SOME REMARKS ON INTEGRAL OPERATORS AND EQUIMEASURABLE SETS

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Abstract: We give a characterisation of equimeasurable sets in terms of the difference between the notions of almost everywhere convergence and convergence in measure. We apply this characterisation to obtain a direct proof of a criterion for integral representability of operators, due to A. V. Bukhvalov (obtained in 1974) by a criterion of the present author (obtained in 1979).

In the second part - following an idea due to A. Costé - we show that convolution with a suitably chosen singular measure defines a positive operator on L^2, which is of trace class p, for p > 2, but fails to be integral. This sharpens a result, due to D. H. Fremlin.

Introduction: This paper is stimulated by the question of characterisation of integral operators. This problem, which was raised by J. v. Neumann [v.N.] in 1935 was solved in 1974 by Bukhvalov [B2] using the theory of order-bounded operators (see theorem 2.3. below). In 1979, the present author - working on problems of [H-S] - obtained independently a different characterisation using methods from the theory of differentiation of vector measures [S1].

In 1981 A. Schep [S2] has indicated that the latter characterisation may be deduced from Bukhvalov's, while in 1982 L. Weis [W] showed (among many other interesting results) how to prove Bukhvalov's characterisation by means of arguments similar to those used in [S1].

In the present paper we show that the notion of equimeasurability, due to A. Grothendieck, on which our characterisation is based, is intimately linked with the difference between convergence in measure and convergence almost everywhere, on which Bukhvalov's criterion is based. This furnishes a better understanding why the two conditions are equivalent.

We start the paper by defining the concept of equimeasurability and pointing out its relevance to the question of representing an operator from $L^1(\mu)$ by a Bochner-derivative (proposition 2.2 below). We then state the characterisations of integral operators due to Bukhvalov and the present author (theorem 2.3 below) and analyse the structure of equimeasurable sets to obtain some equivalent characterisations, one of them involving the difference between convergence in measure and a.e. With the help of this characterisation we then deduce the criterion of Bukhvalov from ours by a very direct argument.

In the last part of the paper we deal with a different topic. In 1975 D. H. Fremlin [F] gave an example of a positive compact operator on L^2 which is not integral. This important and highly non-trivial result was achieved by means of a somewhat ad-hoc-construction.

On the other hand J. J. Uhl [U] noted - based on work of A. Costé - that "convolution with a biased coin" furnishes an example of a completely continuous non-representable operator from L^1 to L^1. Although Fremlin had applied his example to answer a question of Schaefer in the negative (i.e., there exists a positive compact (Grothendieck-)intergral operator from L^∞ to L^1 which is not nuclear - which comes up to the same as a positive completely continuous non-representable operator from L^1 to L^1), apparently the connection between these two examples has not been noticed.

Costé's argument is based on the classical result, due to Menchoff, that there is a Lebesgue-singular measure on the torus such that the Fourier-coefficients tend to zero.

Translating Costé's example into the language of integral operators on L^2 we obtain a considerable sharpening of Fremlin's result: There is a positive operator on $L^2(0,1)$ which is of trace-class p for every $p > 2$, but not integral. The operator is just the convolution with a suitably chosen "sequence of biased coins".

2. Equimeasurable sets

Let (X,Σ,μ) and (Y,T,ν) denote finite measure spaces (the σ-finite case reduces to this case by a simple change of density). For $1 \leq p \leq \infty$, let $L^p(\mu)$ and $L^p(\nu)$ denote the usual Lebesgue-spaces (over the reals) and let $L^0(\mu)$ be the F-space of (equivalence classes of) μ-measurable real-valued functions, equipped with the metric

$$d(f,g) \quad \inf \{\varepsilon \quad \mu(\omega \quad |f(\omega)-g(\omega)| \geq \varepsilon) < \varepsilon\}.$$

<u>2.1. Definition (Grothendieck, [G]):</u> A subset M of $L^0(\mu)$ is called equi-measurable if, for $\varepsilon > 0$, there is $X_\varepsilon \subset X$ with $\mu(X \setminus X_\varepsilon) < \varepsilon$ such that M restricted to X_ε is relatively norm-compact in $L^\infty(X_\varepsilon, \mu|_X$

The following proposition, which goes back to Grothendieck's memoir [G] shows the importance of this notion for the representability of operators.

