MARTINGALE MEASURES FOR DISCRETE TIME PROCESSES WITH INFINITE HORIZON

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Revised Version November 1992

Abstract. Let $(S_t)_{t\in I}$ be an $\mathbb{R}^d$–valued adapted stochastic process on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in I}, P)$. A basic problem, occuring notably in the analysis of securities markets, is to decide whether there is a probability measure Q on $\mathcal{F}$ equivalent to P such that $(S_t)_{t\in I}$ is a martingale with respect to Q.

It is known since the fundamental papers of Harrison–Kreps (79), Harrison–Pliska(81) and Kreps(81) that there is an intimate relation of this problem with the notions of "no arbitrage" and "no free lunch" in financial economics.

We introduce the intermediate concept of "no free lunch with bounded risk". This is a somewhat more precise version of the notion of "no free lunch": It requires that there should be an absolute bound of the maximal loss occuring in the trading strategies considered in the definition of "no free lunch". We shall give an argument why the condition of "no free lunch with bounded risk" should be satisfied by a reasonable model of the price process $(S_t)_{t\in I}$ of a securities market.

We can establish the equivalence of the condition of "no free lunch with bounded risk" with the existence of an equivalent martingale measure in the case when the index set I is discrete but (possibly) infinite. A similar theorem was recently obtained by Delbaen (92) for the case of continuous time processes with continuous paths. We can combine these two theorems to get a similar result for the continuous time case when the process $(S_t)_{t\in \mathbb{R}_+}$ is bounded and — roughly speaking — the jumps occur at predictable times.

1. Introduction

Let $(S_t)_{t\in I}$ be an $\mathbb{R}^d$–valued martingale defined on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in I}, P)$ - (precise definitions and notations will be given below). For $s, t \in I$, $s < t$ and a bounded $\mathbb{R}^d$–valued $\mathcal{F}_s$–measurable function h we have

$$E_P(h, S_t - S_s) = 0 \tag{1}$$

This is essentially the defining property of a martingale and reflects the intuitive idea behind this concept: "One can't win systematically by betting on a martingale".

We investigate a kind of converse to the fundamental fact (1): Let an $\mathbb{R}^d$–valued adapted stochastic process $(S_t)_{t\in I}$ on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in I}, P)$ be given. Under what conditions does there exist a probability measure Q on $\mathcal{F}$, equivalent to P, such that $(S_t)_{t\in I}$ is a martingale with respect to $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in I}, Q)$?

This question arose in particular in the analysis of stochastic models of securities markets. In this context the random variables

$$(h(\omega), S_t(\omega) - S_s(\omega)) \tag{2}$$

Acknowledgement: We thank F. Delbaen, H. Föllmer, R. Haydon, B. Maurey, J. Orihuela, D. Preiss, M. Schweizer, W. Stummer and M. Talagrand for helpful conversations during the preparation of this paper

as above have an obvious interpretation as the net gain of an elementary trading operation. In the fundamental papers of Harrison-Kreps (79), Harrison-Pliska (81) and Kreps (81) the concepts of "no arbitrage" and "no free lunch" were investigated. Intuitively they state that there should be no non-negative element —except for the zero-function — among the functions appearing in (2). It was shown that — under appropriate conditions — these concepts coincide with the existence of an equivalent martingale measure for the process (S_t) and this result is sometimes referred to as the *fundamental theorem of asset pricing* (see Dybvig-Ross (87)).

However, we claim that for general processes $(S_t)_{t\in I}$ the "best possible characterisation" for the existence of an equivalent martingale measure is not yet completely understood although there has been in recent years a lot of research activity and a number of important steps in this direction (Duffie-Huang (86), Back-Pliska (90), Stricker (90), Ansel-Stricker (90) and (92), Dalang-Morton-Willinger (90), Delbaen (92),Mcbeth (91), Lakner (92b), Schweizer (92b)).

The aim of the present paper is to introduce the concept of "no free lunch with bounded risk" and to investigate whether this concept is equivalent to the existence of an equivalent martingale measure. This concept was also considered (under different names) in the work of Delbaen (92) and Mcbeth (91). We claim that this concept has a more precise economic interpretation than that of "no free lunch" (see 1.5 below) and that it is of primary interest to understand its precise relation to the existence of an equivalent martingale measure.

For the case of finite discrete time I the relation between the existence of an equivalent martingale measure and the absence of arbitrage opportunities is completely clear by the work of Dalang–Morton–Willinger (90) (compare also Back–Pliska (90) and Schachermayer (92)). In the present paper we shall show that for the case of discrete (but possibly infinite) time (i.e., $I = \mathbb{N}_0$) the situation is clarified too: We can establish the equivalence of "No free lunch with bounded risk" with the existence of an equivalent martingale measure (theorem A below) in a completely general setting (no boundedness or integrability conditions have to be imposed on the process $(S_t)_{t\in\mathbb{N}_0}$).

In the case of finite continuous time (i.e., $I = [0,1]$) an analogous theorem has been proved in the remarkable paper of Delbaen (92) for processes with continuous paths. We can combine these two results to obtain in theorem B the equivalence of "no free lunch with bounded risk" with the existence of an equivalent martingale measure for a fairly general class of continuous time processes (roughly speaking the jumps of the process must occur at predictable times; but in this case we do need a boundedness assumption).

Let us now start to be more precise. We adopt the following setting: Let I be a subset of $\mathbb{R}_+$, $(\Omega, (\mathcal{F}_t)_{t\in I}, \mathcal{F}, P)$ a filtered probability space, and $(S_t)_{t\in I}$ a family of $\mathcal{F}_t$–measurable $\mathbb{R}^d$-valued random variables. We shall consider the case $I = \mathbb{N}_0$ in the first part of the paper and then the case $I = [0,1]$ or $I = \mathbb{R}_+$. We shall always assume without loss of generality that $S_0 \equiv 0$ and that $\mathcal{F}$ is generated by $(\mathcal{F}_t)_{t\in I}$. Note that at this stage we do not impose the "usual conditions" on the filtration $(\mathcal{F}_t)_{t\in I}$, nor the requirement that the process S_t is cadlag, nor any integrability conditions.

1.1 Definition. *We say that $(S_t)_{t\in I}$ satisfies (EMM) (which stands for "equivalent martingale measure") if there is a probability measure Q on $\mathcal{F}$ equivalent to P such that $(S_t)_{t\in I}$ is a martingale with respect to $(\Omega, (\mathcal{F}_t)_{t\in I}, Q)$, i.e., each S_t is Q-integrable and formula (1) above holds true for each bounded $\mathcal{F}_s$–measurable function h with P replaced by Q.*

Similarly as in Stricker (90) define K_0 to be the vector space of easy stochastic integrals

$$K_0 = \text{ span } \{(h(\omega), S_t(\omega) - S_s(\omega))\}, \tag{3}$$

where $(\cdot,\cdot)$ denotes the inner product in $\mathbb{R}^d$, s, t runs through the pairs in I with $s < t$ and h is an $\mathbb{R}^d$–valued $\mathcal{F}_s$–measurable function. Note that K_0 is a subspace of $L^0(\Omega, \mathcal{F}, P)$, which is the space of

$\mathcal{F}$-measurable, real-valued functions. The economic interpretation of the random variable $(h(\omega), S_t(\omega) - S_s(\omega))$ is that it describes the net gain of the trading operation of bying $h(\omega)$ units of the stock at time s and selling these stocks again at time t. The requirement that h is $\mathcal{F}_s$-measurable corresponds to the fact that at time s the economic agent posesses only the information modelled by $\mathcal{F}_s$.

Throughout this paper we shall denote by C_0 the convex cone $(K_0 - L^0_+(\Omega, \mathcal{F}, P))$, i. e. , those elements of L^0 that are dominated by some $f \in K_0$. Denote by K (resp. C) the linear space (resp. the convex cone) $K_0 \cap L^\infty$ (resp. $C_0 \cap L^\infty$). Note that C consists of those elements of L^∞ that are dominated by some $f \in K_0$. We shall denote by $\overline{C}$ the closure of C with respect to the σ^*–topology of L^∞ and by $\widetilde{C}$ the set of all limits of σ^*–convergent *sequences* in C. Clearly $\overline{C}$ and $\widetilde{C}$ are convex cones in L^∞.

We now may define the key concept of this paper:

1.2 Definition.

(a) *We say that $(S_t)_{t\in I}$ satisfies (NA) (which stands for "no arbitrage") if*

$$C \cap L^\infty_+ = \{0\}.$$

(b) *We say that $(S_t)_{t\in I}$ satisfies $(NFLBR)$ (which stands for "no free lunch with bounded risk") if*

$$\widetilde{C} \cap L^\infty_+ = \{0\}.$$

(c) *We say that $(S_t)_{t\in I}$ satisfies (NFL) (which stands for "no free lunch") if*

$$\overline{C} \cap L^\infty_+ = \{0\}.$$

Obviously $(NFL) \Rightarrow (NFLBR) \Rightarrow (NA)$ and it is almost as obvious that $(EMM) \Rightarrow (NFL)$ (see 3.1 below).

Note that the condition (NA) of "no arbitrage" is the same as to require that $K_0 \cap L^0_+ = \{0\}$. This concept has the obvious economic interpretation that there should be no easy trading strategy which allows to create positive expectation for a gain with zero investment and without bearing any risk. It was proved by Dalang–Morton–Willinger (90) — generalising previous work of Harrison–Kreps (79), Harrison–Pliska (81) and Back–Pliska (90) — that in the case of finite discrete time (NA) is equivalent to (EMM). But unfortunatly this equivalence breaks down if the time index set I becomes infinite as is shown by easy examples (see Back–Pliska (90) or Dalang–Morton–Willinger (90)).

It was already noted by Kreps (81) that in this case (i.e., I being infinite) a topological condition has to be added which led him to the notion of (NFL). The definition (c) is taken from Kreps (81), where general pairs of dual vector spaces $\langle E, F\rangle$ are considered. In the present setting the pair $\langle L^\infty, L^1\rangle$ and the σ^*–topology on L^∞ are natural as each P–absolutely continuous measure Q may be identified with its Radon–Nikodym derivative $dQ/dP \in L^1(\Omega, \mathcal{F}, P)$ and therefore defines a σ^*–continuous linear functional on L^∞.

1.3 Theorem (Kreps–Yan). *Suppose that, for each $t \in I$, S_t is bounded. Then (NFL) is equivalent to (EMM).*

This remarkable result has been proved (under a mild but irrelevant separability assumption) in Kreps (81). We take the liberty to refer to it as Kreps–Yan theorem as in Yan (80) a similar result was proved independently and in a different context. Both authors had the decisive idea that – in order to get the good notion – one has to consider the closure of the convex cone $C = (K_0 - L^0_+) \cap L^\infty$ and not just that of $K = K_0 \cap L^\infty$ (see example 3.3 below). Once this key idea is established the proof reduces to a combination of a Hahn–Banach and an exhaustion argument (see 3.1 below).

The relevance of Yan's work was noted by Stricker (90) and Ansel–Stricker (90), who used this result of Yan (or rather its proof) to obtain analogous theorems characterising the existence of equivalent martingale measures with finite q–th moments, where $q > 1$. This setting has the advantage that one may state a condition in terms of the norm closure of C in L^p instead of the more delicate σ^*–closure in L^∞ as in 1.2 (c) above. On the other hand, to impose finite q–th moments is somewhat unnatural in the present context and, in particular, not invariant under changes of measure.

Let us turn to the economic interpretation of the concept of (NFL) as given by
Kreps (81):

1.4 Proposition. *$(S_t)_{t\in I}$ satisfies (NFL) iff there does not exist a nonnegative function $f_0 \in L^\infty_+, f_0 \not\equiv 0$, a net $(f_\alpha)_{\alpha\in J}$ in K_0 and a net $(h_\alpha)_{\alpha\in J}$ in L^0_+such that each $f_\alpha - h_\alpha$ is uniformly bounded and $(f_\alpha - h_\alpha)_{\alpha\in J}$ converges to f_0 with respect to the σ^*-topology of L^∞.*

The proof of the above proposition is rather obvious. Let us discuss the economic interpretation: The usual argument as to why a reasonable model $(S_t)_{t\in I}$ of a stock price process should satisfy the no arbitrage condition (NA) goes as follows: If there exists an arbitrage opportunity, i. e. , a random variable $f_0 \in C \cap L^\infty_+, f_0 \not\equiv 0$, then there should be at least one economic agent quickly taking advantage of this opportunity until — by the law of supply and demand — the opportunity quickly disappears. Hence a reasonable model of a financial market on which there are potential arbitrageurs should not provide any arbitrage opportunities from the very beginning.

To argue that a reasonable model $(S_t)_{t\in I}$ should in fact satisfy the stronger condition of (NFL), one might argue as follows: If (NFL) is violated then an arbitrageur may still find a nonnegative $f_0 \in L^\infty_+\backslash\{0\}$, which — possibly — is not quite in C but may be approximated by elements of C in the following sense: There is a net $(f_\alpha)_{\alpha\in J}$ in K_0 such that if the agent "throws away" the amount of money $h_\alpha \in L^0_+$ the random variable $f_\alpha - h_\alpha$ becomes close to f_0 with respect to the σ^*–topology of L^∞. Whence, similarly as above, there should be an arbitrageur who takes advantage of the "almost arbitrage opportunity" f_α for some $\alpha \in J$ which, in turn, would quickly make this opportunity disappear.

We believe that this argument is not very convincing: It requires the existence of rather imprudent arbitrageurs as — although $(f_\alpha - h_\alpha)$ is in some sense close to f_0 — there is no control on the maximal loss obtained when using the trading strategy which gives the gain f_α. Let us also note that a similar remark applies to the conditions used by Stricker (90), as a control of the L^p–norm of $(f_\alpha - h_\alpha) - f_0$ for some $p < \infty$ does not give a control for the respective maximal loss.

