

In this case $a = \mathbf{E}_{\mathbf{Q}}[f]$, the stochastic integral $(H \cdot S)$ is unique and we have that

$$\mathbf{E}_{\mathbf{Q}}[f|\mathcal{F}_t] = \mathbf{E}_{\mathbf{Q}}[f] + (H \cdot S)_t, \quad t = 0, \ldots, T. \tag{2.17}$$

Proof The implication (i) ⇒ (ii) immediately follows from the preceding theorem; for the implication (ii) ⇒ (i), note that, (2.16) implies that, for elements $\mathbf{Q}_1$, $\mathbf{Q}_2 \in \mathcal{M}^a(S)$, we have $\mathbf{E}_{\mathbf{Q}_1}[f] = a = \mathbf{E}_{\mathbf{Q}_2}[f]$; hence it suffices to note that. if $\mathcal{M}^e(S)$ contains two different elements $\mathbf{Q}_1$, $\mathbf{Q}_2$. we may find $f \in L^\infty(\Omega, \mathcal{F}, \mathbf{P})$ s.t. $\mathbf{E}_{\mathbf{Q}_1}[f] \neq \mathbf{E}_{\mathbf{Q}_2}[f]$. ■

2.1 Utility Maximization

We are now ready to study utility maximization problems with the convex duality approach.

2.1.1 The complete case (Arrow)

As a first case we analyze the situation of a *complete* financial market (Corollary 2.12 above), i.e., the set $\mathcal{M}^e(S)$ of equivalent probability measures under which S is a martingale is reduced to a singleton $\{\mathbf{Q}\}$. In this setting consider the *Arrow assets* $\mathbf{1}_{\{\omega_n\}}$, which pay 1 unit of the numéraire at time T, when ω_n turns out to be the true state of the world, and 0 otherwise. In view of our normalization of the numéraire $S^0_t \equiv 1$, we get for the price of the Arrow assets at time $t = 0$ the relation

$$\mathbf{E_Q}\left[\mathbf{1}_{\{\omega_n\}}\right] = \mathbf{Q}[\omega_n] = q_n, \tag{2.18}$$

and by 2.12 each Arrow asset $\mathbf{1}_{\{\omega_n\}}$ may be represented as $\mathbf{1}_{\{\omega_n\}} = \mathbf{Q}[\omega_n] + (H \cdot S)_T$, for some predictable trading strategy $H \in \mathcal{H}$.

Hence, for fixed initial endowment $x \in \mathrm{dom}(U)$, the utility maximization problem (1.16) above may simply be written as

$$\mathbf{E_P}\left[U(X_T)\right] = \sum_{n=1}^{N} p_n U(\xi_n) \to \max! \tag{2.19}$$

$$\mathbf{E_Q}[X_T] = \sum_{n=1}^{N} q_n \xi_n \quad \leq \; x. \tag{2.20}$$

To verify that (2.19) and (2.20) indeed are equivalent to the original problem (1.16) above (in the present finite, complete case), note that by Theorem 2.11 a random variable $(X_T(\omega_n))_{n=1}^N = (\xi_n)_{n=1}^N$ can be dominated by a random variable of the form $x + (H \cdot S)_T = x + \sum_{t=1}^{T} H_t \Delta S_t$ iff $\mathbf{E_Q}[X_T] = \sum_{n=1}^{N} q_n \xi_n \leq x$. This basic relation has a particularly evident interpretation in the present setting, as q_n is simply the price of the Arrow asset $\mathbf{1}_{\{\omega_n\}}$.

We have written ξ_n for $X_T(\omega_n)$ to stress that (2.19) simply is a concave maximization problem in $\mathbb{R}^N$ with one linear constraint. To solve it, we form the Lagrangian

$$\begin{aligned} L(\xi_1, \ldots, \xi_N, y) &= \sum_{n=1}^{N} p_n U(\xi_n) - y\left(\sum_{n=1}^{N} q_n \xi_n - x\right) && (2.21) \\ &= \sum_{n=1}^{N} p_n \left(U(\xi_n) - y\tfrac{q_n}{p_n}\xi_n\right) + yx. && (2.22) \end{aligned}$$

We have used the letter $y \geq 0$ instead of the usual $\lambda \geq 0$ for the Lagrange multiplier; the reason is the dual relation between x and y which will become apparent in a moment.

Write

$$\Phi(\xi_1, \ldots, \xi_N) = \inf_{y>0} L(\xi_1, \ldots, \xi_N, y), \quad \xi_n \in \mathrm{dom}(U), \tag{2.23}$$

and

$$\Psi(y) = \sup_{\xi_1,\ldots,\xi_N} L(\xi_1, \ldots, \xi_N, y), \quad y \geq 0. \tag{2.24}$$

Note that we have

$$\sup_{\xi_1,\ldots,\xi_N} \Phi(\xi_1, \ldots, \xi_N) = \sup_{\substack{\xi_1,\ldots,\xi_N \\ \sum_{n=1}^N q_n\xi_n \leq x}} \sum_{n=1}^N p_n U(\xi_n) = u(x). \tag{2.25}$$

Indeed, if $(\xi_1, \ldots, \xi_N)$ is in the admissible region $\sum_{n=1}^N q_n\xi_n \leq x$ then $\Phi(\xi_1, \ldots, \xi_N) = L(\xi_1, \ldots, \xi_N, 0) = \sum_{n=1}^N p_n U(\xi_n)$. On the other hand, if $(\xi_1, \ldots, \xi_N)$ satisfies $\sum_{n=1}^N q_n\xi_n > x$, then by letting $y \to \infty$ in (2.23) we note that $\Phi(\xi_1, \ldots, \xi_N) = -\infty$.

As regards the function $\Psi(y)$ we make the following pleasant observation which is the basic reason for the efficiency of the duality approach: using the form (2.22) of the Lagrangian and fixing $y > 0$, the optimization problem appearing in (2.24) splits into N independent optimization problems over $\mathbb{R}$

$$U(\xi_n) - y\tfrac{q_n}{p_n}\xi_n \to \max!, \quad \xi_n \in \mathbb{R}. \tag{2.26}$$

In fact, these one-dimensional optimization problems are of a very convenient form: recall (see, e.g., [R 70], [ET 76] or [KLSX 91]) that, for a concave function $U : \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$, the *conjugate function* V (which is just the Legendre-transform of $x \mapsto -U(-x)$) is defined by

$$V(\eta) = \sup_{\xi \in \mathbb{R}} \left[U(\xi) - \eta\xi\right], \quad \eta > 0. \tag{2.27}$$

Definition 2.13 *We say that the function $V : \mathbb{R} \to \mathbb{R}$, conjugate to the function U, satisfies the* usual regularity assumptions, *if V is finitely valued, differentiable, strictly convex on $]0, \infty[$, and satisfies*

$$V'(0) := \lim_{y \searrow 0} V'(y) = -\infty. \tag{2.28}$$

As regards the behavior of V at infinity, we have to distinguish between case 1 and case 2 in Assumption 1.2 above:

$$\textit{case 1:} \qquad \lim_{y\to\infty} V(y) = \lim_{x\to 0} U(x) \qquad \textit{and} \qquad \lim_{y\to\infty} V'(y) = 0 \tag{2.29}$$

$$\textit{case 2:} \qquad \lim_{y\to\infty} V(y) = \infty \qquad\quad\; \textit{and} \qquad \lim_{y\to\infty} V'(y) = \infty \tag{2.30}$$

We have the following wellknown fact (see [R 70] or [ET 76]).

Proposition 2.14 *If U satisfies Assumption 1.2, then its conjugate function V satisfies the the inversion formula*

$$U(\xi) = \inf_{\eta} \left[V(\eta) + \eta\xi \right], \quad \xi \in \mathrm{dom}(U) \tag{2.31}$$

and satisfies the regularity assumptions in Definition 2.13. In addition, $-V'(y)$ is the inverse function of $U'(x)$. Conversely, if V satisfies the regulatory assumptions of Definition 2.13, then U defined by (2.31) *satisfies Assumption 1.2.*

Following [KLS 87] we denote $-V' = I$ (for "inverse" function).

Proof It follows from Assumption 1.2 that V is finitely valued on $]0, \infty[$. Note that we have that

$$U(x) \leq a + yx \quad \forall x \in \mathrm{dom}(U) \qquad \Longleftrightarrow \qquad V(y) \leq a \tag{2.32}$$

which implies the inversion formula above. In turn, this formula shows that V is the supremum of affine functions, and therefore convex. Since U is strictly concave and differentiable, the maximizer $\widehat{\xi} = \xi(\mu)$ in (2.27) solves the first-order condition $U'(\xi(\eta)) = \eta$. Also, we have that U' is a continuous bijection between $\{U > -\infty\}$ and $\mathbb{R}_+$. This observation and the inversion formula show that V is both strictly convex, differentiable, and that $-V'$ is the inverse of U'. ■

Remark 2.15 Of course, U' has a good economic interpretation as the *marginal utility* of an economic agent modeled by the utility function U.

Here are some concrete examples of pairs of conjugate functions:

$$\begin{array}{ll} U(x) = \ln(x), \quad x > 0, & V(y) = -\ln(y) - 1, \\ U(x) = -\frac{e^{-\gamma x}}{\gamma}, \quad x \in \mathbb{R}, & V(y) = \frac{y}{\gamma}(\ln(y) - 1), \quad \gamma > 0 \\ U(x) = \frac{x^\alpha}{\alpha}, \quad x > 0, & V(y) = \frac{1-\alpha}{\alpha} y^{\frac{\alpha}{\alpha-1}}, \quad \alpha \in (-\infty, 1) \setminus \{0\}. \end{array}$$

We now apply these general facts about the Legendre transformation to calculate $\Psi(y)$. Using definition (2.27) of the conjugate function V and (2.22), formula (2.24) becomes

$$\begin{aligned} \Psi(y) &= \sum_{n=1}^{N} p_n V\left(y \tfrac{q_n}{p_n}\right) + yx & (2.33) \\ &= \mathbf{E_P}\left[V\left(y \tfrac{d\mathbf{Q}}{d\mathbf{P}}\right)\right] + yx. & (2.34) \end{aligned}$$

Denoting by $v(y)$ the dual value function

$$v(y) := \mathbf{E_P}\left[V\left(y\tfrac{d\mathbf{Q}}{d\mathbf{P}}\right)\right] = \sum_{n=1}^{N} p_n V\left(y\tfrac{q_n}{p_n}\right), \quad y > 0, \tag{2.35}$$

the function v has the same qualitative properties as the function V listed in Definition 2.13, since it is a convex combination of V calculated on linearly scaled arguments.