<u>2.2. Proposition ([S2], [S1]):</u> An operator T from $L^1(\mu)$ to a Banach space E is representable by a Bochner-integrable function $F \quad X \to E$, i.e. for $f \in L^1(\mu)$

$$Tf \quad \int_X F(\omega)f(\omega)d\mu(\omega),$$

iff T^* maps the unitball of E^* into an equimeasurable subset of $L^\infty(\mu)$.

The proof of the theorem in fact essentially goes back to Dunford and Pettis (see e.g. [D-U], prop. III. 2, 21, p. 78). It reduces quickly to the correspondence between compact operators $T \quad L^1(\mu) \to E$ and relatively compact valued μ-measurable functions $F \quad X \to E$. Note in passing that for a weak-star dense subset C of the unitball of E^* we have that $T^*(C)$ is equimeasurable iff T^* (unitball (E^*)) is equimeasurable.

Despite its simplicity proposition 2.2 seems to us to be a key result in understanding questions about representability of operators by Bochner-derivatives or - what is closely related - by kernel functions.

We shall now state the two criteria for integral representation. For simplicity we restrict ourselves to the classical L^2-case. See, however, the subsequent remark 2.9.

2.3. Theorem: Let $T \quad L^2(\nu) \to L^2(\mu)$ be a linear map. T.f.a.e.

(i) T is an integral operator (for a definition see, e.g. [B1], [H-S] or [S1]).

(ii) For $(g_n)_{n=1}^\infty \in L^2(\nu)$ such that there is $g \in L^2_+(\nu)$ with $|g_n| \leqq g$ and $g_n \to 0$ in measure, the sequence $(Tg_n)_{n=1}^\infty$ converges to zero μ-a.e. (Bukhvalov, 1974).

(iii) T transforms order bounded subsets of $L^2(\nu)$ into equimeasurable sets (Schachermayer, 1979).

Of course, the preceding theorem is well-known: The equivalence of (i) and (ii) was shown in [B2], that of (i) and (iii) in [S1]. The question we are now dealing with is somewhat aesthetical: How direct a proof of (ii) $\Rightarrow$ (iii) and (iii) $\Rightarrow$ (ii) may be given? More precisely: What is the relation between equimeasurable sets and the distinction between convergence in measure and convergence almost everywhere?

The subsequent result shows that there is in fact an intimate relation for <u>convex</u> sets (absolute convexity is assumed just for the convenience of the formulation).

<u>2.4. Proposition:</u> Let M be an absolutely convex subset of $L^0(\mu)$. T.f.a.e.

(i) There is a function $\varphi \in L^{\infty}_{+}(\mu)$, $\varphi > 0$ μ-a.e., such that

$$\varphi.M \quad \{\varphi.f \quad f \in M\}$$

is relatively compact in $(L^{\infty}(\mu), \|.\|_{\infty})$.

(ii) M is order-bounded and every sequence $(f_n)_{n=1}^{\infty} \subset M$ which converges to zero in measure converges to zero μ-a.e.

(iii) M is equimeasurable.

<u>Proof:</u>

(iii) $\Rightarrow$ (i): If M is equimeasurable we may find a μ-a.e. partition of X into disjoint sets $(A_k)_{k=1}^{\infty}$ in Σ such that M restricted to A_k is relatively $\|.\|_{\infty}$-compact. In particular there are constants $a_k \in \mathbb{R}_+$ such that M restricted to A_k is $\|.\|_{\infty}$-bounded by the constant a_k. It is obvious that

$$\sum_{k=1}^{\infty} (k.a_k)^{-1} \chi_{A_k}$$

does the desired job.