This drawback of the notion of (NFL) is not shared by the notion of "no free lunch with bounded risk" as will be shown by the subsequent proposition. This is the main attraction of this notion and should explain its name. The proof of proposition 1.5 is a consequence of the Banach–Steinhaus theorem and will be given in 3.6 below.

1.5 Proposition:. *The process $(S_t)_{t\in I}$ satisfies $(NFLBR)$ iff there does not exist a $[0,\infty]$-valued random variable f_0, $f_0 \not\equiv 0$, and a sequence $(f_n)_{n=1}^\infty$ in K_0 such that*

(a) $f_n(\omega) \geq -1$ *for P–a.e. $\omega \in \Omega$ and $n \in \mathbb{N}$*

(b) $\lim\limits_{n\to\infty} f_n(\omega) = f_0(\omega)$ *for P–a.e. $\omega \in \Omega$.*

We claim that a "free lunch with bounded risk" is much more appealing to an arbitrageur than just a "free lunch": He or she knows that in any case (in the sense of P–almost everywhere) he or she can at most lose one unit of money while — as n becomes big — the net gain $f_n(\omega)$ becomes pointwise arbitrarily close to $f_0(\omega)$. Hence he or she should choose some big n and do the trading operation that yields $f_n(\omega)$.

Note that we did not impose any bound from above on f_0 and we even allowed it to take the value $+\infty$. This curious fact of "allowing the agents to become arbitrarily rich" is in fact crucial for the above proposition to hold true and corresponds to the fact that in prop. 1.4 we had to allow the agents to "throw away money".

The main result of this paper reads as follows:

1.6 Theorem A. *Let $(S_t)_{t\in\mathbb{N}_0}$ be an adapted stochastic process on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbb{N}_0}, P)$. Then $(NFLBR)$ is equivalent to (EMM).*

The proof of this theorem will be given in section 4. Let us stress that we did not impose any boundedness or integrability assumptions on the process $(S_t)_{t\in\mathbb{N}_0}$, i.e., we only assume that $(S_t)_{t\in\mathbb{N}_0}$ is a sequence of $\mathcal{F}_t$-measurable R^d-valued functions.

Theorem A should be compared with the subsequent theorem of Delbaen (92):

1.7 Theorem. *(Delbaen)*

Let $(S_t)_{t\in[0,1]}$ be a bounded adapted stochastic process on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,1]}, P)$ with continuous paths. Then again $(NFLBR)$ is equivalent to (EMM).

For Delbaen's theorem to be true one has to use slightly more general "easy" stochastic integrals than those appearing in formula (2) above: the deterministic times $s < t$ have to be replaced by stopping times $U < V$ (see section 5 below).

Using yet a slightly more general notion of elementary stochastic integrals (see again section 5 below) we may combine these two theorems to obtain a fairly general result:

1.8 Theorem B. *Let $(S_t)_{t\in\mathbb{R}_+}$ be an adapted cadlag stochastic process defined on the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbb{R}_+}, P)$ satisfying the usual conditions. Suppose that*

(i) for each $t \in \mathbb{R}_+$, S_t is bounded, and

(ii) there is a sequence $(T_n)_{n=1}^{\infty}$ of predictable stopping times increasing to infinity such that the jumps of $(S_t)_{t\in\mathbb{R}_+}$ are contained in $\bigcup_{n=1}^{\infty} [\![T_n]\!]$.

Then again $(NFLBR)$ is equivalent to (EMM).

There are natural examples of the situation encountered in theorem B: suppose the (discounted) price process $(S_t)_{t\in\mathbb{R}_+}$ of n stocks is modelled to develop continuously except for some jumps occuring at predictable moments (e.g. when elections are held or earnings announcements are given), where of course the size of the jump (or its sign) need not be known in advance. This is the situation described by theorem B and the message of the theorem is that in this setting there is a crisp economic characterisation for the existence of an equivalent martingale measure.[1]

Finally we note that one may also view theorem A as a help for determining the existence of a sequence of trading strategies yielding a free lunch with bounded risk. Let us illustrate this with an easy example.

Let $(\epsilon_n)_{n=1}^{\infty}$ be a sequence of independent random veariables defined on $(\Omega, \mathcal{F}, P)$ such that

[1]Note added in revising the paper: There remains the obvious problem whether theorem B may be generalized by dropping the assumption that the jumps of the process occur at predictable times only. This question turns out to have a negative answer in the framework of "easy integrands" considered in the present paper. However by passing to general stochastic integration it is possible to obtain a theorem analogous to theorem B above which applies to the general case. These questions will be dealt with in the forthcoming paper Delbaen-Schachermayer (93).

$$P\{\epsilon_n = 1\} = P\{\epsilon_n = -1\} = \frac{1}{2}.$$

We assume that the σ-algebra $\mathcal{F}$ is generated by $(\epsilon_n)_{n=1}^{\infty}$.

Fix a sequence $(\alpha_n)_{n=1}^{\infty}$ of numbers in $]0,1[$ and define the process $(S_n)_{n=1}^{\infty}$ by $S_0 \equiv 0$ and

$$S_n - S_{n-1} = \epsilon_n + \alpha_n \qquad \text{for } n \in \mathbb{N}.$$

Interpreting the process S as the (discounted) price of a risky asset we see the α_n is the expected change in price, while the residual ϵ_n is either plus or minus one with probability one-half each.

It is straightforward to verify that there is a unique probability measure Q on $\mathcal{F}$ which turns S into a martingale, i.e., a unique "risk-neutral probability": Under this measure Q the sequence $(\epsilon_n)_{n=1}^{\infty}$ is a sequence of independent random variables such that

$$P\{\epsilon_n = 1\} = \frac{1+\alpha_n}{2} \quad \text{and} \quad P\{\epsilon_n = -1\} = \frac{1-\alpha_n}{2}.$$

A classical theorem of Kakutani (see, e.g., Williams (91) 14.17, page 150) asserts that Q is either equivalent to P, or Q and P are mutually singular, depending on whether the sequence $(\alpha_n)_{n=1}^{\infty}$ is in l^2 or not.

This implies – in order to insure the existence of an equivalent martingale measure – the risk premia of the process S have to go to zero over time and at a sufficiently high rate . This may be regarded as an unpalatable assumption and shows that in the infinite horizon setting the process S has to be already "almost a martingale" in order to allow an equivalent martingale measure.

It is obvious that the process S above does not permit arbitrage possibilities if we allow only trading strategies with a finite horizon. On the other hand Kakutani's theorem in tandem with theorem A above implies that there is a free lunch with bounded risk if and only if the sequence $(\alpha_n)_{n=1}^{\infty}$ fails to be square integrable.

In order to achieve some intuitive understanding of the situation we pass to an even simpler situation: Suppose that all α_n equal a fixed α and let us pass to the familiar "geometric" version of the process: Let $\tilde{S}_0 \equiv 1$ and

$$\frac{\tilde{S}_n - \tilde{S}_{n-1}}{\tilde{S}_{n-1}} = \epsilon_n + \alpha.$$

If $\alpha \geq \sqrt{2} - 1$ – which we shall assume in order to make things even simpler – the process $\tilde{S}$ tends almost surely to infinity, whence an investor has an obvious strategy yielding a free lunch with bounded risk: Simply buy the risky security at time 0 and sell it at the first time t when $\tilde{S}_t \geq 2\tilde{S}_0 = 2$ or at a given time n, if n is reached first. Now let $n \to \infty$ to obtain a sequence of simple strategies which have bounded risk and converge to an arbitrage opportunity. Such a sequence is by definition a free lunch with bounded risk.

Of course, this was a particularly easy setting. But if we choose, for example, $\alpha_n = (n+1)^{-1/2}$ it is already more challenging to directly construct a sequence of trading strategies yielding the desired free lunch with bounded risk.

We now give an outline of the paper. After fixing definitions and notations in section 2 we establish some preliminary results in section 3. We first reproduce the proof of the Kreps–Yan theorem, which is fundamental. In example 3.3 we show that in general it is indispensable to consider in the Kreps–Yan theorem the convex cone of functions *dominated* by the gains of trading operations and not just the vector space formed by these gains functions. We then introduce the notion of *Fatou–convergence*, which alludes to Fatou's lemma and is tailor–made for the convex cones encountered in the setting of the Kreps–Yan theorem. This notion allows for a kind of compactness result for sequences of functions bounded from below (lemma 3.5) which will turn out to be a very useful tool.

In section 4 we prove theorem A. We introduce the notion of *admissible* integrands which essentially appears already inMcbeth (91), the idea and the name going back to a remark in Harrison-Pliska (81). This notion is also tailor–made to fit into the framework of the Kreps–Yan theorem. After establishing some technical results we first prove Theorem A for the case of bounded processes $(S_t)_{t\in\mathbb{N}_0}$ and finally for the general case which is more delicate and involves the use of Frechet– rather than Banach–spaces. In both cases the Krein–Smulian theorem will be the decisive ingredient to the proof.

In section 5 we prove theorem B. Unfortunatly, this is not just a straightforward corollary of Delbaen's theorem and theorem A. Instead we have to work quite hard for the proof. First we reformulate Delbaen's theorem for our setting and then relate it to the notion of admissible integrands. This notion of admissible integrands may then be extended to processes $(S_t)_{t\in\mathbb{R}_+}$ satisfying the assumptions of theorem B, and this allows us to adapt the arguments used for theorem A to prove theorem B.

2. Definitions and Notations

$(\Omega,\mathcal{F},P)$ will denote a probability space. Let I be a subset of $\mathbb{R}_+$ containing zero. (We adopt this degree of generality mainly to cover the continuous and discrete time cases simultaneously). An increasing family $(\mathcal{F}_t)_{t\in I}$ of sub–σ–algebras of $\mathcal{F}$ will denote a filtration.

$(S_t)_{t\in I}$ will denote an adapted $\mathbb{R}^d$–valued process, i.e. a family of $\mathbb{R}^d$–valued functions such that each S_t is $\mathcal{F}_t$–measurable. As is usual, we shall identify functions with their equivalence classes (modulo functions vanishing almost everywhere). It will be clear that in the context of the present paper no confusion can arise.

We denote by $L^0(\Omega,\mathcal{F},P;\mathbb{R}^d)$ the space of (equivalence classes of) $\mathcal{F}$–measurable $\mathbb{R}^d$–valued functions and, for $1\le p\le\infty$, by $L^p(\Omega,\mathcal{F},P;\mathbb{R}^d)$ the subspace of functions with finite p–th moments. $\mathbb{R}^d$ will be equipped with its canonical inner product $(\cdot,\cdot)$ and euclidean norm $\|.\|$.

Let Q be a probability measure on $\mathcal{F}$. We say that Q is equivalent to P if Q and P have the same nullsets or, equivalently, if the mutual Radon–Nikodym derivatives $\frac{dQ}{dP}$ and $\frac{dP}{dQ}$ exist. We denote, for $f\in L^1(\Omega,\mathcal{F},P;\mathbb{R}^d)$, by $E_P(f)$ the expectation of f with respect to P.

If Q is a probability measure on $\mathcal{F}$ equivalent to P and $(S_t)_{t\in I}$ is an $\mathbb{R}^d$–valued process adapted to $(\mathcal{F}_t)_{t\in I}$, we say that $(S_t)_{t\in I}$ is a martingale with respect to Q if each S_t is in $L^1(\Omega,\mathcal{F}_t,Q;\mathbb{R}^d)$ and, for $s,t\in I$, $s<t$ and $h\in L^\infty(\Omega,\mathcal{F}_s,P;\mathbb{R}^d)$, we have

$$E_Q(h,S_t-S_s)=0\ .$$

In this case we say that Q is a martingale measure for $(S_t)_{t\in I}$, and we say that Q is an equivalent martingale measure if, in addition, Q is equivalent to$P$.

By $L^p(\Omega,\mathcal{F},P)_+$ or L^p_+ we denote the positive cone of L^p.

An *easy integrand* will be a linear combination of functions of the form

$$H(\omega,t)=h(\omega)\chi_{]u,v]}(t)$$

where $u<v$ are elements of I, $\chi_{]u,v]}(t)$ denotes the indicator function of the interval $]u,v]$, and $h\in L^0(\Omega,\mathcal{F}_u,P;\mathbb{R}^d)$. To be precise, we shall use this notion of an easy integrand in sections 3 and 4, while in section 5 we shall use a slightly more general concept.

An easy integrand H gives rise to an *easy stochastic integral*, which is a linear combination of processese of the form

$$(H.S)_t(\omega) = \begin{cases} 0, & \text{for } t \le u \\ (h(\omega), S_t(\omega) - S_u(\omega)), & \text{for } u \le t \le v \\ (h(\omega), S_v(\omega) - S_u(\omega)), & \text{for } v \le t. \end{cases}$$

For an easy integrand H we may also define

$$(H.S)_\infty = \lim_{t \in I} (H.S)_t(\omega),$$

where $\lim_{t\in I} (H.S)_t(\omega)$ either equals $(H.S)_{t_1}(\omega)$ if there is a maximal element $t_1 \in I$ or, if I contains no maximal element, the limit as t tends to the supremum of I. Note that in any case there are no convergence problems arising in the above definitions.

3. Preliminary Results

In this section we develop the necessary machinery for the proof of the main theorems.

We start by proving some results which were mentioned without proofs in the introduction. First we give a proof of the fundamental Kreps–Yan theorem (compare Yan (80), Kreps (81), Stricker (90)):

3.1 Proof of theorem 1.3. (EMM) $\Rightarrow$ (NFL): This is the easy part. Suppose there is an equivalent martingale measure Q and denote by g its Radon–Nikodym derivative $\frac{dQ}{dP}$. It essentially follows from the definition of a martingale that for each $f \in K$ we have

$$E_Q(f) = \langle f, g \rangle = \int f(\omega) g(\omega) dP(\omega) = 0 \ ,$$

and therefore, for each $f \in C$

$$E_Q(f) = \langle f, g \rangle = \int f(\omega) g(\omega) dP(\omega) \le 0 \ ,$$

By the weak-star continuity of g this inequality remains valid for each $f \in \overline{C}$.