Hence by (2.28), (2.29), and (2.30) we find, for fixed $x \in \mathrm{dom}(U)$, a unique $\widehat{y} = \widehat{y}(x) > 0$ such that $v'(\widehat{y}(x)) = -x$, which therefore is the unique minimizer to the dual problem

$$\Psi(y) = \mathbf{E_P}\left[V\left(y\tfrac{d\mathbf{Q}}{d\mathbf{P}}\right)\right] + yx = \min! \tag{2.36}$$

Fixing the critical value $\widehat{y}(x)$, the concave function

$$(\xi_1, \ldots, \xi_N) \mapsto L(\xi_1, \ldots, \xi_N, \widehat{y}(x)) \tag{2.37}$$

defined in (2.22) assumes its unique maximum at the point $(\widehat{\xi}_1, \ldots, \widehat{\xi}_N)$ satisfying

$$U'(\widehat{\xi}_n) = \widehat{y}(x)\tfrac{q_n}{p_n} \quad \text{or, equivalently,} \quad \widehat{\xi}_n = I\left(\widehat{y}(x)\tfrac{q_n}{p_n}\right), \tag{2.38}$$

so that we have

$$\begin{aligned} \inf_{y>0} \Psi(y) &= \inf_{y>0} (v(y) + xy) && (2.39)\\ &= v(\widehat{y}(x)) + x\widehat{y}(x) && (2.40)\\ &= L(\widehat{\xi}_1, \ldots, \widehat{\xi}_N, \widehat{y}(x)). && (2.41) \end{aligned}$$

Note that $\widehat{\xi}_n$ are in $\mathrm{dom}(U)$, for $1 \le n \le N$, so that L is continuously differentiable at $(\widehat{\xi}_1, \ldots, \widehat{\xi}_N, \widehat{y}(x))$, which implies that the gradient of L vanishes at $(\widehat{\xi}_1, \ldots, \widehat{\xi}_N, \widehat{y}(x))$ and, in particular, that $\frac{\partial}{\partial y} L(\xi_1, \ldots, \xi_N, y)|_{(\widehat{\xi}_1, \ldots, \widehat{\xi}_N, \widehat{y}(x))} = 0$. Hence we infer from (2.21) and the fact that $\widehat{y}(x) > 0$ that the constraint (2.20) is binding, i.e.,

$$\sum_{n=1}^{N} q_n \widehat{\xi}_n = x, \tag{2.42}$$

and that

$$\sum_{n=1}^{N} p_n U(\widehat{\xi}_n) = L(\widehat{\xi}_1, \ldots, \widehat{\xi}_N, \widehat{y}(x)). \tag{2.43}$$

In particular, we obtain that

$$u(x) = \sum_{n=1}^{N} p_n U(\widehat{\xi}_n). \tag{2.44}$$

Indeed, the inequality $u(x) \geq \sum_{n=1}^{N} p_n U(\widehat{\xi}_n)$ follows from (2.42) and (2.25), while the reverse inequality follows from (2.43) and the fact that for all $\xi_1, \ldots, \xi_N$ verifying the constraint (2.20)

$$\sum_{n=1}^{N} p_n U(\xi_n) \leq L(\xi_1, \ldots, \xi_N, \widehat{y}(x)) \leq L(\widehat{\xi}_1, \ldots, \widehat{\xi}_N, \widehat{y}(x)). \tag{2.45}$$

We shall write $\widehat{X}_T(x) \in C(x)$ for the optimizer $\widehat{X}_T(x)(\omega_n) = \widehat{\xi}_n$, $n = 1, \ldots, N$.

Combining (2.39), (2.43) and (2.44) we note that the value functions u and v are conjugate:

$$\inf_{y>0} (v(y) + xy) = v(\widehat{y}(x)) + x\widehat{y}(x) = u(x), \quad x \in \mathrm{dom}(U), \tag{2.46}$$

which, by Proposition 2.14 the remarks after equation (2.35), implies that u inherits the properties of U listed in Assumption 1.2. The relation $v'(\widehat{y}(x)) = -x$ which was used to define $\widehat{y}(x)$, therefore translates into

$$u'(x) = \widehat{y}(x), \quad \text{for } x \in \mathrm{dom}(U). \tag{2.47}$$

Let us summarize what we have proved:

Theorem 2.16 (finite Ω, complete market) *Let the financial market $S = (S_t)_{t=0}^T$ be defined over the finite filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F})_{t=0}^T, \mathbf{P})$ and satisfy $\mathcal{M}^e(S) = \{\mathbf{Q}\}$, and let the utility function U satisfy Assumption 1.2.*

Denote by $u(x)$ and $v(y)$ the value functions

$$u(x) = \sup_{X_T \in C(x)} \mathbf{E}[U(X_T)], \qquad x \in \mathrm{dom}(U), \tag{2.48}$$

$$v(y) = \mathbf{E}\left[V\left(y\tfrac{d\mathbf{Q}}{d\mathbf{P}}\right)\right], \qquad y > 0. \tag{2.49}$$

We then have:

(i) The value functions $u(x)$ and $v(y)$ are conjugate and u inherits the qualitative properties of U listed in Assumption 1.2.

(ii) The optimizer $\widehat{X}_T(x)$ in (2.48) exists, is unique and satisfies

$$\widehat{X}_T(x) = I(y\tfrac{d\mathbf{Q}}{d\mathbf{P}}), \quad \text{or, equivalently,} \quad y\tfrac{d\mathbf{Q}}{d\mathbf{P}} = U'(\widehat{X}_T(x)), \tag{2.50}$$

where $x \in \mathrm{dom}(U)$ and $y > 0$ are related via $u'(x) = y$ or, equivalently, $x = -v'(y)$.

(iii) The following formulae for u' and v' hold true:

$$u'(x) = \mathbf{E_P}[U'(\widehat{X}_T(x))], \qquad v'(y) = \mathbf{E_Q}\left[V'\left(y\tfrac{d\mathbf{Q}}{d\mathbf{P}}\right)\right] \tag{2.51}$$

$$xu'(x) = \mathbf{E_P}\left[\widehat{X}_T(x)U'(\widehat{X}_T(x))\right], \qquad yv'(y) = \mathbf{E_P}\left[y\tfrac{d\mathbf{Q}}{d\mathbf{P}}V'\left(y\tfrac{d\mathbf{Q}}{d\mathbf{P}}\right)\right]. \tag{2.52}$$

Proof Items (i) and (ii) have been shown in the preceding discussion, hence we only have to show (iii). The formulae for $v'(y)$ in (2.51) and (2.52) immediately follow by differentiating the relation

$$v(y) = \mathbf{E_P}\left[V\left(y\tfrac{d\mathbf{Q}}{d\mathbf{P}}\right)\right] = \sum_{n=1}^{N} p_n V\left(y\tfrac{q_n}{p_n}\right). \tag{2.53}$$

Of course, the formula for v' in (2.52) is an obvious reformulation of the one in (2.51). But we write both of them to stress their symmetry with the formulae for $u'(x)$.

The formula for u' in (2.51) translates via the relations exhibited in (ii) into the identity

$$y = \mathbf{E_P}\left[y\tfrac{d\mathbf{Q}}{d\mathbf{P}}\right], \tag{2.54}$$

while the formula for $u'(x)$ in (2.52) translates into

$$v'(y)y = \mathbf{E_P}\left[V'\left(y\tfrac{d\mathbf{Q}}{d\mathbf{P}}\right)y\tfrac{d\mathbf{Q}}{d\mathbf{P}}\right], \tag{2.55}$$

which we just have seen to hold true. ■

Remark 2.17 Firstly, let us recall the economic interpretation of (2.50)

$$U'\left(\widehat{X}_T(x)(\omega_n)\right) = y\frac{q_n}{p_n}, \quad n = 1, \dots, N. \tag{2.56}$$

This equality means that, in every possible state of the world ω_n, the *marginal utility* $U'(\widehat{X}_T(x)(\omega_n))$ of the wealth of an optimally investing agent at time T is *proportional to the ratio of the price* q_n *of the corresponding Arrow security* $\mathbf{1}_{\{\omega_n\}}$ *and the probability of its success* $p_n = \mathbf{P}[\omega_n]$. This basic relation was analyzed in the fundamental work of K. Arrow and allows for a convincing economic interpretation: considering for a moment the situation where this proportionality relation fails to hold true, one immediately deduces from a marginal variation argument that the investment of the agent cannot be optimal. Indeed, by investing a little more in the more favorable asset and a little less in the less favorable the economic agent can strictly increase expected utility under the same budget constraint. Hence for the optimal investment the proportionality must hold true. The above result also

identifies the proportionality factor as $y = u'(x)$, where x is the initial endowment of the investor. This also allows for an economic interpretation.

Theorem 2.16 indicates an easy way to solve the utility maximization at hand: calculate $v(y)$ by (2.49), which reduces to a simple one-dimensional computation; once we know $v(y)$, the theorem provides easy formulae to calculate all the other quantities of interest, e.g., $\widehat{X}_T(x)$, $u(x)$, $u'(x)$ etc.

Another message of the above theorem is that the value function $x \mapsto u(x)$ may be viewed as a utility function as well, sharing all the qualitative features of the original utility function U. This makes sense economically, as the "indirect utility" function $u(x)$ denotes the expected utility at time T of an agent with initial endowment x, after having optimally invested in the financial market S.