(i) ⇒ (ii): The order-boundedness of M is obvious. If $(f_n)_{n=1}^{\infty} \subset M$ converges to zero in measure then $(\varphi f_n)_{n=1}^{\infty}$ does so too. As convergence in measure defines a coarser Hausdorff-topology on M than the $\|.\|$ - topology, we infer from the relative $\|.\|_{\infty}$-compactness of M that $(\varphi . f_n)_{n=1}^{\infty}$ converges to zero uniformly. Hence $(f_n)_{n=1}^{\infty}$ converges to zero μ-a.e.

(ii) ⇒ (iii): This is the (relatively) non-trivial part of the proposition and we shall delay its proof to the subsequent proposition, which states a more general "local version", as the proof for the present case is not easier than the general one.

<u>2.5. Remark:</u> The conditions (i), (ii) and (iii) of the above proposition 2.4 correspond essentially to the conditions of theorem 2.3. The implication (iii) ⇒ (ii) of 2.4 was noted by A. Schep [S2] and from this he easily deduced the implication (iii) ⇒ (ii) of 2.3. But it is worth remarking that (ii) and (iii) of 2.4 are in fact equivalent (for absolutely convex sets). The reader should note that a subset $M \subset L^{0}(\mu)$ is equimeasurable iff its absolutely convex, closed hull has this property.

The reformulation 2.4 (i) of the concept of equimeasurability was stressed out to us by J. B. Cooper and it emphazises that - up to multiplication with a weight function - the equimeasurable sets are just the relatively compact subsets of $L^{\infty}(\mu)$. Representing $L^{\infty}(\mu)$ as a $C(K)$-space they correspond to the equicontinuous sets (by the Ascoli-Arzela theorem). Hence "equimeasurability" may be viewed as a kind of analogue to "equicontinuity", the former applying to measurable the latter to continuous functions.

Let us recall in this context that the original definition of

A. Grothendieck [G] was given in the "french style", i.e. μ is a Radon-measure, X_ε a compact subset of X and M, restricted to X_ε, an equicontinuous subset of $C(X_\varepsilon)$.

We now pass to the announced "local version" of proposition 2.4. It will be convenient to state it for bounded subsets of $L^\infty(\mu)$ (instead of order-bounded subsets of $L^0(\mu)$) and to assume that the measure space (X,Σ,μ) is separable, i.e. that $L^1(\mu)$ is a separable Banach space.

Both restrictions are inessential: Passing from order-bounded subsets of $L^0(\mu)$ to bounded subsets of $L^\infty(\mu)$ is a matter of multiplication with a weight function. Also the conditions of 2.6 are easily seen to be separably determined which allows the reduction of the case of a general (finite) measure space to the separable one. However, for a separable measure space (X,Σ,μ) the weak-star-topology is metrisable on bounded subsets of $L^\infty(\mu)$, which allows us to formulate the subsequent proposition more elegantly: Fix δ to be a metric on $L^\infty(\mu)$ inducing the weak-star-topology on its bounded subsets.

<u>2.6. Proposition:</u> Let M be an absolutely convex bounded subset of and let $A \in \Sigma$. T.f.a.e.

(a) Let $F_n = \sup\{f \in M : d(f,0) \leq n^{-1}\}$ and $\lim F_n$. Then F equals zero μ-a.e. on A.

(a') Let $G_n = \sup\{f \in M : \delta(f,0) \leq n^{-1}\}$ and G $\lim G_n$. Then G equals zero μ-a.e. on A.

(b) If $(f_n)_{n=1}^\infty \subset M$ converges to zero in measure, then $(f_n)_{n=1}^\infty$ converges to zero μ-a.e. on A.

(b') If $(f_n)_{n=1}^\infty \subset M$ weak-star-convergerges to zero, then $(f_n)_{n=1}^\infty$ converges to zero μ-a.e. on A.

(c) The restriction of M to A is equimeasurable.

Note that the suprema appearing in (a) and (a') (taken over a (possibly) uncountably set of functions) have to be interpreted in the lattice-sense.