On the other hand, for $f \in L^\infty_+$, $f \not\equiv 0$,

$$E_Q(f) = \langle f, g \rangle > 0 \ ,$$

which clearly implies that $\overline{C}$ is disjoint from $L^\infty_+ \setminus \{0\}$.

Note that for this implication we did not need the boundedness assumption on $(S_t)_{t\in I}$.

We now pass to the reverse implication.

(NFL) $\Rightarrow$(EMM) Step 1 (Hahn–Banach argument):
We claim that, for fixed $f \in L^\infty_+$, $f \not\equiv 0$, there is $g \in L^1_+$ which — viewed as a linear functional on L^∞ — is less than or equal to zero on $\overline{C}$ such that

$$\langle f, g \rangle > 0 \ .$$

To see this, apply the separation theorem (e.g., Schaefer (71), th. II, 9.2) to the σ^*–closed convex set $\overline{C}$ and the compact set $\{f\}$ to find $g \in L^1$ and $\alpha < \beta$ such that

$$g \mid_{\overline{C}} \ \le \alpha \text{ and } \langle f, g \rangle > \beta \ .$$

As $0 \in C$ we have $\alpha \geq 0$. This implies that g is zero or negative on $\overline{C}$ and, in particular, non-negative on L^∞_+, i.e. $g \in L^1_+$. Noting that $\beta > 0$ we proved step 1.

Step 2 (Exhaustion Argument): Denote by $\mathcal{G}$ the set of all $g \in L^1_+$, g being less than or equal to zero on C. As $0 \in \mathcal{G}$ (or by Step 1), $\mathcal{G}$ is nonempty.

Let $\mathcal{S}$ be the family of (equivalence classes of) subsets of Ω formed by the supports of the elements $g \in \mathcal{G}$. Note that $\mathcal{S}$ is closed under countable unions, as for a sequence $(g_n)_{n=1}^\infty \in \mathcal{G}$ we may find strictly positive scalars $(\alpha_n)_{n=1}^\infty$, such that $\sum\limits_{n=1}^{\infty} \alpha_n g_n \in \mathcal{G}$.

Hence there is $g_0 \in \mathcal{G}$ such that for $S_0 = \{g_0 > 0\}$ we have

$$P(S_0) = \sup\{P(S) : S \in \mathcal{G}\}.$$

We now claim that $P(S_0) = 1$, which readily shows that g_0 is strictly positive almost surely. If $P(S_0) < 1$ then we could apply step 1 to $f = \chi_{(\Omega \setminus S_0)}$ to find $g_1 \in \mathcal{G}$ with

$$\langle f, g_1 \rangle = \int\limits_{\Omega \setminus S_0} g_1(\omega) dP(\omega) > 0$$

Hence $g_0 + g_1$ would be an element of $\mathcal{G}$ whose support has P–measure strictly bigger than $P(S_0)$, a contradiction.

Normalize g_0 so that $||g_0||_1 = 1$ and let Q be the measure on $\mathcal{F}$ with Radon–Nikodym derivative $dQ/dP = g_0$. By our boundedness assumption, for $s < t$ and $h \in L^\infty(\Omega, \mathcal{F}_s, P; \mathbb{R}^d)$ we have that the random variable $(h, S_t - S_s)$ is bounded, and therefore $(h, S_t - S_s)$ as well as $-(h, S_t - S_s)$ are in C. Hence $E_Q(h, S_t - S_s) = 0$ thus proving (EMM).

q.e.d.

3.2 REMARK. We have given the proof for the pair of dual spaces $\langle L^\infty, L^1 \rangle$. But it is clear that the same proof applies to any pair $\langle E, F \rangle$ of dual vector spaces of measurable functions on Ω, provided we choose topologies on E and F compatible with this pairing and if, for any sequence $(g_n)_{n=1}^\infty$ there are strictly positive scalars $(\alpha_n)_{n=1}^\infty$ such that $\sum_{n=1}^\infty \alpha_n g_n$ converges in F. Compare Kreps (81) and also 4.10 below where we apply the above proof to a setting where E is a Frechet space.

Let us recast the above theorem in a more abstract version: Let C be a convex cone in L^∞ such that

(i) $C = C - L^\infty_+$
(ii) C is weak star closed and
(iii) $C \cap L^\infty_+ = \{0\}$.

Then there is $g \in L^1$ with $g|_C \leq 0$ and g being strictly positive almost surely.

We now give an example showing that in the definition of (NFL) – and therefore in the Kreps-Yan theorem – the σ^*-closure of $C = (K_0 - L^0_+) \cap L^\infty$ may not be replaced by the σ^*–closure of $K = (K_0 \cap L^\infty)$. A similar example, displaying the same phenomenon, was given by Mcbeth ((92), example 5.2).

3.3 EXAMPLE. There is a uniformly bounded $\mathbb{R}$–valued process $(S_t)_{t \in \mathbb{N}_0}$ such that with the above notation and letting $K = K_0 \cap L^\infty$ we have

$$\overline{C} = L^\infty$$

while

$$\overline{K} \cap L^\infty_+ = \{0\},$$

where $\overline{}$ denotes the closure with respect to the σ^*–topology of L^∞.

Proof. Let $(A_n)_{n=1}^\infty$ be a partition of Ω into sets of probability $P(A_n) = 2^{-n}$. Split each A_n into two disjoint sets A_n^+ and A_n^- of probability $2^{-(n+1)}$. Let $B_n = \bigcup_{k=n+1}^\infty A_k$. Define

$$f_n = 2^{5n}\chi_{A_n^+} + 2^n\chi_{A_n^-} - 2^{-n}\chi_{B_n}$$

and let K be the span of (f_n) in L^∞.

Let us first show that the constant function 1 is in the σ^*–closure of $K - L_+^\infty$. The same analysis readily shows that $K - L_+^\infty$ is σ^*–dense in L^∞. Indeed, $g_n = \sum_{k=1}^n f_k$ is bounded from below by -1 and bigger than 1 on $\Omega \setminus B_n$. Hence $(g_n \wedge 1)_{n=1}^\infty$ tends almost surely and therefore σ^* to the constant function 1.

To show the second assertion, suppose to the contrary that there is $g_0 \in L_+^\infty \setminus \{0\}$ which is in the σ^*–closure of K. As K is convex, g_0 is in fact in the closure of K with respect to the Mackey-topology and a fortiori in the closure of K with respect to the L^1–norm (Observe that the Mackey-topology on L^∞ is the topology of uniform convergence on weakly compact subsets of L^1; as the unit-ball of L^∞, viewed as a subset of L^1, is weakly compact, one readily observes that the Mackey-topology is finer than the topology of uniform convergence on the unit-ball of L^∞, i.e., the norm-topology induced by the L^1-norm. For details, see *e.g.* Schaefer (71)).

Clearly g_0 is constant on each A_n^+ and A_n^-. Let n_0 be the first number such that g_0 does not vanish on A_{n_0}. One easily verifies that there is $a > 0$ such that g_0 equals $2^{n_0}a$ on $A_{n_0}^-$ and $2^{5n_0}a$ on $A_{n_0}^+$.

Let $n > n_0$ and consider $g \in K$ such that $g \geq 0$ on $\bigcup_{k=1}^n A_k$ and $g = g_0$ on $\bigcup_{k=1}^{n_0} A_k$. Glancing at the definition of f_n one verifies inductively that, for $n \geq k > n_0$, g is bigger than $2^{2(k-n_0)}a$ on A_k^+, whence the L^1–norm of g is bigger than $2^{2(n-n_0)-(n+1)}a$.

It follows easily that for $M \in \mathbb{R}_+$ there is $n \in \mathbb{N}$ and $\epsilon > 0$ such that for each $g \in K$ with $\|(g - g_0)\chi_{\bigcup_{k=1}^{n_0} A_k}\|_1 < \epsilon$ and $g \geq -\epsilon$ on the set $\bigcup_{k=1}^n A_k$ we have $\|g\|_1 > M$. This contradiction readily proves the second assertion about K.

We still have to show that K may be constructed as $K = K_0 \cap L^\infty$ where K_0 is the space of the elementary stochastic integrals of a process $(S_t)_{t\in\mathbb{N}_0}$. Let $S_0 \equiv 0$ and define, for $n \in \mathbb{N}$, $S_n - S_{n-1} = 2^{-6n}f_n$. The numbers 2^{-6n} are choosen sufficiently small such that the process $(S_t)_{t\in\mathbb{N}_0}$ stays uniformly bounded. If we define the σ–algebras $\mathcal{F}_n$ to be generated by $(S_0, \ldots, S_n)$ we obtain $K = K_0$ as the space of elementary stochastic integrals of the process $(S_t)_{t\in\mathbb{N}_0}$.

q.e.d.

In order to deal with the cone $C = (K_0 - L_+^0) \cap L^\infty$ we introduce the subsequent concept which is similar to the notion of $*-$*convergence* considered by Mcbeth (92) and will be crucial to deal with the "one-sided boundedness" situations in the sequel.

3.4 Definition. *Denote by $F(\Omega, \mathcal{F}, P)$ the cone of $\mathbb{R} \cup \{+\infty\}$-valued $\mathcal{F}$-measurable functions. We say that a sequence $(f_n)_{n=1}^\infty \in F(\Omega, \mathcal{F}, P)$ Fatou-converges to $f_0 \in F(\Omega, \mathcal{F}, P)$ if*

(i) *there is $M \in \mathbb{R}_+$ such that*
$f_n(\omega) \geqslant -M \qquad n \in \mathbb{N}, P-a.s.$

(ii) $\lim_{n\to\infty} f_n(\omega) = f_0(\omega) \qquad P-a.s.$

To pass from σ^*–convergence in L^∞ to almost sure convergence we shall repeatedly use the following easy fact: If a net $(f_\alpha)_{\alpha\in J} \in L^\infty(\Omega, \mathcal{F}, P)$ converges weak star to $f_0 \in L^\infty(\Omega, \mathcal{F}, P)$ (or, more generally, if $(f_\alpha)_{\alpha\in J} \in L^1(\Omega, \mathcal{F}, P)$ converges with respect to the weak topology of L^1 to $f_0 \in L^1(\Omega, \mathcal{F}, P)$) there are convex combinations $g_n \in \operatorname{conv}(f_\alpha)_{\alpha\in J}$ such that $(g_n)_{n=1}^\infty$ converges a.s. to f_0. Indeed, by the Hahn–Banach theorem there are convex combinations $g_n \in \operatorname{conv}(f_\alpha)_{\alpha\in J}$ converging to f_0 with respect to the norm of L^1. Hence there is a subsequence — which, of course, again is a sequence of convex combinations of $(f_\alpha)_{\alpha\in J}$ — converging almost surely.

Also note that, conversely, if $(f_n)_{n=1}^\infty \in L^\infty(\Omega,\mathcal{F},P)$ is uniformly bounded and $(f_n)_{n=1}^\infty$ converges a.s. to $f_0 \in L^\infty(\Omega,\mathcal{F},P)$ then, by Lebesgue's theorem, $(f_n)_{n=1}^\infty$ converges weak star to f_0.

The next lemma, whose proof is somewhat long but only uses standard arguments, extends this kind of situation to sequences of functions in $F(\Omega,\mathcal{F},P)$.

3.5 Lemma. *If $(f_n)_{n=1}^\infty \in F(\Omega,\mathcal{F},P)$ is uniformly bounded from below there is a sequence $g_n \in \operatorname{conv}(f_n, f_{n+1},\dots)$ such that $(g_n)_{n=1}^\infty$ Fatou-converges to some $g_0 \in F(\Omega,\mathcal{F},P)$.*

Proof. First observe the following

Fact : If $(h_n)_{n=1}^\infty \in F(\Omega,\mathcal{F},P)$ and $h_0(\omega) = \liminf h_n(\omega)$ then at least one of the following assertions holds true:

(i) there is a subsequence $(h_{n_k})_{k=1}^\infty$ of $(h_n)_{n=1}^\infty$ converging a.s. to h_0 , or

(ii) there is a sequence $k_n \in \operatorname{conv}(h_n, h_{n+1},\dots)$ s. t. for $k_0(\omega) = \liminf k_n(\omega)$ we have $k_0 \geqslant h_0$ and $k_0(\omega) > h_0(\omega)$ on a set of positive P-measure.

To see this consider, for $\varepsilon > 0$,

$$a_\varepsilon(n) = P\{h_n \geqslant h_0 + \varepsilon\}.$$

If, for each $\varepsilon > 0$, the sequence $(a_\varepsilon(n))_{n=1}^\infty$ tends to zero, then it is easy to produce a subsequence $(h_{n_k})_{k=1}^\infty$ satisfying (i).

So suppose that there is $\varepsilon > 0$ and $\alpha > 0$ such that $\limsup a_\varepsilon(n) = 2\alpha > 0$. By passing to a subsequence we may assume $a_\varepsilon(n) \geqslant \alpha$ for all $n \in \mathbb{N}$. The sequence $(\chi_{\{h_n \geq h_0+\varepsilon\}})_{n=1}^\infty$ is bounded in $L^\infty(\Omega,\mathcal{F},P)$, whence there is a sequence $r_n \in \operatorname{conv}(\chi_{\{h_n\geq h_0+\varepsilon\}}, \chi_{\{h_{n+1}\geq h_0+\varepsilon\}},\dots)$ converging a.s. to $r_0 \in L_+^\infty(\Omega,\mathcal{F},P)$. Clearly $E(r_0) \geqslant \alpha$.

Let $k_n \in \operatorname{conv}(h_n, h_{n+1},\dots)$ be obtained by using the same weights on $(n, n+1,\dots)$ as $(r_n)_{n=1}^\infty$. Then

$$k_0 = \liminf k_n \geqslant h_0 + \varepsilon r_0,$$

hence $(k_n)_{n=1}^\infty$ satisfies (ii), thus proving the "fact".