Let us also give an economic interpretation of the formulae for $u'(x)$ in item (iii) along these lines: suppose the initial endowment x is varied to $x+h$, for some small real number h. The economic agent may use the additional endowment h to finance, in addition to the optimal pay-off function $\widehat{X}_T(x)$, h units of the cash account, thus ending up with the pay-off function $\widehat{X}_T(x)+h$ at time T. Comparing this investment strategy to the optimal one corresponding to the initial endowment $x+h$, which is $\widehat{X}_T(x+h)$, we obtain

$$\begin{aligned} \lim_{h\to 0} \frac{u(x+h)-u(x)}{h} &= \lim_{h\to 0} \frac{\mathbf{E}[U(\widehat{X}_T(x+h)) - U(\widehat{X}_T(x))]}{h} && (2.57)\\ &\geq \lim_{h\to 0} \frac{\mathbf{E}[U(\widehat{X}_T(x)+h) - U(\widehat{X}_T(x))]}{h} && (2.58)\\ &= \mathbf{E}[U'(\widehat{X}_T(x))]. && (2.59) \end{aligned}$$

Using the fact that u is differentiable, and that h may be positive as well as negative, we must have equality in (2.58) and therefore have found another proof of formula (2.51) for $u'(x)$; the economic interpretation of this proof is that the economic agent, who is optimally investing, is indifferent of first order towards a (small) additional investment into the cash account.

Playing the same game as above, but using the additional endowment $h \in \mathbb{R}$ to finance an additional investment into the optimal portfolio $\widehat{X}_T(x)$ (assuming, for simplicity, $x \neq 0$), we arrive at the pay-off function $\frac{x+h}{x}\widehat{X}_T(x)$. Comparing this investment with $\widehat{X}_T(x+h)$, an analogous calculation as in (2.57) leads to the formula for $u'(x)$ displayed in (2.52). The interpretation now is, that the optimally investing economic agent is indifferent of first order towards a marginal variation of the investment into the optimal portfolio.

It now becomes clear that formulae (2.51) and (2.52) for $u'(x)$ are just special cases of a more general principle: for each $f \in L^\infty(\Omega, \mathcal{F}, \mathbf{P})$ we have

$$\mathbf{E}_\mathbf{Q}[f]u'(x) = \lim_{h\to 0} \frac{\mathbf{E}_\mathbf{P}[U(\widehat{X}_T(x)+hf) - U(\widehat{X}_T(x))]}{h}. \tag{2.60}$$

The proof of this formula again is along the lines of (2.57) and the interpretation is the following: by investing an additional endowment $h\mathbf{E}_{\mathbf{Q}}[f]$ to finance the contingent claim hf, the increase in expected utility is of first order equal to $h\mathbf{E}_{\mathbf{Q}}[f]u'(x)$; hence again the economic agent is of first order indifferent towards an additional investment into the contingent claim f.

2.1.2 The Incomplete Case

We now drop the assumption that the set $\mathcal{M}^e(S)$ of equivalent martingale measures is reduced to a singleton (but we still remain in the framework of a finite probability space Ω) and replace it by Assumption 1.1 requiring that $\mathcal{M}^e(S) \neq \emptyset$.

In this setting it follows from Theorem 2.11 that a random variable $X_T(\omega_n) = \xi_n$ may be dominated by a random variable of the form $x + (H \cdot S)_T$ iff $\mathbf{E}_{\mathbf{Q}}[X_T] = \sum_{n=1}^N q_n \xi_n \leq x$, for each $\mathbf{Q} = (q_1 \ldots, q_N) \in \mathcal{M}^a(S)$ (or equivalently, for every $\mathbf{Q} \in \mathcal{M}^e(S)$).

In order to reduce the infinitely many constraints, where $\mathbf{Q}$ runs through $\mathcal{M}^a(S)$, to a finite number, make the easy observation that $\mathcal{M}^a(S)$ is a bounded, closed, convex polytope in $\mathbb{R}^N$ and therefore the convex hull of its finitely many extreme points $\{\mathbf{Q}^1, \ldots, \mathbf{Q}^M\}$. Indeed, $\mathcal{M}^a(S)$ is given by finitely many linear constraints. For $1 \leq m \leq M$, we identify $\mathbf{Q}^m$ with the probabilites $(q_1^m, \ldots, q_N^m)$.

Fixing the initial endowment $x \in \mathrm{dom}(U)$, we therefore may write the utility maximization problem (1.16) similarly as in (2.19) as a concave optimization problem over $\mathbb{R}^N$ with finitely many linear constraints:

$$\mathbf{E}_{\mathbf{P}}\left[U(X_T)\right] = \sum_{n=1}^N p_n U(\xi_n) \to \max! \tag{2.61}$$

$$\mathbf{E}_{\mathbf{Q}^m}[X_T] = \sum_{n=1}^N q_n^m \xi_n \leq x, \quad \text{for } m = 1, \ldots, M. \tag{2.62}$$

Writing again

$$C(x) = \left\{X_T \in L^0(\Omega, \mathcal{F}, \mathbf{P}) : \mathbf{E}_{\mathbf{Q}}[X_T] \leq x, \text{ for all } \mathbf{Q} \in \mathcal{M}^a(S)\right\} \tag{2.63}$$

we define the value function, for $x \in \mathrm{dom}(U)$,

$$u(x) = \sup_{H \in \mathcal{H}} \mathbf{E}\left[U\left(x + (H \cdot S)_T\right)\right] = \sup_{X_T \in C(x)} \mathbf{E}[U(X_T)]. \tag{2.64}$$

The Lagrangian now is given by

$$
\begin{aligned}
& L(\xi_1, \ldots, \xi_N, \eta_1, \ldots, \eta_M) && (2.65)\\
&= \sum_{n=1}^{N} p_n U(\xi_n) - \sum_{m=1}^{M} \eta_m \left(\sum_{n=1}^{N} q_n^m \xi_n - x \right) && (2.66)\\
&= \sum_{n=1}^{N} p_n \left(U(\xi_n) - \sum_{m=1}^{M} \frac{\eta_m q_n^m}{p_n} \xi_n \right) + \sum_{m=1}^{M} \eta_m x, && (2.67)\\
& \quad \text{where} \quad (\xi_1, \ldots, \xi_N) \in \mathrm{dom}(U)^N, \quad (\eta_1, \ldots, \eta_M) \in \mathbb{R}_+^M. && (2.68)
\end{aligned}
$$

Writing $y = \eta_1 + \ldots + \eta_M$, $\mu_m = \frac{\eta_m}{y}$, $\mu = (\mu_1, \ldots, \mu_M)$ and

$$
\mathbf{Q}^\mu = \sum_{m=1}^{M} \mu_m \mathbf{Q}^m, \tag{2.69}
$$

note that, when $(\eta_1, \ldots, \eta_M)$ runs trough $\mathbb{R}_+^M$, the pairs $(y, \mathbf{Q}^\mu)$ run through $\mathbb{R}_+ \times \mathcal{M}^a(S)$. Hence we may write the Lagrangian as

$$
\begin{aligned}
L(\xi_1, \ldots, \xi_N, y, \mathbf{Q}) = &= \mathbf{E}_\mathbf{P}[U(X_T)] - y \left(\mathbf{E}_\mathbf{Q}[X_T - x] \right) \\
&= \sum_{n=1}^{N} p_n \left(U(\xi_n) - \frac{y q_n}{p_n} \xi_n \right) + yx, && (2.70)
\end{aligned}
$$

where $\xi_n \in \mathrm{dom}(U)$, $y > 0$, $\mathbf{Q} = (q_1, \ldots, q_N) \in \mathcal{M}^a(S)$.

This expression is entirely analogous to (2.22), the only difference now being that $\mathbf{Q}$ runs through the set $\mathcal{M}^a(S)$ instead of being a fixed probability measure. Defining again

$$
\Phi(\xi_1, \ldots, \xi_n) = \inf_{y>0, \mathbf{Q} \in \mathcal{M}^a(S)} L(\xi_1, \ldots, \xi_N, y, \mathbf{Q}), \tag{2.71}
$$

and

$$
\Psi(y, \mathbf{Q}) = \sup_{\xi_1, \ldots, \xi_N} L(\xi_1, \ldots, \xi_N, y, \mathbf{Q}), \tag{2.72}
$$

we obtain, just as in the complete case,

$$
\sup_{\xi_1, \ldots, \xi_N} \Phi(\xi_1, \ldots, \xi_N) = u(x), \quad x \in \mathrm{dom}(U), \tag{2.73}
$$

and

$$
\Psi(y, \mathbf{Q}) = \sum_{n=1}^{N} p_n V \left(\frac{y q_n}{p_n} \right) + yx, \qquad y > 0, \quad \mathbf{Q} \in \mathcal{M}^a(S), \tag{2.74}
$$

where $(q_1, \ldots, q_N)$ denotes the probabilities of $\mathbf{Q} \in \mathcal{M}^a(S)$. The minimization of Ψ will be done in two steps: first we fix $y > 0$ and minimize over $\mathcal{M}^a(S)$, i.e.,

$$\Psi(y) := \inf_{\mathbf{Q} \in \mathcal{M}^a(S)} \Psi(y, \mathbf{Q}), \quad y > 0. \tag{2.75}$$

For fixed $y > 0$, the continuous function $\mathbf{Q} \to \Psi(y, \mathbf{Q})$ attains its minimum on the compact set $\mathcal{M}^a(S)$, and the minimizer $\widehat{\mathbf{Q}}(y)$ is unique by the strict convexity of V. Writing $\widehat{\mathbf{Q}}(y) = (\widehat{q}_1(y), \ldots, \widehat{q}_N(y))$ for the minimizer, it follows from $V'(0) = -\infty$ that $\widehat{q}_n(y) > 0$, for each $n = 1, \ldots, N$; Indeed, suppose that $\widehat{q}_n(y) = 0$, for some $1 \leq n \leq N$ and fix any equivalent martingale measure $\mathbf{Q} \in \mathcal{M}^e(S)$. Letting $\mathbf{Q}^\epsilon = \epsilon \mathbf{Q} + (1-\epsilon)\widehat{\mathbf{Q}}$ we have that $\mathbf{Q}^\epsilon \in \mathcal{M}^e(S)$, for $0 < \epsilon < 1$, and $\Psi(y, \mathbf{Q}^\epsilon) < \Psi(y, \widehat{\mathbf{Q}})$ for $\epsilon > 0$ sufficiently small, a contradiction. In other words, $\widehat{\mathbf{Q}}(y)$ is an equivalent martingale measure for S.