<u>Proof:</u> (c) $\Rightarrow$ (b'): Let $(A_k)_{k=1}^{\infty}$ be a μ-a.e. partition of A such that M restricted to A_k is relatively $\|.\|_\infty$-compact. The operator

$$\begin{aligned} L^\infty(\mu) &\to L^\infty(\mu) \\ f &\to f.\chi_{A_k} \end{aligned}$$

is weak-star-continuous. On $\chi_{A_k} M$ the σ -topology and the $\|.\|_\infty$-topology coincide, hence the operator χ_{A_k} is σ^*-$\|.\|_\infty$-continuous on M. So, if $(f_n)_{n=1}^{\infty} \subset M$ converges σ^* to zero, then it converges to zero (uniformly) on every A_k, hence μ-a.e. on A.

(a') $\Rightarrow$ (c): If G equals zero μ-a.e. on A then Egoroff's theorem tells us that there is a μ-a.e. partition $(A_k)_{k=1}^{\infty}$ of A such that G_n converges to zero uniformly on every A_k. This means - as above - that the restriction of the operator χ_A to M is σ -$\|.\|_\infty$-continuous at the point $0 \in M$. From the absolute convexity of M we conclude that the restriction of to M is σ -$\|.\|_\infty$ uniformly continuous. As M is relatively σ - compact we deduce the relative $\|.\|_\infty$-compactness of the restriction of M to A_k.

(b') $\Rightarrow$ (a'): If (a') fails then $d(\chi_A G,0) = \alpha > 0$. As $G \geq G \geq 0$ we may find, for every $n \in \mathbb{N}$, a finite sequence $f_1^n,\ldots,f_m^n$ in M such that

$$\delta(f_i^n,0) \leq n^{-1} \qquad\qquad i \quad 1,\ldots,m_n$$

while

The sequence converges weak-star to zero, while it cannot converge to zero μ-a.e. on A as the lim sup of the sequence has a distance from zero (in measure) of at least $\alpha/2$.

(b) $\Rightarrow$ (a) uses the same argument as (b') $\Rightarrow$ (a') (replacing G by F and δ by d).

(b') $\Rightarrow$ (b) and (a') $\Rightarrow$ (a) are obvious.

(a) $\Rightarrow$ (a'): We shall in fact show that F equals G μ-a.e. As we shall use a pointwise reasoning it will be convenient to argue with (countably many) representants of the equivalence-classes of functions: For $n \in \mathbb{N}$, find sequences $(\tilde{g}_i^n)_{i=1}^{\infty}$ of representants of elements of M with $\delta(\tilde{g}_i^n, 0) \leq n^{-1}$ and such that the function

$$\tilde{G}_n(\omega) \quad \sup_i \tilde{g}_i^n(\omega)$$

is a representant of G_n. We may assume that the sequence $(\tilde{G}_n)_{n=1}^{\infty}$ is pointwise decreasing at every $\omega \in X$. Hence

$$\tilde{G}(\omega) \quad \lim_n \tilde{G}_n(\omega)$$

exists for every $\omega \in X$ and is a representant of G. Let C be the collection of all functions of the form

$$C \quad \{k^{-1}(\tilde{g}_{i_1}^{n_1} + \ldots + \tilde{g}_{i_k}^{n_k})\}$$

which is a countable set of representants of elements of M. Define

$$\tilde{H}_n(\omega) \quad \sup\{\tilde{f}(\omega) \quad \tilde{f} \in C, \ \|\tilde{f}\|_2 \leq n^{-1}\}$$

and

$$\tilde{H}(\omega) \quad \lim \tilde{H}_n(\omega).$$

We shall show that

$$\tilde{H}(\omega) \geq \tilde{G}(\omega) \qquad\qquad \forall\omega \in X.$$

Indeed, fixing $\omega_o \in X$, we may find for every $n \in \mathbb{N}$ an $i \in \mathbb{N}$ such that $\tilde{g}_i^n(\omega_o) \geq \tilde{G}(\omega_o)$ and $\delta(\tilde{g}_i^n, O)$ Given $\varepsilon > 0$, let $k \in \mathbb{N}$ s.t.