Now we apply an inductive procedure: if $(f_n)_{n=1}^\infty$ satisfies condition (i) then – by passing to a subsequence – we have proved the lemma. If (i) fails then there are $f_n^{(1)} \in \operatorname{conv}(f_n, f_{n+1},\dots)$ such that with

$$f_0^{(1)} = \liminf f_n^{(1)}$$

the conditions of (ii) are satisfied. In addition, we choose $(f_n^{(1)})_{n=1}^\infty$ such that

$$E(\arctan(f_0^{(1)})) \geqslant \sup E(\arctan(\liminf(h_n))) - 1,$$

where the sup is taken over all sequences of convex combinations

$$h_n \in \operatorname{conv}(f_n, f_{n+1},\dots).$$

Continuing in an obvious way we either come to the situation where (i) holds true – in which case we are finished – or we get sequences $f_n^{(k)} \in \operatorname{conv}(f_n^{(k-1)}, f_{n+1}^{(k-1)},\dots)$ such that with

$$f_0^{(k)} = \liminf f_n^{(k)}$$

we have

$$E(\arctan(f_0^{(k)})) \geqslant E(\arctan(\liminf h_n)) - k^{-1}$$

for each sequence $(h_n)_{n=1}^{\infty}, h_n \in \operatorname{conv}(f_n^{(k-1)}, f_{n+1}^{(k-1)}, \dots)$. Choose a diagonal sequence $g_k = f_{n_k}^{(k)}$ such that

$$P\{\arctan(f_{n_k}^{(k)}) < \arctan(f_0^{(k)}) - k^{-1}\} < 2^{-k}.$$

Then for every sequence $h_k \in \operatorname{conv}(g_k, g_{k+1}, \dots)$

$$E(\arctan(\liminf g_k)) = E(\arctan(\liminf h_k))$$

whence $(g_k)_{k=1}^{\infty}$ must satisfy assertion (i) and – by passing once again to a subsequence – we have proved the lemma.

q.e.d.

The above proposition will be of constant use in the sequel. First of all it allows one to prove proposition 1.5.

3.6 Proof of proposition 1.5. Obviously the condition formulated in 1.5 implies the existence of a free lunch with bounded risk (consider $f_n \wedge 1$).

Conversely suppose (NFLBR) fails, i.e. there are sequences $(g_n)_{n=1}^{\infty} \in K_0$ and $(r_n)_{n=1}^{\infty} \in L_+^0$ such that $(g_n - r_n)_{n=1}^{\infty}$ σ^*–converges to $g_0 \in L_+^{\infty} \backslash \{0\}$. By Banach-Steinhaus $(\|g_n - r_n\|_{\infty})_{n=1}^{\infty}$ is bounded, whence $(g_n)_{n=1}^{\infty}$ is uniformly bounded from below and – by multiplying g_n, r_n and g_0 by a suitable scalar – there is no loss of generality to assume that $(g_n)_{n=1}^{\infty}$ is bounded from below by -1.

By the remark after definition 3.4 we may choose a sequence $(h_n - s_n)_{n=1}^{\infty}$ with $(h_n - s_n) \in \operatorname{conv}((g_j - r_j)_{j=1}^{\infty})$, $h_n \in K_0, s_n \in L_+^0$, such that $(h_n(\omega) - s_n(\omega))_{n=1}^{\infty}$ converges a.s. to g_0. Hence

$$\liminf_{n\to\infty} h_n(\omega) \geqslant g_0(\omega) \qquad P - a.s.$$

By lemma 3.5 there is a sequence $f_n \in \operatorname{conv}(h_n, h_{n+1}, \dots)$ and $f_0 \in F(\Omega, \mathcal{F}, P)$ such that

$$\lim_{n\to\infty} f_n(\omega) = f_0 \qquad P - a.s.$$

As $f_n \geq -1$ for each $n \in \mathbb{N}$ and $f_0 \geqslant g_0$ we finished the proof.

q.e.d.

The next result will be useful to avoid problems arising from the above discussed phenomenon of "embarras de richesse".

3.7 Lemma. *If $(S_t)_{t\in I}$ verifies (NFLBR) and $(f_n)_{n=1}^{\infty} \in K_0$ Fatou-converges to some $f_0 \in F(\Omega, \mathcal{F}, P)$, then $f_0(\omega) < \infty$ a.s.*

Proof. Let $A = \{f_0 = \infty\}$ and suppose $P(A) > 0$. Find a subsequence $(n_k)_{k=1}^{\infty}$ such that

$$P\{A \cap \{f_{n_k} > k\}\} > P(A) - 2^{-k}$$

and note that

$$\lim_{k\to\infty} (f_{n_k}(\omega)/k) \wedge \chi_A(\omega) = \chi_A(\omega) \qquad P - a.s.,$$

a contradiction to (NFLBR).

q.e.d.

To end this section we note in the subsequent proposition that – similarly as in the case of martingales – condition (NA) allows one to recover the process $(H.S)_{t\in I}$ from the random variable $(H.S)_\infty$.

3.8 Proposition. *Suppose that $(S_t)_{t\in I}$ satisfies (NA) and let H_1, H_2 be easy integrands such that*

$$(H_1.S)_\infty(\omega) = (H_2.S)_\infty(\omega) \qquad P\text{-a.s.}$$

Then, for any $t \in I$, we have

$$(H_1.S)_t(\omega) = (H_2.S)_t(\omega) \qquad P\text{-a.s.}$$

Hence we may associate unambigously to each $f = (H.S)_\infty \in K_0$ the stochastic process $(f_t)_{t\in I} = (H.S)_{t\in I}$. In addition we have for each $t \in I$

$$\operatorname{ess\,inf}(f_t) \geqslant \operatorname{ess\,inf}(f).$$

Proof. If the first assertion fails we may find H_1, H_2 and $t \in I$ such that

$$(H_1.S)_\infty(\omega) = (H_2.S)_\infty(\omega) \qquad P\text{-a.s.}$$

while

$$P\{(H_1.S)_t < (H_2.S)_t\} > 0.$$

Note that

$$((H_1.S)_\infty - (H_1.S)_t - ((H_2.S)_\infty - (H_2.S)_t)).\chi_{\{(H_1.S)_t<(H_2.S)_t\}}$$

is an element of K_0 and contained in $L^0_+(\Omega, \mathcal{F}, P) \setminus \{0\}$, a contradiction to (NA).

To prove the last assertion, suppose again to the contrary that there is $f \in K_0$ and $t \in I$ such that

$$A = \{f_t < \operatorname{ess\,inf}(f)\}$$

has positive P-measure. Then $(f - f_t).\chi_A$ again is in $K_0 \cap (L^0_+ \setminus \{0\})$, contradicting (NA).

q.e.d.

4. The proof of Theorem A

We now turn to the proof of theorem A (see 1.6 above). Throughout this section the index set I will equal $\mathbb{N}_0$.

The following concept already appears essentially in Mcbeth (92), the idea and the name going back to a remark (3.27) in Harrison-Pliska (81). It is intimately related to the concept of (NFLBR): The underlying motivation is that an economic agent has an initial wealth of M units of money and is only allowed to perform trading operations which cannot result in a negative wealth.

4.1 Definition. *Let $(S_t)_{t\in\mathbb{N}_0}$ be an R^d-valued process defined on the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbb{N}_0}, P)$. A general integrand will be a function $H(t,\omega)$ of the form*

$$H(t,\omega) = \sum_{n=1}^{\infty} g_n(\omega).\chi_{\{n\}}(t)$$

where, for $n \in \mathbb{N}$, g_n is an $\mathbb{R}^d$-valued $\mathcal{F}_{n-1}$-measurable function. We then may define the stochastic integral as the process

$$(H.S)_t(\omega) = \sum_{n=1}^{t} (g_n(\omega), S_n(\omega) - S_{n-1}(\omega)) \qquad t \in \mathbb{N}_0 .$$

We say that a general integrand $H(t,\omega)$ is an admissible integrand if there is $M \in \mathbb{R}_+$ such that, for $t \in \mathbb{N}_0$,

$$(H.S)_t(\omega) \geq -M \qquad P - a.s.$$

Our program for the proof of theorem A will be as follows: we first show that — assuming the hypothesis of (NFLBR) — we may define $(H.S)_\infty$ for admissible integrands (prop.4.2). Then we define C^{adm} as the convex cone in L^∞ of functions dominated by $(H.S)_\infty$, where H is an admissible integrand, and finally we show that the cone C^{adm} is weak star closed in L^∞ (prop.4.6) and meets L^∞_+ only in $\{0\}$. An application of the Kreps-Yan theorem will then prove theorem A.

In fact, we shall go through this scheme — with variations — several times in the sequel, namely twice in this section and twice in the next sections, to cover the cases of theorem A and B.

4.2 Proposition. *If $(S_t)_{t\in\mathbb{N}_0}$ satisfies (NFLBR) and $H(t,\omega)$ is an admissible integrand then*

$$(H.S)_\infty(\omega) \quad = \quad \lim_{t\to\infty} (H.S)_t(\omega)$$

exists almost surely. In addition, for each $t \in \mathbb{N}_0$

$$\operatorname{ess\,inf}(H.S)_\infty \quad \leq \quad \operatorname{ess\,inf}(H.S)_t.$$

Proof. Fix $M \in \mathbb{R}_+$ such that $(H.S)_t \geq -M$ almost surely for each $t \in \mathbb{N}_0$. If $((H.S)_t)_{t\in\mathbb{N}_0}$ does not converge almost surely (in the compact interval $[-M,\infty]$) there are $A \in \mathcal{F}$ with $P(A) = \alpha > 0$, and real numbers $\beta < \gamma$ such that, for $\omega \in A$

$$\liminf_{t\to\infty} (H.S)_t(\omega) < \beta \qquad \text{while} \qquad \limsup_{t\to\infty} (H.S)_t(\omega) > \gamma.$$

Noting that $\mathcal{F}$ is generated by $(\mathcal{F}_t)_{t\in\mathbb{N}_0}$ we may find, for $\epsilon > 0$, some $t_0 \in \mathbb{N}$ and $A_0 \in \mathcal{F}_{t_0}$ such that, for the symmetric difference $A \bigtriangleup A_0$, we have $P(A \bigtriangleup A_0) < \epsilon$. Define the stopping times U and V by

$$U(\omega) = \min\{t \geq t_0 : (H.S)_t(\omega) < \beta\}$$

and

$$V(\omega) = \min\{t \geq U(\omega) : (H.S)_t(\omega) > \gamma\},$$

where $U(\omega)$ and $V(\omega)$ equal $+\infty$ if the respective sets above are empty.

Since $V(\omega) < \infty$ for $\omega \in A$, we may find $t_1 \in \mathbb{N}$ such that, for $A_1 = A_0 \cap \{V \leq t_1\}$, we have

$$P(A_1) > \alpha - 2\epsilon.$$

Note that the random variable

$$f(\omega) = \chi_{A_0}(\omega).((H.S)_{V\wedge t_1}(\omega) - (H.S)_{U\wedge t_1}(\omega))$$

is of the form $f(\omega) = (L.S)_\infty$, where L is an easy integrand. The function f is bounded from below by $-(M+\beta)$, vanishes outside of A_0, and is bigger than $\gamma - \beta$ on A_1.

Repeating the above construction for $\epsilon_k = 2^{-k}$ and applying lemma 3.5 we obtain a sequence $(f_k)_{k=1}^\infty$ in K_0, bounded from below by $-(M+\beta)$, converging pointwise almost surely to zero outside of A and to a value bigger than or equal to $\gamma - \beta$ on A. This gives the desired contradiction to (NFLBR).

Hence we have shown that $(H.S)_\infty(\omega) = \lim_{t\to\infty}(H.S)_t(\omega)$ exists almost surely, where we – a priori – allowed $(H.S)_\infty(\omega)$ to take the value $+\infty$. But a glance at proposition 3.7 reveals that $(H.S)_\infty(\omega)$ is necessarily finite almost surely.

To show the last assertion, suppose to the contrary that there is $t_0 \in \mathbb{N}_0$ and $\epsilon > 0$ such that, for $A = \{(H.S)_{t_0} < \operatorname{ess\,inf}(H.S)_\infty - \epsilon\}$, we have $P(A) > 0$. Then the random variables

$$f_t = \chi_A.((H.S)_t - (H.S)_{t_0})$$

for $t \geq t_0$ give rise to a contradiction to (NFLBR).

q.e.d.

The above proof may be extended to a more general situation which is described by the subsequent proposition.

4.3 Proposition. *Suppose that $(S_t)_{t\in\mathbb{N}_0}$ satisfies (NFLBR) and that $H_n(t,\omega)$ is a sequence of admissible integrands such that $(g_n)_{n=1}^\infty = ((H_n.S)_\infty)_{n=1}^\infty$ Fatou-converges to some $g_0 \in L^0(\Omega,\mathcal{F},P)$ and, for each $t \in \mathbb{N}_0$, the sequence $(g_{n,t})_{n=1}^\infty = ((H_n.S)_t)_{n=1}^\infty$ Fatou-converges to some $g_{0,t} \in L^0(\Omega,\mathcal{F}_t,P)$. Then we still have*

$$g_0(\omega) \leq \liminf_{t\to\infty} g_{0,t}(\omega)$$

almost surely.

The reader should note, however, that in this more general setting we only claimed an inequality. In fact, one may construct examples such that the equality

$$g_0(\omega) = \lim_{t\to\infty} g_{0,t}(\omega)$$

does not hold true.

Proof of proposition 4.3. First note that there is $M \in \mathbb{R}_+$ such that $(H_n.S)_\infty \geq -M$ and therefore $(H_n.S)_t \geq -M$ for all t and n. If the above inequality were false we could proceed similarly as in 4.2 above: again we could find $A \in \mathcal{F}$ with $P(A) = \alpha > 0$ and $\beta < \gamma$ such that

$$g_0(\omega) > \gamma \qquad \text{while} \qquad \liminf_{t\to\infty} g_{0,t}(\omega) < \beta$$

for $\omega \in A$.