Defining the dual value function $v(y)$ by

$$\begin{aligned} v(y) &= \inf_{\mathbf{Q} \in \mathcal{M}^a(S)} \sum_{n=1}^N p_n V\left(y \frac{q_n}{p_n}\right) && (2.76) \\ &= \sum_{n=1}^N p_n V\left(y \frac{\widehat{q}_n(y)}{p_n}\right) && (2.77) \end{aligned}$$

we find ourselves in an analogous situation as in the complete case above: defining again $\widehat{y}(x)$ by $v'(\widehat{y}(x)) = -x$ and

$$\widehat{\xi}_n = I\left(\widehat{y}(x) \frac{\widehat{q}_n(y)}{p_n}\right), \tag{2.78}$$

similar arguments as above apply to show that $(\widehat{\xi}_1, \ldots, \widehat{\xi}_N, \widehat{y}(x), \widehat{\mathbf{Q}}(y))$ is the unique saddle-point of the Lagrangian (2.70) and that the value functions u and v are conjugate.

Let us summarize what we have found in the incomplete case:

Theorem 2.18 (finite Ω, incomplete market) *Let the financial market $S = (S_t)_{t=0}^T$ defined over the finite filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F})_{t=0}^T, \mathbf{P})$ and let $\mathcal{M}^e(S) \neq \emptyset$, and the utility function U satisfies Assumptions 1.2.*

Denote by $u(x)$ and $v(y)$ the value functions

$$\begin{aligned} u(x) &= \sup_{X_T \in C(x)} \mathbf{E}[U(X_T)], && x \in \mathrm{dom}(U), && (2.79) \\ v(y) &= \inf_{\mathbf{Q} \in \mathcal{M}^a(S)} \mathbf{E}\left[V\left(y \tfrac{d\mathbf{Q}}{d\mathbf{P}}\right)\right], && y > 0. && (2.80) \end{aligned}$$

We then have:

(i) The value functions $u(x)$ and $v(y)$ are conjugate and u shares the qualitative properties of U listed in Assumption 1.2.

(ii) The optimizers $\widehat{X}_T(x)$ and $\widehat{\mathbf{Q}}(y)$ in (2.79) and (2.80) exist, are unique, $\widehat{\mathbf{Q}}(y) \in \mathcal{M}^e(S)$, and satisfy

$$\widehat{X}_T(x) = I\left(y\frac{d\widehat{\mathbf{Q}}(y)}{d\mathbf{P}}\right), \qquad y\frac{d\widehat{\mathbf{Q}}(y)}{d\mathbf{P}} = U'(\widehat{X}_T(x)), \tag{2.81}$$

where $x \in \mathrm{dom}(U)$ and $y > 0$ are related via $u'(x) = y$ or, equivalently, $x = -v'(y)$.

(iii) The following formulae for u' and v' hold true:

$$u'(x) = \mathbf{E}_{\mathbf{P}}[U'(\widehat{X}_T(x))], \qquad v'(y) = \mathbf{E}_{\widehat{\mathbf{Q}}}\left[V'\left(y\tfrac{d\widehat{\mathbf{Q}}(y)}{d\mathbf{P}}\right)\right] \tag{2.82}$$

$$xu'(x) = \mathbf{E}_{\mathbf{P}}[\widehat{X}_T(x)U'(\widehat{X}_T(x))], \qquad yv'(y) = \mathbf{E}_{\mathbf{P}}\left[y\tfrac{d\widehat{\mathbf{Q}}(y)}{d\mathbf{P}}V'\left(y\tfrac{d\widehat{\mathbf{Q}}(y)}{d\mathbf{P}}\right)\right] \tag{2.83}$$

Remark 2.19 Let us again interpret the formulae (2.82), (2.83) for $u'(x)$ similarly as in Remark 2.17 above. In fact, the interpretations of these formulae as well as their derivations remain in the incomplete case exactly the same.

But a new and interesting phenomenon arises when we pass to the variation of the optimal pay-off function $\widehat{X}_T(x)$ by a small unit of an arbitrary pay-off function $f \in L^\infty(\Omega, \mathcal{F}, \mathbf{P})$. Similarly as in (2.60) we have the formula

$$\mathbf{E}_{\widehat{\mathbf{Q}}(y)}[f]u'(x) = \lim_{h\to 0}\frac{\mathbf{E}_{\mathbf{P}}[U(\widehat{X}_T(x)+hf) - U(\widehat{X}_T(x))]}{h}, \tag{2.84}$$

the only difference being that $\mathbf{Q}$ has been replaced by $\widehat{\mathbf{Q}}(y)$ (recall that x and y are related via $u'(x) = y$).

The remarkable feature of this formula is that it does not only pertain to variations of the form $f = x + (H \cdot S)_T$, i.e, contingent claims attainable at price x, but to arbitrary contingent claims f, for which — in general — we cannot derive the price from no arbitrage considerations.

The economic interpretation of formula (2.84) is the following: the pricing rule $f \mapsto \mathbf{E}_{\widehat{\mathbf{Q}}(y)}[f]$ yields precisely those prices, at which an economic agent with initial endowment x, utility function U and investing optimally, is indifferent of first order towards adding a (small) unit of the contingent claim f to her portfolio $\widehat{X}_T(x)$.

In fact, one may turn the view around, and this was done by M. Davis [D 97] (compare also the work of L. Foldes [F 90]): one may *define* $\widehat{\mathbf{Q}}(y)$ by (2.84), verify that this indeed is an equivalent martingale measure for S, and interpret this pricing rule as "pricing by marginal utility", which is, of course, a classical and basic paradigm in economics.

Let us give a proof for (2.84) (under the hypotheses of Theorem 2.18). One possible strategy of proof, which also has the advantage of a nice economic interpretation, is the idea of introducing "fictitious securities" as developed in [KLSX 91]: fix $x \in \mathrm{dom}(U)$ and $y = u'(x)$ and let $(f^1, \dots, f^k)$ be finitely many elements of $L^\infty(\Omega, \mathcal{F}, \mathbf{P})$ such that the space $K = \{(H \cdot S)_T : H \in \mathcal{H}\}$, the constant function $\mathbf{1}$, and $(f^1, \dots, f^k)$ linearly span $L^\infty(\Omega, \mathcal{F}, \mathbf{P})$. Define the k processes

$$S_t^{d+j} = \mathbf{E}_{\widehat{\mathbf{Q}}(y)}[f^j | \mathcal{F}_t], \quad j = 1, \dots, k, \quad t = 0, \dots, T. \tag{2.85}$$

Now extend the $\mathbb{R}^{d+1}$-valued process $S = (S^0, S^1, \dots, S^d)$ to the $\mathbb{R}^{d+k+1}$-valued process $\overline{S} = (S^0, S^1, \dots, S^d, S^{d+1}, \dots, S^{d+k})$ by adding these new coordinates. By (2.85) we still have that $\overline{S}$ is a martingale under $\widehat{\mathbf{Q}}(y)$, which now is the unique probability under which $\overline{S}$ is a martingale, by our choice of $(f^1, \dots, f^k)$ and Corollary 2.12.

Hence we find ourselves in the situation of Theorem 2.16. By comparing (2.50) and (2.81) we observe that the optimal pay-off function $\widehat{X}_T(x)$ has not changed. Economically speaking this means that in the "completed" market $\overline{S}$ the optimal investment may still be achieved by trading only in the first $d+1$ assets and without touching the "fictitious" securities $S^{d+1}, \dots, S^{d+k}$.

In particular, we now may apply formula (2.60) to $\mathbf{Q} = \widehat{\mathbf{Q}}(y)$ to obtain (2.84).

Finally we remark that the pricing rule induced by $\widehat{\mathbf{Q}}(y)$ is precisely such that the interpretation of the optimal investment $\widehat{X}_T(x)$ defined in (2.81) (given in Remark 2.17 in terms of marginal utility and the ratio of Arrow prices $\widehat{q}_n(y)$ and probabilities p_n) carries over to the present incomplete setting. The above completion of the market by introducing "fictious securities" allows for an economic interpretation of this fact.

Chapter 3

The general case

In the previous chapter we have analyzed the duality theory of the utility maximization problem in detail and with full proofs, for the case when the underlying probability space is finite.

We now pass to the question under which conditions the crucial features of the above Theorem 2.18 carry over to the general setting. In particular one is naturally led to ask: under which conditions

- are the optimizers $\widehat{X}_T(x)$ and $\widehat{\mathbf{Q}}(y)$ of the value functions $u(x)$ and $v(y)$ attained?
- does the basic duality formula

$$U'\left(\widehat{X}_T(x)\right) = \widehat{y}(x)\frac{d\widehat{\mathbf{Q}}(\widehat{y}(x))}{d\mathbf{P}} \tag{3.1}$$

  or, equivalently

$$\widehat{X}_T(x) = I\left(\widehat{y}(x)\frac{d\widehat{\mathbf{Q}}(\widehat{y}(x))}{d\mathbf{P}}\right) \tag{3.2}$$

  hold true?
- are the value functions $u(x)$ and $v(y)$ conjugate?
- does the value function $u(x)$ still inherit the qualitative properties of U listed in Assumption 1.2?
- do the formulae for $u'(x)$ and $v'(y)$ still hold true?

We shall see that we get affirmative answers to these questions under two provisos: firstly, one has to make an appropriate choice of the sets in which X_T and $\mathbf{Q}$ are allowed to vary. This choice will be different for case 1, where $\mathrm{dom}(U) = \mathbb{R}_+$, and

case 2, where $\mathrm{dom}(U) = \mathbb{R}$. Secondly, the utility function U has to satisfy — in addition to Assumption 1.2 — a mild regularity condition, namely the property of *"reasonable asymptotic elasticity"*.

3.1 The reasonable asymptotic elasticity condition

The essential message of the theorems below is that, assuming that U has *"reasonable asymptotic elasticity"*, the duality theory works just as well as in the case of finite Ω. On the other hand, we shall see that we do not have to impose any regularity conditions on the underlying stochastic process S, except for its arbitrage-freeness in the sense made precise by Assumption 1.1. We shall also see that the assumption of reasonable asymptotic elasticity on the utility function U cannot be relaxed, even if we impose very strong assumptions on the process S (e.g., having continuous paths and defining a complete financial market), as we shall see below.