$$k \geq 2(\sup \{\|f\|_2 : f \in M\}/\varepsilon)^2$$

and, for $\leq l \leq k$, find inductively functions $\tilde{g}_{i_l}^{n_l}$, such that

$$\tilde{g}_{i_l}^{n_l}(\omega_o) \geq \tilde{G}(\omega_o)$$

and such that the $\tilde{g}_{i_l}^{n_l}$'s are mutually almost orthogonal, say

$$(\tilde{g}_{i_l}^{n_l}, \tilde{g}_{i_m}^{n_m})| < \varepsilon^2/2 \qquad\qquad \forall l \neq m.$$

Then

$$\tilde{f} \quad k^{-1}(\tilde{g}_{i_i}^{n_1} + \ldots + \tilde{g}_{i_k}^{n_k})$$

is an element of C with

$$\tilde{f}(\omega_o) \geq \tilde{G}(\omega_o)$$

and we may estimate its $\|.\|_2$-norm as follows:

$$(\tilde{f}, \tilde{f}) \leq k^{-1}[k.\sup \{\|f\|_2 . f \in M\}^2 + k^2.\frac{\varepsilon^2}{2}] \leq \frac{\varepsilon^2}{2} + \frac{\varepsilon^2}{2}.$$

Thus we have found an $\tilde{f} \in C$ of $\|.\|_2$-norm less than ε, that majorizes G at the point ω_o. This readily shows that $\tilde{H}(\omega) \geq \tilde{G}(\omega)$ for all $\omega \in X$. If H denotes the equivalence class of the function $\tilde{H}$ then $F \geq H \geq G$; on the other hand the inequality $F \leq G$ is obvious. Hence $F \quad G$ (as equivalence classes of functions). □

2.7. Remark: It seems worth noting that the proof of (a) ⇒ (a') really shows that the functions F, G (as defined in the statement of 2.6) as well as the function

$H \quad \lim H_n$

for

$H_n = \sup \{f \quad \|f\|_2 \leq n^{-1}\}$

are identical.

This gives a connection to a result of Mokobodzki [M] from 1972 stating that $T : L^1(\nu) \to L^1(\mu)$ is representable by a Bochner-integrable function (equivalently: is an integral operator) iff T maps dominated $\sigma(L^1(\nu), L^\infty(\nu))$-convergent sequences to almost everywhere convergent sequences. This resembles (for the special case of L^1) Bukhvalov's criterion with convergence in measure replaced by weak convergence. Proposition 2.6 clarifies why both conditions are equivalent.

Let us also note that taking in proposition 2.6 $A = X$ the implication (b) $\Rightarrow$ (c) of 2.4 together with the remarks preceding 2.6 furnishes the missing proof of the (ii) $\Rightarrow$ (iii) of 2.4.

2.8.: Let us now show how Bukhvalov's criterion may be deduced directly from ours (i.e. (ii) $\Rightarrow$ (iii) of th. 2.3) with the help of the above proposition 2.6. Consider the direct sum of the two measure spaces (X,Σ,μ) and (Y,T,ν), i.e. $(X \cup Y, \Sigma \oplus T, \mu \oplus \nu)$.

It is easily seen that 2.3 (ii) implies the order-continuity of T. Thus if 2.3 (ii) holds while 2.3 (iii) fails, we can find $\psi \in L^2_+(\nu)$, say $\psi \geq 1$, such that there is $\varphi \in L^\infty(\mu)$, $\varphi > 0$ μ-a.e. with

$$T([-\psi,\psi]) \subset [-\varphi^{-1},\varphi^{-1}]$$

but such that $T([-\psi,\psi])$ is not equimeasurable. If M_φ and $M_{\psi^{-1}}$ denote the multiplication operator with φ and ψ^{-1} resp. define

S

The set $S([-\psi,\psi])$ is an absolutely convex, bounded subset of $L^{\infty}(\mu\otimes\nu)$, such that the restriction to X fails to be equimeasurable and we infer from 2.8 (b) that there is a sequence $(g_n)_{n=1}^{\infty}$ in $[-\psi,\psi]$ such that $(M_{\varphi}Tg_n, M_{\psi^{-1}}g_n)_{n=1}^{\infty}$ tends to zero in measure but $(M_{\varphi}Tg_n)_{n=1}^{\infty}$ does not converge μ-a.e. Hence $(g_n)_{n=1}^{\infty}$ is a sequence in $L^2(\nu)$, $|g_n| \leq \psi$ for $n \in \mathbb{N}$, which converges to zero in measure while Tg_n does not converge to zero μ-a.e.; with this contradiction we are done.