Hence for $\epsilon > 0$ there is $t_0 \in \mathbb{N}$ and $A_0 \in \mathcal{F}_{t_0}$ such that $P(A \mathbin{\triangle} A_0) < \epsilon$ and $n_0 \in \mathbb{N}$ such that, for $n \geq n_0$

$$P(A \cap \{g_n(\omega) \leq \gamma\}) < \epsilon.$$

Next find $t_0 < s_1 < s_2 < \cdots < s_k$ such that

$$P(A \cap \{\min_{1 \le i \le k} g_{0,s_i} \ge \beta\}) < \epsilon,$$

and $n \ge n_0$ such that

$$P(A \cap \{\min_{1 \le i \le k} g_{n,s_i} \ge \beta\}) < \epsilon.$$

Now define the stopping times U and V by

$$U(\omega) = \min\{t \ge t_0 : g_{n,t}(\omega) = (H_n.S)_t(\omega) < \beta\}$$

and

$$V(\omega) = \min\{t \ge U(\omega) : g_{n,t}(\omega) = (H_n.S)_t(\omega) > \gamma\},$$

where $U(\omega)$ and $V(\omega)$ equal $+\infty$ if the respective sets above are empty.

Note that $A \cap \{V(\omega) < \infty\}$ has measure bigger than $\alpha - 2\epsilon$, hence we may find $t_1 \in \mathbb{N}$ such that, for $A_1 = A_0 \cap \{V \le t_1\}$, we have

$$P(A_1) > \alpha - 3\epsilon.$$

Note again that the random variable

$$f(\omega) = \chi_{A_0}(\omega).((H_n.S)_{V \wedge t_1}(\omega) - (H_n.S)_{U \wedge t_1}(\omega))$$

is of the form $f(\omega) = (L.S)_\infty$, where L is an easy integrand. The function f is bounded from below by $-(M+\beta)$, vanishes outside of A_0, and is bigger than $\gamma - \beta$ on A_1. Hence by applying the same argument as in 4.2 above we arrive at a contradiction to (NFLBR).

q.e.d.

We now can fix some notation related to the notion of admissible integrands.

4.4 Definition. *Let $(S_t)_{t \in \mathbb{N}_0}$ satisfy (NFLBR). Denote by K_0^{adm} the convex cone in $L^0(\Omega, \mathcal{F}, P)$ spanned by the random variables $(H.S)_\infty$, where H runs through the admissible integrands. Denote by C_0^{adm} the convex cone $K_0^{adm} - L^0_+(\Omega, \mathcal{F}, P)$ and by C^{adm} its intersection with L^∞.*

4.5 Proposition. *Let $(S_t)_{t \in \mathbb{N}_0}$ satisfy (NFLBR). Then $C^{adm} \cap L^\infty_+ = \{0\}$.*

Proof. Let $f \in C^{adm} \cap L^\infty_+$ and find an admissible integrand H such that $(H.S)_\infty \ge f$. By proposition 4.2 $(H.S)_t \ge 0$ almost surely for each $t \in \mathbb{N}_0$ and $((H.S)_t)_{t=0}^\infty$ converges almost surely to $(H.S)_\infty$. By the assumption of (NFLBR) (in fact (NA) would suffice here) we have that $(H.S)_t \equiv 0$ for each $t \in \mathbb{N}_0$ and therefore $(H.S)_\infty \equiv 0$.

q.e.d.

The next proposition is a crucial step. It relies essentially on a classical result from functional analysis, the Krein-Smulian theorem. For the convenience of the reader we restate this theorem below (see, e.g., Horvath (66), p.246). Let us point out that similar arguments as in the proof of proposition 4.6 below were used by Delbaen (92); in fact, Delbaen applied the Banach-Dieudonne theorem, a close relative of the Krein-Smulian theorem.

Theorem of Krein-Smulian. *Let E be a Fréchet space and E^* its dual. A convex subset C of E^* is $\sigma(E^*,E)$-closed iff, for each balanced, convex $\sigma(E^*,E)$-closed set M of E^*, the intersection $C \cap M$ is $\sigma(E^*,E)$-closed.*

We deduce that, in particular, if E is a Banach space and C a convex cone in E^*, then C is $\sigma(E^*,E)$-closed iff the intersection of C with the closed unit ball of E^* is $\sigma(E^*,E)$-closed.

We also recall a lemma from Stricker (90) which is similar in spirit to the subsequent proposition 4.6. For a discussion of Stricker's lemma, its relation to the theorem of Dalang-Morton-Willinger and an alternative proof we refer to Schachermayer (92).

Stricker's Lemma. *(Stricker (90), prop. 2)*

Let $(\Omega,\mathcal{F}_1,P)$ be a probability space, $Y \in L^0(\Omega,\mathcal{F}_1,P;\mathbb{R}^d)$, $\mathcal{F}_0$ a sub–σ–algebra of $\mathcal{F}_1$ and denote by K_0 the subspace of $L^0(\Omega,\mathcal{F}_1,P)$

$$K_0 = \{(h,Y) : h \in L^0(\Omega,\mathcal{F}_0,P;\mathbb{R}^d)\}\ .$$

Then K_0 is closed in $L^0(\Omega,\mathcal{F}_1,P)$ with respect to the topology of convergence in measure.

Finally let us specify some technicalities pertaining to the formation of convex combinations. Given a sequence $(g_n)_{n=1}^\infty$ in a vector space X we say that $(g_n^{(1)})_{n=1}^\infty$ is a *sequence of convex combinations of* $(g_n)_{n=1}^\infty$ if, for each $n \in \mathbb{N}$, $g_n^{(1)} \in conv(g_n, g_{n+1}, \dots)$, i.e., there are nonnegative scalars $(\lambda_k)_{k=n}^{N_n}$, $\sum_{k=n}^{N_n} \lambda_k = 1$, such that $g_n^{(1)} = \sum_{k=n}^{N_n} \lambda_k g_k$.

If $(h_n)_{n=1}^\infty$ is a sequence in a vector space Y we say that a sequence of convex combinations $(h_n^{(1)})_{n=1}^\infty$ is obtained *by using the same weights as* $(g_n^{(1)})_{n=1}^\infty$ if, for each $n \in \mathbb{N}$, $h_n^{(1)} = \sum_{k=n}^{N_n} \lambda_k h_k$.

Note that if $((g_n^{(j)})_{n=1}^\infty)_{j=0}^\infty$ is a sequence of sequences in X such that, for each $j \in \mathbb{N}_0$, $(g_n^{(j+1)})_{n=1}^\infty$ is a sequence of convex combinations of $(g_n^{(j)})_{n=1}^\infty$, then the diagonal sequence $(g_n^{(n)})_{n=1}^\infty$ is a sequence of convex combinations of $(g_n^{(0)})_{n=1}^\infty$.

In particular, if $(g_n^{(0)})_{n=1}^\infty$ is a sequence of random variables converging almost surely to a random valiable g_0 then $(g_n^{(n)})_{n=1}^\infty$ converges almost surely to g_0 too.

4.6 Proposition. *Let $(S_t)_{t\in\mathbb{N}_0}$ satisfy (NFLBR). Then the convex cone C^{adm} is weak star closed in L^∞.*

Proof. By the Krein–Smulian theorem and the above remark it suffices to show that $C^{adm} \cap ball(L^\infty)$ is weak star closed. Let $(f_\alpha)_{\alpha\in J}$ be a net in $C^{adm} \cap ball(L^\infty)$ converging weak star to f_0. By the remark preceding lemma 3.5 there exists a sequence $(f_n)_{n\in\mathbb{N}}$, $f_n \in \mathrm{conv}((f_\alpha)_{\alpha\in J})$ converging almost surely to f_0. Clearly each f_n is in $C^{adm} \cap ball(L^\infty)$ and therefore we may find a sequence $(H_n)_{n\in\mathbb{N}}$ of admissible integrands such that $f_n \le (H_n.S)_\infty$. Let $g_n = (H_n.S)_\infty$ and, for $t \in \mathbb{N}_0$, $g_{n,t} = (H_n.S)_t$. Note that $g_n \ge -1$ and therefore, by prop. 4.2, $g_{n,t} \ge -1$ for $n \in \mathbb{N}, t \in \mathbb{N}_0$.

By the lemmata 3.5 and 3.7 we may find a sequence $(g_n^{(1)})_{n=1}^\infty$ of convex combinations of $(g_n)_{n=1}^\infty$ converging almost surely to some $g_0 \in L^0$ for which we clearly have that $g_0 \ge f_0 \ge -1$ almost surely. Denote by $(H_n^{(1)})_{n=1}^\infty$ the sequence of admissible integrands obtained from $(H_n)_{n=1}^\infty$ by using the same weights as $(g_n^{(1)})_{n=1}^\infty$. We again apply lemmata 3.5 and 3.7 to find a sequence $(g_n^{(2)})_{n=1}^\infty$ of convex combinations of $(g_n^{(1)})_{n=1}^\infty$ and the corresponding sequence $(H_n^{(2)})_{n=1}^\infty$ of admissible integrands, obtained from $(H_n^{(1)})_{n=1}^\infty$ by using the same weights as $(g_n^{(2)})_{n=1}^\infty$, such that $(g_{n,0}^{(2)})_{n=1}^\infty = (H_n^{(2)}.S)_0$ converges almost surely to some $g_{0,0} \in L^0$.

Continuing in an obvious way and applying the diagonalisation procedure explained in the paragraph preceding prop. 4.6, we may assume that $(g_n)_{n=1}^\infty$ converges almost surely to g_0 and, for each $t \in \mathbb{N}_0$, $(g_{n,t})_{n=1}^\infty$ to some $g_{0,t} \in L^0$.

We shall show that there is an admissible integrand H_0 such that $g_{0,t} = (H_0.S)_t$ for each $t \in \mathbb{N}_0$. Admitting this for the moment we can finish the proof as follows: By proposition 4.3 we have

$$g_0(\omega) \le \liminf_{t\to\infty} g_{0,t}(\omega) = (H_0.S)_\infty.$$

Hence g_0 and *a fortiori* f_0 is dominated by an element of K_0^{adm} showing that $f_0 \in C^{adm}$, thus finishing the proof.

To show the existence of the admissible integrand H_0 we apply Stricker's lemma. Consider, for $t \in \mathbb{N}_0$, the sequence $(H_n(t,\omega))_{n=1}^\infty$ of $\mathcal{F}_{t-1}$-measurable random variables and let $Y(\omega) = S_t(\omega) - S_{t-1}(\omega) \in L^0(\Omega, \mathcal{F}_t, P; \mathbb{R}^d)$. We know that

$$(H_n(t,\omega), S_t(\omega) - S_{t-1}(\omega))_{n=1}^\infty = (g_{n,t}(\omega) - g_{n,t-1}(\omega))_{n=1}^\infty$$

converges almost surely to $g_{0,t} - g_{0,t-1}$, whence by Stricker's lemma there is an $\mathcal{F}_{t-1}$-measurable random variable $H_0(t,\omega)$ such that

$$(H_0(t,\omega), S_t(\omega) - S_{t-1}(\omega)) = (g_{0,t}(\omega) - g_{0,t-1}(\omega)).$$

Defining $H_0(t,\omega)$ in this way for each $t \in \mathbb{N}$, this means precisely that $g_{0,t} = (H_0.S)_t$ for all $t \in \mathbb{N}_0$, which finishes the proof.

q.e.d.

We now have assembled all the ingredients for the proof of theorem A in the case when the process $(S_t)_{t\in\mathbb{N}_0}$ is such that each S_t is bounded. For expository reasons we present this case first as the idea of the proof should become more transparent than in the proof of the general case, where we have to deal with some additional technicalities.

4.7 Proof of theorem A (special case).

In addition to the assumptions of theorem A stated in the introduction we assume that each S_t is bounded.

By propositions 4.5 and 4.6 C^{adm} is a weak star closed convex cone in $L^\infty(\Omega, \mathcal{F}, P)$, such that $C^{adm} \cap L_+^\infty = \{0\}$. Hence by the abstract version of the Kreps-Yan theorem (see remark 3.2) there is an element $g \in L^1(\Omega, \mathcal{F}, P)$, $g > 0$ almost surely, such that g — viewed as a linear functional on L^∞ — is less than or equal to zero on C^{adm}. Note that by our boundedness assumption each function of the form

$$(h(\omega), S_t(\omega) - S_s(\omega)),$$

with $s < t$ and h being $\mathcal{F}_s$-measurable and bounded, is bounded and therefore in C^{adm}. Whence

$$E_Q(h(\omega), S_t(\omega) - S_s(\omega)) \le 0,$$

where Q denotes the measure on $\mathcal{F}$ with Radon–Nikodym derivative equal to g. By passing to $-h$ we conclude that equality holds above which means precisely that $(S_t)_{t\in\mathbb{N}_0}$ is a martingale under Q.

q.e.d.

To extend the above proof to the general case (*i.e.*, without assuming any boundedness or integrability assumptions on the process $(S_t)_{t\in\mathbb{N}_0}$) we have to develop some more concepts. For $t \in \mathbb{N}_0$ define the weight function

$$w_t(\omega) = \max(1, \|S_1(\omega)\|, \ldots, \|S_t(\omega)\|).$$

Clearly w_t is $\mathcal{F}_t$-measurable and takes its values in $[1, \infty[$ almost surely. Denote by $\mathcal{W}$ the sequence $(w_t)_{t=0}^{\infty}$.

For $t_0 \in \mathbb{N}_0$, a general integrand $H(t,\omega)$ is called w_{t_0}-admissible if there is $M \in \mathbb{R}_+$ such that $(H.S)_t(\omega) \geq -Mw_{t_0}(\omega)$ almost surely for all $t \geq t_0$. We call $H(t,\omega)$ $\mathcal{W}$-admissible if it is w_t-admissible for some $t \in \mathbb{N}_0$.