Before passing to the positive results we first analyze the notion of "reasonable asymptotic elasticity" and sketch the announced counterexample.

Definition 3.1 *A utility function U satisfying Assumption 1.2 is said to have* "reasonable asymptotic elasticity" *if*

$$AE_{+\infty}(U) = \limsup_{x\to\infty} \frac{xU'(x)}{U(x)} < 1, \tag{3.3}$$

and, in case 2 of Assumption 1.2, we also have

$$AE_{-\infty}(U) = \liminf_{x\to-\infty} \frac{xU'(x)}{U(x)} > 1. \tag{3.4}$$

We recall the following lemma from [KS99, Lemma 6.1], from which it follows that, for any concave function U such that the right hand side makes sense, we always have that $AE_{+\infty}(U) \leq 1$. Note that, the asymptotic elasticity assumption requires that the strict inequality holds.

Lemma 3.2 *For a strictly concave, increasing, real-valued differentiable function U the asymptotic elasticity $AE(U)$ is well-defined and, depending on $U(\infty) = \lim_{x\to\infty} U(x)$, takes its values in the following sets:*

(i) For $U(\infty) = \infty$ we have $AE(U) \in [0, 1]$,
(ii) For $0 < U(\infty) < \infty$ we have $AE(U) = 0$,
(iii) For $-\infty < U(\infty) \leq 0$ we have $AE(U) \in [-\infty, 0]$.

Proof (i) Using the monotonicity and positivity of U' we may estimate

$$\begin{aligned} 0 \leq xU'(x) =& (x-1)U'(x) + U'(x) \\ \leq& [U(x) - U(1)] + U'(1) \end{aligned}$$

hence, in the case $U(\infty) = \infty$,

$$0 \leq \limsup_{x\to\infty} \frac{xU'(x)}{U(x)} \leq \limsup_{x\to\infty} \frac{U(x) - U(1) + U'(1)}{U(x)} = 1.$$

(ii) For each $x_0 > 0$ we have

$$\begin{aligned} \limsup_{x\to\infty} xU'(x) =& \limsup_{x\to\infty} (x - x_0)U'(x) \\ \leq& \limsup_{x\to\infty} (U(x) - U(x_0)). \end{aligned}$$

If $U(\infty) < \infty$ we may choose x_0 such that the right hand side becomes arbitrary small.

(iii) We infer from $U(\infty) \leq 0$ that $U(x) < 0$, for $x \in \mathbb{R}_+$, so that $\frac{xU'(x)}{U(x)} < 0$, for all $x \in \mathbb{R}_+$. ■

Example 3.3

- For $U(x) = \log x$, we have $AE_{+\infty}(U) = 0$.
- For $U(x) = \frac{x^\alpha}{\alpha}$, we have $AE_{+\infty}(U) = \alpha$, for $\alpha \in (-\infty, 1) \setminus \{0\}$.
- For $U(x) = \frac{x}{\log x}$ for $x \geq x_0$, we have $AE_{+\infty}(U) = 1$.

The asymptotic elasticity compares as follows with other conditions used in the literature [KLSX 91]:

Lemma 3.4 *Let U be a utility function, and consider the following conditions:*

i) There exists $x_0 > 0$, $\alpha < 1$, $\beta > 1$ such that $U'(\beta x) < \alpha U'(x)$ for all $x \geq x_0$.

ii) $AE_{+\infty}(U) < 1$

iii) There exist k_1, k_2 and $\gamma < 1$ such that $U(x) \leq k_1 + k_2 x^\gamma$ for all $x \geq 0$.

Then we have that i) $\Rightarrow$ ii) $\Rightarrow$ iii). The reverse implications do not hold true in general.

Proof $(i) \Rightarrow (ii)$ Assume (i) and let $a = \alpha\beta$ and $b = \frac{1}{\alpha} > 1$ and estimate, for $x > ax_0$:

$$\begin{aligned} U(bx) =& U(\beta x_0) + \int_{\beta x_0}^{bx} U'(t)dt && (3.5)\\ =& U(\beta x_0) + \beta \int_{x_0}^{x/a} U'(\beta t)dt && (3.6)\\ \leq& U(\beta x_0) + \alpha\beta \int_{x_0}^{x/a} U'(t)dt && (3.7)\\ =& U(\beta x_0) + aU(\frac{x}{a}) - aU(x_0). && (3.8) \end{aligned}$$

It follows that criterion (ii) of corollary 6.1 in [KS99] is satisfied, hence $AE(U) < 1$.

$(ii) \Rightarrow (iii)$ is immediate from assertion (i) of lemma 6.3 in [KS99].

$(ii) \not\Rightarrow (i)$: For $n \in \mathbb{N}$, let $x_n = 2^{2^n}$ and define the function $U(x)$ by letting $U(x_n) = 1 - \frac{1}{n}$ and to be linear on the intervals $[x_{n-1}, x_n]$; (for $0 < x \leq x_1$ continue $U(x)$ in an arbitrary way, so that U satisfies (2.4)).

Clearly $U(x)$ fails (i) as for any $\beta > 1$ there are arbitrary large $x \in \mathbb{R}$ with $U'(\beta x) = U'(x)$. On the other hand, we have $U(\infty) = 1$ so that $AE(U) = 0$ by Lemma 3.2. Finally, note that in this counterexample the limit $\lim_{x\to\infty} \frac{xU'(x)}{U(x)}$ exists and equals zero.

The attentive reader might object that $U(x)$ is neither strictly concave nor differentiable. But it is obvious that one can slightly change the function to "smooth out" the kinks and to "strictly concavify" the straight lines so that the above conclusion still holds true.

$(iii) \not\Rightarrow (ii)$: Let again $x_n = 2^{2^n}$ and consider the utility function $\widetilde{U}(x) = x^{1/2}$. Define $U(x)$ by letting $U(x_n) = \widetilde{U}(x_n)$, for $n = 0, 1, 2...$ and to be linear on the intervals $[x_n, x_{n+1}]$; (for $0 < x \leq x_1$ again continue $U(x)$ in an arbitrary way, so that U satisfies (2.4)).

Clearly $U(x)$ satisfies condition (iii) as U is dominated by $\widetilde{U}(x) = x^{1/2}$.

To show that $AE(U) = 1$ let $x \in]x_{n-1}, x_n[$ and calculate the marginal utility U' at x:

$$U'(x) = \frac{U(x_n) - U(x_{n-1})}{x_n - x_{n-1}} = \frac{2^{2^{n-1}} - 2^{2^{n-2}}}{2^{2^n} - 2^{2^{n-1}}} = \frac{2^{2^{n-1}}(1 - 2^{-2^{n-2}})}{2^{2^n}(1 - 2^{-2^{n-1}})} = 2^{-2^{n-1}}(1 + o(1)).$$

On the other hand we calculate the average utility at $x = x_n$:

$$\frac{U(x_n)}{x_n} = \frac{2^{2^{n-1}}}{2^{2^n}} = 2^{-2^{n-1}}.$$

Hence

$$AE_{+\infty}(U) = \limsup_{x\to\infty} \frac{xU'(x)}{U(x)} = 1.$$

As regards the lack of smoothness and strict concavity of U a similar remark applies as in $(ii) \not\Rightarrow (i)$ above. ■

Let us discuss the economic meaning of the notion of reasonable asymptotic elasticity: as H.-U. Gerber ponted out to us, the quantity $\frac{xU'(x)}{U(x)}$ is the elasticity of the function U at x. We are interested in its asymptotic behaviour. It easily follows from Assumption 1.2 that the limits in (3.3) and (3.4) are less (resp. bigger) than or equal to one (compare Lemma 3.2). What does it mean that $\frac{xU'(x)}{U(x)}$ tends to one, for $x \to \infty$? It means that the ratio between the *marginal utility* $U'(x)$ and the *average utility* $\frac{U(x)}{x}$ tends to one. A typical example is a function $U(x)$ which equals $\frac{x}{\ln(x)}$, for x large enough; note however, that in this example Assumption 1.2 is not violated insofar as the marginal utility still decreases to zero for $x \to \infty$, i.e., $\lim_{x\to\infty} U'(x) = 0$.

If the marginal utility $U'(x)$ is approximately equal to the average utility $\frac{U(x)}{x}$ for large x, this means that for an economic agent, modeled by the utility function U, the increase in utility by varying wealth from x to $x+1$, when x is large, is approximately equal to the average of the increase of utility by changing wealth from n to $n+1$, where n runs through $1, 2, \ldots, x-1$ (we assume in this argument that x is a large natural number and, w.l.o.g., that $U(1) \approx 0$). We feel that the economic intuition behind decreasing marginal utility suggests that, for large x, the marginal utility $U'(x)$ should be substantially smaller than the average utility $\frac{U(x)}{x}$. Therefore we have denoted a utility function, where the ratio of $U'(x)$ and $\frac{U(x)}{x}$ becomes arbitrarily close to one if x tends either to $+\infty$ or $-\infty$, as being "unreasonable".

P. Guasoni observed, that there is a close connection between the asymptotic behaviour of the *elasticity* of U, and the asymptotic behaviour of the *relative risk aversion* associated to U. Recall (see, e.g., [HL88]) that the relative risk aversion of an agent with endowment x, whose preferences are described by the utility function U, equals

$$RRA(U)(x) = -\frac{xU''(x)}{U'(x)}. \tag{3.9}$$

A formal application of de l'Hôpital's rule yields

$$\lim_{x\to\infty} \frac{xU'(x)}{U(x)} = \lim_{x\to\infty} \frac{U'(x) + xU''(x)}{U'(x)} = 1 - \lim_{x\to\infty} \left(-\frac{xU''(x)}{U'(x)}\right) \tag{3.10}$$

which insinuates that the asymptotic elasticity of U is less than one iff the "*asymptotic relative risk aversion*" is strictly positive.

Turning the above formal argument into a precise statement, one easily proves the following result: if $\lim_{x\to\infty}(-\frac{xU''(x)}{U'(x)})$ exists, then $\lim_{x\to\infty}\frac{xU'(x)}{U(x)}$ exists too, and the former is strictly positive iff the latter is less than one (for details see [S 04a]). Hence "*essentially*" these two concepts coincide.