2.9. Remark: We have stated theorem 2.3 for the case of L^2-spaces but the arguments carry over to operators from F to E, where E and F are general order ideal spaces (on the finite measure spaces (X,Σ,μ) and $(Y,\mathcal{T},\nu)$ resp.). Indeed, note first that $T : F \to E$ is integral iff T is integral as an operator from F to $L^{0}(\mu)$, hence the question of integral representability does not depend on the space E on the right hand side. In fact, it only depends on the collection of order-intervals of F. Precisely the same arguments as in the L^2-case work in the general case and again proposition 2.6 gives the link between Bukhvalov's and our criterion.

3. An example of a positive, compact operator on L^2, which is not integral

We now turn to a different question: We shall show that convolution with a suitably chosen "sequence of biased coins" furnishes an example of a positive, compact operator on L^2 which fails to be integral. The example is to a large extent just a translation of an example due to A. Costé ([C] and [D-U], p. 90). It seems more natural than D. H. Fremlin's construction [F] and gives a sharper result: The operator is not only compact, if is even of trace class p, $p > 2$. I would like to thank V. Losert, who pointed out to me the use of an infinite product to obtain the estimate relevant for the S -norm.

Let X be the compact group $\Delta = \{-1,1\}^{\mathbb{N}}$, equipped with normalized Haar-measure μ on the Borel-σ-algebra Σ. For $1/2 < \alpha < 1$ let $\lambda(a)$ be the measure on the two-point-set $\{-1,1\}$ given by

$$\lambda(\alpha)(\{1\}) = \alpha$$
$$\lambda(\alpha)(\{-1\}) \quad 1-\alpha.$$

Given a sequence $(\alpha_n)_{n=1}^{\infty}$ in $]1/2,1[$ define the probability measure $\lambda((\alpha_n)_{n=1}^{\infty})$ on Δ as the product of the $\lambda(\alpha_n)$, i.e.

$$\lambda((\alpha_n)_{n=1}^{\infty}) = \bigotimes_{n=1} \lambda(\alpha_n).$$

We have the following dichotomy result:

3.1. Proposition (Kakutani [K], [Ü]): We have $\lambda \perp \mu$ or $\lambda \ll \mu$ according as $\sum_{n=1}^{\infty} (2\alpha_n - 1)^2$ diverges or converges.

We now fix a sequence $(\alpha_n)_{n=1}^{\infty}$ in $]1/2,1[$ such that

$$\sum_{n=1} (2\alpha_n - 1)^2$$

while

$$\sum_{n=1}^{\infty} (2\alpha_n - 1)^p < \infty \quad \text{for} \quad p > 2.$$

The proceeding proposition tells us that the probability measure

is singular with respect to μ. Let

be the operator of convolution with λ. Clearly T is positive (in the lattice sense) since λ is positive. The fact that λ is singular with respect to μ corresponds to the fact that T_λ is not an integral operator. Indeed, viewing T_λ as an operator from $C(\Delta)$ to $C(\Delta)$ the restriction of the adjoint operator $T_\lambda^* : M(\Delta) \to M(\Delta)$ to $L^1(\mu)$ is represented by a μ-essentially uniquely weak-star-measurable function $F : \Delta \to \mathcal{M}(\Delta)$ (see [D-S], p. 503). It is obvious from the definition of the convolution that this F is given by

$$F \quad \omega \to \lambda_\omega$$

where λ_ω denotes the translate of λ by $\omega \in \Omega$; hence F takes its values in $\mathcal{M}(\Delta) \setminus L^1(\mu)$. So there can not exist a Halmos-function $\gamma : \Delta \to L^1(\mu)$ representing T_λ (for the definition see [S1]) as γ would have to equal F μ-a.e.