Under the assumption (NFLBR), for $\mathcal{W}$-admissible integrands a similar theory may be established as for admissible integrands. This will be done in the subsequent propositions which are technical variants of the corresponding propositions in the first part of this section.

4.8 Proposition. *Suppose that $(S_t)_{t\in\mathbb{N}_0}$ satisfies (NFLBR) and let $H(t,\omega)$ be a $\mathcal{W}$-admissible integrand. Then*

$$(H.S)_\infty = \lim_{t\to\infty} (H.S)_t$$

exists almost surely. If $t_0 \in \mathbb{N}_0$ and $M \in \mathbb{R}_+$ are such that

$$(H.S)_\infty(\omega) \geq -Mw_{t_0}(\omega) \qquad P-\ a.s.,$$

then for each $t \geq t_0$

$$(H.S)_t(\omega) \geq -Mw_{t_0}(\omega) \qquad P-\ a.s.$$

Proof. If H is $\mathcal{W}$-admissible there is $t_1 \in \mathbb{N}_0$ and $M_1 \in \mathbb{R}_+$ such that, for $t \geq t_1$

$$(H.S)_t(\omega) \geq -M_1 w_{t_1}(\omega) \qquad\qquad P-\ a.s.$$

In order to show that $((H.S)_t)_{t=0}^{\infty}$ converges almost surely, it suffices to show that, for each $L \in \mathbb{R}_+$, the sequence

$$(((H.S)_t - (H.S)_{t_1})\chi_{\{(H.S)_{t_1} < Lw_{t_1}\}})_{t=t_1}^{\infty}$$

converges almost surely.

Consider the general integrand

$$\tilde{H}(t,\omega) = w_{t_1}^{-1} H(t,\omega)\chi_{]t_1,\infty[}(t)\chi_{\{(H.S)_{t_1} < Lw_{t_1}\}}(\omega).$$

Note that $(\tilde{H}.S)_t \geq -(M_1 + L)$ for all $t \in \mathbb{N}_0$. Hence $\tilde{H}$ is an admissible integrand and we may conclude by prop.4.2 that $(\tilde{H}.S)_{t=0}^{\infty}$ and therefore $(H.S)_{t=0}^{\infty}$ converges almost surely.

For the second assertion let again H be $\mathcal{W}$-admissible and suppose in addition that there are $t_0 \in \mathbb{N}_0$ and $M \in \mathbb{R}_+$ such that $(H.S)_\infty \geq -Mw_{t_0}$. By the $\mathcal{W}$-admissibility of H there are t_1 and M_1 as above and we may assume that $t_0 \leq t_1$.

We first show that for $t \geq t_1$ we have $(H.S)_t \geq -Mw_{t_0}$. Indeed, otherwise we could find $t_2 \geq t_1$ and $\epsilon > 0$ such that

$$A = \{(H.S)_{t_2} < -(M+\epsilon)w_{t_0}\}$$

has measure $P(A) > 0$. Letting, for $t \geq t_2$,

$$f_t = w_{t_1}^{-1}((H.S)_t - (H.S)_{t_2})\chi_A,$$

we find a sequence $(f_t)_{t=t_2}^{\infty}$ of easy integrals on the process $(S_t)_{t\in\mathbb{N}_0}$ bounded from below by $-M_1$ and such that, for $t \to \infty$, $(f_t)_{t=t_2}^{\infty}$ converges almost surely to a function dominating the function $\epsilon w_{t_0} w_{t_1}^{-1}\chi_A$, a contradiction to (NFLBR).

To show that the same result holds true for $t_0 \leq t < t_1$, suppose again to the contrary that there is $t_0 \leq t_2 < t_1$ such that

$$A = \{(H.S)_{t_2} < -(M+\epsilon)w_{t_0}\}$$

has measure $P(A) > 0$. Considering

$$((H.S)_{t_1} - (H.S)_{t_2})\chi_A,$$

we obtain a contradiction to (NA) and therefore to (NFLBR).

q.e.d.

We now may define $K_0^{\mathcal{W}-adm}$ as the convex cone in $L^0(\Omega,\mathcal{F},P)$ formed by the random variables of the form $(H.S)_\infty$, where H is a $\mathcal{W}$-admissible integrand. $C_0^{\mathcal{W}-adm}$ will denote the convex cone $K_0^{\mathcal{W}-adm} - L_+^0(\Omega,\mathcal{F},P)$. For $t \in \mathbb{N}$ we denote by $w_t^{-1}C_0^{\mathcal{W}-adm}$ the convex cone of functions of the form $w_t^{-1}f$ with $f \in C_0^{\mathcal{W}-adm}$.

4.9 Proposition. *If $(S_t)_{t\in\mathbb{N}_0}$ satisfies (NFLBR) then $C_0^{\mathcal{W}-adm} \cap L_+^0(\Omega,\mathcal{F},P) = \{0\}$ and, for each $t \in \mathbb{N}_0$, the convex cone $w_t^{-1}C_0^{\mathcal{W}-adm} \cap F(\Omega,\mathcal{F},P)$ is closed with respect to Fatou-convergence.*

Proof. The first assertion follows immediatly from prop. 4.8 above. If H is $\mathcal{W}$-admissible and $(H.S)_\infty \geq 0$, then $(H.S)_t \geq 0$ for all $t \in \mathbb{N}_0$, whence $(H.S)_\infty \equiv (H.S)_t \equiv 0$.

For the Fatou–closedness of $w_t^{-1}C_0^{\mathcal{W}-adm} \cap F(\Omega,\mathcal{F},P)$, fix $t_0 \in \mathbb{N}_0$ and let $(f_n)_{n=1}^{\infty}$ be a sequence in $w_{t_0}^{-1}C_0^{\mathcal{W}-adm}$ with $f_n \geq -M$ that converges almost surely to $f_0 \in F(\Omega,\mathcal{F},P)$. Find $\mathcal{W}$-admissible integrands H_n such that for $g_n = (H_n.S)_\infty$ we have $g_n \geq w_{t_0}f_n$. Denoting $(H_n.S)_t$ by $g_{n,t}$ we infer from prop. 4.8 that $g_{n,t} \geq -Mw_{t_0}$ for $t \geq t_0$.

Similarly as in the proof of prop. 4.6 we may assume — by passing to convex combinations of the sequence $(g_n)_{n=1}^{\infty}$ — that $(g_n)_{n=1}^{\infty}$ converges almost surely to some $g_0 \in L^0(\Omega,\mathcal{F},P)$ and, for $t \in \mathbb{N}_0$, $(g_{n,t})_{n=1}^{\infty}$ converges almost surely to some $g_{0,t} \in L^0(\Omega,\mathcal{F}_t,P)$. Indeed, as regards the convergence of the sequence $(g_n)_{n=1}^{\infty}$ and of the sequences $(g_{n,t})_{n=1}^{\infty}$ for $t \geq t_0$, this may be deduced from the lemmata 3.5 and 3.7 by considering $(w_{t_0}^{-1}g_n)_{n=1}^{\infty}$ and $(w_{t_0}^{-1}g_{n,t})_{n=1}^{\infty}$.

To obtain the same result for $t < t_0$ some extra care is needed: We deduce from the Dalang–Morton–Willinger theorem that there is a measure Q on $\mathcal{F}_{t_0}$ equivalent to the restriction of P to $\mathcal{F}_{t_0}$ such that $(S_t)_{t=0}^{t_0}$ is a martingale under Q. In particular, w_{t_0} is Q-integrable and $(g_{n,t_0})_{n=1}^{\infty}$ therefore is bounded in $L^1(Q)$. It follows that, for each $t < t_0$, $(g_{n,t})_{n=1}^{\infty}$ is bounded in $L^1(Q)$ and we now may apply Komlos' theorem (Komlos (67)). Recall that this theorem implies that, for a bounded sequence in L^1, there is a sequence of convex combinations converging almost surely. Hence we may find convex combinations of

$(g_n)_{n=1}^{\infty}$ such that the corresponding sequences of convex combinations $(g_{n,t})_{n=1}^{\infty}$ converge almost surely to some $g_{0,t}$ for each $0 \le t < t_0$.

Now we may again apply Stricker's (90) lemma to obtain an integrand $H_0(t,\omega)$ such that $(H_0.S)_t = g_{0,t}$ for each $t \in \mathbb{N}_0$. Clearly H_0 is w_{t_0}-admissible and therefore $(H_0.S)_\infty = \lim(H_0.S)_t$ exists almost surely. We shall show that $(H_0.S)_\infty \ge g_0$, which will finish the proof as $g_0 \ge w_{t_0} f_0$ and therefore $f_0 \in w_{t_0}^{-1} C_0^{\mathcal{W}-adm}$.

To show that $(H_0.S)_\infty \ge g_0$, denote by v the weight function

$$v(\omega) = \max(M w_{t_0}(\omega), \sup_n g_{n,t_0}(\omega)),$$

which is $\mathcal{F}_{t_0}$-measurable and finite almost everywhere. For $n \in \mathbb{N}_0$ let

$$\tilde{H}_n = v^{-1} H_n \chi_{]t_0,\infty[}.$$

Note that $(\tilde{H}_n.S)_t \ge -2$ for all n and t. Hence by proposition 4.3

$$\lim_{t\to\infty} (\tilde{H}_0.S)_t \ge \lim_{n\to\infty} (\tilde{H}_n.S)_\infty,$$

which means that

$$\lim_{t\to\infty} v^{-1}(g_{0,t} - g_{0,t_0}) \ge \lim_{n\to\infty} v^{-1}(g_n - g_{n,t_0}).$$

Noting that $(g_{n,t_0})_{n=1}^{\infty}$ converges almost surely to g_{0,t_0}, we conclude that

$$\lim_{t\to\infty} g_{0,t} \ge \lim_{n\to\infty} g_n = g_0,$$

which finishes the proof.

q.e.d.

4.10 Proof of Theorem A. Define the vector space

$$L^1_{\mathcal{W}}(\Omega,\mathcal{F},P) = \{g \in L^1(\Omega,\mathcal{F},P) : E(w_t|g|) < \infty \text{ for } t \in \mathbb{N}_0\}.$$

$L^1_{\mathcal{W}}$ is a Frechet space, the topology given by the increasing sequence of seminorms

$$\|g\|_t = E(w_t|g|).$$

We denote by $L^\infty_{\mathcal{W}}$ the dual of $L^1_{\mathcal{W}}$ which is given by (see, *e.g.* Schaefer (71) or Horvath (66))

$$L^\infty_{\mathcal{W}}(\Omega,\mathcal{F},P) = \{f \in L^0(\Omega,\mathcal{F},P) : \text{there is } t \in \mathbb{N}_0, M \in \mathbb{R}_+ \text{ s.t. } |f| \le M w_t\}.$$

Note that $L^1_{\mathcal{W}}$ separates points of $L^\infty_{\mathcal{W}}$, *i.e.*, the topology $\sigma(L^\infty_{\mathcal{W}}, L^1_{\mathcal{W}})$ is Hausdorff. Indeed, for $\epsilon > 0$ find sets $A_{t,\epsilon} \in \mathcal{F}_t$, $P(A_{t,\epsilon}) > 1 - \epsilon 2^{-t}$ such that w_t is bounded on $A_{t,\epsilon}$. Then $A_\epsilon = \bigcap_{t=1}^{\infty} A_{t,\epsilon}$ has measure bigger than $1-\epsilon$ and $\chi_{A_\epsilon} \in L^1_{\mathcal{W}}$, which quickly implies the assertion.

A fundamental sequence of equimeasurable sets of $L^\infty_{\mathcal{W}}$ is given by

$$B_t = \{f \in L^\infty_{\mathcal{W}} : |f| \le w_t\} \qquad t \in \mathbb{N}_0 .$$

Note that there is an equivalent martingale measure Q for $(S_t)_{t\in\mathbb{N}_0}$ iff there is $g = dQ/dP \in L^1_{\mathcal{W}}$ such that $g(\omega) > 0$ almost surely and, for each bounded $\mathcal{F}_{t-1}$-measurable h, we have

$$E_Q(h, S_t - S_{t-1}) = 0.$$

As $(h, S_t - S_{t-1}) \in K_0^{\mathcal{W}-adm} \cap L^\infty_{\mathcal{W}}$, this will be the case if g is less than or equal to 0 on $C^{\mathcal{W}-adm} = C_0^{\mathcal{W}-adm} \cap L^\infty_{\mathcal{W}}$.

By prop. 4.9 $C^{\mathcal{W}-adm} \cap (L^\infty_{\mathcal{W}})_+ = \{0\}$ and $C^{\mathcal{W}-adm} \cap B_t$ is $\sigma(L^\infty_{\mathcal{W}}, L^1_{\mathcal{W}})$-closed for each $t \in \mathbb{N}_0$. Hence by the Krein–Smulian theorem in its version for Frechet spaces (see, e. g. Horvath (66)) we may conclude that $C^{\mathcal{W}-adm} \cap L^\infty_{\mathcal{W}}$ is $\sigma(L^\infty_{\mathcal{W}}, L^1_{\mathcal{W}})$-closed. The proof of the Kreps–Yan theorem reveals the existence of an element $g \in L^1_{\mathcal{W}}$, $g > 0$ a.s. such that $g|_{C^{\mathcal{W}-adm}} \le 0$. This finishes the proof of theorem A.

q.e.d.

4.11 Remark. Observe a curious feature of theorem A. Let $(S_t)_{t\in\mathbb{N}_0}$ be such that $C = -L^\infty_+$, *i.e.*, there are no easy integrands (except for $H \equiv 0$) such that the stochastic integral is uniformly bounded from below. In this case (NFLBR) is trivially satisfied, whence by theorem A we get that $(S_t)_{t\in\mathbb{N}_0}$ satisfies (EMM).