On the other hand, in general (i.e. without assuming that the above limit exists), there is no way to characterize the condition $\limsup_{x\to\infty}\frac{xU'(x)}{U(x)} < 1$ in terms of the asymptotic behaviour of $-\frac{xU''(x)}{U'(x)}$, as $x \to \infty$.

3.2 Existence Theorems

Let us now move to the positive results in the spirit of Theorem 2.16 and Theorem 2.18 above. We first consider the case where U satisfies case 1 of Assumption 1.2, which was studied in [KS 99].

Case 1: $\mathrm{dom}(U) = \mathbb{R}_+$.

The heart of the argument in the proof of Theorem 2.18 (which we now want to extend to the general case) is to find a saddlepoint for the Lagrangian. In more general situations we have to apply the minimax theorem, which is crucial in the theory of Lagrange multipliers. We want to extend the applicability of the minimax theorem to the present situation. The infinite-dimensional versions of the minimax theorem available in the literature (see, e.g, [ET 76] or [St 85]) are along the following lines: Let $\langle E, F\rangle$ be a pair of locally convex vector spaces in separating duality, $C \subseteq E$, $D \subseteq F$ a pair of convex subsets, and $L(x, y)$ a function defined on $C \times D$, concave in the first and convex in the second variable, having some (semi-)continuity property compatible with the topologies of E and F (which in turn should be compatible with the duality between E and F). If (at least) one of the sets C and D is compact and the other is complete, then one may assert the existence of a saddle point $(\widehat{\xi}, \widehat{\eta}) \in C \times D$ such that

$$L(\widehat{\xi}, \widehat{\eta}) = \sup_{\xi\in C}\inf_{\eta\in D} L(\xi, \eta) = \inf_{\eta\in D}\sup_{\xi\in C} L(\xi, \eta). \tag{3.11}$$

We try to apply this theorem to the analogue of the Lagrangian encountered in the proof of Theorem 2.18 above. Fixing $x > 0$ and $y > 0$ let us formally write the Lagrangian (2.70) in the infinite-dimensional setting,

$$\begin{aligned} L^{x,y}(X_T, \mathbf{Q}) &= \mathbf{E}_{\mathbf{P}}[U(X_T)] - y(\mathbf{E}_{\mathbf{Q}}[X_T - x]) && (3.12)\\ &= \mathbf{E}_{\mathbf{P}}\left[U(X_T) - y\tfrac{d\mathbf{Q}}{d\mathbf{P}}X_T\right] + yx, && (3.13)\end{aligned}$$

where X_T runs through "all" non-negative $\mathcal{F}_T$-measurable functions and $\mathbf{Q}$ through the set $\mathcal{M}^a(S)$ of absolutely continuous local martingale measures.

To restrict the set of "all" nonnegative functions to a more amenable one, note that $\inf_{y>0,\mathbf{Q}\in\mathcal{M}^a(S)} L^{x,y}(X_T,\mathbf{Q}) > -\infty$ iff

$$\mathbf{E}_{\mathbf{Q}}[X_T] \leq x, \quad \text{for all} \quad \mathbf{Q} \in \mathcal{M}^a(S). \tag{3.14}$$

Using the basic result on the super-replicability of the contingent claim X_T (see [KQ 95], [J 92], [AS 94], [DS 94], and [DS 98]), we have — as encountered in Theorem 2.11 for the finite dimensional case — that a non-negative $\mathcal{F}_T$-measurable random variable X_T satisfies (3.14) iff there is an admissible trading strategy H such that

$$X_T \leq x + (H \cdot S)_T. \tag{3.15}$$

Hence let

$$\begin{aligned} C(x) &= \big\{X_T \in L^0_+(\Omega,\mathcal{F}_T,\mathbf{P}) : \\ &\qquad X_T \leq x + (H\cdot S)_T, \text{ for some admissible } H\big\} \qquad (3.16)\\ &= \big\{X_T \in L^0_+(\Omega,\mathcal{F}_T,\mathbf{P}) : \\ &\qquad \mathbf{E}_{\mathbf{Q}}[X_T] \leq x, \text{ for all } \mathbf{Q}\in\mathcal{M}^a(S)\big\} \qquad (3.17) \end{aligned}$$

and simply write C for $C(1)$ (observe that $C(x) = xC$).

We thus have found a natural set $C(x)$ in which X_T should vary when we are mini-maxing the Lagrangian $L^{x,y}$. Dually, the set $\mathcal{M}^a(S)$ seems to be the natural domain where the measure $\mathbf{Q}$ is allowed to vary (in fact, we shall see later, that this set still has to be slightly enlarged). But what are the locally convex vector spaces E and F in separating duality into which C and $\mathcal{M}^a(S)$ are naturally embedded? As regards $\mathcal{M}^a(S)$ the natural choice seems to be $L^1(\mathbf{P})$ (by identifying a measure $\mathbf{Q} \in \mathcal{M}^a(S)$ with its Radon-Nikodym derivative $\frac{d\mathbf{Q}}{d\mathbf{P}}$); note that $\mathcal{M}^a(S)$ is a *closed* subset of $L^1(\mathbf{P})$, which is good news. On the other hand, there is no reason for C to be contained in $L^\infty(\mathbf{P})$, or even in $L^p(\mathbf{P})$, for any $p > 0$; the natural space in which C is embedded is just $L^0(\Omega,\mathcal{F}_T,\mathbf{P})$, the space of all real-valued $\mathcal{F}_T$-measurable functions endowed with the topology of convergence in probability.

The situation now seems hopeless (if we don't want to impose artificial $\mathbf{P}$-integrability assumptions on X_T and/or $\frac{d\mathbf{Q}}{d\mathbf{P}}$), as $L^0(\mathbf{P})$ and $L^1(\mathbf{P})$ are not in any reasonable duality; in fact, $L^0(\mathbf{P})$ is not even a locally convex space, hence there seems to be no hope for a good duality theory, which could serve as a basis for the application of the minimax theorem. But the good news is that the sets C and $\mathcal{M}^a(S)$ are in the *positive orthant* of $L^0(\mathbf{P})$ and $L^1(\mathbf{P})$ respectively; the crucial observation is, that for $f \in L^0_+(\mathbf{P})$ and $g \in L^1_+(\mathbf{P})$, it is possible to well-define

$$\langle f, g\rangle := \mathbf{E}_{\mathbf{P}}[fg] \in [0,\infty]. \tag{3.18}$$

The spirit here is similar as in the very foundation of Lebesgue integration theory: For positive measurable functions the integral is always well-defined, but possibly $+\infty$. This does not cause any logical inconsistency.

Similarly the bracket $\langle\,.\,,\,.\,\rangle$ defined in (3.18) shares many of the usual properties of a scalar product. The difference is that $\langle f, g\rangle$ now may assume the value $+\infty$ and that the map $(f, g) \mapsto \langle f, g\rangle$ is not continuous on $L^0_+(\mathbf{P}) \times L^1_+(\mathbf{P})$, but only lower semi-continuous (this immediately follows from Fatou's lemma).

At this stage it becomes clear that the role of $L^1_+(\mathbf{P})$ is somewhat artificial, and it is more natural to define (3.18) in the general setting where f and g are both allowed to vary in $L^0_+(\mathbf{P})$. The pleasant feature of the space $L^0(\mathbf{P})$ in the context of Mathematical Finance is, that it is invariant under the passage to an equivalent measure $\mathbf{Q}$, a property only shared by $L^\infty(\mathbf{P})$, but by no other $L^p(\mathbf{P})$, for $0 < p < \infty$.

We now can turn to the polar relation between the sets C and $\mathcal{M}^a(S)$. By (3.15) we have, for an element $X_T \in L^0_+(\Omega, \mathcal{F}, \mathbf{P})$,

$$X_T \in C \Longleftrightarrow \mathbf{E_Q}[X_T] = \mathbf{E_P}[X_T \tfrac{d\mathbf{Q}}{d\mathbf{P}}] \leq 1, \quad \text{for} \quad \mathbf{Q} \in \mathcal{M}^a(S). \tag{3.19}$$

Denote by D the closed, convex, solid hull of $\mathcal{M}^a(S)$ in $L^0_+(\mathbf{P})$. It is easy to show (using, e.g., Lemma 3.5 below), that D equals

$$\begin{aligned} D \;\; = \;\; & \{Y_T \in L^0_+(\Omega, \mathcal{F}_T, \mathbf{P}) : \text{ there is} \\ & (\mathbf{Q}_n)_{n=1}^\infty \in \mathcal{M}^a(S) \text{ s.t. } Y_T \leq \lim_{n\to\infty} \tfrac{d\mathbf{Q}_n}{d\mathbf{P}}\}, \end{aligned} \tag{3.20}$$

where the $\lim_{n\to\infty} \frac{d\mathbf{Q}_n}{d\mathbf{P}}$ is understood in the sense of almost sure convergence. We have used the letter Y_T for the elements of D to stress the dual relation to the elements X_T in C. In further analogy we write, for $y > 0$, $D(y)$ for yD, so that $D = D(1)$. By (3.20) and Fatou's lemma we again find that, for $X_T \in L^0_+(\Omega, \mathcal{F}, \mathbf{P})$

$$X_T \in C \Longleftrightarrow \mathbf{E_P}[X_T Y_T] \leq 1, \quad \text{for} \quad Y_T \in D. \tag{3.21}$$

Why did we pass to this enlargement D of the set $\mathcal{M}^a(S)$? The reason is that we now obtain a more symmetric relation between C and D: for $Y_T \in L^0_+(\Omega, \mathcal{F}, \mathbf{P})$ we have

$$Y_T \in D \Longleftrightarrow \mathbf{E_P}[X_T Y_T] \leq 1, \quad \text{for} \quad X_T \in C. \tag{3.22}$$

The proof of (3.22) relies on an adaption of the "bipolar theorem" from the theory of locally convex spaces (see, e.g., [S 66]) to the present duality $\langle L^0_+(\mathbf{P}), L^0_+(\mathbf{P})\rangle$, which was worked out in [BS 99].