This shows that T is not an integral operator. Let

$$\varepsilon_n : \{-1,1\}^{\mathbb{N}} \to \{-1,1\}$$

be the projection onto the n'th coordinate and, for a finite subset $A \subset \mathbb{N}$, define the Walsh-function

$$w_A(\omega) = \prod_{n \in A} \varepsilon_n(\omega).$$

It is wellknown (e.g., [K2]) that the Walsh-functions are the characters of the group Δ and that T_λ is a diagonal operator with respect to the Walsh-basis. The corresponding eigenvalues are given by

$$\int w_A(\omega)\,d\lambda(\omega) \quad \int \Big(\prod_{n \in A} \varepsilon_n(\omega)\Big)\,d\lambda(\omega) \quad \prod_{n \in A} (1.\alpha_n + (-1).(1-\alpha_n))$$
$$\prod_{n \in A} (2\alpha_n - 1).$$

The norm of T_λ with respect to the trace class p, for $p > 2$, can therefore be estimated by

$$\|T_\lambda\|_p^p \quad \sum_{A \subset \mathbb{N}} \Big(\prod_{n \in A} (2\alpha_n - 1)\Big)^p \quad \prod_{n=1}^{\infty} (1 + (2\alpha_n - 1)^p)$$
$$\exp \sum_{n=1}^{\infty} \ln\,(1 + (2\alpha_n - 1)^p)$$
$$\leqq \exp\Big(\sum_{n=1} (2\alpha_n - 1)^p\Big) < \infty.$$

This shows that T_λ is of trace class p, for every $p > 2$, and finishes the presentation of the example.

3.2. Remark: To point out the flavour of the different criteria of theorem 2.3 we shall show how to use our or Bukhvalov's integral representability criterion to see that the above operator T_λ is not integral. The alert reader will notice that these arguments are just different aspects of the same issue.

a) We shall show that T_λ transforms the unit-ball of $C(\Delta)$ into a non-equi-measurable set. Indeed if T_λ (ball $C(\Delta)$) were equi-measurable then - by translation-invariance - it would be relatively $\|.\|$ - compact, i.e. T_λ would induce a compact operator from $C(\Delta)$ to $C(\Delta)$. The adjoint $T_\lambda : M(\Delta) \to M(\Delta)$ would also be compact and, since $T_\lambda^*(L^1(\mu)) = T_\lambda(L^1(\mu)) \subset L^1(\mu)$, this would imply that T_λ^* maps $M(\Delta)$ into $L^1(\mu)$. But if δ denotes the Dirac-measure located at the unit-element e of the group Δ, then $T_\lambda(\delta_e) = \lambda$, which is in $M(\Delta) \setminus L^1(\mu)$; this furnishes the desired contradiction.

b) To apply Bukhvalov's criterion note that for every $n \in \mathbb{N}$ there is a compact set $K_n \subset \Delta$ such that $\lambda(K_n) > 1/2$ while $\mu(K_n) \leqq n^{-1}$ Let f^n be a $[0,1]$-valued continuous function on Δ, which equals on K_n and zero on a set of μ-measure greater than $1-2n^{-1}$. Note that is a continuous function on Δ, s.t.

$$T_\lambda(f^n)(e) = \int_\Delta f^n(\omega)\,d\lambda(\omega) > 1/2,$$

hence there is a neighbourhood of the unit-element e on which is greater than $1/2$. By the compactness of Δ we may find finitely many translates of such that the supremum is greater than $1/2$ on all of Δ. As T_λ commutes with the translation, there are finitely many translates $f_1^n,\ldots,f_m^n$ of f^n such that, for every $\omega \in \Delta$,

$$\sup\{T_\lambda f_i^n(\omega) : 1 \leqq i \leqq m_n\} > 1/2.$$

The sequence $((f_i^n)_{i=1}^{m_n})_{n=1}^{\infty}$ is dominated by the constant function 1, converges to zero measure, while $((T_\lambda f_i^n)_{i=1}^{m_n})_{n=1}^{\infty}$ does not converge to zero at any point of Δ; this gives the desired contradiction to Bukhvalov's criterion.

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