For example, this is the case when $(S_t - S_{t-1})_{t=1}^\infty$ is a sequence of real–valued independent random variables which are neither bounded from below nor from above. In this case the existence of an equivalent martingale measure was proved by Mcbeth (92), who also showed that $C = -L^\infty_+$ implies the existence of a local martingale measure for $(S_t)_{t\in\mathbb{N}_0}$ and correctly conjectured that it implies in fact (EMM).

Finally we note that one may also view theorem A as a help for determining the existence of a sequence of trading strategies yielding a free lunch with bounded risk. Let us illustrate this with an easy example.

Suppose that $\Omega = \{-1, +1\}^{\mathbb{N}}$ equipped with normalized Haar measure P; denote by ϵ_n the projection on the n'th coordinate of Ω and by $\mathcal{F}_n$ the sigma-algebra generated by $\{\epsilon_1, \ldots, \epsilon_n\}$. $\mathcal{F}$ will denote the Borel-sigma-algebra of Ω. Fix a sequence $(\alpha_n)_{n=1}^\infty$ of numbers in $]0,1[$ and define the process $(S_n)_{n=1}^\infty$ by $S_0 \equiv 0$ and

$$S_n - S_{n-1} = \epsilon_n + \alpha_n \qquad \text{for } n \in \mathbb{N}.$$

It is straightforward to verify that there is a unique probability measure Q on $\mathcal{F}$ which turns S into a martingale, namely

$$Q = \bigotimes_{n=1}^\infty \left(\frac{1+\alpha_n}{2}\delta_{-1} + \frac{1-\alpha_n}{2}\delta_1\right)$$

where δ denotes the Dirac-measure. A classical theorem of Kakutani (see, e.g., Williams (91) 14.17, page 150) asserts that Q is either equivalent to P or Q and P are mutually singular depending on whether the sequence $(\alpha_n)_{n=1}^\infty$ is in l^2 or not.

For example, if $\alpha_n = (n+1)^{-1/2}$, we deduce that there is no equivalent martingale measure for the process S. Theorem A tells us that there is a free lunch with bounded risk; but it is quite a challenging task – at least to the author – to directly construct a sequence of trading strategies yielding the desired free lunch with bounded risk.

5. The proof of Theorem B

We now turn to the case of continuous time I. In the mathematical finance literature the case $I = [0,1]$ has usually been considered, but in the present context it is more natural to work with the general case

$I = \mathbb{R}_+$. So we shall consider in this section a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbb{R}_+}, P)$ satisfying the usual conditions and an adapted cadlag process $(S_t)_{t\in\mathbb{R}_+}$.

Similarly as in Delbaen (92) we define in this section *easy integrands* to be linear combinations of functions of the form

$$H = h\chi_{]\!]U,V]\!]},$$

where $U \le V$ are stopping times taking finite values almost surely and h is an $\mathcal{F}_U$-measurable $\mathbb{R}^d$-valued function. We then may define the stochastic integral with respect to easy integrands as processes which are linear combinations of processes of the form

$$(H.S)_t = (h, (S_{V\wedge t} - S_{U\wedge t})) \qquad t \in \mathbb{R}_+.$$

Note that for easy integrands H we may define

$$(H.S)_\infty = \lim_{t\to\infty} (H.S)_t = (h, (S_V - S_U)).$$

As we shall also deal with processes $(S_t)_{t\in\mathbb{R}_+}$ with jumps we shall also consider easy integrands of a second type, namely linear combinations of functions of the form

$$H = h\chi_{[\![T]\!]}$$

where T is a predictable stopping time taking finite values almost surely and h is an $\mathcal{F}_{T-}$-measurable $\mathbb{R}^d$-valued function. There is an obvious economic interpretation of these integrands: If the agent knows in advance that there is a possibility of a jump at time T (e.g., when earnings announcements are given), he or she should be able to bet on this jump using all the information prior to T (which is reflected by the requirement $h \in \mathcal{F}_{T-}$). Again we may define

$$(H.S)_t = (h, \Delta S_T \chi_{\{T\le t\}}) \qquad \text{and} \qquad (H.S)_\infty = (h, \Delta S_T),$$

and throughout this section an easy integrand will refer to a linear combination of processes H of the two kinds considered above.

Note that the definition of easy integrands has been chosen such that the easy integrals do not involve any limiting procedure and therefore are well defined for any process $(S_t)_{t\in\mathbb{R}_+}$.

We shall adopt in this section the following notation: K_0 will denote the subspace of $L^0(\Omega, \mathcal{F}, P)$ spanned by $(H.S)_\infty$, where H runs through the easy integrands as defined above. Again we denote by C_0 the cone $K_0 - L^0_+(\Omega, \mathcal{F}, P)$ and by C the convex cone $C_0 \cap L^\infty$. Similarly as in definition 1.2 we say that the continuous time process $(S_t)_{t\in\mathbb{R}_+}$ satisfies (NFL) (resp. (NFLBR) or (NA)) if $\overline{C} \cap L^\infty_+ = \{0\}$ (resp. $\widetilde{C} \cap L^\infty_+ = \{0\}$ or $C \cap L^\infty_+ = \{0\}$).

With this notation we may formulate a version of Delbaen's theorem appropriate for our setting:

5.1 Delbaen's Theorem. *Let $(S_t)_{t\in\mathbb{R}_+}$ be a process with continuous paths. If $(S_t)_{t\in\mathbb{R}_+}$ satisfies (NFLBR) then there is a measure Q on $\mathcal{F}$ equivalent to P such that $(S_t)_{t\in\mathbb{R}_+}$ is a local martingale with respect to Q. If, in addition, each S_t is bounded, then Q is in fact a martingale measure for $(S_t)_{t\in\mathbb{R}_+}$.*

5.2 Remark. Contrary to Delbaen's (92) original formulation of his theorem we can only assure the existence of an equivalent *local* martingale measure as we dropped the assumption that $(S_t)_{t\in\mathbb{R}_+}$ is bounded. We refer to Delbaen-Schachermayer (92) for examples showing that in this general setting one only obtains an equivalent *local* martingale measure Q.

Also note that for the above case of processes with continuous paths there is no need to consider easy integrands of the form $H = h\chi_{[\![T]\!]}$.

Proof of proposition 5.1. Let $T_0 \equiv 0$ and T_n be the first moment where $\|S_t(\omega)\|$ equals for the first time n if this occurs before $t = n$ and $T_n = n$ otherwise.

Define the process $(X_t)_{t \in \mathbb{R}_+}$ inductively on the stochastic intervals $[\![T_{n-1}, T_n]\!]$: for $t \in [\![T_{n-1}, T_n]\!]$ let

$$X_t - X_{T_{n-1}} = 2^{-n}(S_t - S_{T_{n-1}}).$$

Clearly $(X_t)_{t \in \mathbb{R}_+}$ is a well defined, adapted, continuous and uniformly bounded process such that $\lim_{t\to\infty} X_t$ converges uniformly to a random variable X_∞. One easily verifies that one may apply Delbaen's theorem to the process $(X_t)_{t\in[0,\infty]}$ to conclude that there is an equivalent martingale measure Q for the process $(X_t)_{t\in[0,\infty]}$.

Hence, for each $n \in \mathbb{N}$, the process $(S_t^{T_n})_{t \in \mathbb{R}_+}$ is a martingale under Q which readily implies that $(S_t)_{t \in \mathbb{R}_+}$ is a local martingale under Q.

For the final assertion note that it follows from easy no-arbitrage arguments (compare prop.3.8) that $(\|S_t\|_\infty)_{t \in \mathbb{R}_+}$ is increasing. Hence, for each $t_0 \in \mathbb{R}_+$ the process $(S_t)_{t\in[0,t_0]}$ is a uniformly bounded local martingale and therefore a martingale under Q, which implies the assertion.

q.e.d.

Proposition 5.1 shows in particular that a process $(S_t)_{t \in \mathbb{R}_+}$ with continuous paths and satisfying (NFLBR) is a semimartingale, and we therefore may apply the general stochastic integration theory available for semimartingales.

5.3 Definition. *Given an R^d-valued semimartingale $(S_t)_{t \in \mathbb{R}_+}$ we call a predictable $\mathbb{R}^d$-valued process $(H_t)_{t \in \mathbb{R}_+}$ an <u>admissible integrand</u> if H is S-integrable (see Protter(90), p.134 for a definition) and there is $M \in \mathbb{R}_+$ such that*

$$(H.S)_t \geq -M \qquad \text{for } t \in \mathbb{R}_+, \quad P \text{ a.s.}$$

We shall now develop a theory of stochastic integrals for admissible integrands in the continuous time case similarly as we did in the previous section for the discrete time case.

5.4 Proposition. *If $(S_t)_{t \in \mathbb{R}_+}$ has continuous paths and satisfies (NFLBR) then for each admissible integrand H the limit*

$$(H.S)_\infty = \lim_{t\to\infty} (H.S)_t$$

exists almost surely and, for each $t \in \mathbb{R}_+$,

$$\operatorname{ess\,inf}(H.S)_\infty \leq \operatorname{ess\,inf}(H.S)_t.$$

Proof. Let Q be an equivalent local martingale measure for $(S_t)_{t \in \mathbb{R}_+}$. The process $(H.S)_t$ is a well defined continuous local martingale with respect to Q (see, e.g. Protter (90) th. IV 22 and 30). As it is uniformly bounded from below by $-M$, it is a supermartingale which is bounded in the norm of $L^1(Q)$ and therefore converges almost surely. The final assertion now follows from the fact that $E_Q((H.S)_\infty|\mathcal{F}_t) \leq (H.S)_t$.

q.e.d.

5.5 Definition. *Given a continuous process $(S_t)_{t\in\mathbb{R}_+}$ satisfying (NFLBR) denote by K_0^{adm} the convex cone in $L^0(\Omega,\mathcal{F},P)$ spanned by the functions $(H.S)_\infty$ where H runs through the admissible integrands. Again denote by C_0^{adm} the convex cone $K_0^{adm} - L^0_+(\Omega,\mathcal{F},P)$ and by C^{adm} its intersection with L^∞.*

We come to the crucial proposition which will give the link of Delbaen's theorem to the setting of theorem B.

5.6 Proposition. *If $(S_t)_{t\in\mathbb{R}_+}$ has continuous paths and satisfies (NFLBR), then C^{adm} is weak star closed in L^∞ and $C^{adm}\cap L^\infty_+ = \{0\}$.*

Proof. Again let Q be an equivalent local martingale measure for $(S_t)_{t\in\mathbb{R}_+}$. By the Krein–Smulian theorem it suffices to show that $C^{adm}\cap ball(L^\infty)$ is weak star closed.

As in 4.6 above let $(f_n)_{n=1}^\infty$ be a sequence in $C^{adm}\cap ball(L^\infty)$ converging weak star to $f_0\in ball(L^\infty)$. Find a sequence $g_n = (H_n.S)_\infty$ in K_0^{adm} such that $g_n \geq f_n$.

By lemma 5.4 we know that, for $n\in\mathbb{N}$ and $t\in\mathbb{R}_+$, we have $(H_n.S)_t \geq -1$ almost surely; by stopping at the first moment when $(H_n.S)_t = 1$ we also may assume that $(H_n.S)_t \leq 1$ and therefore $g_n\in ball(L^\infty)$. By passing to convex combinations we may assume that $(g_n)_{n=1}^\infty$ converges almost surely — and therefore with respect to the norm of $L^2(Q)$ — to some $g_0\in ball(L^\infty)$.

This implies that the sequence of admissible integrands $(H_n)_{n=1}^\infty$ is a Cauchy sequence with respect to the $\mathcal{H}_2$-norm

$$\|H\|_2 = E_Q\Big(\int_0^\infty d[H.S,H.S]\Big).$$

As the space formed by the predictable processes H such that $\|H\|_2 < \infty$ is complete (i.e., a Hilbert space), there is a predictable S-integrable process H_0 with $\lim\|H_0 - H_n\|_2 = 0$. Clearly H_0 is an admissible integrand and $(H_0.S)_\infty = g_0 \geq f_0$, which readily shows that $f_0\in C^{adm}$.

The final assertion $C^{adm}\cap L^\infty_+ = \{0\}$ quickly follows from proposition 5.4: if H is an admissible integrand with $(H.S)_\infty \geq 0$ then $(H.S)_t \geq 0$ for all t. Hence $(H.S)_t$ is a nonnegative local Q-martingale with $(H.S)_0 \equiv 0$ and therefore identically equal to zero.

q.e.d.

We now turn to the setting of theorem B: let $(S_t)_{t\in\mathbb{R}_+}$ be an adapted cadlag process such that each S_t is bounded and such that there is an increasing sequence $(T_n)_{n=1}^\infty$ of predictable stopping times tending to $+\infty$ such that the jumps of $(S_t)_{t\in\mathbb{R}_+}$ are contained in $\bigcup_{n=1}^\infty [\![T_n]\!]$. Let $T_0\equiv 0$. By passing to $T_n\wedge n$ we may suppose that $(T_n)_{n=1}^\infty$ is finite almost surely.

We may decompose $(S_t)_{t\in\mathbb{R}_+}$ into its discontinuous part

$$S_t^d = \sum_{n=1}^\infty \Delta S_{T_n}\chi_{([\![T_n,\infty[\setminus[\![T_{n-1}]\!])}(t)$$

and its continuous part

$$S_t^c = S_t - S_t^d \quad .$$

From now on we shall assume that $(S_t)_{t\in\mathbb{R}_+}$ satisfies (NFLBR) with respect to the easy integrands introduced in the beginning of this section. Glancing at the definition of these integrands it is clear that $(S_t^d)_{t\le T_n}$ and $(S_t^c)_{t\le T_n}$ satisfy (NFLBR) too. As (S_t^d) is, of course, a semimartingale it follows again from 5.1 that $(S_t)_{t\in\mathbb{R}_+}$ is a semimartingale, so definition 5.3 of admissible integrands does make sense for $(S_t)_{t\in\mathbb{R}_+}$.