Why is it important to define the enlargement D of $\mathcal{M}^a(S)$ in such a way that (3.22) holds true? After all, $\mathcal{M}^a(S)$ is a nice, convex, closed (w.r.t. the norm of $L^1(\mathbf{P})$) set and one may prove that, for $g \in L^1(\mathbf{P})$ *such that* $\mathbf{E_P}[g] = 1$,

$$g \in \mathcal{M}^a(S) \Longleftrightarrow \mathbf{E_P}[X_T g] \leq 1, \quad \text{for} \quad X_T \in C. \tag{3.23}$$

The reason is that, in general, the saddle point $(\widehat{X}_T, \widehat{\mathbf{Q}})$ of the Lagrangian will *not* be such that $\widehat{\mathbf{Q}}$ is a probability measure; it will only satisfy $\mathbf{E}\left[\frac{d\widehat{\mathbf{Q}}}{d\mathbf{P}}\right] \leq 1$, the

inequality possibly being strict. But it will turn out that $\widehat{\mathbf{Q}}$, which we identify with $\frac{d\widehat{\mathbf{Q}}}{d\mathbf{P}}$, is always in D. In fact, the passage from $\mathcal{M}^a(S)$ to D is the *crucial feature* in order to make the duality work in the present setting: even for nice utility functions U, such as the logarithm, and for nice processes, such as a continuous process $(S_t)_{0\leq t\leq T}$ based on the filtration of two Brownian motions, the above described phenomenon can occur: the saddle point of the Lagrangian leads out of $\mathcal{M}^a(S)$.

The set D can be characterized in several equivalent manners. We have defined D above in the abstract way as the convex, closed, solid hull of $\mathcal{M}^a(S)$ and mentioned the description (3.20). Equivalently, one may define D as the set of random variables $Y_T \in L^0_+(\Omega, \mathcal{F}, \mathbf{P})$ such that there is a process $(Y_t)_{0\leq t\leq T}$ starting at $Y_0 = 1$ with $(Y_t X_t)_{0\leq t\leq T}$ a $\mathbf{P}$-supermartingale, for every non-negative process $(X_t)_{0\leq t\leq T} = (x + (H \cdot S)_t)_{0\leq t\leq T}$, where $x > 0$ and H is predictable and S-integrable. This definition was used in [KS 99]. Another equivalent characterization was used in [CSW 01]: Consider the convex, solid hull of $\mathcal{M}^a(S)$, and embed this subset of $L^1(\mathbf{P})$ into the bidual $L^1(\mathbf{P})^{**} = L^\infty(\mathbf{P})^*$; denote by $\overline{\mathcal{M}^a(S)}$ the weak-star closure of the convex solid hull of $\mathcal{M}^a(S)$ in $L^\infty(\mathbf{P})^*$. Each element of $\overline{\mathcal{M}^a(S)}$ may be decomposed into its regular part $\mu^r \in L^1(\mathbf{P})$ and its purely singular part $\mu^s \in L^\infty(\mathbf{P})^*$. It turns out that D equals the set $\{\mu^r \in L^1(\mathbf{P}) : \mu \in \overline{\mathcal{M}^a(S)}\}$, i.e. consists of the regular parts of the elements of $\overline{\mathcal{M}^a(S)}$. This description has the advantage that we may associate to the elements $\mu^r \in D$ a singular part μ^s, and it is this extra information which is crucial when extending the present results to the case of random endowment as in [CSW 01]. Compare also [HK 02], where the case of random endowment is analyzed in full generality without using the space $L^\infty(\mathbf{P})^*$.

Why are the sets C and D hopeful candidates for the minimax theorem to work out properly for a function L defined on $C \times D$? Both are closed, convex and bounded subsets of $L^0_+(\mathbf{P})$. But recall that we still need some compactness property to be able to localize the mini-maximizers (resp. maxi-minimizers) on C (resp. D). In general, neither C nor D is compact (w.r.t. the topology of convergence in measure), i.e., for a sequence $(f_n)_{n=1}^\infty$ in C (resp. $(g_n)_{n=1}^\infty$ in D) we cannot pass to a subsequence converging in measure. But C and D have a property which is close to compactness and in many applications turns out to serve just as well.

Lemma 3.5 *Let A be a closed, convex, bounded subset of $L^0_+(\Omega, \mathcal{F}, \mathbf{P})$. Then for each sequence $(h_n)_{n=1}^\infty \in A$ there exists a sequence of convex combinations $k_n \in \operatorname{conv}(h_n, h_{n+1}, \ldots)$ which converges almost surely to a function $k \in A$.*

This easy lemma (see, e.g., [DS 94, Lemma A.1.1], for a proof) is in the spirit of the celebrated theorem of Komlos [Kom 67], stating that for a bounded sequence $(h_n)_{n=1}^\infty$ in $L^1(\mathbf{P})$ there is a subsequence converging in Cesaro-mean almost surely. The methodology of finding pointwise limits by using convex combinations has turned out to be extremely useful as a surrogate for compactness. For an extensive

discussion of more refined versions of the above lemma and their applications to Mathematical Finance we refer to [DS 99].

The application of the above lemma is the following: by passing to convex combinations of optimizing sequences $(f_n)_{n=1}^{\infty}$ in C (resp. $(g_n)_{n=1}^{\infty}$ in D), we can always find limits $f \in C$ (resp. $g \in D$) w.r.t. almost sure convergence. Note that the passage to convex combinations does not cost more than passing to a subsequence in the application to convex optimization.

We have now given sufficient motivation to state the central result of [KS 99], which is the generalization of Theorem 2.18 to the semi-martingale setting under Assumption 1.2, case 1, and having reasonable asymptotic elasticity.

Theorem 3.6 ([KS 99], Theorem 2.2) *Let the semi-martingale $S = (S_t)_{0\leq t\leq T}$ and the utility function U satisfy Assumptions 1.1, 1.2 case 1 and 1.3; suppose in addition that U has reasonable asymptotic elasticity. Define*

$$u(x) = \sup_{X_T \in C(x)} \mathbf{E}[U(X_T)], \qquad v(y) = \inf_{Y_T \in D(y)} \mathbf{E}[V(Y_T)]. \tag{3.24}$$

Then we have:

(i) The value functions $u(x)$ and $v(y)$ are conjugate; they are continuously differentiable, strictly concave (resp. convex) on $]0, \infty[$ and satisfy

$$u'(0) = -v'(0) = \infty, \qquad u'(\infty) = v'(\infty) = 0. \tag{3.25}$$

(ii) The optimizers $\widehat{X}_T(x)$ and $\widehat{Y}_T(y)$ in (3.24) exist, are unique and satisfy

$$\widehat{X}_T(x) = I(\widehat{Y}_T(y)), \qquad \widehat{Y}_T(y) = U'(\widehat{X}_T(x)), \tag{3.26}$$

where $x > 0$, $y > 0$ are related via $u'(x) = y$ or equivalently $x = -v'(y)$.

(iii) We have the following relations between u', v' and $\widehat{X}_T, \widehat{Y}_T$ respectively:

$$u'(x) = \mathbf{E}\left[\frac{\widehat{X}_T(x)U'(\widehat{X}_T(x))}{x}\right], \ x > 0, \quad v'(y) = \mathbf{E}\left[\frac{\widehat{Y}_T(y)V'(\widehat{Y}_T(y))}{y}\right], \ y > 0. \tag{3.27}$$

For the full proof of the theorem we refer to [KS 99].

How severe is the fact that the dual optimizer $\widehat{Y}_T(1)$ may fail to be the density of a probability measure (or that $\mathbf{E}[\widehat{Y}_T(y)] < y$, for $y > 0$, which amounts to the same thing)? In fact, in many respects it does not bother us at all: we still have the basic duality relation between the primal and the dual optimizer displayed in Theorem 3.6 (ii). Even more is true: using the terminology from [KS 99] the product

$(\widehat{X}_t(x)\widehat{Y}_t(y))_{0\le t\le T}$, where x and y satisfy $u'(x) = y$, is a uniformly integrable martingale. This fact can be interpreted in the following way: by taking the optimal portfolio $(\widehat{X}_t(x))_{0\le t\le T}$ as numéraire instead of the original cash account, the pricing rule obtained from the dual optimizer $\widehat{Y}_T(y)$ then is induced by an equivalent martingale measure. We refer to ([KS 99], p. 912) for a thorough discussion of this argument.

Finally we want to draw the attention of the reader to the fact that — comparing item (iii) of Theorem 3.6 to the corresponding item of Theorem 2.18 — we only asserted one pair of formulas for $u'(x)$ and $v'(y)$. The reason is that, in general, the formulae (2.82) do not hold true any more, the reason again being precisely that for the dual optimizer $\widehat{Y}_T(y)$ we may have $\mathbf{E}[\widehat{Y}_T(y)] < y$. Indeed, the validity of $u'(x) = \mathbf{E}[U'(\widehat{X}_T(x))]$ is tantamount to the validity of $y = \mathbf{E}[\widehat{Y}_T(y)]$.

Case 2: $\mathrm{dom}(U) = \mathbb{R}$

We now pass to the case of a utility function U satisfying Assumption 1.2 case 2 which is defined and finitely valued on all of $\mathbb{R}$. The reader should have in mind the exponential utility $U(x) = -e^{-\gamma x}$, for $\gamma > 0$, as the typical example.

We want to obtain a result analogous to Theorem 3.6 also in this setting. Roughly speaking, we get the same theorem, but the sets C and D considered above have to be chosen in a somewhat different way, as the optimal portfolio $\widehat{X}_T$ now may assume negative values too.

Firstly, we have to assume throughout the rest of this section that the semimartingale S is *locally bounded*. The case of non locally bounded processes is not yet understood and waiting for future research.

Next we turn to the question; what is the proper definition of the set $C(x)$ of terminal values X_T dominated by a random variable $x + (H \cdot S)_T$, where H is an "allowed" trading strategy? On the one hand we cannot be too liberal in the choice of "allowed" trading strategies as we have to exclude doubling strategies and similar schemes. We therefore maintain the definition of the value function $u(x)$ unchanged

$$u(x) = \sup_{H\in\mathcal{H}} \mathbf{E}\left[U\left(x + (H \cdot S)_T\right)\right], \quad x \in \mathbb{R}, \tag{3.28}$$

where we still confine H to run through the set $\mathcal{H}$ of admissible trading strategies, i.e., such that the process $((H \cdot S)_t)_{0\le t\le T}$ is uniformly bounded from below. This notion makes good sense economically as it describes the strategies possible for an agent having a finite credit line.

On the other hand, in general, we have no chance to find the minimizer $\widehat{H}$ in (3.28) within the set of admissible strategies: already in the classical cases studied by Merton ([M 69] and [M 71] where, in particular, the case of exponential utility is solved for the Black-Scholes model) the optimal solution $x + (\widehat{H} \cdot S)_T$ to (3.28) is *not* uniformly bounded from below; this random variable typically assumes low values with very small probability, but its essential infimum typically is minus infinity.

In [S 01] the following approach was used to cope with this difficulty: fix the utility function $U : \mathbb{R} \to \mathbb{R}$ and first define the set $C_U^b(x)$ to consist of all random variables G_T dominated by $x + (H \cdot S)_T$, for some *admissible* trading strategy H and such that $\mathbf{E}[U(G_T)]$ makes sense:

$$C_U^b(x) = \big\{ G_T \in L^0(\Omega, \mathcal{F}_T, \mathbf{P}) : \text{ there is } H \text{ admissible s.t.} \tag{3.29}$$

$$G_T \leq x + (H \cdot S)_T \text{ and } \mathbf{E}[|U(G_T)|] < \infty \big\}. \tag{3.30}$$

Next we define $C_U(x)$ as the set of $\mathbb{R} \cup \{+\infty\}$-valued random variables X_T such that $U(X_T)$ can be approximated by $U(G_T)$ in the norm of $L^1(\mathbf{P})$, when G_T runs through $C_U^b(x)$:

$$C_U(x) = \big\{ X_T \in L^0(\Omega, \mathcal{F}_T, \mathbf{P}; \mathbb{R} \cup \{+\infty\}) : U(X_T) \text{ is in} \tag{3.31}$$

$$L^1(\mathbf{P})\text{-closure of } \{U(G_T) : G_T \in C_U^b(x)\} \big\}. \tag{3.32}$$

The optimization problem (3.28) now reads

$$u(x) = \sup_{X_T \in C_U(x)} \mathbf{E}[U(X_T)], \quad x \in \mathbb{R}. \tag{3.33}$$

The set $C_U(x)$ was chosen in such a way that the value functions $u(x)$ defined in (3.28) and (3.33) coincide; but now we have much better chances to find the maximizer to (3.33) in the set $C_U(x)$.

Two features of the definition of $C_U(x)$ merit some comment: firstly, we have allowed $X_T \in C_U(x)$ to attain the value $+\infty$; indeed, in the case when $U(\infty) < \infty$ (e.g., the case of exponential utility), this is natural, as the set $\{U(X_T) : X_T \in C_U(x)\}$ should equal the $L^1(\mathbf{P})$-closure of the set $\{U(G_T) : G_T \in C_U^b(x)\}$. But we shall see that — under appropriate assumptions — the optimizer $\widehat{X}_T$, which we are going to find in $C_U(x)$, will almost surely be finite.

Secondly, the elements X_T of $C_U(x)$ are only *random variables* and, at this stage, they are not related to a *process* of the form $x + (H \cdot S)$. Of course, we finally want to find for each $X_T \in C_U(x)$, or at least for the optimizer $\widehat{X}_T$, a predictable, S-integrable process H having "allowable" properties (in order to exclude doubling strategies) and such that $X_T \leq x + (H \cdot S)_T$. We shall prove later that — under appropriate assumptions — this is possible and give a precise meaning to the word "allowable".

After having specified the proper domain $C_U(x)$ for the primal optimization problem (3.33), we now pass to the question of finding the proper domain for the dual optimization problem. Here we find a pleasant surprise: contrary to case 1 above, where we had to pass from the set $\mathcal{M}^a(S)$ to its closed, solid hull D, it turns out that, in the present case 2, the dual optimizer always lies in $\mathcal{M}^a(S)$. This fact was first proved by F. Bellini and M. Fritelli ([BF 02]).

We now can state the main result of [S 01]:

Theorem 3.7 *[S 01, Theorem 2.2] Let the locally bounded semi-martingale $S = (S_t)_{0\le t\le T}$ and the utility function U satisfy Assumptions 1.1, 1.2 case 2 and 1.3; suppose in addition that U has reasonable asymptotic elasticity. Define*

$$u(x) = \sup_{X_T \in C_U(x)} \mathbf{E}[U(X_T)], \qquad v(y) = \inf_{\mathbf{Q}\in\mathcal{M}^a(S)} \mathbf{E}\left[V\left(y\tfrac{d\mathbf{Q}}{d\mathbf{P}}\right)\right]. \tag{3.34}$$

Then we have:

(i) The value functions $u(x)$ and $v(y)$ are conjugate; they are continuously differentiable, strictly concave (resp. convex) on $\mathbb{R}$ (resp. on $]0,\infty[$) and satisfy

$$u'(-\infty) = -v'(0) = v'(\infty) = \infty, \qquad u'(\infty) = 0. \tag{3.35}$$

(ii) The optimizers $\widehat{X}_T(x)$ and $\widehat{\mathbf{Q}}(y)$ in (3.34) exist, are unique and satisfy

$$\widehat{X}_T(x) = I\left(y\frac{d\widehat{\mathbf{Q}}(y)}{d\mathbf{P}}\right), \qquad y\frac{d\widehat{\mathbf{Q}}(y)}{d\mathbf{P}} = U'(\widehat{X}_T(x)), \tag{3.36}$$

where $x \in \mathbb{R}$ and $y > 0$ are related via $u'(x) = y$ or equivalently $x = -v'(y)$.

(iii) We have the following relations between u', v' and $\widehat{X}, \widehat{\mathbf{Q}}$ respectively:

$$u'(x) = \mathbf{E}_{\mathbf{P}}[U'(\widehat{X}_T(x))], \qquad v'(y) = \mathbf{E}_{\widehat{\mathbf{Q}}}\left[V'\left(y\tfrac{d\widehat{\mathbf{Q}}(y)}{d\mathbf{P}}\right)\right] \tag{3.37}$$

$$xu'(x) = \mathbf{E}_{\mathbf{P}}[\widehat{X}_T(x)U'(\widehat{X}_T(x))], \qquad yv'(y) = \mathbf{E}_{\mathbf{P}}\left[y\tfrac{d\widehat{\mathbf{Q}}(y)}{d\mathbf{P}}V'\left(y\tfrac{d\widehat{\mathbf{Q}}(y)}{d\mathbf{P}}\right)\right] \tag{3.38}$$

(iv) If $\widehat{\mathbf{Q}}(y) \in \mathcal{M}^e(S)$ and $x = -v'(y)$, then $\widehat{X}_T(x)$ equals the terminal value of a process of the form $\widehat{X}_t(x) = x + (H \cdot S)_t$, where H is predictable and S-integrable, and such that $\widehat{X}$ is a uniformly integrable martingale under $\widehat{\mathbf{Q}}(y)$.

We refer to [S 01] for a proof of this theorem and further related results. We cannot go into the technicalities here, but a few comments on the proof of the above theorem are in order: the technique is to reduce case 2 to case 1 by approximating the utility function $U : \mathbb{R} \to \mathbb{R}$ by a sequence $(U^{(n)})_{n=1}^{\infty}$ of utility functions $U^{(n)} : \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$ such that $U^{(n)}$ coincides with U on $[-n, \infty[$ and equals $-\infty$ on $]-\infty, -(n+1)]$. For fixed initial endowment $x \in \mathbb{R}$, we then apply Theorem 3.6 to find for each $U^{(n)}$ the saddle-point $(\widehat{X}_T^{(n)}(x), \widehat{Y}_T^{(n)}(\widehat{y}_n)) \in C_U^b(x) \times D(\widehat{y}_n)$; finally we show that this sequence converges to some $(\widehat{X}_T(x), \widehat{y}\widehat{\mathbf{Q}}_T) \in C_U(x) \times \widehat{y}\mathcal{M}^a(S)$, which then is shown to be the saddle-point for the present problem. The details of this construction are rather technical and lengthy (see [S 01]).

We have assumed in item (iv) that $\widehat{\mathbf{Q}}(y)$ is equivalent to $\mathbf{P}$ and left open the case when $\widehat{\mathbf{Q}}(y)$ is only absolutely continuous to $\mathbf{P}$. F. Bellini and M. Fritelli have observed ([BF 02]) that, in the case $U(\infty) = \infty$ (or, equivalently, $V(0) = \infty$), it follows from (3.34) that $\widehat{\mathbf{Q}}(y)$ is equivalent to $\mathbf{P}$. But there are also other important cases where we can assert that $\widehat{\mathbf{Q}}(y)$ is equivalent to $\mathbf{P}$: for example, for the case of the exponential utility $U(x) = -e^{-\gamma x}$, in which case the dual optimization becomes the problem of finding $\widehat{\mathbf{Q}} \in \mathcal{M}^a(S)$ minimizing the relative entropy with respect $\mathbf{P}$, it follows from the work of Csiszar [C 75] (compare also [R 84], [F 00], [GR 01]) that the dual optimizer $\widehat{\mathbf{Q}}(y)$ is equivalent to $\mathbf{P}$, provided only that there is at least one $\mathbf{Q} \in \mathcal{M}^e(S)$ with finite relative entropy.

Under the condition $\widehat{\mathbf{Q}}(y) \in \mathcal{M}^e(S)$, item (iv) tells us that the optimizer $\widehat{X}_T \in C_U(x)$ is almost surely finite and equals the terminal value of a process $x + (H \cdot S)$, which is a uniformly integrable martingale under $\widehat{\mathbf{Q}}(y)$; this property qualifies H to be a "allowable", as it certainly excludes doubling strategies and related schemes. One may turn the point of view around and take this as the *definition* of the "allowable" trading strategies; this was done in [DGRSSS 02] for the case of exponential utility, where this approach is thoroughly studied and some other definitions of "allowable" trading strategies, over which the primal problem may be optimized, are also investigated. Further results on these lines were obtained in [KS 02] for the case of exponential utility, and in [S 03a] for general utility functions.

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