We now shall develop similar results for the processes $(S_t)_{t\in\mathbb{R}_+}$ satisfying the assumptions of theorem B as those which we have obtained for the case of continuous processes $(S_t)_{t\in\mathbb{R}_+}$ in the first part of this section.

5.7 Proposition. *Assume that the process $(S_t)_{t\in\mathbb{R}_+}$ satisfies the assumptions of theorem B. Let H be an admissible integrand. Then*

$$(H.S)_\infty(\omega) \quad = \quad \lim_{t\to\infty}(H.S)_t(\omega)$$

exists almost surely. In addition, for each $t \in \mathbb{N}_0$

$$\operatorname{ess\,inf}(H.S)_\infty \quad \le \quad \operatorname{ess\,inf}(H.S)_t.$$

Proof. We mimic the proof of proposition 4.2. Fix again $M \in \mathbb{R}_+$ such that $(H.S)_t \ge -M$ almost surely for each $t \in \mathbb{R}_+$. If $((H.S)_t)_{t\in\mathbb{R}_+}$ does not converge almost surely there are $A \in \mathcal{F}, \alpha > 0$ with $P(A) = \alpha > 0$ and real numbers $\beta < \gamma$ such that for $\omega \in A$

$$\liminf(H.S)_t(\omega) < \beta \qquad \text{while} \qquad \limsup(H.S)_t(\omega) > \gamma.$$

Noting that $\mathcal{F}$ is generated by $(\mathcal{F}_{T_k})_{k\in\mathbb{N}}$, we may find, for $\epsilon > 0$, some $k_0 \in \mathbb{N}$ and $A_0 \in \mathcal{F}_{T_{k_0}}$ such that for the symmetric difference $A \vartriangle A_0$ we have $P(A \vartriangle A_0) < \epsilon$. Define the stopping times U and V by

$$U(\omega) = \inf\{t \ge T_{k_0} : (H.S)_t(\omega) < \beta\}$$

and

$$V(\omega) = \inf\{t \ge U(\omega) : (H.S)_t(\omega) > \gamma\},$$

where $U(\omega)$ and $V(\omega)$ equal $+\infty$ if the respective sets above are empty.

Since $V(\omega) < \infty$ for $\omega \in A$ we may find $k_1 \in \mathbb{N}$ such that for $A_1 = A_0 \cap \{V \le T_{k_1}\}$ we have

$$P(A_1) > \alpha - 2\epsilon.$$

Note that the random variable

$$f(\omega) = \chi_{A_0}(\omega).((H.S)_{V\wedge T_{k_1}}(\omega) - (H.S)_{U\wedge T_{k_1}}(\omega))$$

is of the form $f(\omega) = (L.S)_\infty$, where L is an admissible integrand supported by the stochastic intervall $]\!]T_{k_0}, T_{k_1}]\!]$. The function f is bounded from below by $-(M + \beta)$, vanishes outside of A_0, and is bigger than $\beta - \gamma$ on A_1.

We shall approximate L by an easy integrand L^{easy} such that $(L^{\text{easy}}.S)_\infty$ is bounded from below by $-(M+\beta+\epsilon)$ and bigger than $\beta-\gamma-\epsilon$ on a subset of A_1 of measure bigger than $\alpha-3\epsilon$.

If we have done this we may finish the proof as in 4.2 above: we repeat the above construction for a sequence $(\epsilon_k)_{k=1}^\infty$ tending to zero which will give us a contradiction to (NFLBR).

To construct L^{easy} find first a constant $D \geq M+\beta$ such that, if we stop at the first moment W when either $(|S_t|)_{t\in\mathbb{R}_+}$ or the process $((L.S)_t)_{t\leq T_{k_1}}$ is bigger than or equal to D, we have that $\{T_{k_1}\wedge W = T_{k_1}\}$ is a set of measure bigger than $1-\epsilon/2$.

For each $k_0 \leq j < k_1$ consider the stochastic interval $]\!]T_j\wedge W, T_{j+1}\wedge W[\![$. The process

$$(S_t - S_{T_j\wedge W})_{T_j\wedge W\leq t<T_{j+1}\wedge W}$$

is continuous, bounded, and satisfies (NFLBR), hence it is a martingale under some equivalent measure Q_j.

The process $(L.(S_t - S_{T_j\wedge t}))_{T_j\wedge W\leq t<T_{j+1}\wedge W}$ is uniformly bounded and therefore in $L^2(Q_j)$. Hence it may be approximated by easy integrands L^j supported by $]\!]T_j\wedge W, T_{j+1}\wedge W[\![$ with respect to the norm of $L^2(Q_j)$. In particular we may find, for $k_0\leq j<k_1$, easy integrands L^j supported by $]\!]T_j\wedge W, T_{j+1}\wedge W[\![$ verifying the estimate

$$P\{\sup|(L-L^j, S_t - S_{T_j\wedge t})|_{T_j\wedge W\leq t<T_{j+1}\wedge W} > \epsilon/2k_1\} < \epsilon/2k_1.$$

Define L^{easy} to equal L^j on $]\!]T_j\wedge W, T_{j+1}\wedge W[\![$ and L on $[\![T_{j+1}]\!]$, for each $k_0\leq j<k_1$, and zero elsewhere. Then L^{easy} is an easy integrand and

$$P\{\sup|(L-L^{\text{easy}}, S_t)|_{0\leq t\leq T_{k_1}\wedge W} > \epsilon/2\} < \epsilon/2.$$

Defining the stopping time W_1 to be the first moment before W when $|(L-L^{\text{easy}}, S)_t|$ is at least ϵ (and therefore equal to ϵ as L_t and L_t^{easy} agree where $(S_t)_{t\leq W}$ has jumps) and defining L^{easy} to equal zero after $W\wedge W_1$ we have completed our construction of the desired easy integrand.

To show the last assertion we first show that, for each stopping time T with $T\leq T_{k_0}$ for some $k_0\in\mathbb{N}$, we have

$$\operatorname{ess\,inf}(H.S)_\infty \quad \leq \quad \operatorname{ess\,inf}(H.S)_T.$$

Indeed, if there is $\epsilon>0$ such that, for $A=\{(H.S)_T < \operatorname{ess\,inf}(H.S)_\infty - \epsilon\}$ we have $P(A)>0$, then the random variables

$$f_k = \chi_A.((H.S)_{T_k} - (H.S)_T)$$

for $k\geq k_0$ give rise to a contradiction to (NFLBR) if we approximate the admissible integrand $H(\omega,t)\chi_{A\cap\{(H.S)_T\leq D\}}(\omega)\chi_{]\!]T,T_k]\!]}$ by easy integrands as above. As $(T_k)_{k=1}^\infty$ tends to infinity, this readily implies the last assertion of the proposition.

q.e.d.

Similarly as in proposition 4.3, the above proof may be extended to a more general situation:

5.8 Proposition. *Under the assumptions of proposition 5.7 let $(H_n)_{n=1}^\infty$ be a sequence of admissible integrands such that the sequence $(g_n)_{n=1}^\infty = ((H_n.S)_\infty)_{n=1}^\infty$ Fatou-converges to some $g_0\in L^0(\Omega,\mathcal{F},P)$*

and, for each $k \in \mathbb{N}$, the sequence $(g_{n,T_k})_{n=1}^\infty = ((H_n.S)_{T_k})_{n=1}^\infty$ converges to some $g_{0,T_k} \in L^0(\Omega, \mathcal{F}_{T_k}, P)$. Then we still have

$$g_0(\omega) \leq \liminf_{k\to\infty} g_{0,T_k}(\omega)$$

almost surely.

Proof. Indeed, note first that there is $M \in \mathbb{R}_+$ such that $(H_n.S)_\infty \geq -M$ and therefore $(H_n.S)_{T_k} \geq -M$ for all k and n. If the above inequality were false, again we could find $A \in \mathcal{F}$ with $P(A) = \alpha > 0$ and $\beta < \gamma$ such that

$$g_0(\omega) > \gamma \qquad \text{while} \qquad \liminf_{k\to\infty} g_{0,T_k}(\omega) < \beta$$

for $\omega \in A$.

Hence for each $k_0 \in \mathbb{N}$ we have

$$g_n(\omega) > \gamma \qquad \text{while} \qquad \inf_{k \geq k_0} g_{n,T_k}(\omega) < \beta$$

for almost each $\omega \in A$, provided that n is sufficiently big (*i.e.*, for $n > n_0$, where n_0 depends on $\omega \in A$ and $k \in \mathbb{N}$).

Hence we may combine the arguments of the proof of proposition 5.7 with the arguments of the proof of proposition 4.3 to obtain a contradiction to (NFLBR).

q.e.d.

We now can proceed in an analogous way as we did in section 4 for the case of discrete time.

5.9 Definition. *Let $(S_t)_{t\in\mathbb{R}_+}$ satisfy the assumptions of theorem B. Similarly as in 4.4 and 5.5, denote by K_0^{adm} the convex cone in $L^0(\Omega, \mathcal{F}, P)$ spanned by the random variables $(H.S)_\infty$, where H runs through the admissible integrands. Denote by C_0^{adm} the convex cone $K_0^{adm} - L_+^0(\Omega, \mathcal{F}, P)$ and by C^{adm} its intersection with L^∞.*

5.10 Proposition. *If $(S_t)_{t\in\mathbb{R}_+}$ satisfies the assumptions of theorem B, then C^{adm} is weak star closed in L^∞ and $C^{adm} \cap L_+^\infty = \{0\}$.*

Proof. As in the proof of proposition 4.6 it suffices to show for the first assertion the following: Let $(f_n)_{n=1}^\infty$ be a sequence in $C^{adm} \cap ball(L^\infty)$ converging almost surely to f_0. Then $f_0 \in C^{adm}$.

Find admissible integrands $(H_n)_{n=1}^\infty$ such that $g_n = (H_n.S)_\infty$ satisfies $g_n \geq f_n$. We may assume that H_n equals zero after the first moment when $(H_n.S)_t \geq 1$. This implies that, for each $k \in \mathbb{N}$, the continuous process $((H_n.S)_t - (H_n.S)_{T_{k-1}})_{T_{k-1} \leq t < T_k}$ is bounded by 2.

By passing to convex combinations of the sequence $(g_n)_{n=1}^\infty$ in a similar way as in 4.6 above we may assume that $(g_n)_{n=1}^\infty$ converges to some $g_0 \in L^0$ and, for each $k \in \mathbb{N}$, $(g_{n,T_k})_{n=1}^\infty = ((H_n.S)_{T_k})_{n=1}^\infty$ and $(g_{n,T_k^-})_{n=1}^\infty = ((H_n.S)_{T_k^-})_{n=1}^\infty$ converge to some g_{0,T_k} and g_{0,T_k^-} in L^0 respectively.

We shall show that there is an admissible integrand H_0 such that, for each $k \in \mathbb{N}$,

$$g_{0,T_k^-} \leq (H_0.S)_{T_k^-} \qquad \text{and} \qquad g_{0,T_k} \leq (H_0.S)_{T_k}.$$

This again will finish the proof as by proposition 5.8

$$g_0(\omega) \leq \liminf_{k\to\infty} g_{0,T_k}(\omega) \leq \lim_{k\to\infty} (H_0.S)_{T_k} = (H_0.S)_\infty.$$

Hence g_0 and, *a fortiori*, f_0 are dominated by an element of K_0^{adm}.

We now shall construct for $k \in \mathbb{N}$ the integrand H_0 on the stochastic intervals $]\!]T_{k-1}, T_k[\![$ and $[\![T_k]\!]$. For the intervals $[\![T_k]\!]$ we may apply — just as in the proof of 4.6 — Stricker's lemma to obtain an integrand $H_0^{d,k}$ supported by $[\![T_k]\!]$ such that

$$(H_0^{d,k}.S)_\infty = (H_0^{d,k}.S)_{T_k} - (H_0^{d,k}.S)_{T_k^-} = g_{0,T_k} - g_{0,T_k^-}.$$

For the intervals $]\!]T_{k-1}, T_k[\![$ we may apply prop. 5.6 to find an admissible integrand $H_0^{c,k}$ supported by $]\!]T_{k-1}, T_k[\![$ such that

$$(H_0^{c,k}.S)_\infty = (H_0^{c,k}.S)_{T_k^-} - (H_0^{c,k}.S)_{T_{k-1}} \geq g_{0,T_k^-} - g_{0,T_{k-1}}.$$

Defining H_0 to equal $H_0^{c,k}$ and $H_0^{d,k}$ on the intervals $]\!]T_{k-1}, T_k[\![$ and $[\![T_k]\!] \setminus [\![T_{k-1}]\!]$ respectively, we obtain the desired admissible integrand H_0, thus finishing the proof.

q.e.d.

5.11 Proof of theorem B. If $(S_t)_{t\in\mathbb{R}_+}$ satisfies the assumptions of theorem B, then by proposition 5.9 C^{adm} is a weak star closed convex cone in L^∞ that satisfies $C^{adm} \cap L_+^\infty = \{0\}$. A glance at the abstract version of the Kreps–Yan theorem (3.2 above) reveals the existence of an element $g \in L^1(\Omega, \mathcal{F}, P)$, $g > 0$ a.s., such that for the measure Q on $\mathcal{F}$ with Radon–Nikodym derivative $dQ/dP = g$ and $f \in C^{adm}$ we have

$$E_Q(f) \leq 0.$$

Note that by our boundedness assumption on $(S_t)_{t\in\mathbb{R}_+}$ we have that, for each $s < t$ and each R^d-valued bounded $\mathcal{F}_s$-measurable function h, the integrand $h\chi_{]s,t]}$ is admissible and therefore

$$E_Q(h, S_t - S_s) \leq 0.$$

By passing to $-h$ we conclude that equality holds above which readily implies that $(S_t)_{t\in\mathbb{R}_+}$ is a martingale under Q thus finishing the proof of theorem B.

q.e.d.

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